The Abstract Index Notation

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1. Introduction

We write down precise meanings for the expressions arising in the abstract index notation developed by Penrose [3] for manipulating tensors and tensor fields. This document is not useful for performing computations—only for figuring out what the notation means.

Tensor fields on a manifold $\mathcal{M}$ are often described in terms of coordinate vector fields e_α and their duals E^α (where $\alpha = 1, \ldots, \dim \mathcal{M}$), and sums of tensor products of these. A moderately complicated tensor might be $S = \sum_{\alpha,\beta,\gamma=1}^{n} S^{\alpha\beta}{}_\gamma e_\alpha \otimes e_\beta \otimes E^\gamma$, where the $S^{\alpha\beta}{}_\gamma$'s are real-valued functions. Differential geometers usually suppress both the summation symbol and the expression $e_\alpha \otimes e_\beta \otimes E^\gamma$, writing only $S^{\alpha\beta}{}_\gamma$ and calling it a tensor. There are operations on such expressions. One is called contraction and yields symbols like $S^{\alpha\beta}{}_\alpha$ with repeated indices. Another is the operation of raising and lowering indices in the presence of a riemannian metric, which yields expressions like $S_\alpha{}^\beta{}_\gamma$ and $S^{\alpha\beta\gamma}$ and $S_{\alpha\beta\gamma}$. Although these operations have coordinate-independent interpretations, the notation requires a coordinate chart to even have meaning. Certain equations are called "tensor equations", a term which does not seem to be defined precisely anywhere. Other equations are "not tensorial" and it is not clear what meaning these equations have, or exactly how they are less meaningful than tensorial equations.

Penrose's notation is a dramatic improvement. Letting $\mathfrak{S}^\diamond$ (resp $\mathfrak{S}_\diamond$) denotes the space of vector fields (resp. one-forms) on $\mathcal{M}$, he introduces infinitely many "copies" of it, with names $\mathfrak{S}^a$, $\mathfrak{S}^b$,... (resp. $\mathfrak{S}_a$, $\mathfrak{S}_b$,...). There are also spaces like $\mathfrak{S}^{ab}$ which are copies of spaces like $\mathfrak{S}^{\diamond\diamond} = \mathfrak{S}^\diamond \otimes \mathfrak{S}^\diamond$. If v is a vector field on $\mathcal{M}$, then it doesn't need any indices to exists, and if we like we may use mathematicians' index-free notation (see [1] for one version of such). But if we want to use indices, we can: by v^a we do not mean the components of v with respect to some coordinate system, but simply a copy of v that lies in $\mathfrak{S}^a$. Similarly, if g is a riemannian metric then g_{cd} is the copy of g in $\mathfrak{S}_{cd}$, and there are other copies of g in other copies of $\mathfrak{S}_{\diamond\diamond}$. We call each string of superscript and subscript letters an "index label". To each index label, there corresponds a module over the ring $C^\infty(\mathcal{M})$ and a rule used to interpret "honest" tensors (i.e., those in $\mathfrak{S}^\diamond$, $\mathfrak{S}_{\diamond\diamond}$, etc.) of appropriate sorts as members of this module. Sometimes the map is not an isomorphism and the module is smaller than might be expected. For example, $\mathfrak{S}_a{}^a$ is just $C^\infty(\mathcal{M})$, not a copy of $\mathfrak{S}_\diamond{}^\diamond$. This allows the notation to describe contraction of indices. For example, if $S \in \mathfrak{S}_\diamond{}^\diamond$ then S may be regarded as a bundle self-map of $T\mathcal{M}$ and $S_a{}^a$ is the trace of this map. However, $S_b{}^c$ is a completely different beast, being the copy of S in $\mathfrak{S}_b{}^c$ (which *is* a copy of $\mathfrak{S}_\diamond{}^\diamond$). Certain expressions like g_{aa} are denied meaning. In this case, the identical lower indices disqualify the expression: there is no module $\mathfrak{S}_{aa}$, and the index label ${}_{aa}$ is "illegal" and cannot be used to interpret any tensors at all.

If v and w are vector fields then $v \otimes w \in \mathfrak{S}^{\diamond\diamond}$, and so should be interpretable by the index label ^{ab}, so we should have $(v \otimes w)^{ab} \in \mathfrak{S}^{ab}$. However, differential geometers simply write $v^a w^b$ for $(v \otimes w)^{ab}$. A good way to handle this is for there to be a bilinear map $\mathfrak{S}^a \times \mathfrak{S}^b \to \mathfrak{S}^{ab}$ which is essentially the tensor product map. This product map is implicitly invoked whenever a symbol representing an element of $\mathfrak{S}^a$ is followed immediately by a symbol representing an element of $\mathfrak{S}^b$. Sometimes this bilinear map is not the natural map into the tensor product of the spaces. For example, the map $\mathfrak{S}_a \times \mathfrak{S}^a \to \mathfrak{S}_a{}^a \cong C^\infty(\mathcal{M})$ is the natural pairing between one-forms and vector fields.

We require several compatibility conditions on these maps. First, we require that they be "associative", so that equations like

$$(S_{ab}T_{cd})U_{efg} = S_{ab}(T_{cd}U_{efg}) \tag{1.1}$$

hold, giving expressions like $S_{ab}T_{cd}U_{efg}$ unambiguous meaning. Second, we require that they be compatible with the interpretation maps, yielding equalities like

$$(v \otimes w)^{ab} = v^a w^b. \tag{1.2}$$

The last ingredient of a formalism is the scope of its interpretation rules. Each index label can interpret certain sorts of tensors but not others. The choices made in this regard are subtle and important. For example, if $S \in \mathfrak{S}_\diamond{}^\diamond$ then $S_a{}^b \in \mathfrak{S}_a{}^b$. But it might also be reasonable to allow ${}^b{}_a$ to interpret S, so that $S^b{}_a \in \mathfrak{S}^b{}_a$; after all, $\mathfrak{S}_\diamond{}^\diamond$ and $\mathfrak{S}^\diamond{}_\diamond$ are canonically isomorphic, by the map $v \otimes w \mapsto w \otimes v$. Use of this convention is essentially the decision to regard $e_\alpha \otimes E^\beta$ as being "the same as" $E^\beta \otimes e_\alpha$. It natural extension is that indices may be reordered so long as the sets of upper and lower indices each keep their own orderings.

However, this convention is incompatible with the standard riemannian geometry convention of raising and lowering indices using the metric tensor. If v is a vector field then we may convert it into a one-form by contracting it with one index of the metric tensor g, obtaining (say) $g_{ab}v^b$. Geometers usually write this as v_a. Thus, the index label ${}_a$ can interpret one-forms (which it does in the usual manner) and also vector fields (which it does using the canonical isomorphism $T\mathcal{M} \to T^*\mathcal{M}$ induced by g). One can also "raise" indices using the inverse metric $g^{-1} \in \mathfrak{S}^{\diamond\diamond}$, so upper indices may interpret one-forms. For example, if $w \in \mathfrak{S}_\diamond$ then $w^a = (g^{-1})^{ab}w_b$. Together the raising and lowering operators induce an isomorphism $\mathfrak{S}^\diamond{}_\diamond \to \mathfrak{S}_\diamond{}^\diamond$ which carries $v \otimes w$ to the tensor product of the 1-form associated to v with the vector field associate to w. This isomorphism $\mathfrak{S}^\diamond \otimes \mathfrak{S}_\diamond \to \mathfrak{S}_\diamond \otimes \mathfrak{S}^\diamond$ is not the same as the one described in the previous paragraph. That is, in riemannian geometry one usually cannot move upper and lower indices past each other. In the absence of a metric, the assertion $T_a{}^b = T^b{}_a$ is tautologically true for $T \in \mathfrak{S}_\diamond{}^\diamond$, whereas in the presence of a metric it is the assertion that $T_a{}^b = T_c{}^d(g^{-1})^{cb}g_{ad}$, which turns out to express the condition that T be self-adjoint with respect to g when regarded as a field of linear transformations of the tangent spaces of $\mathcal{M}$. This example shows that the choice of interpretation schemes greatly affects the meaning of the notation.

Many differential geometry references implicitly or explicitly use the "smooth manifolds" convention, and write $T^{a\ldots b}{}_{c\ldots d}$ or even $T^{a\ldots b}_{c\ldots d}$ for a generic tensor. Oddly, no reference I know of emphasizes the need to stagger indices when using the raising and lowering conventions of metric geometry. (They typically stagger the indices without mentioning why they are doing so.)

The considerations above suggest that an index formalism should have the following components, which we have numbered so that they correspond to the parts of the formal definition in section 4.

1, 2. A commutative ring and some modules over it,

3, 4, 5. A set of index labels (some of which might be "illegal") and a way to concatenate them,

6. A module for each legal index label,

7. Pairings between these modules, satisfying conditions like (1.1).

8, 9. For each legal index label, a list of "shapes" of tensor that it can interpret, and a rule for interpreting such tensors. The interpretation rules should satisfy conditions like (1.2).

In the setting above, the ring was $C^\infty(\mathcal{M})$, the modules were the vector fields and one-forms, and the index labels were strings of superscripts and subscripts with the obvious concatenation

operation. The modules $\mathfrak{S}^{ab}$ etc., pairings $\mathfrak{S}^a \times \mathfrak{S}^b \to \mathfrak{S}^{ab}$ etc., and interpretation maps were not really defined but only hinted at.

We have written everything in considerable generality, to avoid ever having to do this again. One could probably go further and generalize the notation for work with sheaves of commutative rings and modules rather than just rings and modules. One might also be able to introduce some sort of index formalism for bimodules over (possibly noncommutative) rings, which might be useful for work with supermanifolds.

2. Foundations

We define here various concepts that are so commonly used that one might lose track of the formal definitions. We will need certain properties of these definitions, so we state them here.

We define 0 to be the empty set $\emptyset$ and $n+1$ to be $\{0, \ldots, n\}$. The set $\{0, 1, 2, \ldots\}$ is denoted ω_0. If X is a set then $|X|$ denotes its cardinality (or size). Sets of size 1 and 2 are sometimes called singletons and doubletons, respectively. If X and Y are sets then $X \smallsetminus Y$ will denote their set-theoretic difference $\{ x \in X \mid x \notin Y \}$.

In section 3 we will define the product $X \times Y$ of two sets in terms of functions from $\{0, 1\}$ to $X \cup Y$. Therefore we do not define a function from X to Y as a certain kind of subset of $X \times Y$. Instead we use a standard trick and introduce the abbreviation (x, y) for the set $\{\{x\}, \{x, y\}\}$. It is easy to see that $(x, y) = (x', y')$ if and only if $x = x'$ and $y = y'$. We refer to (x, y) as an ordered pair and to x (resp. y) as the abcissa (resp. ordinate) of (x, y).

A function (or map) from X to Y (or one "on X with values in Y") is a set f of ordered pairs (x, y) with $x \in X$ and $y \in Y$ such that for each $x \in X$ there is a unique element of f whose abcissa is x; the ordinate of this element of f is called the value of f at x and denoted $f(x)$. The expression $f : X \to Y$ indicates that f is a function from X to Y. The domain (resp. image) of f is the set of abcissas (resp. ordinates) of its elements. If $X' \subseteq X$ then the restriction $f|_{X'}$ of f to X' is the set $\{ (x, y) \in f \mid x \in X' \}$. If X is a set then we denote id_X the identity map on X, namely, $\{ (x, x) \mid x \in X \}$. If $f : X \to Y$ and $X' \subseteq X$ then we write $f(X')$ for the image of X' under f, namely the set $\{ f(x) \mid x \in X' \}$. This is an abuse of notation because it is possible for X' to be both a subset and an element of X, in which case the notation $f(X')$ is ambiguous. In our applications this will not cause confusion. If Y' is a set then we write $f^{-1}(Y')$ for the set $\{ x \in X \mid f(x) \in Y' \}$ and call it the inverse image (under f) of Y'.

We will sometimes use the fact that a function is a certain kind of set in order to perform the obscure operation of taking the union of two functions. If f and f' are functions with domains X and X', respectively, and $f|_{X \cap X'} = f'|_{X \cap X'}$ then $f \cup f'$ is the function on $X \cup X'$ whose restriction to X (resp. X') is f (resp. f').

A sequence is a function whose domain is either ω_0 or an element of ω_0. Its length is defined to be the cardinality of its domain, and it is called finite or infinite accordingly. We indicate a sequence x with domain n by the notation $(x_i)_n$ or $(x(i))_n$, where x_i or $x(i)$ is the image of $i \in n$; sometimes we refer to x_i as the ith term of the sequence. Note that what is commonly called the first term of a sequence is the image of 0. We sometimes specify a finite sequence $(x_i)_n$ by explictly listing its terms: if $n \neq 0$ then we write $(x_0, \ldots, x_{n-1})$ for $(x_i)_n$. Any sequence with domain 0 is the empty set $\emptyset$; we call this the empty sequence and sometimes denote it $()$.

A sequence of length two expresses the same concept as does an ordered pair. In fact, our notation (x, y) is ambiguous. There is a bijection between these two concepts: to the ordered pair p, we associate the sequence whose value at 0 (resp. 1) is the abcissa (resp. ordinate) of p. One may regard this bijection as being induced by the notation; in light of it we will not worry about the formal differences between these concepts.

Suppose f and g are finite sequences with respective domains m and n. We define the con-

catenation fg of the sequences to be the function $f \cup g'$, where g' if the function whose domain is the set $\{ m+i \mid i \in n \}$ and whose value at $m+i$ is $g(i)$ for each $i \in n$. The empty sequence () is a 2-sided identity element for this operation. Concatenation is associative. For example,

$$(p,d,q)(b,a)(c,h) = (p,d,q,b,a,c,h).$$

3. Products

Suppose Y is a function on a set X whose value $Y(x)$ at each $x \in X$ is a set. The union $\bigcup_{x \in X} Y(x)$ is the set $\{ y \mid y \in Y(x) \text{ for some } x \in X \}$. In particular, if $X = \emptyset$ then the union is also empty. We define the product $\prod_{x \in X} Y(x)$ to be the set of functions f from X to $\cup_{x \in X} Y(x)$ satisfying $f(x) \in Y(x)$ for each $x \in X$. X is called the parameter set for the product and the sets $Y(x)$ are called the factors. A special case of this definition is that if a pair X and X' of disjoint sets are the respective domains for two set-valued functions Y and Y' then

$$\prod_{z \in (X \cup X')} (Y \cup Y')(z) = \left\{ f \cup f' \;\middle|\; f \in \prod_{x \in X} Y(x),\ f' \in \prod_{x' \in X'} Y'(x') \right\}.$$

Here are some extreme cases. If $X = \emptyset$ then $\prod_{x \in X} Y(x)$ contains just one function, $\emptyset$. If (instead) $Y(x) = \emptyset$ for some $x \in X$ then the product *is* the empty set, containing no functions.

If X and Z are sets then Z^X is the set of all functions from X to Z. This is a special case of the construction $\prod_{x \in X} Y(x)$, which arises when $Y(x) = Z$ for all $x \in X$. Certain familiar typographical constructs have their usual meanings: for example, a point of $\mathbb{R}^n$ is a map from $\{0, 1, \ldots, n-1\}$ to $\mathbb{R}$. The extreme cases of the definition of Z^X are similar to those above. If $X = \emptyset$ then $Z^X = \{\emptyset\}$; if $X \neq \emptyset$ but $Z = \emptyset$ then $Z^X = \emptyset$.

If $(Y_i)_n$ is a finite sequence of sets then we sometimes write the product $\prod_{i \in n} Y_i$ as $Y_0 \times \cdots \times Y_{n-1}$. Conversely, the latter notation indicates that we regard the spaces Y_i as implictly parameterized (from left to right) as a sequence. Then the above definition of products of sets applies, so the elements of the product are sequences with values in the appropriate factors. For example,

$$Z \times X \times Y = \{ f : \{0,1,2\} \to Z \cup X \cup Y \mid f(0) \in Z,\ f(1) \in X,\ f(2) \in Y \}.$$

Note that a product $X_0 \times \cdots \times X_{m-1} \times Y_0 \times \cdots \times Y_{n-1}$ consists of the concatenations ff' of sequences $f \in X_0 \times \cdots \times X_{m-1}$ and $f' \in Y_0 \times \cdots \times Y_{n-1}$. We sometimes identify $(X_0 \times \cdots \times X_{m-1}) \times (Y_0 \times \cdots \times Y_{n-1})$ with $X_0 \times \cdots \times X_{m-1} \times Y_0 \times \cdots \times Y_{n-1}$ in this way without comment.

If X, X', Y and Y' are sets and $f : X \to X'$ and $g : Y \to Y'$ then by $f \times g$ we mean the function from $X \times Y$ to $X' \times Y'$ defined by $(f \times g)(x,y) = (f(x), g(y))$. Because the functions f and g are just sets of a particular sort and we have already given a definition of the product of two sets, this is an abuse of notation. In any given context, it will be clear which definition of $f \times g$ we intend.

We restrict our attention to commutatuve rings with identity, and to modules over such rings on which the identity element acts by the identity transformation.

If $\mathcal{R}$ is a ring and X is a set then the free $\mathcal{R}$-module generated by X, denoted $\mathcal{R}[X]$, is defined to be the set of functions from X to $\mathcal{R}$ which vanish except at finitely many points. Addition is pointwise addition of functions and the action of $\mathcal{R}$ is the obvious one: if $f : X \to \mathcal{R}$ lies in $\mathcal{R}[X]$ and $r \in \mathcal{R}$ then $(rf)(x) = r \cdot (f(x))$ for all $x \in X$. The set X embeds in $\mathcal{R}[X]$ in the obvious way, by characteristic functions of singletons. We denote this embedding by e.

