AFFINE MANIFOLDS WITH PARALLEL TORSION AND CURVATURE

WERNER BALLMANN

In these lecture notes I discuss the theorem of Ambrose and Hicks on parallel translation of torsion and curvature [Amb56], [Hic59] and the Lie theoretic description of affine manifolds with parallel torsion and curvature by Nomizu [Nom54]. These results are based on earlier work of E. Cartan on the corresponding problems for locally symmetric Riemannian spaces. Some immediate applications concern results of Kostant [Kos60] and Ambrose and Singer [AS58]. A related discussion and more references can be found in [KN63], [KN69] and [TV83].

This is not the final version of the notes, comments and criticism are welcome. I would like to thank Gregor Weingart for his valuable comments which led to an improvement of an earlier version.

Contents

1. Preliminaries about Affine Manifolds 2
2. Parallel Translation of Torsion and Curvature 5
3. Homogeneous Structures: Uniqueness 8
4. Homogeneous Structures: Existence 10
5. Riemannian Homogeneous Spaces 16
References 18

Date: November 16, 1998.

1. Preliminaries about Affine Manifolds

We assume some familiarity with connections on manifolds. Recall that an *affine manifold* is a manifold together with a connection.

For the convenience of the reader I recall some definitions and general facts. Let M be an affine manifold with connection D, torsion T and curvature R. One of the formulas we need is as follows,

$$\mathcal{S}\{R(X,Y)Z\} = -\mathcal{S}\{T(X,T(Y,Z))\} + \mathcal{S}\{D_X T(Y,Z)\}\,. \tag{1.1}$$

where $\mathcal{S}$ denotes the sum over all cyclic permutations of X, Y and Z. Another formula we need is the *Jacobi equation*,

$$J'' + R(J,\dot c)\dot c = -T(J',\dot c) + T'(J,\dot c)\,, \tag{1.2}$$

where c is a geodesic and the prime denotes covariant differentiation along c.

A continous curve $c : I \to M$ is called a *geodesic polygon* if there is a subdivision $\dots t_{i-1} < t_i < t_{i+1} \dots$ of I such that $c|[t_{i-1}, t_i]$ is a geodesic for all i. For a point $p \in M$, we denote by Π_p the space of all geodesic polygons $c : [0,1] \to M$ with $c(0) = p$, endowed with the compact–open topology.

Let $p \in M$ and $c \in \Pi_p$. Let $0 = t_0 < t_1 < \cdots < t_k = 1$ be a subdivision of $[0,1]$ such that $c|[t_{i-1}, t_i]$ is a geodesic, $1 \le i \le k$. Let $v_i \in T_pM$ be the parallel translate of $\dot c(t_i + 0)$ along $c|[t_0, t_i]$, $0 \le i \le k-1$. Then c is completely determined by the data

$$(v_0, t_1), \dots, (v_{k-1}, t_k) \in T_pM \times [0,1]\,.$$

If M is geodesically complete, any such family of pairs in $T_pM \times [0,1]$ determines a geodesic polygon in Π_p. Since subdivision points may also occur in smooth points of c, the correspondence between c and the data is not one to one. The data correspond to geodesic polygons together with a subdivision of $[0,1]$.

1.3. Lemma. *Let M be a connected affine manifold. Then for any two points $p, q \in M$, there is a geodesic polygon $c : [0,1] \to M$ from p to q.*

Proof. Say that two points $p, q \in M$ are equivalent if there is a geodesic polygon connecting p and q. This defines an equivalence relation on M. The equivalence classes are open and non-empty. The claim now follows since M is connected. □

Let M and $\tilde M$ be affine manifolds of the same dimension and let $f : M \to \tilde M$ be a smooth map. Then f is *affine* if f is a local diffeomorphism and if for all f-related vector fields $X, \tilde X$ and $Y, \tilde Y$ on M and $\tilde M$ respectively, the covariant derivatives $D_X Y$ and $\tilde D_{\tilde X} \tilde Y$ are also f-related. With respect to local coordinate systems (x, U) on M and $(\tilde x, \tilde U)$ on $\tilde M$, this means that in corresponding points

$$\Gamma^k_{ij} \partial_k f^l = \tilde\Gamma^l_{mn} \cdot \partial_i f^m \cdot \partial_j f^n + \partial^2_{ij} f^l\,. \tag{1.4}$$

In particular, if $x = \tilde x \circ f$ — since f is a local diffeomorphism such a choice of coordinates is always possible —, then

$$\Gamma^k_{ij} = \tilde\Gamma^k_{ij}. \tag{1.5}$$

The identity, composition of affine maps and inverses to affine diffeomorphisms are affine. In particular, the space of affine diffeomorphism of M is a group, denoted here by $\mathfrak{A}(M)$. We call elements of $\mathfrak{A}(M)$ also *affine transformations* of manifold M.