Now suppose $\mathcal{R}$ is commutative and that $\mathfrak{X}$ is a function on a finite set X taking values in some set of $\mathcal{R}$-modules. Then the product $P = \prod_{x \in X} \mathfrak{X}(x)$ may be embedded via e (as a set, not

an $\mathcal{R}$-module) in $F = \mathcal{R}[P]$. We define two classes of elements of F. The first consists of those of the form $e(f) + e(f') - e(f'')$ where $f, f', f'' \in P$ satisfy the condition that there be $x \in X$ such that $f(x) + f'(x) = f''(x)$ and that for all $y \in X \setminus \{x\}$ we have $f(y) = f'(y) = f''(y)$. The second class consists of those of the form $re(g) - e(g')$ where $g, g' \in P$ satisfy the condition that there be $x \in X$ such that $rg(x) = g'(x)$ and for all $y \in X \setminus \{x\}$ we have $g(y) = g'(y)$. Let K (for "kernel") be the submodule of F generated by 0 and the elements of these two classes. We define the tensor product $\bigotimes_{x \in X} \mathfrak{X}(x)$ as the quotient module F/K. We denote the natural map $F \to F/K$ by q and the image under $q \circ e$ of $f \in P$ by $\otimes_{x \in X} f(x)$. (When the tensor product symbol appears with a parameterizing index, as here, its smaller version '$\otimes$' usually indicates an element of a tensor product module, which is itself indicated by the larger '$\bigotimes$'.) The set X is called the parameter set for the tensor product and the modules $\mathfrak{X}(x)$ are called the factors. If Z is some fixed $\mathcal{R}$-module then we sometimes denote $\bigotimes_{x \in X} Z$ by $Z^{\otimes X}$.

When $X = n \in \omega_0$ then we sometimes use alternate (and more standard) notation. We write $\bigotimes_{i \in n} \mathfrak{X}_i$ as $\mathfrak{X}_0 \otimes \cdots \otimes \mathfrak{X}_{n-1}$ and an element $\otimes_{i \in n} y_i$ thereof as $y_0 \otimes \cdots \otimes y_{n-1}$. A consequence of these conventions is that if Z is an $\mathcal{R}$-module then $Z^{\otimes n}$ is the $\mathcal{R}$-module $Z \otimes \cdots \otimes Z$, where there are n factors.

The slight difference between our notion of a tensor product and the usual one is that there is no "ordering of the factors" in our $\bigotimes_{x \in X} \mathfrak{X}(x)$ unless X is an ordered set. In the usual case, $X = n \in \omega_0$ and so the usual definition applies. From our perspective, the ordering is an extra structure on X which is sometimes useful and sometimes annoying. For general sets X and Y there is no natural definition of a tensor product of two tensor products $\bigotimes_{x \in X} \mathfrak{X}(x)$ and $\bigotimes_{y \in Y} \mathfrak{Y}(y)$. However, if $X \cap Y = \emptyset$ then we may construct a tensor product simply as $\bigotimes_{z \in X \cup Y} (\mathfrak{X} \cup \mathfrak{Y})(z)$. There is a natural bilinear map $\bigotimes_{x \in X} \mathfrak{X}(x) \otimes \bigotimes_{y \in Y} \mathfrak{Y}(y) \to \bigotimes_{z \in X \cup Y} (\mathfrak{X} \cup \mathfrak{Y})(z)$ that is characterized by its bilinearity and the condition that if $f = \otimes_{x \in X} f(x) \in \bigotimes_{x \in X} \mathfrak{X}(x)$ and $g = \otimes_{y \in Y} g(y) \in \bigotimes_{y \in Y} \mathfrak{Y}(y)$ then the image of (f, g) is $\otimes_{z \in X \cup Y} (f \cup g)(z)$. In this definition, there is no issue about which of the two factors "comes first"—the elements of X and Y are treated completely symmetrically.

On the other hand, the orderings on $m, n \in \omega_0$ are useful because one can define a reasonable tensor product of tensor products parameterized by m and n, even though $m \cap n \neq \emptyset$, by using concatenations of sequences. That is, the natural isomorphism of $(X_0 \times \cdots \times X_{m-1}) \times (Y_0 \times \cdots \times Y_{n-1})$ with $X_0 \times \cdots \times X_{m-1} \times Y_0 \times \cdots \times Y_{n-1}$ induces a natural isomorphism of $(X_0 \otimes \cdots \otimes X_{m-1}) \otimes (Y_0 \otimes \cdots \otimes Y_{n-1})$ with $X_0 \otimes \cdots \otimes X_{m-1} \otimes Y_0 \otimes \cdots \otimes Y_{n-1}$. We will sometimes regard without comment the latter two spaces as identified with each other via this isomorphism.

Here are some extreme cases of the tensor product. If $X = \emptyset$ then $P = \{\emptyset\}$, so $F = \mathcal{R}^{\{\emptyset\}}$. Since $K \cong \{0\}$ we have for each module-valued function $\mathfrak{X}$ on $\emptyset$, the $\mathcal{R}$-module isomorphism $\bigotimes_{x \in \emptyset} \mathfrak{X}(x) \cong \mathcal{R}$. Similarly, $Z^{\otimes \emptyset} \cong \mathcal{R}$ as an $\mathcal{R}$-module for any $\mathcal{R}$-module Z. Since $\emptyset \cap \emptyset = \emptyset \cup \emptyset = \emptyset$, the tensor product definition above defines a ring structure on $\bigotimes_{x \in \emptyset} \mathfrak{X}(x)$, which is naturally isomorphic to $\mathcal{R}$. One can wade through the formalism and check that the image under $q \circ e$ of the unique element $\emptyset$ of P is the identity element of this ring, in compliance with the usual convention that an empty product of ring elements is the identity. On the other hand, if for some $x \in X$ we have $\mathfrak{X}(x) \cong \{0\}$, then $\bigotimes_{x \in X} \mathfrak{X}(x) \cong \{0\}$.

4. Index formalisms

An index formalism is a 9-tuple $(\mathcal{R}, \mathcal{X}, \mathcal{I}, \cdot, \mathcal{L}, \mathfrak{S}, \mathcal{G}, \mathcal{D}, \mathcal{F})$ of objects of the following sorts. In practice (see later sections), all of these symbols except for $\mathfrak{S}$ will be suppressed. The reader should compare the steps below with the explanation of their roles given in the introduction.

1. $\mathcal{R}$ is a commutative ring.
2. $\mathcal{X}$ is a set of $\mathcal{R}$-modules.

Before continuing, we make some definitions. A shape is a finite sequence with values in $\mathfrak{X}$. Shapes are sequences and so can be concatenated; as usual we write st for the concatenation of shapes s and t. A tensor of shape $(\mathfrak{X}_i)_n$ is an element of $\mathfrak{X}_0 \otimes \cdots \otimes \mathfrak{X}_{n-1}$. The tensors of any given shape s form an $\mathcal{R}$-module, denoted $\mathfrak{T}(s)$; this is called the tensor space of s. Suppose s and t are shapes of respective lengths m and n. The spaces $\mathfrak{T}(s)$ and $\mathfrak{T}(t)$ are tensor products parameterized by the sets m and n, respectively. We denote the natural bilinear map $\mathfrak{T}(s) \times \mathfrak{T}(t)$ described in section 3 by the symbol $\mathcal{B}_{s,t}$.

We now continue the definition of an index formalism.

3. $\mathcal{I}$ is a set, called the set of index labels.

4. $\cdot$ is an associative product on $\mathcal{I}$, called concatenation.

5. $\mathcal{L}$ is a subset of $\mathcal{I}$, called the set of legal index labels.

6. $\mathfrak{S}$ is a function on $\mathcal{L}$ taking values in some set of $\mathcal{R}$-modules. $\mathfrak{S}(L)$ for $L \in \mathcal{L}$ is called the tensor space of L.

7. $\mathcal{G}$ is a function on $\left\{ (L, M) \in \mathcal{L}^2 \;\middle|\; L \cdot M \in \mathcal{L} \right\}$, whose value at (L, M), denoted $\mathcal{G}_{L,M}$, is an $\mathcal{R}$-bilinear function $\mathcal{G}_{L,M} : \mathfrak{S}(L) \times \mathfrak{S}(M) \to \mathfrak{S}(L \cdot M)$ such that if L, M, N, $L \cdot M$, $M \cdot N$ and $L \cdot M \cdot N$ all lie in $\mathcal{L}$ then

$$\mathcal{G}_{L\cdot M,N} \circ \left(\mathcal{G}_{L,M} \times \mathrm{id}_{\mathfrak{S}(N)}\right) = \mathcal{G}_{L,M\cdot N} \circ \left(\mathrm{id}_{\mathfrak{S}(L)} \times \mathcal{G}_{M,N}\right) \tag{4.1}$$

as functions on $\mathfrak{S}(L) \times \mathfrak{S}(M) \times \mathfrak{S}(N)$. The informal meaning of this is that (1.1) holds.

8. $\mathcal{D}$ is a function from $\mathcal{L}$ to sets of shapes such that if L, M and $L \cdot M$ are legal index labels then $s \in \mathcal{D}(L)$ and $t \in \mathcal{D}(M)$ imply $st \in \mathcal{D}(L \cdot M)$. ($\mathcal{D}$ suggests the word "domain"; $\mathcal{D}(L)$ describes the domain of $\mathcal{F}_L$, whose properties are given below.)

9. $\mathcal{F}$ is a function on $\mathcal{L}$ whose value at $L \in \mathcal{L}$, denoted $\mathcal{F}_L$, is a function with domain $\cup_{s\in\mathcal{D}(L)} \mathfrak{T}(s)$ whose restriction to $\mathfrak{T}(s)$ is $\mathcal{R}$-linear for each $s \in \mathcal{D}(L)$ and which satisfies the condition

$$\mathcal{G}_{L,M} \circ (\mathcal{F}_L \times \mathcal{F}_M) = \mathcal{F}_{L\cdot M} \circ \mathcal{B}_{s,t} \tag{4.2}$$

for all $L, M \in \mathcal{L}$ with $L \cdot M \in \mathcal{L}$ and all $s \in \mathcal{D}(L)$ and $t \in \mathcal{D}(M)$. This is just is a fancy way of asserting (1.2). Note that the composition on the right hand side is defined, because $st \in \mathcal{D}(L \cdot M)$ by condition 8 above. If $S \in \mathfrak{T}(s)$ with $s \in \mathcal{D}(L)$ then we refer to $\mathcal{F}_L(S)$ as the interpretation of S by L.

Here are some comments on the definition.

All of our index formalisms arewill be "commutative" in the sense that (i) if $L, M \in \mathcal{L}$ then $L \cdot M$ is legal if and only if $M \cdot L$ is, and (ii) in this case we have $\mathcal{G}_{L,M}(S, T) = \mathcal{G}_{M,L}(T, S)$ for all $S \in \mathfrak{S}(L)$ and $T \in \mathfrak{S}(M)$. Informally, this is the condition that equations like $v^a w^{bc} = w^{bc} v^a$ hold.

If a legal index label L happened to also be a shape s, then the phrase "the tensor space of L" would be ambiguous, meaning one of $\mathfrak{S}(L)$ or $\mathfrak{T}(s)$. This possible ambiguity is the reason we use distinct symbols $\mathfrak{S}$ and $\mathfrak{T}$. In practice this will never be an issue. In all of our formalisms the only index label which is also a shape will be the empty set, and $\mathfrak{S}(\emptyset)$ and $\mathfrak{T}(\emptyset)$ will coincide. In light of this, section 5 introduces a convention for writing $\mathfrak{S}(s)$ for $\mathfrak{T}(s)$ for a shape s; this reduces notational clutter.

It is conceivable that in some index formalism we could have $\mathfrak{T}(s) \cap \mathfrak{T}(t) \neq \emptyset$ for some pair of distinct shapes s and t. If this holds for some $s, t \in \mathcal{D}(L)$ for some $L \in \mathcal{L}$ then the fact that $\mathcal{F}_L$ is a function on $\cup_{u\in\mathcal{D}(L)} \mathfrak{T}(u)$ and not defined "separately on $\mathfrak{T}(s)$ and $\mathfrak{T}(t)$" means that $\mathcal{F}_L(S)$ is unambiguously defined on $\mathfrak{T}(s) \cap \mathfrak{T}(t)$.

A concept that is useful for some formalisms but not essential in what follows is that of the type (or valence) of a tensor. If s is a shape then its type is the function on $\mathfrak{X}$ whose value at $\mathfrak{X}$ is

$|s^{-1}(\{\mathfrak{X}\})|$. If a shape s has type k then each element of $\mathfrak{T}(s)$ is said to have type k or be of type k. For smooth manifolds, this is a familiar concept and the type is usually written $\binom{k^+}{k^-}$ where $k^\pm$ is the value of the function at $\mathfrak{X}^\pm$. In this context the shape ${}_\diamond{}^\diamond{}_{\diamond\diamond}$ has type $\binom{1}{3}$.

5. Conventions for index formalisms

In this section and the next two we will explain a general procedure for constructing index formalisms, illustrated throughout with examples. We will simultaneously treat the 3 main examples, those of a smooth manifold with no metric, of a riemannian manifold, and of a 4-manifold with a spinor structure. The "input" into the conventions is fairly straightforward, but a bit more involved than would ideally be the case. To simplify matters when constructing formalisms on the fly (e.g., "consider a compact hausdorff space with 10 different vector bundles over it, etc. etc."), in sections 8 and 10 we describe further conventions for applying the conventions here more easily.

STEP 1: We suppose given a commutative ring $\mathcal{R}$ and a set $\mathcal{X}$ of $\mathcal{R}$-modules. *Smooth or riemannian case*: Let $\mathcal{M}$ be a smooth manifold and $\mathcal{R} = C^\infty(\mathcal{M})$. Let $\mathfrak{X}^+$ (resp. $\mathfrak{X}^-$) be the space of vector fields (resp. one-forms) on $\mathcal{M}$, and let $\mathcal{X} = \{\mathfrak{X}^+, \mathfrak{X}^-\}$. If $\mathcal{M}$ is Riemann then let $g \in \mathfrak{X}^- \otimes \mathfrak{X}^-$ be its metric tensor. *Spinor case*: Let $\mathcal{M}$ be as above, let $\mathcal{R} = C^\infty(\mathcal{M}, \mathbb{C})$, and let $\mathfrak{X}^+$ (resp. $\mathfrak{X}^-$) be the space of sections of the complexified tangent (resp. contangent) bundle $\mathbb{C}T\mathcal{M}$ (resp. $\mathbb{C}T^*\mathcal{M}$). Let S^+ be a 2-dimensional complex vector bundle over $\mathcal{M}$ with dual bundle S^- and let the complex conjugates of these bundles be denoted $\bar{S}^+$ and $\bar{S}^-$, respectively. Let $\mathfrak{Y}^+$ (resp. $\mathfrak{Y}^-$, $\bar{\mathfrak{Y}}^+$, $\bar{\mathfrak{Y}}^-$) be the space of smooth sections of S^+ (resp. S^-, $\bar{S}^+$, $\bar{S}^-$). Define $\mathcal{X} = \{\mathfrak{X}^\pm, \mathfrak{Y}^\pm, \bar{\mathfrak{Y}}^\pm\}$. A spinor structure on $\mathcal{M}$ is a choice of a bundle S^+ as above and a bundle isomorphism $S^+ \otimes \bar{S}^+ \to T\mathcal{M}$ that respects complex conjugation. (Complex conjuation on $T\mathcal{M}$ is the obvious operation, while on $S^+ \otimes \bar{S}^+$ it exchanges the tensor factors.)

STEP 2: We suppose given, for each $\mathfrak{X} \in \mathcal{X}$, a typographical symbol $\sigma(\mathfrak{X})$, called the "shape symbol" of $\mathfrak{X}$; these should all be distinct. *Smooth or riemannian case*: The shape symbols for $\mathfrak{X}^+$ and $\mathfrak{X}^-$ are $^\diamond$ and $_\diamond$, respectively. *Spinor case*: The shape symbols for $\mathfrak{X}^+$ and $\mathfrak{X}^-$ are as above, and those for $\mathfrak{Y}^+$, $\mathfrak{Y}^-$, $\bar{\mathfrak{Y}}^+$ and $\bar{\mathfrak{Y}}^-$ are $^\circ$, $_\circ$, $^\bullet$ and $_\bullet$, respectively. Note that the similarities and differences between the shape symbols reflect relationships between the modules.

It is convenient to introduce a brief notation for shapes—this is the purpose of the shape symbols. We abbreviate a shape $(\mathfrak{X}_i)_n$ by writing the shape symbols $\sigma(\mathfrak{X}_i)$ from left to right in incresing order of i. Concatenation of shapes corresponds to the obvious typographic operation. In any of our examples, the shape $(\mathfrak{X}^+, \mathfrak{X}^+, \mathfrak{X}^-, \mathfrak{X}^-, \mathfrak{X}^-, \mathfrak{X}^+)$ would be written ${}^{\diamond\diamond}{}_{\diamond\diamond\diamond}{}^\diamond$, and the concatentation of this shape followed by ${}^\diamond{}_{\diamond\diamond}{}^\diamond$ is ${}^{\diamond\diamond}{}_{\diamond\diamond\diamond}{}^{\diamond\diamond}{}_{\diamond\diamond}{}^\diamond$. In this brief notation the empty shape is represented by invisible notation. This is usually desirable; when it is not we refer to is as () of $\emptyset$.

STEP 3: We suppose given a set $\mathcal{A}$ called the alphabet, satisfying a minor set-theoretic condition given below. Elements of $\mathcal{A}$ are called letters. For each $\alpha \in \mathcal{A}$ we assume given a subset $\phi(\alpha)$ of $\mathcal{X}$. One may indicate that $\mathfrak{X} \in \phi(\alpha)$ by a phrase like "α can stand for $\mathfrak{X}$".

Smooth or riemannian case: $\mathcal{A}$ is the set of roman lowercase letters a, b, c,... together with copies of these letters equipped with numerical subscripts a_0, b_0, c_2, etc. Each letter may stand for either $\mathfrak{X}^+$ of $\mathfrak{X}^-$. *Spinor case*: $\mathcal{A}$ is the union of the letters as above, each of which may stand for $\mathfrak{X}^+$ or $\mathfrak{X}^-$, with the set of roman uppercase letters (possibly with numerical subscripts) A, B, C_2, etc., Each of the uppercase letters may stand for any of $\mathfrak{Y}^\pm$ or $\bar{\mathfrak{Y}}^\pm$.

An index is a pair $(\alpha, \mathfrak{X})$ where $\alpha \in \mathcal{A}$ and $\mathfrak{X} \in \phi(\alpha)$; an index label is a finite sequence $I = ((\alpha_i, \mathfrak{X}_i))_n$ of indices. The indices $(\alpha_i, \mathfrak{X}_i)$ are called the indices of I and the shape of I is the sequence $(\mathfrak{X}_i)_n$. The "minor set-theoretic condition" referred to above is that $\mathcal{A}$ must be such that the set of indices be disjoint from $\mathcal{X}$. The purpose of this condition is to make the sets of shapes and index labels be disjoint (except that both contain the empty sequence); this is required for

some of the constructions and conventions below to make sense.

We take the third compenent $\mathcal{I}$ of the index formalism to be the set of index labels, with the operation being that of concatenation of sequences.

STEP 4: We assume given a "decoration scheme" that assigns to each index $(\alpha, \mathfrak{X})$ a typographical symbol $\tau((\alpha, \mathfrak{X}))$. These symbols (called index symbols) should be distinct from each other and from the shape symbols. *Smooth or riemannian case*: For $\alpha \in \mathcal{A}$ the index symbols of $(\alpha, \mathfrak{X}^+)$ and $(\alpha, \mathfrak{X}^-)$ are ${}^{\alpha}$ and ${}_{\alpha}$, respectively. *Spinor case*: For lowercase letters the index symbols are as above. For a capital letter $\alpha \in \mathcal{A}$, the index symbols of $(\alpha, \mathfrak{Y}^+)$, $(\alpha, \mathfrak{Y}^-)$, $(\alpha, \bar{\mathfrak{Y}}^+)$ and $(\alpha, \bar{\mathfrak{Y}}^-)$ are ${}^{\alpha}$, ${}_{\alpha}$, ${}^{\alpha'}$ and ${}_{\alpha'}$, respectively.

As for shapes, we use a compact notation for index labels. If $((\alpha_i, \mathfrak{X}_i))_n$ is an index label then we simply write the index symbols for the $(\alpha_i, \mathfrak{X}_i)$ from left to right in increasing order of i. In any of our examples, we would write ${}^{ab}{}_{b_1 b}{}^{dc}$ for the index label

$$((a, \mathfrak{X}^+), (b, \mathfrak{X}^+), (b_1, \mathfrak{X}^-), (b, \mathfrak{X}^-), (d, \mathfrak{X}^+), (c, \mathfrak{X}^+)).$$

In the case of spinors, one would write ${}_a{}^{BC'}{}_C{}^b$ for the index label

$$((a, \mathfrak{X}^-), (B, \mathfrak{Y}^+), (C, \bar{\mathfrak{Y}}^+), (C, \mathfrak{Y}^-), (b, \mathfrak{X}^+)).$$

As for shapes, concatenation of index labels corresponds to the obvious typographic operation.

STEP 5: We suppose given a (possibly empty) set of relators $I = J$ in the monoid $\mathcal{I}$. This is an obscure ingrdient in our conventions and may be ignored for normal work with manifolds. *Smooth or riemannian case*: We take the set of relations to be empty. *Spinor case*: We impose the relations ${}^a = {}^{AA'}$, ${}_a = {}_{AA'}$, ${}^b = {}^{BB'}$, ${}_b = {}_{BB'}$, ${}^{c_3} = {}^{C_3 C_3'}$, ${}_{c_3} = {}_{C_3 C_3'}$, etc.

An index label is called legal (i.e., lies in $\mathcal{L}$) if each index label to which it is equivalent under the given relations is injective as a function (as a sequence, it is a function). *Smooth or riemannian case*: An index label is legal just if it is injective. In terms of the compact notation this means that no letter appears twice as a superscript or twice as a subscript. *Spinor case*: The index label ${}^{aA}{}_{A'b}$ is injective, but it is not legal because it is equiavlent to ${}^{AA'A}{}_{A'b}$, which is not injective. It is easy to tell whether any given index label is legal—just convert all the lowercase letters to pairs of uppercase ones and check whether the resulting index label is injective.

We will now construct the heart of the formalism: the tensor spaces $\mathfrak{S}(L)$ for $L \in \mathcal{L}$ and the various maps to and between them. We will build several versions of each of these objects, each more flexible and useful than the last. The first versions are very easy to define. For $L \in \mathcal{L}$ we define L^0 to be the set of its indices, and

$${}^0\mathfrak{S}(L) = \bigotimes_{\ell \in L^0} \chi(\ell),$$

where χ is the map assigning to an index $(\alpha, \mathfrak{X})$ the module $\mathfrak{X}$. We will typically write ${}^0\mathfrak{S}(L)$ simply as ${}^0\mathfrak{S}L$ and use our compact notation for L. In any of our examples, we have

$${}^0\mathfrak{S}^{ab}{}_{ab} = {}^0\mathfrak{S}({}^{ab}{}_{ab}) \cong \mathfrak{X}^+ \otimes \mathfrak{X}^+ \otimes \mathfrak{X}^- \otimes \mathfrak{X}^-.$$

Note that ${}^0\mathfrak{S}(L)$ depends only on the function $\ell \mapsto \chi(\ell)$ on L^0, which is a set with no ordering. This means that if the indices of L are reordered to obtain a new index label L', and if L' is also legal, then ${}^0\mathfrak{S}(L) = {}^0\mathfrak{S}(L')$. In any of our examples, we have ${}^0\mathfrak{S}^{abc}{}_{df}{}^j = {}^0\mathfrak{S}^{bajc}{}_{df}$. Furthermore, in any of our examples, L' will automatically be legal if L is.

If $L, M \in \mathcal{L}$ such that LM is also legal, then $L^0 \cap M^0 = \emptyset$ and $L^0 \cup M^0 = (LM)^0$. Then we define

$${}^0\mathcal{G}_{L,M} : {}^0\mathfrak{S}(L) \times {}^0\mathfrak{S}(M) \to {}^0\mathfrak{S}(LM)$$

by the natural bilinear map given in section 3. Specifically, if

$$S = \bigotimes_{\ell \in L^0} s(\ell) \in {}^0\mathfrak{S}(L) \qquad \text{and} \qquad T = \bigotimes_{m \in M^0} t(m) \in {}^0\mathfrak{S}(M)$$

then

$${}^0\mathcal{G}_{L,M}(S,T) = \bigotimes_{\ell \in (LM)^0} (s \cup t)(\ell).$$

Finally, we define ${}^0\mathcal{D}(L)$ to be the set containing just one shape S—that of L. We define ${}^0\mathcal{F}_L$ on the shape space $\mathfrak{T}(s)$ as follows. Formally, $L = (\ell_i)_n$ is a map from n to the set of indices, and its shape $S = (s_i)_n$ is a map from n to $\mathcal{X}$. For

$$X = \bigotimes_{i=0}^{n-1} x(i) \in \bigotimes_{i=0}^{n-1} s_i = \mathfrak{T}(S)$$

we define

$${}^0\mathcal{F}_L(X) = \bigotimes_{\ell \in L^0} x(L^{-1}(\ell)).$$

What is going on here is that L establishes a bijection between n and L^0. Since $\mathfrak{T}(S)$ and ${}^0\mathfrak{T}(L)$ are tensor products parameterized by these sets, L induces an isomorphism between them, which is just ${}^0\mathcal{F}_L$.

The 9-tuple $(\mathcal{R}, \mathcal{X}, \mathcal{I}, \cdot, \mathcal{L}, {}^0\mathfrak{S}, {}^0\mathcal{G}, {}^0\mathcal{D}, {}^0\mathcal{F})$ is an index formalism, but not very flexible or useful. We will improve it in two stages in sections 6 and 7. Before continuing, however, we discuss how the conventions apply to the empty set, and introduce notation to suppress explicit mention of $\mathfrak{T}$, $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{F}$. We will apply the notational conventions developed here (and those for ${}^0\mathfrak{S}$ above) to the formalisms devloped in later sections, as well as the current formalism. In light of this we suppress the superscripts from $\mathfrak{S}$, $\mathcal{G}$, $\mathcal{D}$ and $\mathcal{F}$.

Observes that $\mathfrak{S}(())$ and $\mathfrak{T}(())$ are both $\mathcal{R}$-module tensor products with parameter set $\emptyset$ and hence are equal to each other. We will no longer use the symbols $\mathfrak{T}(s)$ for shapes s, and instead conform to Penrose's notation by writing $\mathfrak{S}(s)$. Since $()$ is the only index label that is also a shape, and $\mathfrak{S}(()) = \mathfrak{T}(())$, this notation is unambiguous. Naturally, we will also suppress the parentheses from $\mathfrak{S}(s)$ and write s using our compact notation, so that in work with smooth manifolds one will see symbols like $\mathfrak{S}^{\diamond\diamond}{}_{\diamond}{}^{\diamond}$.

A special case of this is that we write $\mathfrak{S}$ for $\mathfrak{S}(())$, which is naturally a ring isomorphic to $\mathcal{R}$, with multiplication given by $\mathcal{G}_{(),()}$. For any shape s, the concatenations $()s$ and $s()$ both equal s, and so $\mathfrak{S}(s)$ is a module over $\mathfrak{S}$ under $\mathcal{B}_{s,()}$ and under $\mathcal{B}_{(),s}$; these two actions of $\mathfrak{S}$ are the same, and both coincide with the action of $\mathcal{R}$, under the natural isomorphism $\mathcal{R} \cong \mathfrak{S}$. Similarly, if L is an index symbol then $()L$ and $L()$ both equal L, and if L is legal then $\mathfrak{S}(L)$ is a module over $\mathfrak{S}$ under $\mathcal{G}_{(),L}$ and under $\mathcal{G}_{L,()}$. Again, both these $\mathfrak{S}$-actions coincide with the action of $\mathcal{R}$. Consequently, we may regard all of the tensor spaces of the index formalism as being modules of $\mathfrak{S}$ rather than $\mathcal{R}$. In light of this we will follow Penrose and usually suppress the symbol $\mathcal{R}$ in favor of $\mathfrak{S}$.

Now suppose s is a shape in $\mathcal{D}(L)$ for some $L \in \mathcal{L}$, and suppose $X \in \mathfrak{T}(s)$. Then we will usually indicate $\mathcal{F}_L(X)$ by writing X followed by the compact notation for L. For example, on a smooth manifold, the image in $\mathfrak{S}^a$ of a vector field $X \in \mathfrak{S}^\diamond$ under the map $\mathcal{F}_L$ for $L = {}^a$ is denoted simply by X^a. Similarly, if g is a riemannian metric and $M = {}_{cd}$ then $\mathcal{F}_M(g)$ is denoted g_{cd}. If $L, M \in \mathcal{L}$ such that $LM \in \mathcal{L}$, and $X \in \mathfrak{S}(L)$ and $Y \in \mathfrak{S}(M)$, then we indicate $\mathcal{G}_{L,M}(X,Y)$ by simply juxtaposing X and Y—just writing XY. Continuing the example above, we have $X^a g_{cd} = \mathcal{G}_{L,M}(X^a, g_{cd})$. By (4.1), expressions like $X^a g_{cd} Y^c$ are unambiguous. Our notation finally resembles standard notation!

The conventions of the preceding paragraph apply only when they make sense—that is, S followed by the compact notation for L is only defined when $S \in \mathfrak{S}(s)$ for some $s \in \mathcal{D}(L)$. One might have a family of tensors, indexed by (say) greek letters, on some manifold. Then T_α would be one of these tensors, of whatever shape, and could be interpreted by following it with an index symbol. So $T_{\alpha bgh}$ might denote the tensor T_α interpreted as an element of $\mathfrak{S}_{bgh}$. One could conceivably consider families of tensors indexed by roman letters, but anyone who tries this deserves what he gets. Furthermore, we will meet some symbols (such a covariant derivative operators ∇) that do not represent tensors but which sometimes still are written with indices (e.g., ∇_a or ∇_b). The "interpretations" of these symbols are not defined here, so must be defined elsewhere before the symbols have meaning. (See section 13.)

6. Conventions for interpretation rules

For smooth manifolds, it seems reasonable for a tensor to be interpreted by some index labels that don't have quite the same shape. For example, a tensor $T = e_\alpha \otimes e_\beta \otimes E^\gamma$ could be interpreted by the index ${}^a{}_c{}^b$ as well as by ${}^{ab}{}_c$, the former interpretation obtained by first replacing T by $e_\alpha \otimes E^\gamma \otimes e_\beta$. So the formalism should "know how to shuffle the tensor factors around." However, one cannot slide two indices past each other if both are subscripts or both are superscripts. For example, one should never confuse T with $e_\beta \otimes e_\alpha \otimes E^\gamma$.

On a riemannian manifold there is a natural isomorphism $\Phi : \mathfrak{X}^+ \to \mathfrak{X}^-$ induced by the metric, and it is used to "raise and lower indices". For example the tensor $T = e_\alpha \otimes E^\beta$ may be interpreted either as $T^a{}_b$ or as $T_b{}^a$ but the meanings are different from those above. $T^a{}_b$ is obtained by "substituting e_α for a and E^β for b." But $T_b{}^a$ is not obtained (as it would be above) by first converting $e_\alpha \otimes E^\beta$ into $E^\beta \otimes e_\alpha$ and then "substituting". Instead, $e_\alpha \otimes E^\beta$ is converted to $\Phi(e_\alpha) \otimes \Phi^{-1}(E^\beta) \in \mathfrak{S}_\diamond{}^\diamond$ and only then interpreted by $T_b{}^a$, by "substitution". Thus in working with riemannian manifolds one cannot slide indices past each other, even if one is a superscript and the other a subscript.

There are more complicated "mixed" situations. In work with complex vector bundles (see section 10), there are 4 modules and 4 kinds of decoration. Each letter of the alphabet can appear as either a subscript or superscript and with or without a prime. Changing the location of a letter corresponds to changing to the dual module and adding or removing a prime corresponds to taking the complex conjugate module. We might be given a nondegenerate bilinear pairing on one of the modules (hence on all 4 of them) and then we would be able to raise and lower letters as in riemannian geometry. But now, primed and unprimed letters can still be moved past each other. On the other hand, if we were given a nondegenerate sesquilinear pairing on one of the modules then each raising or lowering operation would also introduce or remove a prime from the letter raised or lowered. We could then move a pair of letters past each other unless they had opposite positions and opposite "primed status".

In work with principal bundles one meets Lie algebra-valued differential forms, and one might or might not be able to move the "Lie algebra indices" past the "tensor indices". Naturally, one would also have "dual Lie algebra indices" and one might or might not be able to raise and lower these to convert them to Lie algebra indices. If one could (e.g., if the Killing form were nondegenerate) then one wouldn't be able to move these two kinds of indices past each other. And on spacetime one can make everything even more complicated by using spinors to describe "normal" tensors.

The common theme in all of these examples is that some index labels may interpret tensors of a variety of shapes. There are two main ways this can happen—by permuting the tensor factors (as for smooth manifolds) and by applying given isomorphisms between various of the given modules of the formalism (as for riemannian manifolds). One can mix the two processes, for example if a

manifold were equipped with several vector bundles, some of which were equipped with metrics and others of which were not. In this section we will provide conventions for constructing a new and more flexible formalism from $(\mathcal{R}, \mathcal{X}, \mathcal{I}, \cdot, \mathcal{L}, {}^0\mathfrak{S}, {}^0\mathcal{G}, {}^0\mathcal{D}, {}^0\mathcal{F})$ by replacing ${}^0\mathcal{D}$ and ${}^0\mathcal{F}$ by ${}^1\mathcal{D}$ and ${}^1\mathcal{F}$; our conventions are more than sufficient to express precisely any of the ideas above.

We write $\mathcal{S}$ for the set of shapes. The basic idea is to define for each shape s a set of shapes $\Sigma(s)$ containing s, and for each pair (s,t) with $s \in \mathcal{S}$ and $t \in \Sigma(s)$ a map $\Omega_{s,t} : \mathfrak{S}(t) \to \mathfrak{S}(s)$, such that $\Omega_{s,s} = \mathrm{id}_{\mathfrak{S}(s)}$. We indicate that $t \in \Sigma(s)$ by saying that s can be converted into t. Given such data, the construction of the new interpretation rules is easy. If $L \in \mathcal{L}$ has shape t then we define

$$ {}^1\mathcal{D}(L) = \{\, s \in \mathcal{S} \mid t \in \Sigma(s) \,\}\,. $$

Then we define ${}^1\mathcal{F}_L$ on $\mathfrak{S}(t)$ for $s \in {}^1\mathcal{D}(L)$ by ${}^1\mathcal{F}_L|_{\mathfrak{S}(s)} = {}^0\mathcal{F}_L \circ \Omega_{s,t}$. That is, by means of the given map $\Omega_{s,t}$, we first convert an element of $\mathfrak{S}(s)$ into an element of $\mathfrak{S}(t)$, and then we interpret it in the usual way. Note that ${}^1\mathcal{F}_L$ and ${}^0\mathcal{F}_L$ agree on $\mathfrak{S}(s)$. Furthermore, if for each shape s we have $\Sigma(s) = \{s\}$ then ${}^0\mathcal{D} = {}^1\mathcal{D}$ and ${}^0\mathcal{F} = {}^1\mathcal{F}$.

In our applications all of the maps $\Omega_{s,t}$ will be isomorphisms and the sets $\Sigma(s)$ will be the equivalence classes of an equivalence relation. But the additional generality might be useful for (say) a spinor structure on a lorentzian 3-manifold $\mathcal{M}$, by which we mean a 2-dimensional real vector bundle S over $\mathcal{M}$ together with a bundle map $S \otimes S \to T\mathcal{M}$ that is equivariant with respect to actions of the group $SL_2\mathbb{R}$ on the fibers at each point, the action on the fiber of S being the standard one and that on the tangent space being the standard representation of $SO(2,1) \cong PSL_2\mathbb{R}$. Such a bundle map is not an isomorphism; it has a 1-dimensional kernel.

Here are conventions for describing the sets $\Sigma(s)$ and maps $\Omega_{s,t}$ in terms of simpler data. For current purposes, we define a relator to be an ordered triple (s,t,Φ) where $s,t \in \mathcal{S}$ and $\Phi : \mathfrak{S}(s) \to \mathfrak{S}(t)$ is a linear map. If $r = (s,t,\Phi)$ is a relator and Φ is an isomorphism then we call the relator (t,s,Φ^{-1}) the inverse of r.

STEP 6: We assume given some set R (possibly empty) of relators, satisfying a compatibility condition given below. *Smooth case*: The relators are $({}^\diamond{}_\diamond, {}_\diamond{}^\diamond, \Phi)$ and its inverse, where $\Phi : \mathfrak{S}^\diamond{}_\diamond \to \mathfrak{S}_\diamond{}^\diamond$ is given by $a \otimes b \mapsto b \otimes a$. These relators indicate exactly what it means to "slide upper and lower indices past each other". *Riemannian case*: The relators are $({}^\diamond{}_\diamond, \Phi)$ and its inverse, where $\Phi : \mathfrak{S}^\diamond \to \mathfrak{S}_\diamond$ is the isomorphism that carries a vector field v to the 1-form given by taking inner products with v under the metric g. These relators indicate precisely what it means to raise and lower indices. This is the first (and last) place that the formalism for a smooth manifold depends on whether a metric is distinguished.

Spinor case: The most interesting relators are $({}^{\circ\bullet}{}, {}^\diamond, \Phi)$ and its inverse, where $\Phi : \mathfrak{S}^{\circ\bullet} \to \mathfrak{S}^\diamond$ is the map induced by the bundle isomorphism $S^+ \otimes \bar{S}^+ \to T\mathcal{M}$ given as part of the spin structure. We also take as relators the $({}_\diamond{}_{\circ\bullet}, \Phi^\dagger)$ and its inverse, where $\Phi^\dagger$ is the adjoint of Φ. These relators indicate that we are identifying $\mathcal{Y}^+ \otimes \bar{\mathcal{Y}}^+$ with $\mathcal{X}^+$ and precisely what we mean by this. The remaining relators are similar to those of the smooth case above. When we declare that two shapes $s = (\mathfrak{X})$ and $t = (\mathfrak{Y})$ of length 1 may slide past each other we mean that we are introducing the relator (s,t,Φ) and its inverse, where $\Phi : \mathfrak{X} \otimes \mathfrak{Y} \to \mathfrak{Y} \otimes \mathfrak{X}$ simply by exchanging the tensor factors. In terms of this convention, we declare that any two distinct shapes may slide past each other except that $^\diamond$ may not slide past $^\circ$ or $^\bullet$ and that $_\diamond$ may not slide past $_\circ$ or $_\bullet$. This specification of relators explicitly is tedious and repetitive, even with extra conventions such as that introduced in the spinor example. Section 8 introduces conventions for specifiying the relators with a minimum of fuss.

We define an elementary move to be an ordered 4-tuple (u,v,r,x), where $u,v,x \in \mathcal{S}$, $r = (s,t,\Phi)$ is a relator, and there exists a shape z such that $u = xsz$ and $v = xtz$. In this case we say that u can be transformed to v by applying relator r after the subword w. When we want to

express less detail we simply say that u can be transformed to v by an elementary move. We say that u can be transformed to v (written $u \succ v$) if $u = v$ or if there is a sequence $(u_i)_n$ of shapes with $u_0 = u$ and $u_{n-1} = v$ such such that for each $i = 0, \ldots, n-2$, u_i can be transformed to u_{i+1} by an elementary move. Observe that $\succ$ is a transitive and reflexive relation. For $L \in \mathcal{L}$ with shape s, we define ${}^1\mathcal{D}_L$ to be the set of shapes which can be transformed to s.

Now suppose that (u, v, r, x) is an elementary move, with z and $r = (s, t, \Phi)$ as above, with $r \in R$. Then we define a linear map $\Psi(u, v, r, x) : \mathfrak{S}(u) \to \mathfrak{S}(v)$ by

$$\Psi(u, v, r, x) = \mathrm{id}_{\mathfrak{S}(x)} \otimes \Phi \otimes \mathrm{id}_{\mathfrak{S}(z)} .$$

If $K = (k_i)_n = ((u_i, v_i, r_i, x_i))_n$ is a sequence of elementary moves with $u_i = v_{i-1}$ for $i = 1, \ldots, n-1$ then we say that K transforms u to v. In this case we define a map $\hat{\Psi}_K : \mathfrak{S}(u_0) \to \mathfrak{S}(v_{n-1})$ by

$$\hat{\Psi}_K = \Psi(k_{n-1}) \circ \Psi(k_{n-2}) \circ \cdots \circ \Psi(k_0). \tag{6.1}$$

The conditions on the maps Φ that are mentioned above in the input for step are that this map should be independent of K. By this, we mean two things. First, we require that if K transforms u to u then $\hat{\Psi}_K = \mathrm{id}_{\mathfrak{S}(u)}$. Second, we require that if both K and K' are sequences of elementary moves transforming u to v then $\hat{\Psi}_K = \hat{\Psi}_{K'}$. In general these would be very difficult conditions to check, but in all of our examples, the verification is easy. In light of this condition, we may unambiguously define the map $\Omega_{u,v}$ to be any of the maps $\hat{\Psi}_K$ for K transforming u to v. We have now defined $\Sigma(s)$ and the maps $\Omega_{s,t}$, so we may apply the procedure given above. This provides a new interpretation rules ${}^1\mathcal{D}$ and ${}^1\mathcal{F}$. We then have a new formalism $(\mathcal{R}, \mathcal{X}, \mathcal{I}, \cdot, \mathcal{L}, {}^0\mathfrak{S}, {}^0\mathcal{G}, {}^1\mathcal{D}, {}^1\mathcal{F})$.

Another use of the maps $\Omega_{s,t}$ is the following. In section 5 we introduced the convention that if $S \in \mathfrak{S}(s)$ for some shape s, then we would sometimes denote S by writing its symbol S followed by the compact notation for s. We generalize this here, so that if $S \in \mathfrak{S}(s)$ and t is a shape to which s may be transformed, then the symbol S followed by the compact notation for t indicates $\Omega_{s,t}(S)$. Since $\Omega_{s,s}(S) = S$, this is consistent with the previous notation. For example, on a riemannian manifold the metric tensor g lies in $\mathfrak{S}_{\diamond\diamond}$, so $g = g_{\diamond\diamond}$. But now we can also write (say) $g^{\diamond\diamond}$ for the metric tensor "with both indices raised", even though no indices are present.

7. Relations in formalisms

In this section we complete the work of §§5–6. The chief fault of the formalisms constructed so far is that they fail to implement the usual contraction on repeated indices. In any of our examples, if $v \in \mathfrak{S}^\diamond$ and $w \in \mathfrak{S}_\diamond$ then one would like to write $v^a w_a$ for their natural pairing—a smooth function on $\mathcal{M}$. But under the formalisms so far constructed, $\mathfrak{S}^\diamond{}_\diamond$ is a tensor product parameterized by the set $\{{}^a{}_a\}$ of indices and is isomorphic to $\mathfrak{X}^+ \otimes \mathfrak{X}^-$. It is easy to "add relations" to the formalism to crush $\mathfrak{S}^\diamond{}_\diamond$ down to be a copy of $\mathfrak{S}$. There is also an important application to spinors.

We define $\hat{\mathfrak{S}} = \oplus_{L \in \mathcal{L}} {}^0\mathfrak{S}(L)$. On this space there is a partial multiplication defined. Namely, if $x \in {}^0\mathfrak{S}(L) \subseteq \hat{\mathfrak{S}}$ and $y \in {}^0\mathfrak{S}(M) \subseteq \hat{\mathfrak{S}}$ with LM legal then we have $xy = {}^0\mathcal{G}_{L,M}(x, y) \in {}^0\mathfrak{S}(LM) \subseteq \hat{\mathfrak{S}}$. Suppose $X = \sum_{L \in \mathcal{L}} x_L$ and $Y = \sum_{L \in \mathcal{L}} y_L$, with each sum having only finitely many nonzero terms and $x_L, y_L \in {}^0\mathfrak{S}(L)$ for each $L \in \mathcal{L}$. Then XY is defined just if LM is legal for each pair $L, M \in \mathcal{L}$ for which $x_L \neq 0$ and $y_M \neq 0$. Then XY is just $\sum_{L,M \in \mathcal{L}} x_L y_M$. We can now describe the last ingredient in our recipe for formalisms.

STEP 7: We assume given a set (possibly empty) of elements of $\hat{\mathfrak{S}}$. One should think of these as relations that we will impose on $\hat{\mathfrak{S}}$. *Smooth or riemannian case*: Our relations are $v_\alpha w^\alpha = \langle\, v \mid w \,\rangle$ for all $\alpha \in \mathcal{A}$ and all $v \in \mathfrak{S}^\diamond$, $w \in \mathfrak{S}_\diamond$. This will implement the convention of contraction on repeated indices. Formally, the elements of $\hat{\mathfrak{S}}$ that we mean are the $v_\alpha w^\alpha - \langle\, v \mid w \,\rangle$

with α, v and w as above. According to customary abuse of notation, one often uses the "equals" notation.

Spinor case: We impose the relations above for each lowercase letter α, plus the similar relations

$$\sigma_\alpha \tau^\alpha = \langle\, \sigma \mid \tau \,\rangle$$
$$\bar\sigma_\alpha \bar\tau^\alpha = \langle\, \bar\sigma \mid \bar\tau \,\rangle$$

for all capital letters α and all $\sigma \in \mathfrak{S}^\circ$, $\tau \in \mathfrak{S}_\circ$, $\bar\sigma \in \mathfrak{S}^\bullet$ and $\bar\tau \in \mathfrak{S}_\bullet$. As before, $\langle \mid \rangle$ indicates the natural pairing between each pair of mutually dual bundles. Furthermore, we impose the additional relations

$$w^a = w^{AA'}, \qquad w^b = w^{BB'}, \qquad w^{c_3} = w^{C_3 C_3'}, \ldots$$
$$v_a = v_{AA'}, \qquad v_b = v_{BB'}, \qquad v_{c_3} = v_{C_3 C_3'}, \ldots$$

for all $w \in \mathfrak{S}^\diamond$ and $v \in \mathfrak{S}_\diamond$. Note the both sides of each of these relations are defined, because of the extension of the interptetation rules made in section 6. As in the previous example, each of these relations should be regarded as determining an element of $\hat{\mathfrak{S}}$.

Now we define $\hat{\mathfrak{S}}'$ to be the subspace of $\hat{\mathfrak{S}}$ spanned by the given elements, together with all of their products with other elements of $\hat{\mathfrak{S}}$. (That is, all products that are defined.) Then we define ${}^1\mathfrak{S}(L)$ for $L \in \mathfrak{L}$ to be the image of ${}^0\mathfrak{S}(L)$ in $\hat{\mathfrak{S}}/\hat{\mathfrak{S}}'$. Next, because $\hat{\mathfrak{S}}'$ is an "ideal" in $\hat{\mathfrak{S}}$, for each pair (L, M) of legal indices such that LM is also legal, the map ${}^0\mathcal{G}_{L,M}$ induces a map ${}^1\mathfrak{S}(L) \times {}^1\mathfrak{S}(M) \to {}^1\mathfrak{S}(LM)$. We take ${}^1\mathcal{G}_{L,M}$ to be this map. Finally, we define ${}^2\mathcal{F}(L)$ to be the map ${}^1\mathcal{F}(L)$ followed by the natural map ${}^0\mathfrak{S}(L) \to {}^1\mathfrak{S}(L)$. This yields a new formalism $(\mathcal{R}, \mathcal{X}, \mathcal{I}, \cdot, \mathfrak{L}, {}^1\mathfrak{S}, {}^1\mathcal{G}, {}^1\mathcal{D}, {}^2\mathcal{F})$, which is our "standard formalism". Henceforth, provided that suitable data are provided (steps 1–7 in this section and the preceeding two), we will regard this as the formalism defined, and suppress the superscripts from the last 4 components of the formalism.

Smooth or riemannian case: Modulo $\hat{\mathfrak{S}}'$, we have $v_a w^a = \langle\, v \mid w \,\rangle$ for all $v \in \mathfrak{S}^\diamond$ and $w \in \mathfrak{S}_\diamond$. Similar relations hold with other lowercase letters in place of a. A consquence of this is that $v_a w^a x^b = \langle\, v \mid w \,\rangle\, x^b$, so that ${\mathfrak{S}_a}^{ab}$ and $\mathfrak{S}^b$ are identified. This accords with the property of the usual notation that repeated indices are irrelevant to the shape of a tensor.

Spinor case: The extra "spinor" conditions implement the usual custom of writing $v^a = v^{AA'}$ for $v \in \mathfrak{S}^\diamond$. Without the extra relations, the different sides of this equation would lie in different vector spaces, and so couldn't be sensibly compared. After the identifications, one can consider equalities like $v^b = x^B y^{B'}$ for $v \in \mathfrak{S}^\diamond$, $x \in \mathfrak{S}^\circ$ and $y \in \mathfrak{S}^\bullet$. We can now explain the obscure step 5 of our recipe for formalisms. This ingredient is there mainly to allow us to describe the spinor formalism. The point is that ${}^{AAA'}$ is not a legal index symbol by anyone's standards, and since we want an "equality of index labels" ${}^a = {}^{AA'}$, we had better call ^{Aa} illegal also.

8. A real manifold equipped with real vector bundles

This section and section 10 provide conventions for quickly constructing index formalisms for the motivating case: that of a smooth manifold possibly equipped with several vector bundles. Here we treat the case in which all the vector bundles are real and later we treat the case involving complex bundles. We take an unusually complicated example as an illustration here, namely that of a real manifold equipped with two additional vector bundles, one equipped with a metric and the other with a symplectic "inner product". For any vector bundle W we write $\Gamma(W)$ for the space of smooth section of W. The steps of the construction given here correspond to the steps of sections 5–7.

STEP 1: We assume given a smooth real manifold $\mathcal{M}$ and a collection $\mathcal{W}_0$ of real vector bundles on $\mathcal{M}$. We set $\mathcal{R} = C^\infty(\mathcal{M}, \mathbb{R})$. Let $\mathcal{W}$ be the set of vector bundles consisisting of the elements of $\mathcal{W}_0$ and their duals, and let $\mathcal{X} = \{\, \Gamma(W) \mid W \in \mathcal{W} \,\}$.

Example: We suppose we are given $\mathcal{M}$, equipped with two vector bundles E and F, as well as the tangent bundle $T\mathcal{M}$, so that $\mathcal{E}_0 = \{E, F, T\mathcal{M}\}$. Then $\mathcal{R} = C^\infty(\mathcal{M}, \mathbb{R})$ and $\mathcal{X} = \{\Gamma(E), \Gamma(F), \Gamma(E^*), \Gamma(F^*), \mathfrak{X}^+, \mathfrak{X}^-\}$, where an asterisk denote the dual of a bundle and $\mathfrak{X}^+$ and $\mathfrak{X}^-$ are the space of vector fields an one-forms on $\mathcal{M}$, as usual.

STEP 2: We assume given a "bundle symbol" for each element $\mathcal{W}_0$. These should all be distinct, and should be neither superscripts nor subscripts. Then we define the shape symbols for the elements of $\mathcal{X}$ as follows. For $W \in \mathcal{W}_0$, the shape symbol for $\Gamma(W)$ (resp. $\Gamma(W^*)$) is the bundle symbol for W written as a superscript (resp. subscript).

Example: We take the bundle symbol for $T\mathcal{M}$ to be $\diamond$ (the standard use for this symbol), and those for E and F to be some other symbols, say $\heartsuit$ and $\triangle$. Then the shape symbols for $\Gamma(E)$, $\Gamma(F)$, $\Gamma(E^*)$, $\Gamma(F^*)$, $\mathfrak{X}^+$ and $\mathfrak{X}^-$ are ${}^{\heartsuit}$, ${}^{\triangle}$, ${}_{\heartsuit}$, ${}_{\triangle}$, ${}^{\diamond}$ and ${}_{\diamond}$, respectively.

STEP 3: We suppose given a prealphabet $\mathcal{A}_0$ (a set) containing no lowercase greek letters (so we have some letters that we know won't take on unexpected meanings) or bundle symbols, and for each $\alpha \in \mathcal{A}_0$ an element W_α of $\mathcal{W}_0$. One can says that α can stand for W_α.

Example: We take the prealphabet to consist of the lowercase and uppercase roman letters and the uppercase greek letters. Each lowercase roman letter can stand for $T\mathcal{M}$ (this is the standard use of these letters) and each uppercase roman letter can stand for E. Each uppercase greek letter that is not a roman letter (i.e., the letters Γ, Δ, Θ, Λ, Ξ, Π, Σ, Υ, Φ, Ψ and Ω) can stand for F.

We define the alphabet $\mathcal{A}$ of the formalism to be the prealphabet, together with copies of these letters equipped with numerical subscripts. We declare that if an element α of $\mathcal{A}_0$ can stand for W_α, then each of the letters of $\mathcal{A}$ associated to α (i.e., α itself, possibly with a numeric subscript) can stand for either of the $\mathcal{R}$-modules $\Gamma(W_\alpha)$ and $\Gamma(W_\alpha^*)$.

Example: The letters a, b, c, a_0, b_0, c_2, etc. can stand for either $\mathfrak{X}^+$ or $\mathfrak{X}^-$, the letters A, B, C, A_1, B_2, C_0, etc. can stand for $\Gamma(E)$ or $\Gamma(E^*)$, and the letters Γ, Δ, Λ, Γ_0, Δ_1, Λ_2, etc. can stand for either $\Gamma(F)$ or $\Gamma(F^*)$.

STEP 4: This step provides the index symbols, and requires no input from the designer. We define the index symbols as follows. If $\beta \in \mathcal{A}$ is a letter associated to $\alpha \in \mathcal{A}_0$ then then index symbols for $(\beta, \Gamma(W_\alpha))$ and $(\beta, \Gamma(W_\alpha^*))$ are copies of the letter β written as a superscript and as a subscript, respectively. Since $\mathcal{A}_0$ contains none of the bundle symbols, our index symbols cannot be confused with shape symbols.

Example: Some of our index symbols are ^{a}, ${}_{b_1}$, ^{A_0}, ${}_{Q_2}$, ${}^{\Gamma_4}$ and ${}_{\triangle}$. Index labels, written in the compact notation, are just sequences of such symbols.

STEP 5: This step requires no input from the user. We take the set of relations on the index labels to be empty. (This step in §5 is mainly there to provide a means for describing spinor structures; see §12.) This carries us through the end of §5, providing a formalism $(\mathcal{R}, \mathcal{X}, \mathcal{I}, \cdot, \mathcal{L}, {}^0\mathfrak{S}, {}^0\mathcal{G}, {}^0\mathcal{D}, {}^0\mathcal{F})$.

Example: We have now assigned meanings to the expressions $X_{\diamond\diamond}{}^{\heartsuit}$ and $X_{ab}{}^{\triangle}$, for $X \in \mathfrak{S}_{\diamond\diamond}{}^{\heartsuit}$; these are elements of $\mathfrak{S}_{\diamond\diamond}{}^{\heartsuit}$ and $\mathfrak{S}_{ab}{}^{\triangle}$, respectively.

STEP 6: We assume that for each W in a (possibly empty) subset $\mathcal{W}_{00}$ of $\mathcal{W}_0$ we are given a distinguished nondegenerate bilinear pairing G^W on W.

Example: Suppose E is equipped with a metric (a nondegenerate symmetric bilinear form) $g = G^E \in \mathfrak{S}_{\heartsuit\heartsuit}$ and that F is equipped with a symplectic structure (a nondegenerate antisymmetric bilinear form) $\varepsilon = G^F \in \mathfrak{S}_{\triangle\triangle}$.

We define some relators R, in the terminology of §6. We first define a set of explicit relators and then a set of implicit relators; the latter depend on the former. For $W \in \mathcal{W}_{00}$ we define a relator as follows. Writing s for the shape $(\Gamma(W))$ and t for the shape $(\Gamma(W^*))$, we define a relation (s, t, Φ). Here, $\Phi : \mathfrak{S}(s) \to \mathfrak{S}(t)$ is the map $\Gamma(W) \to \Gamma(W^*)$ whose value on a section X of W is the section of W^* given by the map $Y \mapsto G^W(X, Y)$ for $Y \in \Gamma(W)$. The map Φ is an isomorphism,

so the inverse relator is defined. We take the set of explicit relators to be these relators and their inverses, as W varies over $\mathcal{W}_{00}$.

Example: If v is a section of E then $v^\heartsuit \in \mathfrak{S}^\heartsuit$, as before (at the end of step 4). However, $v_\heartsuit$ is now the section of E^* given by taking inner products of v with respect to g. Furthermore, v_A is the interpretation of v in $\mathfrak{S}_A \cong \mathfrak{S}_\heartsuit$. Assuming for the moment that we have carried out the rest of the construction of this section, including step 7 below, so that the "summation convention" is implemented, we have $v_A = g_{BA} v^B$. That is, $v_\heartsuit$ is $v^\heartsuit$ "contracted with the first index of $g_{\heartsuit\heartsuit}$". Of course, by the symmetry of g we have $g_{BA} = g_{AB}$, so the distinction between first and second indices is irrelevant. However, the difference is important for sections of F, since ε is not symmetric; indeed $\varepsilon_{\Gamma\Delta} = -\varepsilon_{\Delta\Gamma}$. For each $X \in \mathfrak{S}^\triangle$ we have $X_\Delta = \varepsilon_{\Gamma\Delta} X^\Gamma$, again assuming the summation convention introduced below. We sum this up by saying that one always lowers indices by contracting with the first index of whatever bilinear form is present. Some issues arise in raising indices; although we have in this section defined what it means to raise indices, we wait until section 9 to discuss actual calculation.

Having defined the explicit relators we now define the implicit ones. We say that two elements $\mathfrak{X}$ and $\mathfrak{Y}$ of $\mathcal{X}$ are tied if they are equal or if there is an explicit relator (s, t, Φ) where $\mathfrak{X}$ is a term of s or t and $\mathfrak{Y}$ is a term of the other of s and t. For each pair of untied $\mathfrak{X}, \mathfrak{Y} \in \mathcal{X}$, we declare that the shapes $(\mathfrak{X})$ and $(\mathfrak{Y})$ of length one may slide past each other (in the terminology of the spinor example of step 6 of section 6). Then section 6 defines a new index formalism $(\mathcal{R}, \mathcal{X}, \mathcal{I}, \cdot, \mathcal{L}, {}^0\mathfrak{S}, {}^0\mathcal{G}, {}^1\mathcal{D}, {}^1\mathcal{F})$ with more relaxed interpretation rules than the old one.

Example: The only pairs of distinct shapes that are related are $^\heartsuit$ and $_\heartsuit$, and $^\triangle$ and $_\triangle$. The implicit relators then allow us to permute distinct tensor factors of the shape spaces freely, subject to the condition that tensor factors corresponding to tied shapes keep their relative ordering. In particular, we have identified $\mathfrak{S}_\diamond{}^\diamond$ with $\mathfrak{S}^\diamond{}_\diamond$, and indeed we have $X_a{}^b = X^b{}_a$ for all $X \in \mathfrak{S}_\diamond{}^\diamond$. This provides one of the features of the usual notation in the absence of a distinguished metric.

STEP 7: This requires no input from the user; we simply impose some standard relations on $\hat{\mathfrak{S}}$ in order to define ${}^1\mathfrak{S}$, ${}^1\mathcal{G}$ and ${}^2\mathcal{F}$, as in section 7. The relations just implement the summation convention. Suppose $W \in \mathcal{W}_0$ and $\alpha \in \mathcal{A}$ is a letter which can stand for $\Gamma(W) \in \mathcal{X}$ (and thus can also stand for $\Gamma(W^*)$), and that $v \in \Gamma(W)$ and $w \in \Gamma(W^*)$. Let L and M be the index labels $((\alpha, \Gamma(W)))$ and $((\alpha, \Gamma(W^*)))$ respectively; L, M and LM are all legal. Then we define an element of $\hat{\mathfrak{S}}$:

$$ {}^0\mathcal{G}_{L,M}\left({}^1\mathcal{F}_L(v), {}^1\mathcal{F}_M(w)\right) - \langle\, v \mid w \,\rangle, $$

where $\langle\, v \mid w \,\rangle$ is the value (a smooth function) of the natural pairing of v and w. We take the elements of $\hat{\mathfrak{S}}$ required in section 7 to be the set of these elements as W, α, v and w vary over all possibilities values subject to the supposition above. This implements the summation convention and completes our description of the formalism.

Example: The messy formula above, written in terms of the compact notation, simply reads (for α the letter 'a' and $W = T\mathcal{M}$) $v^a w_a = \langle\, v \mid w \,\rangle$, just as in §7. Similar relations hold for sections of E and F. In particular, if $v, w \in \mathfrak{S}^\heartsuit$ then their inner product is $g_{AB} v^A w^B$. Furthermore, we have $g_{AB} v^A = v_B$, so we can also write the inner product $v_B w^B$; the norm of v is then $v_A v^A$. If $v \in \mathfrak{S}^\triangle$ then we have $v_\Gamma = \varepsilon_{\Delta\Gamma} v^\Delta$. Note that $\varepsilon_{\Gamma\Delta} v^\Gamma v^\Delta$ vanishes by the antisymmetry of ε, so $v_\Delta v^\Delta = \varepsilon_{\Gamma\Delta} v^\Gamma v^\Delta = 0$. That is, the "symplectic norm" of v is zero, which is only natural. There are also more complicated consequences of these relations. For example, if $X \in \mathfrak{S}_{\diamond\diamond}{}^{\heartsuit\triangle}$ and $Y \in \mathfrak{S}_{\heartsuit\heartsuit}$ then $X_{ab}{}^{A\Gamma} Y_{BA}$ is the element of $\mathfrak{S}_{ab}{}^\Gamma{}_B$ which is the interpretation by ${}_{ab}{}^\Gamma{}_B$ of the contraction of X with Y at their second and third indices, respectively.

In summary, we have provided a means to specify an index formalism, given a manifold, some real vector bundles over it, a collection of letters (the prealphabet), the shape symbols, and some distinguished nondegenerate bilinear pairings on some of the given bundles. Except for the

addition of the shape symbols, these are precisely the data used in [3] to construct formalisms. Our formalisms are precisely defined and mirror the properties of the formalisms defined less precisely there.

We will now discuss "contraction" and then the "Kronecker delta" tensor. If $L = ((\alpha_i, \mathfrak{X}_i))_n$ is a legal index label, then we say that an index $(\alpha, \mathfrak{X})$ of L s repeated if $(\alpha, \mathfrak{X}^*)$ is also an index of L. We write L^{00} for the set of nonrepeated indices of L. By the relations imposed in step 7 above, $\mathfrak{S}(L)$ is the tensor product $\bigotimes_{\ell \in L^{00}} \chi(\ell)$, where we recall that $\chi((\alpha, \mathfrak{X})) = \mathfrak{X}$. For example, with $L = {}_{ab}{}^{abc}$, the only nonrepeated index is ^{c}, and $\mathfrak{S}_{ab}{}^{abc} = \mathfrak{S}^c \cong \mathfrak{X}^+$. If $I = (\beta, \mathfrak{X})$ and $J = (\gamma, \mathfrak{X}^*)$ are nonrepeated indices of L then we define "contraction on I and J" to as follows. It is simply the map

$$\bigotimes_{\ell \in L^{00}} \chi(\ell) \to \bigotimes_{\ell \in L^{00} \smallsetminus \{I,J\}} \chi(\ell)$$

given by

$$\bigotimes_{\ell \in L^{00}} X(\ell) \mapsto \langle\, X(I) \mid X(J) \,\rangle \bigotimes_{\ell \in L^{00} \smallsetminus \{I,J\}} X(\ell).$$

For example, contraction on ^{a} and ${}_c$ defines a map $\mathfrak{S}^{ab}{}_{cd} \to \mathfrak{S}^b{}_d$ and slo a map $\mathfrak{S}_f{}^{ae}{}_c \to \mathfrak{S}_f{}^e$. Contraction on ^{a} and ${}_c$ coincides with contraction on ${}_c$ and ^{a}, and in the we often refer to either of these as $\binom{a}{c}$-contraction.

A useful tensor field is the identity map $\mathbf{1}_\diamond{}^\diamond \in \mathfrak{S}_\diamond{}^\diamond$ on $T\mathcal{M}$. Once $\mathbf{1}$ is interpreted by an index label, it plays the usual role of the Kronecker delta symbol. For example, $\mathbf{1}_a{}^b v^a = v^b$ and $\mathbf{1}_a{}^b \mathbf{1}_c{}^d w_{db} = w_{ca}$. Of course there are similar tensors given by the identity maps on all the other elements of $\mathcal{W}_0$, each of which (once interpreted by index labels) plays the role of the Kronecker delta for indices associated to that bundle. We will abuse notation by denoting all of these operators by the same symbol $\mathbf{1}$. If greater precision is required we may distinguish them by explicitly indicating their shapes. For example, in the formalism of this section, the identity maps on $T\mathcal{M}$, E and F would all be denoted $\mathbf{1}$. If great precision were required, they could be distinguished by referring to $\mathbf{1}_\diamond{}^\diamond$, $\mathbf{1}_\heartsuit{}^\heartsuit$ and $\mathbf{1}_\triangle{}^\triangle$ instead.

For reference, we recall that the type of a tensor (defined in section 5) is a familiar concept, and that (for example) the tensors in $\mathfrak{S}^\diamond{}_\diamond{}^\diamond{}_\diamond$ and $\mathfrak{S}_{\diamond\diamond\diamond}{}^{\diamond\diamond}$ have types $\binom{2}{2}$ and $\binom{2}{3}$ respectively.

9. Notes on metric and symplectic structures

In this section we make a few comments about the "index gymnastics" available when dealing with bilinear pairings. Thoughout, we will work in the formalism developed as an example in the last section, except with g a metric on $\mathcal{M}$, ε a symplectic pairing on E, and B a generic nondegenerate form on F with no particular symmetry properties. The results of this section, given in terms of this extended example, generalize easily.

The symmetry property of g implies that

$$v^a w_a = v^a g_{ba} w^b = v^a g_{ab} w^b = v_b w^b = v_a w^a$$

for all $v \in \mathfrak{S}_\diamond$ and $w \in \mathfrak{S}^\diamond$. A consequence of this is that if a lowercase roman index is repeated then the positions of its two appearances may be switched simultaneously without changing the meaning of the expression. Applying the same argument to ε, and using its antisymmetry, yields $x^A y_A = -x_A y^A$ for all $x, y \in \mathfrak{S}^\heartsuit$. This is called the "see-saw". In general, simultaneously changing the positions of the two appearances of a repeated uppercase roman letter changes the sign of the quantity expressed. There is no nice description of the see-saw for B, but see below for the best available substitute.

Applying the see-saw , we derive $v_a = g_{ab}v^b = g_a{}^b v_b$ for all $v \in \mathfrak{S}^\diamond$, so we see that $g_\diamond{}^\diamond$, once interpreted by an index label, plays the role of the Kronecker delta. That is, $g_\diamond{}^\diamond = \mathbf{1}_\diamond{}^\diamond$. By the symmetry of g we also have $g^a{}_b = g_b{}^a$ and $v_a = g^b{}_a v_b$, so that $g^\diamond{}_\diamond$, once interpreted, also plays the role of the Kronecker delta. Note, however, that $g^\diamond{}_\diamond \neq g_\diamond{}^\diamond$—indeed, the two sides lie in disjoint vector spaces. Similar considerations apply to ε: we have $x_A = \varepsilon_{BA}x^B = -\varepsilon_{AB}x^B = +\varepsilon_A{}^B x_B$ for all $x \in \mathfrak{S}^\heartsuit$, so that $\varepsilon_\heartsuit{}^\heartsuit = \mathbf{1}_\heartsuit{}^\heartsuit$, just as for g. We also have $\varepsilon_A{}^B = -\varepsilon^B{}_A$, by the antisymmetry of ε, but $\varepsilon_\heartsuit{}^\heartsuit$ and $\varepsilon^\heartsuit{}_\heartsuit$ lie in disjoint vector spaces and cannot be sensibly compared. There seems to be no reasonable relation between $B_\triangle{}^\triangle$ and $\mathbf{1}_\triangle{}^\triangle$.

Now we discuss the raising of indices. Since g, ε and B are nondegenerate, there exist unique tensors $g^{-1} \in \mathfrak{S}^{\diamond\diamond}$, $\varepsilon^{-1} \in \mathfrak{S}^{\heartsuit\heartsuit}$ and $B^{-1} \in \mathfrak{S}^{\triangle\triangle}$ such that

$$g_{ab}(g^{-1})^{cb} = \mathbf{1}_a{}^c, \qquad \varepsilon_{AB}(\varepsilon^{-1})^{CB} = \mathbf{1}_A{}^C, \qquad \text{and} \quad B_{\Gamma\Delta}(B^{-1})^{\Phi\Delta} = \mathbf{1}_\Gamma{}^\Phi.$$

These are called the inverses of g, ε and B, respectively. Then for all $z \in \mathfrak{S}_\triangle$ we have

$$(B^{-1})^{\Delta\Phi}z_\Phi = (B^{-1})^{\Delta\Phi}B_{\Gamma\Phi}z^\Gamma = \mathbf{1}_\Gamma{}^\Delta z^\Gamma = z^\Delta.$$

Similar results hold for raising indices with respect to g and ε: in each case, one raises an index by contracting with the *second* index of the inverse of the tensor used to lower indices. (Recall that one lowers indices using the *first* index of the metric or other tensor.) One seldom uses the explicit superscript -1 because the calculations

$$g_{ab}g^{cb} = g_a{}^b g^c{}_b = g^c{}_a = g_a{}^c = \mathbf{1}_a{}^c$$

and

$$\varepsilon_{AB}\varepsilon^{CB} = -\varepsilon_A{}^B\varepsilon^C{}_B = -\varepsilon^C{}_A = +\varepsilon_A{}^C = \mathbf{1}_A{}^C$$

show that $g^{\diamond\diamond}$ and $\varepsilon^{\heartsuit\heartsuit}$ are the inverses of $g_{\diamond\diamond}$ and $\varepsilon_{\heartsuit\heartsuit}$. This shouldn't be too surprising because the raising and lowering operators are defined in terms of g and ε.

We remark that our condition is defining the inverse of B is that $B_{\Gamma\Delta}(B^{-1})^{\Phi\Delta} = \mathbf{1}_\Gamma{}^\Phi$, rather than the similar condition $B_{\Gamma\Delta}(B^{-1})^{\Delta\Phi} = \mathbf{1}_\Gamma{}^\Phi$. This is because our conventions reproduces the standard conventions in the case that B is antisymmetric. If we used the other conventions then we would raise indices by contracting with the first index of ε^{-1}, and $\varepsilon^{\heartsuit\heartsuit}$ would be the negative of $(\varepsilon^{-1})^{\heartsuit\heartsuit}$.

Sometimes one want to convert a formula written in terms of a formalism that allows raising and lowering of indices into a formula which lacks such freedom. For example, one might want to compare the curvature tensors of two different metrics on the same manifolds, each of which was computed using the raising and lowering operations defined by their respective metrics. Each tensor in any given expression has a shape, which might or might not be apparent from the notation. Simply write each tensor appearing in the equation as a contracted tensor product of its "original" form and factors of $g_{\diamond\diamond}$, $(g^{-1})^{\diamond\diamond}$, $\varepsilon_{\heartsuit\heartsuit}$, $(\varepsilon^{-1})^{\heartsuit\heartsuit}$, $B_{\triangle\triangle}$ and $(B^{-1})^{\triangle\triangle}$. For example, if $R \in \mathfrak{S}_{\diamond\diamond\diamond}{}^\diamond$ and an equation envolves $R_a{}^b{}_{cd}$ then one should replace $R_a{}^b{}_{cd}$ with $R_{aec}{}^f g_{fd}(g^{-1})^{eb}$ or the like. A more complicated example, involving all of the bundles as well as repeated indices, is that for $S \in \mathfrak{S}_{\heartsuit\triangle}{}^{\triangle\diamond}{}_\diamond{}^\heartsuit$, one should replace $S_A{}^\Gamma{}_\Gamma{}^b{}_{bB}$ by $S_{A\Delta}{}^{\Phi b}{}_b{}^C(B^{-1})^{\Gamma\Delta}B_{\Phi\Gamma}\varepsilon_{CB}$. Here, the B^{-1} term raises the Δ to a Γ, the B term lowers the Φ to a Γ, and the ε term lowers the C to a B. For repeated roman indices that are out of place, the see-saw provides an easier means to "get the tensor back to its original shape" than the introduction of pairs of terms g and g^{-1} or ε and ε^{-1}.

Incidentally, we have just implicitly obtained the formula for the see-saw with respect to B:

$$w_\Gamma z^\Gamma = B_{\Delta\Gamma}w^\Delta z_\Phi(B^{-1})^{\Gamma\Phi} = w^\Delta z_\Phi \left(B_{\Delta\Gamma}(B^{-1})^{\Gamma\Phi}\right).$$

The reason the see-saw simplifies with g or ε in place of B is that the parenthesized expression in the right side of the equality simplifies to $\pm\mathbf{1}$.

10. Complex vector bundles on a real manifold

Here we construct formalisms for manifolds equipped with a some set of real or complex vector bundles, just as we treated the case of all real bundles in section 8. The steps enumerated here correspond to those in sections 5–7 and 8.

STEP 1: We suppose given a manifold $\mathcal{M}$ and a set $\mathcal{W}_0$ of real vector bundles over $\mathcal{M}$, and also a set $\mathcal{V}_0$ of complex vector bundles over $\mathcal{M}$. Then we take $\mathcal{R} = C^\infty(\mathcal{M}, \mathbb{C})$ and define $\mathcal{X}$ as follows. Let $\mathcal{W}$ be the set of bundles $\mathbb{C} \otimes W$ and $\mathbb{C} \otimes W^*$ for $W \in \mathcal{W}_0$. Let $\mathcal{V}$ be the set of bundles V, V^*, $\bar{V}$ and $\bar{V}^*$ for $V \in \mathcal{V}_0$, where each bar denotes complex conjugation. Let $\mathcal{U} = \mathcal{V} \cup \mathcal{W}$ and set $\mathcal{X} = \{\, \Gamma(U) \mid U \in \mathcal{U} \,\}$, where $\Gamma(U)$ is the space of smooth sections of U, as in section 8. We also set $\mathcal{U} = \mathcal{V} \cup \mathcal{W}$.

Example: We take a manifold $\mathcal{M}$ with $\mathcal{W}_0 = \{T\mathcal{M}\}$ and $\mathcal{V}_0 = \{S^+\}$, where S^+ is some complex vector bundle. Then $\mathcal{R} = C^\infty(\mathcal{M}, \mathbb{C})$ and $\mathcal{X} = \{\mathfrak{X}^+, \mathfrak{X}^-, \Gamma(S^+), \Gamma(S^-), \Gamma(\bar{S}^+), \Gamma(\bar{S}^-)\}$ where we have written S^- for the dual of S^+ and $\mathfrak{X}^+$ and $\mathfrak{X}^-$ for the spaces of (complex) vector fields and one-forms on $\mathcal{M}$, respectively.

STEP 2: We assume given a "bundle symbol" for each element of $\mathcal{U}_0$; this is just like step 2 of section 8, with the added proviso that the symbol for each element of $\mathcal{V}_0$ should be an "outline" symbol like $\circ$ or $\square$ which can be filled in, to yield symbols like $\bullet$ or $\blacksquare$. Then the shape symbols for the elements of $\mathcal{X}$ of the form $\Gamma(W)$ and $\Gamma(W^*)$ for $W \in \mathcal{W}_0$ are given as in section 8. If $V \in \mathcal{V}_0$ then the shape symbols for $\Gamma(V)$ and $\Gamma(V^*)$ are copies of the bundle symbol for V written as a superscript and as a subscript, respectively. The shape symbols for $\Gamma(\bar{V})$ and $\Gamma(\bar{V}^*)$ are copies of the filled-in version of the bundle symbol, again written as a superscript and as a subscript, respectively. We call the shape symbols for $\Gamma(W)$ real or complex according as $w \in \mathcal{W}$ or $W \in \mathcal{V}$.

Example: We take the bundle symbol for $T\mathcal{M}$ to be $\diamond$, as usual, and that for S^+ to be $\circ$. Then the shape symbols for $\mathfrak{X}^+$, $\mathfrak{X}^-$, $\Gamma(S^+)$, $\Gamma(S^-)$, $\Gamma(\bar{S}^+)$ and $\Gamma(\bar{S}^-)$ are $^\diamond$, $_\diamond$, $^\circ$, $_\circ$, $^\bullet$ and $_\bullet$ respectively.

STEP 3: We suppose given a prealphabet $\mathcal{A}_0$ just as in section 8, except that each letter α may stand for any element U_α of $\mathcal{U}_0$ rather than just elements of $\mathcal{W}_0$. We define the alphabet $\mathcal{A}$ just as in section 8. If $\alpha \in \mathcal{A}_0$ such that $U_\alpha \in \mathcal{W}_0$ then the letters of $\mathcal{A}$ associated to α be stand for either $\Gamma(W_\alpha)$ or $\Gamma(W_\alpha^*)$. If $W_\alpha \in \mathcal{V}_0$ then the letters of $\mathcal{A}$ associated to α may stand for any of $\Gamma(W_\alpha)$, $\Gamma(W_\alpha^*)$, $\Gamma(\bar{W}_\alpha)$ and $\Gamma(\bar{W}_\alpha^*)$.

Example: We take $\mathcal{A}_0$ to contain the lowercase roman letters, which stand for $T\mathcal{M}$ as usual, and the uppercase roman letters, which stand for S^+.

STEP 4: This step requires no input from the user; we define the index symbols as follows. An index of the form $(\alpha, \Gamma(W))$ (resp. $(\alpha, \Gamma(W^*))$) for $W \in \mathcal{W}_0$ has index symbol $^\alpha$ (resp. $_\alpha$); in either of these cases the index is called real. An index of the form $(\alpha, \Gamma(V))$ (resp. $(\alpha, \Gamma(V^*))$, $(\alpha, \Gamma(\bar{V}))$, $(\alpha, \Gamma(\bar{V}^*))$) for $V \in \mathcal{V}_0$ has index symbol $^\alpha$ (resp. $_\alpha$, $^{\alpha'}$, $_{\alpha'}$); in any of these cases the index is called complex. That is, a prime indicates a "complex conjugate index".

Example: Some of out index symbols as ^{a}, $_b$, ^{c_2}, $_{B_4}$, $^{C_1'}$ and $_{A'}$.

STEP 5: This requires no input from the user. We introduce relations just as we did in section 8. This carries us through the end of section 5, providing the preliminary version $(\mathcal{R}, \mathcal{X}, \mathcal{I}, \cdot, \mathcal{L}, ^0\mathfrak{S}, ^0\mathcal{G}, ^0\mathcal{D}, ^0\mathcal{F})$ of our formalism.

Before continuing we say a few words about complex conjugation. We regard conjugation (denoted by an overbar) as an involution on $\mathcal{U}$, assigning to each bundle its conjugate bundle, where the conjugate of one of the real bundles is itself. We also regard conjugation as an involution on $\mathcal{X}$, with $\overline{\Gamma(U)} = \Gamma(\bar{U})$. We define the conjugate of a shape $s = (\mathfrak{X}_i)_n$ to be the shape $\bar{s} = (\bar{\mathfrak{X}}_i)_n$; the compact notation for $\bar{s}$ may be obtained from that for s by leaving each real shape symbol alone and replacing each outline (resp. filled-in) complex shape symbol with the corresponding filled-in (resp. outline) one. Similarly, the conjugate of an index label $L = ((\alpha_i, \mathfrak{X}_i))_n$ is $\bar{L} = ((\alpha_i, \bar{\mathfrak{X}}_i)_n$ and the compact notation for $\bar{L}$ may be obtained from that of L by leaving each real index symbol

alone, adding a prime to each unprimed complex index symbol, and removing the prime from each primed index symbol.

For any shape s and any legal index label L we define the conjugates $\overline{\mathfrak{S}(s)}$ and $\overline{\mathfrak{S}(L)}$ of $\mathfrak{S}(s)$ and $\mathfrak{S}(L)$ to be $\mathfrak{S}(\bar{s})$ and $\mathfrak{S}(\bar{L})$, respectively. For $s = (\mathfrak{X}_i)_n$, we define an antilinear map from $\mathfrak{S}(s)$ to $\mathfrak{S}(\bar{s})$ by sending $v = \otimes_{i\in n} v_i$ to $\bar{v} = \otimes_{i\in n}\bar{v}_i$ and extending by antilinearity. There are similar maps ${}^0\mathfrak{S}(L) \to {}^0\mathfrak{S}(\bar{L})$, and this map will descend to a map ${}^1\mathfrak{S}(L) \to {}^1\mathfrak{S}(\bar{L})$ once ${}^1\mathfrak{S}$ has been defined below. This map ${}^0\mathfrak{S}(L) \to {}^0\mathfrak{S}(\bar{L})$ is characterized by the property that for all $v \in \mathfrak{S}(s)$, where s is the shape of L, we have $\overline{{}^0\mathcal{F}_L(v)} = {}^0\mathcal{F}_{\bar{L}}(\bar{v})$. (Note that $s \in {}^0\mathcal{D}(L)$ implies that $\bar{s} \in {}^0\mathcal{D}(\bar{L})$; the analogous fact will hold for ${}^1\mathcal{D}$, once that is defined.)

Example: we have $\overline{T\mathcal{M}} = T\mathcal{M}$ and $\overline{\bar{S}^\pm} = S^\pm$; we have $\bar{\mathfrak{X}}^\pm = \mathfrak{X}^\pm$ and $\overline{\Gamma(S^\pm)} = \Gamma(\bar{S}^\pm)$; we have $\overline{{}^\diamond{}_{\circ\bullet}{}^{\bullet\bullet}{}_\diamond} = {}^\diamond{}_{\bullet\circ}{}^{\circ\circ}{}_\diamond$ and $\overline{{}^{AB'C}{}_{abD'}} = {}^{A'BC'}{}_{abD}$. If $v \in \mathfrak{S}_\circ{}^\diamond{}_\bullet$ then $\bar{v} \in \overline{\mathfrak{S}_\circ{}^\diamond{}_\bullet} = \mathfrak{S}_\bullet{}^\diamond{}_\circ$; the index label ${}_{AB'}{}^C$ may interpret v, and we have $v_{AB'}{}^C \in \mathfrak{S}_{AB'}{}^C$; then $\overline{v_{AB'}{}^C} = \bar{v}_{AB'}{}^{\overline{C}} = \bar{v}_{A'B}{}^{C'} \in \mathfrak{S}_{A'B}{}^{C'}$.

Now we return to the conventions for the formalism.

STEP 6: We assume that for each W in some subset $\mathcal{W}_{00}$ of $\mathcal{W}_0$ we are given a nondegenerate bilinear pairing G^W on W. (Typically G^W will be real in the sense that it takes real values on pairs of real elements of $\Gamma(W)$, but this is not necessary.) We also assume that for each V in a subset $\mathcal{V}_{00}$ of $\mathcal{V}$ we are given a pairing G^V on V that is either bilinear of sesquilinear.

We define explicit relators exactly as in section 8 for the pairings given on the real bundles. For the bilinear pairings given on the complex bundles we do the same thing, but also construct relators associated to the conjugates of the pairings. For $V \in \mathcal{V}_{00}$ we introduce an explicit relator as in section 8 and also the explicit relator (s, t, Φ) and its inverse. Here, s and t are the shapes $(\Gamma(\bar{V}))$ and $(\Gamma(\bar{V}^*))$, respectively, and $\Phi : \mathfrak{S}(s) \to \mathfrak{S}(t)$ assigns to $X \in \Gamma(\bar{V})$ the section $Y \mapsto \overline{G^V(\bar{X}, \bar{Y})}$ of $\bar{V}^*$, where Y is any section of $\bar{V}$.

Example: Suppose that we distinguish an antisymmetric pairing ε on S^+. Then the procedure above defines raising and lowering operators as in section 8. In particular, $\varepsilon_{AB}X^A = X_B$ for all $X \in \mathfrak{S}^\circ$ and (as explained in section 9) we also have $\varepsilon^{AB}Y_B = Y^A$ for all $Y \in \mathfrak{S}_\circ$. (We are using the contraction convention introduced below in step 7.) The procedure also defines analogous raising and lowering operators $\mathfrak{S}_\bullet \to \mathfrak{S}^\bullet$ and $\mathfrak{S}^\bullet \to \mathfrak{S}_\bullet$ using $\bar{\varepsilon} \in \mathfrak{S}_{\bullet\bullet}$ in place of ε. That is, we will have $X_{B'} = \bar{\varepsilon}_{A'B'}X^{A'}$ and $Y^{B'} = \bar{\varepsilon}^{B'C'}Y_{C'}$ for all $X \in \mathfrak{S}^\bullet$ and $Y \in \mathfrak{S}_\bullet$. The discussion of ε in section 9 applies essentially verbatim to $\bar{\varepsilon}$, and in particular we have $\bar{\varepsilon}_{A'B'} = -\bar{\varepsilon}_{B'A'}$, $\bar{\varepsilon}_\bullet{}^\bullet = \mathbf{1}_\bullet{}^\bullet$, and $\bar{\varepsilon}^{\bullet\bullet} = (\bar{\varepsilon}^{-1})^{\bullet\bullet}$.

We now discuss sesquilinear forms. Such a form on $V \in \mathcal{V}_0$ is a smoothly varying field of nondegenerate bilinear pairings between the fibers of V and the fibers of $\bar{V}$. Such a form induces a nondegenerate bilinear pairing $\Gamma(V) \times \Gamma(\bar{V}) \to \mathcal{R}$. The usual description of a sesquilinear form would be as a pairing $\Gamma(V)\times\Gamma(V) \to \mathcal{R}$ that is linear in one argument (say, its first) and antilinear in the other. Onec an recover this description from ours by taking the "inner product" of $X, Y \in \Gamma(V)$ to be $G^V(X, \bar{Y})$. We introduce a relator (s, t, Φ), where s and t are the shapes $(\Gamma(V))$ and $(\Gamma(\bar{V}^*))$, respectively, and $\Phi : \Gamma(V)\Gamma(\bar{V}^*)$ assigns to $X \in \Gamma(V)$ the section of $\bar{V}^*$ given by $Y \mapsto G^V(X, Y)$ for all $Y \in \Gamma(\bar{V})$. We also introduce the "conjugate" relator $(\bar{s}, \bar{t}, \bar{\Phi})$ where $\bar{\phi} : \mathfrak{S}(\bar{s}) \to \mathfrak{S}(\bar{t})$ carries $X \in \Gamma(\bar{V})$ to the section of V^* given by $Y \mapsto \overline{G^V(\bar{X}, \bar{Y})}$ for $Y \in \Gamma(V)$. Then th eexplicit relators we impose are the the two relators defined here along with their inverses.

Example: Suppose that S^+ is equipped with a sesquilinear form $H = G^{S^+}$ rather than the bilinear form ε used in the example above. Then we may regard H as an element $H_{\circ\bullet}$ of $\mathfrak{S}_{\circ\bullet}$. The contraction conventions of step 7 are set up so that if $X, Y \in \mathfrak{S}^\circ$ then their sesquilinear pairing is $H_{AB'}X^A\bar{Y}^{B'}$. If $X \in \mathfrak{S}^\circ$ then the element of $\mathfrak{S}_\bullet$ given by "lowering" the index on X is the contraction of X with the first index of H. In terms of the contraction conventions introduced in step 7 below, we have $X_{B'} = H_{AB'}X^A$. The raising of primed indices is defined implicitly to be

the inverse of this operation; we discuss this in more detail in section 11. We do *not* lower primed indices by contracting with the second index of H—instead we contract with the first index of $\bar{H} \in \mathfrak{S}_{\bullet\circ}$. That is, $Y_B = \bar{H}_{A'B} Y^{A'}$ for all $Y \in \mathfrak{S}^{\bullet}$, where we are again using the contraction conventions. This implicitly defines the operation of raising unprimed indices. Note that if H is hermitian or skew-hermitian then the distinction between H and $\bar{H}$ vanishes; see section 11.

Having defined the explicit relators, we now follow the conventions of section 8 for defining implicit relators in terms of tied and untied shapes. This lets indices that do not "interact" via the given bilinear and sesquilinear pairings pass over each other. The only new phenomenon here is that if $V \in \mathcal{V}_{00}$ has a distinguished sesquilinear form, then the shapes $(\Gamma(V^*))$ and $(\Gamma(\bar{V}^*))$ are untied and so can slide past each other.

Example: The shapes ${}_{\circ\bullet}$ and ${}_{\bullet\circ}$ can be transformed into each other, as can ${}^{\circ\bullet}$ and ${}^{\bullet\circ}$. However, ${}_{\bullet}$ and ${}^{\circ}$ cannot slide past each other, and neither can ${}_{\circ}$ and ${}^{\bullet}$. A consequence of this is that $H_{AB'} = H_{B'A}$—the order of the primed and unprimed indices on H does not matter. (Although $H_{\circ\bullet} \neq H_{\bullet\circ}$, since these lie in disjoint vector spaces.) Therefore we may rephrase the lowering rules described above as follows: to lower an unprimed (resp. primed) index, contract with the unprimed (resp. primed) index of H (resp. $\bar{H}$).

STEP 7: We define the relations in $\hat{\mathfrak{S}}$ just as in section 8, with pairings between each element of $\mathcal{X}$ and its dual. In our example, this implies contraction on pairs of repeated indices; that is we have $\mathfrak{S}_{A'}{}^{A'} = \mathfrak{S} \neq \mathfrak{S}_A{}^{A'}$ and $\mathfrak{S}_a{}^{ab} = \mathfrak{S}^b$. These relations are also illustrated above in our discussion of step 6. This defines the formalism completely.

We close the section by observing that the discussions of section 8 regarding the contraction maps, the "identity" tensors $\mathbf{1}$, and the "types" of tensors all apply verbatim in the current context.

11. Notes on sesquilinear pairings

This section discusses sesquilinear pairings in the example of the previous section, just as section 9 discussed bilinear pairings in the setting of section 8. Note that the comments in section 9 apply equally well to complex bundles as to real bundles when bilinear forms are present.

We discussed above (step 6 of section 10) how the interpretation conventions imply that one lowers unprimed (resp. primed) indices by contracting with the unprimed (resp. primed) index of H (resp. $\bar{H}$). Some examples of this are

$$X_{A'} = H_{BA'} X^B = H_{A'B} X^B$$

and

$$Y_B = \bar{H}_{BC'} Y^{C'} = \bar{H}_{C'B} Y^{C'}.$$

Note that we have used the facts that $H_{A'B} = H_{BA'}$ and $\bar{H}_{BC'} = \bar{H}_{C'B}$; these follow from the implicit relators on $\hat{\mathfrak{S}}$ given in section 10, which assert that primed and unprimed indices may slide past each other if both are superscripts or both are subscripts.

We define $H^{-1} \in \mathfrak{S}^{\circ\bullet}$ by requiring $H_{AB'}(H^{-1})^{CB'} = \mathbf{1}_A{}^C$, or equivalently (by taking complex conjugates) $\bar{H}_{A'B}(\bar{H}^{-1})^{C'B} = \mathbf{1}_{A'}{}^{C'}$. In the absence of any special symmetry properties of H, I do not know of a good formula for the other obvious contraction of H and H^{-1}, namely $H_{AB'}(H^{-1})^{AC'} \in \mathfrak{S}_{B'}{}^{C'}$. We use the tensors H^{-1} and $\bar{H}^{-1}$ to raise indices, as shown by the following calculation:

$$X^{A'} = X^{B'} \mathbf{1}_{B'}{}^{A'} = X^{B'} \bar{H}_{B'C} (\bar{H}^{-1})^{A'C} = X_C (\bar{H}^{-1})^{A'C}$$

for $X \in \mathfrak{S}_{\circ}$, and the conjugate of this formula, namely that $Y^A = Y_{C'}(H^{-1})^{AC'}$ for $Y \in \mathfrak{S}_{\bullet}$. That is, one raises an unprimed (resp. primed) index by cotracting with the unprimed (resp.

primed) index of $\bar{H}^{-1}$ (resp. H^{-1}). One can summarize all of the raising and lowering rules by saying that lowering uses the original form of H whereas raising uses the inverse version, and that independently of the choice of H or H^{-1}, one uses the barred version just if the operation involves a primed superscript.

The most useful kind of sesquilinear form is a Hermitian form; we assume this for the rest of the section. This means that $H_{AB'}X^A\bar{Y}^{B'} = \overline{H_{AB'}Y^A\bar{X}^{B'}}$ for all $X, Y \in \mathfrak{S}^\circ$. Equivalently, we require $H_{AB'}X^A\bar{Y}^{B'} = \bar{H}_{A'B}\bar{Y}^{A'}X^B$; renaming the repeated indices on the right yields $H_{AB'}X^A\bar{Y}^{B'} = \bar{H}_{B'A}\bar{Y}^{B'}X^A$. Since X and Y are arbitrary, the condition for H to be Hermitian is just $H_{AB'} = \bar{H}_{B'A}$, or simply $H_{AB'} = \bar{H}_{AB'}$. (The condition for H to be skew-Hermitian would be $H_{AB'} = -\bar{H}_{AB'}$.)

In the presence of this condition we have a see-saw rule: if $X \in \mathfrak{S}^\circ$ and $Y \in \mathfrak{S}_\circ$ then

$$X^AY_A = X^A\bar{H}_{B'A}Y^{B'} = X^AH_{B'A}Y^{B'} = X_{B'}Y^{B'} = X_{A'}Y^{A'}.$$

That is, exchanging the places of a repeated letter and simultaneously priming or unpriming both occurences does not change the value of the expression. In the case that H is skew-Hermitian, the same argument shows that the see-saw introduces a change of sign.

Using the see-saw we can deduce that $H_\circ{}^\circ = \mathbf{1}_\circ{}^\circ$. For arbitrary $X \in \mathfrak{S}_\circ$ and $Y \in \mathfrak{S}^\circ$ we have

$$Y_BH_A{}^BX^A = Y^{B'}H_{AB'}X^A = Y^{B'}X_{B'} = Y_BX^B.$$

Since Y is arbitrary, we deduce $H_A{}^BX^A = X^B$ for all X, so

$$H_\circ{}^\circ = \mathbf{1}_\circ{}^\circ. \tag{11.1}$$

The same result holds if H is skew-Hermitian, because there are an even number of applications of the see-saw in the derivation above. Of course we have also shown that $\bar{H}_\bullet{}^\bullet = \mathbf{1}_\bullet{}^\bullet$. Since $\mathbf{1}_A{}^B = \mathbf{1}^B{}_A$ we have $H_A{}^B = H^B{}_A$ and similarly $\bar{H}_{A'}{}^{B'} = \bar{H}^{B'}{}_{A'}$. By raising and lowering indices on each side of (11.1) we obtain

$$H_A{}^B = \mathbf{1}_A{}^B = \mathbf{1}^B{}_A = \bar{H}^B{}_A = \bar{H}_A{}^B,$$

so we see that all of the different versions of H and $\bar{H}$ are closely related.

Finally, we show that $(H^{-1})^{\circ\bullet} = H^{\circ\bullet}$. This follows because

$$H_{AB'}H^{CB'} = H_A{}^BH^C{}_B = \mathbf{1}_A{}^BH^C{}_B = H^C{}_A = H_A{}^C = \mathbf{1}_A{}^C.$$

Since H^{-1} is the only element of $\mathfrak{S}^{\circ\bullet}$ satisfying $H_{AB'}(H^{-1})^{CB'} = \mathbf{1}_A{}^C$, we must have $(H^{-1})^{CB'} = H^{CB'}$. This means that one will seldom see the explicit superscript -1 in calculations. The same result holds if H is skew-Hermitian; the argument is the same except for the introduction of two minus signs, which cancel.

The procedure described in section 9 for converting an expression involving raised and lowered indices into an expression in which each tensor has its "natural shape" obviously applies just as well when given a Hermitian form as when given a bilinear form.

12. $SL_2\mathbb{C}$ spinors

The purpose of this formalism is to simplify certain computations and clarify certain concepts in general relativity. We suppose we are given a 4-manifold $\mathcal{M}$. A spin structure on $\mathcal{M}$ is a choice of 2-dimensional complex vector bundle S^+ over $\mathcal{M}$ together with an isomorphism $S^+ \otimes \bar{S}^+ \to \mathbb{C} \otimes T\mathcal{M}$

that respects complex conjugation (which acts on $\mathbb{C} \otimes T\mathcal{M}$ in the obvious way and on $S^+ \otimes \bar{S}^+$ by exchanging the tensor factors). Other vector bundles, real or complex, may also be given, along with bilinear or sesquilinear pairings on them.

The index formalism is just that of section 10 with a few modifications. We use $\diamond$ and $\circ$ as the bundle symbols for $T\mathcal{M}$ and S^+, and must provide extra bundle symbols for any other bundles which might be present. As usual, we use lowercase roman letters for $T\mathcal{M}$ and uppercase roman letters for S^+; if other bundles are present then we must specify additional letters and associate them to the other bundles.

In step 5 we impose some nontrivial relations in the monoid of shapes, just as in the spinor example of section 5. That is we impose the relations ${}^a = {}^{AA'}$, ${}^b = {}^{BB'}$, ${}_{d_3} = {}_{D_3 D_3'}$, etc. These relations affect the definition of a legal index, as explained in section 5.

In step 6 we specify some additional relators, namely those of the spinor example in section 6. These are the the relators $({}^{\circ\bullet}, {}^{\diamond}, \Phi)$, where $\Phi : \mathfrak{S}^{\circ\bullet} \to \mathfrak{S}^{\diamond}$ is induced by the given isomorphism (the spin structure) $S^+ \otimes \bar{S}^+ \to T\mathcal{M}$, together with its inverse, its adjoint, and the inverse of its adjoint. The introduction of these explicit relators affects the definition of the implicit relators; as an example, the shapes ${}^{\circ}$ and ${}^{\diamond}$ are now tied and cannot slide across each other. Without the extra explicit relators, the implicit relators would have allowed such sliding.

Finally, in step 7 we impose some extra relations on $\hat{\mathfrak{S}}$, namely those of the spinor example of section 7. That is, we impose the relations $v^{AA'} = v^a$, $v^{BB'} = v^b$, $v^{D_4 D_4'} = v^{d_4}$, etc. for all $v \in \mathfrak{S}^{\diamond}$ and the relations $w_{AA'} = w_a$, $w_{BB'} = w_b$, $w_{D_4 D_4'} = w_{d_4}$, etc. for all $w \in \mathfrak{S}_{\diamond}$. This defines the formalism.

The reason this is useful for general relativity is that the elements of $\mathfrak{S}_{\diamond}$ are identified with the elements of $\mathfrak{S}_{\circ\bullet}$, which is to say with sesquilinear forms on $\mathfrak{S}^{\circ}$. Those that are real (i.e., "honest" covector fields, invariant under complex conjugation) correspond to the Hermitian forms on $\mathfrak{S}^{\circ}$. This is just because if $w \in \mathfrak{S}_{\diamond}$ satisfies $\bar{w}_a = w_a$ then we have $\overline{w_{AA'}} = w_{AA'}$, or $\bar{w}_{A'A} = w_{AA'}$, or $\bar{w}_{AA'} = w_{AA'}$, which is to say that w is a Hermitian form on $\mathfrak{S}^{\circ}$. In an open set of $\mathcal{M}$ over which S^+ trivializes, a choice of field of bases of S^+ allows one to compute the determinant of any given Hermitian form on $\mathfrak{S}^{\circ}$; this defines a field of quadratic forms on the fibers of $S^+ \otimes \bar{S}^+$. A choice of a different field of bases yields another quadratic form which differs from the first by a smooth positive function. That is, we obtain a conformal structure on $S^+ \otimes \bar{S}^+$, thus one on $T^*\mathcal{M}$, and thus one on $\mathcal{M}$. The signature of this conformal structure is $+---$, which is just right for general relativity.

One usually equips S^+ with a symplectic structure ε and uses it to raise and lower spinor indices. This also allows us raise and lower world-tensor (lowercase roman) indices, as follows:

$$v_a = v_{AA'} = \varepsilon_{AB} \bar{\varepsilon}_{A'B'} v^{BB'} = \varepsilon_{AB} \bar{\varepsilon}_{A'B'} v^b.$$

One avoids such obfuscation by defining $g \in \mathfrak{S}_{\diamond\diamond}$ by $g_{ab} = \varepsilon_{AB} \bar{\varepsilon}_{A'B'}$ and using g to raise and lower world-tensor indices in the usual way. The conformal class of g lies in the class described above, so g has signature $+---$. Penrose suppresses the bar of $\bar{\varepsilon}$ on the grounds that one can always tell whether ε or $\bar{\varepsilon}$ is intended by looking at the attached index label. However, we prefer to follow Wald [4] and retain the bar, in keeping with the principle that ε and $\bar{\varepsilon}$ have meanings independent of their indices and might appear without any indices at all.

It is sometimes useful to have a "coordinate description" of ε. On an open set of $\mathcal{M}$ over which S^+ trivializes, we may take $o, \imath \in \mathfrak{S}^{\circ}$ so that $o_A \imath^A = 1$. Since S^+ is a two-dimensional bundle, the fact that $o_A \imath_B - o_B \imath_A$ is antisymmetric implies that it must be a scalar multiple of ε_{AB}. Since

$$(o_A \imath_B - o_B \imath_A) o^A \imath^B = 1 = \varepsilon_{AB} o^A \imath^B,$$

by $o_A \imath^A = -o^A \imath_A = 1$ and $o_A o^A = \imath_A \imath^A = 0$, we see that $\varepsilon_{AB} = o_A \imath_B - o_B \imath_A$. Similarly, we have $\mathbf{1}_A{}^B = o_A \imath^B - \imath_A o^B$; this can be checked by simply contracting each side with each of o_B and $\imath_B$.

A useful identity satisfied by ε is the following

$$\varepsilon_{DE}\varepsilon_F{}^C + \varepsilon_{FD}\varepsilon_E{}^C + \varepsilon_{EF}\varepsilon_D{}^C = 0. \tag{12.1}$$

In light of the antisymmetry of ε this may be expressed in the notation of section 15 by $\varepsilon_{[DE}\varepsilon_{F]}{}^C = 0$. The $\varepsilon_{[DE}\varepsilon_{F]}{}^C = 0$. The most straightforward way to verify (12.1) is to express $\varepsilon_{\circ\circ}$ and $\varepsilon_\circ{}^\circ = \mathbf{1}_\circ{}^\circ$ using the expressions of the last paragraph. This fails, though, to explain the identity, which arises from the following obervation: if $u, v, w \in \mathfrak{S}^\circ$ then

$$u_A v^A w^B + w_A u^A v^B + v_A w^A u^B = 0,$$

as can be seen by either the antisymmetry of ε (at points of $\mathcal{M}$ where u, v and w are all proportional) or by contracting the expression on the left side with each of u_B, v_B and w_B (at the other points of $\mathcal{M}$, where u, v and w span the spinor space).

13. Attaching indices to operators

Certain symbols can appear with indices attached even though they do not represent tensors, so that the conventions of section 5 for suppressing the symbols $\mathcal{F}_L$ do not apply. An example is the covariant derivative operator ∇ associated to a connection on (say) the tangent bundle $T\mathcal{M}$ of a manifold $\mathcal{M}$. Such an operator ∇ is a function from $\mathfrak{S}^\diamond$ to $\mathfrak{S}_\diamond{}^\diamond$ satisfying the properties

$$\nabla(S+T) = \nabla S + \nabla T$$

and

$$\nabla(fS) = (df) \otimes S + f \nabla S,$$

where $f \in \mathfrak{S}$ and df is its exterior derivative (a one-form) and $S, T \in \mathfrak{S}^\diamond$. For any $S \in \mathfrak{S}^\diamond$ we have $\nabla S \in \mathfrak{S}_\diamond{}^\diamond$, so $(\nabla S)_a{}^b \in \mathfrak{S}_a{}^b$. It is convenient to attach the index $_a$ to ∇ and write $\nabla_a S^b$ for $(\nabla S)_a{}^b$. We may do this by defining ∇_a to be an operator of $\mathfrak{S}^b$ given by this expression. Similarly, we can define ∇_a on $\mathfrak{S}^c$, $\mathfrak{S}^d$, etc., and even on $\mathfrak{S}^a$. We can also define other operators ∇_b, ∇_c, etc., by requiring "covariance with respect to the attachment of indices". This section provides precise conventions for this procedure, under the assumption that the conventions of sections 5–7 are in force.

Suppose that ∇ is an operator (possibly the covariant derivative operator discussed above) on the tensor space of some shape t, taking values in the tensor space of st, where s is another shape. Then for each legal index label L we define the operator $\nabla[L]$ whose domain is the union of those $\mathfrak{S}(M)$ such that $M, LM \in \mathcal{L}$, $t \in \mathcal{D}(M)$ and $st \in \mathcal{D}(LM)$. The action of $\nabla[L]$ is defined by the requirement that

$$(\nabla[L]) \circ \mathcal{F}_M = \mathcal{F}_{LM} \circ \nabla$$

on $\mathfrak{S}(t)$. In any formalism constructed by sectiond 5–7, $\mathcal{F}_M|_{\mathfrak{S}(t)}$ surjects to $\mathfrak{S}(M)$, so this serves as a definition of $\nabla[L]$. We write $\nabla[L]$ by writing ∇ (or whatever the symbol for the operator is) followed by the compact notation for L.

For most index labels L, the domain of $\nabla[L]$ will usually be empty. The set of L for which the domain is nonempty depends on the index formalism being used. For example, if ∇ is a covariant derivative operator on a manifold with no distinguished metric then $\nabla[L]$ has nonempty domain only for index labels L of shape $_\diamond$. This is to be expected, since ∇ "adds one covariant index" to the tensors on which it operates. On the other hand, if $\mathcal{M}$ were a riemannian manifold and we were

using the raising and lowering conventions then $\nabla[L]$ also has nonempty domain if L has shape $^\diamond$. For example, $\nabla^a S^b = (\nabla S)^{ab}$—that is, $(\nabla S)_a{}^b$ with its first index raised.

If one has an operator $\Box : \mathfrak{S}(t) \to \mathfrak{S}(t)$ for some shape t, then the above considerations apply without change, s being the empty index label. Therefore $\Box$ is automatically viewed as an operator on all those $\mathfrak{S}(L)$ with $L \in \mathfrak{L}$ and $t \in \mathcal{D}(L)$. An example of such an operator is the d'Alembertian on a riemannian manifold, acting on (say) vector fields. This is the operator $\Box$ that satisfies

$$(\Box S)^b = \nabla_a \nabla^a S^b \tag{13.1}$$

for all $S \in \mathfrak{S}^\diamond$, where ∇ is the Levi-Civita connection on $T\mathcal{M}$ associated to the metric. Note that formally, the operator $\Box$ acts on $\mathfrak{S}^\diamond$ whereas the operator on $\mathfrak{S}^b$ is $\Box[\emptyset]$—that is, the d'Alembertian operator "interpreted by the empty index symbol". It is natural to write the empty index symbol using the compact notation, so that it vanishes from formulas. If for some reason one needed to distinguish between $\Box$ and $\Box[\emptyset]$, one could use these longer names.

Finally, we observe that if $S \in \mathfrak{S}(s)$ for some shape s then the map $\mathfrak{S}(t) \to \mathfrak{S}(st)$ given by $T \mapsto S \otimes T$, for each shape t, is an operator and may be treated with this notation. In fact, under the notational conventions established in section 5, it is automatically treated this way.

14. Coordinate interpretation and components of tensors

We motivate the considerations of this section by considering a smooth manifold $\mathcal{M}$ of dimension N; suppose V is an open subset of $\mathcal{M}$ over which $T\mathcal{M}$ trivializes, so that there are vector fields $e_0, \ldots, e_{N-1}$ whose values at each $x \in V$ form a basis for $T_x\mathcal{M}$. Then any vector field S (on V) has the form $S = \sum_{\alpha=0}^{N-1} S^\alpha e_\alpha$ where $S^0, \ldots, S^{N-1}$ lie in $C^\infty(\mathcal{M})$ and will be called the components of S with respect to the given basis. If $E^0, \ldots, E^{N-1}$ are a basis of one-forms dual to $e_0, \ldots e_{N-1}$ then any one-form T on V has the form $T = \sum_{\alpha=0}^{N-1} T_\alpha E^\alpha$ for some smooth functions $T_0, \ldots T_{N-1}$ on V, again called the components of T with respect to the given basis. For more complicated tensors, similar decompositions exists. For example, if $U \in \mathfrak{S}^\diamond{}_\diamond{}^\diamond$ then there exist $U^\alpha{}_\beta{}^\gamma \in C^\infty(\mathcal{M})$ (for $\alpha, \beta, \gamma = 0, \ldots, N-1$) such that

$$U = \sum_{\alpha,\beta,\gamma=0}^{N-1} U^\alpha{}_\beta{}^\gamma \cdot e_\alpha \otimes E^\beta \otimes e_\gamma.$$

More complicated situations can occur, such as when another basis of vector fields is present, say $e_{\hat{0}}, \ldots, e_{\widehat{N-1}}$ with dual basis $E^{\hat{0}}, \ldots, E^{\widehat{N-1}}$. Then one might consider the tensor $A = \sum_{\alpha=0}^{N-1} e_\alpha \otimes E^{\hat{\alpha}} \in \mathfrak{S}^\diamond{}_\diamond$, which is the field of linear transformations carrying the "hatted" basis to the "unhatted" basis. We would say that the components of A are $A_\alpha{}^{\hat{\beta}} = 1$ or 0 according as $\alpha = \beta$ or $\alpha \neq \beta$. In this example, the components of A are being taken with respect to two different bases simultaneously. Another situation we will deal with is when there are more bundles present than just $T\mathcal{M}$, so that sections of these other bundles must also be chosen in order to be able to speak of the components of sections of these bundles (and sections of tensor product bundles involving these bundles). Finally, sometimes one want to take components of only some indices of a given tensor, leaving the rest abstract. Below, we will describe conventions for all of these situations, in the form of an example.

Suppose $\mathcal{M}$ is a smooth N-dimensional manifold equipped with a basis $e_0, \ldots, e_{N-1}$. One could also consider a second basis of vector fields, say $e_{\hat{0}}, \ldots, e_{\widehat{N-1}}$. Consider the legal index label $L = {}^{ab}$ and suppose $S \in \mathfrak{S}^{\diamond\diamond}$. Then the assertion that $S^{ab} \in \mathfrak{S}^{ab}$ has components $S^{\gamma\delta}$ (smooth functions, for $\gamma, \delta = 0, \ldots, N-1$) is the assertion that

$$S^{ab} = \sum_{\gamma,\delta=0}^{N-1} S^{\gamma\delta} (e_\gamma)^a (e_\delta)^b. \tag{14.1}$$

The assertion that S^{ab} has components $S^{\gamma\hat{\delta}}$ (smooth functions, for $\gamma = 0, \ldots, N-1$ and $\hat{\delta} = \hat{0}, \ldots, \widehat{N-1}$ is the statment that

$$S^{ab} = \sum_{\gamma,\delta=0}^{N-1} S^{\gamma\hat{\delta}} (e_\gamma)^a (e_{\hat{\delta}})^b.$$

Such mixed-component expressions are rarely used. Finally, one might take components on only one of the abstract indices. The assertion that S^{ab} has components $S^{\alpha b}$ (elements of $\mathfrak{S}^b$, for $\alpha = 0, \ldots, N-1$) is the statement

$$S^{ab} = \sum_{\alpha=0}^{N-1} S^{\alpha b} (e_\alpha)^a,$$

and the assertion that S^{ab} has components $S^{a\hat{\beta}}$ (elements of $\mathfrak{S}^a$, for $\hat{\beta} = \hat{0}, \ldots, \widehat{N-1}$) is the statement

$$S^{ab} = \sum_{\beta=0}^{N-1} S^{a\hat{\beta}} (e_{\hat{\beta}})^b.$$

The typographic operation used above to represent the taking of components consists of replacing an "abstract index" by an "ordinary index", which will typically be a digit, possibly decorated in some way. Many forms of decoration (bars, primes, etc.) besides "hats" are popular, but one must be careful to not confuse such decorations with notation already in use, as in the context of complex vector bundles (section 10), where bars and primes already have meaning. One can reconstruct the original tensor (an element of $\mathfrak{S}(L)$ for some $L \in \mathfrak{L}$) by taking the sum over an appropriate index set of tensor products of its components with suitable basis tensor fields. Note that we have only defined the components of a tensor that lies in $\mathfrak{S}(L)$ for some legal index L, and not the components of an "honest" tensor (i.e., an element of $\mathfrak{S}(s)$ for a shape s). In certain circumstances one can reconstruct an honest tensor by considering $\mathcal{F}_L^{-1}$, provided $\mathcal{F}_L$ is invertible. In the example above, if $S \in \mathfrak{S}^{\diamond\diamond}$, then the invertibility of $\mathcal{F}_L$ (recall $L = {}^{ab}$) together with (14.1) shows that $S = \sum_{\gamma,\delta=0}^{N-1} S^{\gamma\delta} e_\gamma \otimes e_\delta$. In most treatments, the components of S are defined by this equation, but our approach is more flexibile because it allows one to opt for substitution on only a few indices, so that one need give up only as much abstraction as is needed to perform any given computation.

In the example above, we used lowercase greek letters as dummy variables for the summations. We do this because under the conventions of step 3 of section 8 we know that these letters do not already have meanings as index symbols. Another convenient notation for the dummy variables is to use a boldface version of the letter one would normally have used. (This assumes that boldface type does not already have some meaning as a means of decoration of index symbols.) In the example above, one might write $S^{ab} = \sum_{\mathbf{a},\mathbf{b}=0}^{N-1} S^{\mathbf{ab}} (e_{\mathbf{a}})^a (e_{\mathbf{b}})^b$, using this notation.

A much-used convention for work in differential geometry is the Einstein summation conventions, implicitly summing (over the range of all possible values) on any non-abstract index which appears twice in an expression, once as a superscript and once as a subscript. The contraction rules of section 5 are designed to exactly mimic this convention, and largely obviate Einstein's convention. We prefer to not use Einstein's convention, so that the summation sign will serve as a reminder that the expression relies on a choice of a basis (or bases) for its meaning. (Note also the never-mentioned but constantly used differential geometry convention: for each nonrepeated index appearing in an equation (or other assertion), the equation is implicitly quantified, so that

it holds "for each possbile value" of that index. The abstract index notation also supercedes this convention, giving formulae with nonrepeated indices meaning of their own, so that quantification is not necessary.) Of course, one could use the convention if the notation grew too cluttered; this was presumably Einstein's original motivation for introducing it.

In examples similar to that above, one usually considers a dual basis of sections of the dual of some bundle to be given implicitly by a basis of sections of the original bundle. That is, if $e_0, \ldots, e_{N-1}$ is a basis for V then one simultaneously introduces a basis (say) $E^0, \ldots, E^{N-1}$ with the property that $(e_\gamma)^a (E^\delta)_a = 1$ or 0 according as $\gamma = \delta$ or $\gamma \neq \delta$, for all $\gamma, \delta = 0, \ldots, N-1$. For example, if e_0 and e_1 are a basis of $\mathfrak{X}^+$ and $S \in \mathfrak{S}_\diamond{}^{\diamond\diamond}{}_\diamond$, such that $S_0{}^{00}{}_0 = 5$, $S_1{}^{01}{}_1 = 3$, and all other components of $S_a{}^{bc}{}_d$ vanish, then we deduce

$$\begin{aligned} S_a{}^{bc}{}_d &= S_0{}^{00}{}_0 \, (E^0)_a (e_0)^b (e_0)^c (E^0)_d + S_1{}^{01}{}_1 \, (E^1)_a (e_0)^b (e_1)^c (E^1)_d \\ &= 5 \, (E^0)_a (e_0)^b (e_0)^c (E^0)_d + 3 \, (E^1)_a (e_0)^b (e_1)^c (E^1)_d, \end{aligned}$$

and therefore that

$$S = 5 \; E^0 \otimes e_0 \otimes e_0 \otimes E^0 + 3 \; E^1 \otimes e_0 \otimes e_1 \otimes E^1.$$

15. Symmetry operators

It is customary to write (for example) $S_{[ab]}$ for $(S_{ab} - S_{ba})/2$ and $S_{(ab)}$ for $(S_{ab} + S_{ba})/2$, these being the symmetrization and antisymmetrization of S_{ab}. In this section we will describe (informally) how these "symmetry operators" interact with formalisms. We assume a formalism built along the lines of sections 5–7, and that none of the shape or index symbols can be confused with any of the symmetry operators ${}^{[(|)]}$ or ${}_{[(|)]}$. We don't regard them as part of the formalisms, but instead as merely directions to perform various typographical operations. For example, the equation $S_{[ab]cd} = T_{ac(cd)}$ is the assertion that after certain substitutions have been performed, one will have an equation which holds in whatever formalism is being used. Upon encountering a formula involving symmetry operators, one should first perform the appropriate symbol manipulations and only then attempt to interpret the formula.

We will not dwell on these operators because giving a formal definition is not worth the effort—it seems to require a formal description (computer-science style) of the "language" used in the formalism. Humans have no problem figuring out what the operators mean from the rules below. With the viewpoint of the previous paragraph in mind, our use of the operators is completely standard (see [3, §3.3] and [4, §2.4]). The use of these operators presupposes that one of the main modes of decoration of index symbols is by placement as superscipts and subscripts.

If in an expression, a pair of superscript parentheses is innermost (i.e., encloses no superscript parentheses or brackets), and encloses an even number of superscript appearances of the symbol '|', then we say that an index is affected if it is a superscipt index (either abstract or component) that lies between the parentheses, and '|' appears an even number if times between the index and one (either) of the parentheses. If N indices are affected, then one replaces the expression by $1/N!$ times the sum of all expressions obtained by permuting the affected indices amongst each other, and removing the given parentheses and enclosed '|' symbols from the expression. This operation is called symmetrization. Similar considerations apply for an innermost pair of brackets, replacing (before summing them) each of the $N!$ expressions obtained by permutations by the product of that term with $+1$ or -1, according as the associated permutation is even or odd. This operation is called antisymmetrization. Exactly similar conventions apply (to both symmetrization and antisymmetrization operators) with 'superscript' replaced by 'subscript' throughout the above description. With these conventions, superscript and subscript symmetry operators are completely independent of each other.

Here are some examples of the operators at work:

$$A_{[abc]}B^d = \frac{1}{6}\left(A_{abc} - A_{acb} + A_{cab} - A_{cba} + A_{bca} - A_{bac}\right)B^d$$

$$B_{a[b}C_{c]}{}^d = \frac{1}{2}\left(B_{ab}C_c{}^d - B_{ac}C_b{}^d\right)$$

$$I^{(a}J^b_{[c}K^{d)}_{e]} = \frac{1}{6}\left(I^aJ^b_{[c}K^c{}_{d]} + I^aJ^c_{[c}K^b{}_{d]} + I^cJ^a_{[c}K^b{}_{d]} + I^cJ^b_{[c}K^a{}_{d]} + I^bJ^c_{[c}K^a{}_{d]} + I^bJ^a_{[c}K^c{}_{d]}\right)$$

$$G^{(a|bc|d)} = \frac{1}{2}\left(G^{abcd} + G^{dbca}\right)$$

$$H^{(a|[bc]|d)} = \frac{1}{2}\left(H^{(a|bc|d)} - H^{(a|cb|d)}\right)$$

$$I^{(a|b|c|de|f)} = \frac{1}{6}\left(I^{abcdef} + I^{abfdec} + I^{fbadec} + I^{fbcdea} + I^{cbfdea} + I^{cbadef}\right).$$

Note that we have not assigned meaning to expressions involving symmetry operators that contain no innermost pair of brackets or parentheses, such as $A_{(a[bB_{c)d]}}$ or $A_{(ab]}$. Furthermore, performing the indicated typographical operations may yield an expression that has no meaning in whatever index formalism is in use, say by virtue of one of the resulting index labels being illegal. It is the responsibility of the user of the operators to make sure that the expressions they manipulate have meaning.

Our "typographic" conventions reproduce Penrose's convention for the effect of the operators on "multi-indices". For example, in the formalism of section 12, we have

$$S_{[ab]} = (S_{ab} - S_{ba})/2 = (S_{AA'BB'} - S_{BB'AA'})/2.$$

That is, the indices $_a$ and $_b$ are permuted, and only then are the relationships "$_a = {}_{AA'}$" and "$_b = {}_{BB'}$" invoked.

See §3.3 of [3] for a host of useful properties possessed by the symmetry operators. We also note that our use of the symbol '|' differs from that of [2, box 4.1].

16. Notation to be used elsewhere

This document has concerned itself with describing a system of notation. Most of the notation introduced herein need not be used in actual computation; a lot of it it has been necessary only to explain just what index formalisms are. The only symbols introduced here and intended for export are $\mathfrak{S}$, **1**, the shape and index symbols, and the symmetry operators [], () and |. Of course, one might have several index formalisms and distinguish them by using various symbols in place of $\mathfrak{S}$. Penrose uses $\mathfrak{T}$ when he wants to restrict attention to world-tensors, which form modules over the ring $C^\infty(\mathcal{M}, \mathbb{R})$, and $\mathfrak{S}$ when he wants to also consider spinors, which yield modules over $C^\infty(\mathcal{M}, \mathbb{C})$. The symbols $\mathcal{R}$ and $\langle \ | \ \rangle$ might also be used, but $\mathfrak{S}$ should replace $\mathcal{R}$, and the pairing symbols should be unnecessary in light of the "contraction on repeated indices" conventions.

17. Comparison with Penrose's notation

The main difference between our treatment of the abstract index formalism and that of Penrose is that ours is precisely defined. Beyond that, there are only minor differences. Here we recount those we have noticed. References to Penrose are to [3].

Penrose used "multi-indices" (p. 87) to ease discussion about properties of general tensor and spinor fields. While we haven't formally treated multi-indices, we have made arguments and

definitions in terms of a general (usually legal) index label, often denoted L. This is basically the same thing.

Third, our treatment of symmetry operations differs a little from Penrose's (§3.3). The principal difference is that he provides symbols for spaces like "the elements of $\mathfrak{S}^{abcd}$ that are antisymmetric in a, b and c", namely $\mathfrak{S}^{[abc]d}$. Also, he requires that if multi-indices are subject to symmetry operations then they should have the same "valence". We impose no such restriction, the operators merely indicating that typographical manipulations be made. Note, however, that our interpretation in terms of typography produces the same effect as his treatment when the relevant multi-indices do have the same valence. We also do not require that all indices subject to symmetry operators be "of the same type". For example, we accept $T_{[a\Gamma]}$ as denoting $(T_{a\Gamma} - T_{\Gamma a})/2$ even though a and Γ may stand for different modules. While Penrose doesn't rule out such use of the symbols, we suspect from his rule for multi-indices that he would not consider this a legitimate use of the operators. Finally, our symmetry operator conventions are not powerful enough to interpret expressions such as the right hand side of

$$\kappa^{(C}\Delta_{ab}\kappa^{D)} = \{\varepsilon_{A'B'}X_{ABE}{}^{(C} + \varepsilon_{AB}\Phi_{A'B'E}{}^{(C}\}\kappa^{D)}\kappa^{E}$$

(the displayed equation just prior to Penrose's 4.9.7), but it's not clear that his are either.

Finally we mention two trivial notes. Penrose uses lowercase greek letters for abstract indices in work with manifolds; we have chosen roman letters instead, after [4]. Also, we have denoted the type of a spinor (see section 12) by $\binom{k^+\ \ \bar{k}^+}{k^-\ \ \bar{k}^-}$, while Penrose uses brackets rather than parentheses

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