The following assertions about affine maps are immediate from the definition but (1.5) makes them even more obvious.

1.6. PROPOSITION. *A local diffeomorphism $f : M \to \tilde{M}$ is affine if and only if for all smooth curves c in M and parallel vector fields X along c, the vector field $\tilde{X} = f_*X$ is parallel along $\tilde{c} = f \circ c$. That is, if P denotes parallel translation, then*

$$f_* \circ P_c = P_{\tilde{c}} \circ f_* \,,$$

1.7. COROLLARY. *If $f : M \to \tilde{M}$ is affine and c is a geodesic in M, then $f \circ c$ is a geodesic in $\tilde{M}$. That is,*

$$\widetilde{\exp} \circ f_* = f \circ \exp \ .$$

Let $f : M \to \tilde{M}$ be an affine map, $p \in M$ and $\tilde{p} = f(p) \in \tilde{M}$. Let $c \in \Pi_p$. Then by Corollary 1.7, $\tilde{c} = f \circ c \in \Pi_{\tilde{p}}$. If $(v_0, t_1), \ldots, (v_{k-1}, t_k = 1)$ is data for c as above and $\tilde{v}_i := f_{*p}v_i$, then by Proposition 1.6, $(\tilde{v}_0, t_1), \ldots, (\tilde{v}_{k-1}, t_k)$ is data for $\tilde{c}$. This is the key observation in the construction of global affine maps further below.

We now discuss the behaviour of tensor fields with respect to affine maps. For us the most important tensor fields are torsion and curvature. In local coordinates, their coefficients are expressions involving the Christoffel symbols and their first derivatives. Hence (1.5) gives the following result.

1.8. PROPOSITION. *Suppose $f : M \to \tilde{M}$ is affine. Then $f^*\tilde{T} = T$ and $f^*\tilde{R} = R$. That is, for all $p \in M$ and $u, v, w \in T_pM$, we have*

$$T_p(u, v) = f_{*p}^{-1}(\tilde{T}_{f(p)}(\tilde{u}, \tilde{v})) \quad \text{and} \quad R_p(u, v)w = f_{*p}^{-1}(\tilde{R}_{\tilde{p}}(\tilde{u}, \tilde{v})\tilde{w}) \,,$$

where $\tilde{p} = f(p)$, $\tilde{u} = f_{*p}u$, $\tilde{v} = f_{*p}v$ and $\tilde{w} = f_{*p}w$.

For other types and tensors, a pull back by f is defined similarly. The following assertion is immediate from Proposition 1.6.

1.9. PROPOSITION. *Suppose $f : M \to \tilde{M}$ is affine. Let A be a parallel tensor field on M, $\tilde{A}$ be a parallel tensor field on $\tilde{M}$. Then $f_p^*\tilde{A}_{f(p)} = A_p$ for one $p \in M$ implies that $f_p^*\tilde{A}_{f(p)} = A_p$ for all $p \in M$.*

For us, the most important example is the case where A and $\tilde{A}$ are Riemannian metrics. Another case of interest is that of parallel complex structures.

We now turn to the infinitesimal version of affine transformations. Let X be a vector field X on M and let $f = f_t$ be the flow of X. We say that X is an *affine vector field* if for every t, f_t is an affine map on its domain of definition.

To characterize affine vector fields by a differential equation we introduce the Lie derivative of a connection D,

$$\begin{aligned}(\mathcal{L}_X D)_Y Z &= \lim_{t\to 0}\frac{1}{t}((f_t^* D)_Y Z - D_Y Z)\\ &= [X, D_Y Z] - D_{[X,Y]}Z - D_Y[X,Z]\,,\end{aligned} \tag{1.10}$$

where $f = f_t$ denotes the flow of X. Note that $\mathcal{L}_X D$ is tensorial in Y and Z.

1.11. Proposition. *Vector field X is affine if and only if $\mathcal{L}_X D = 0$. The space of affine vector fields on M is a Lie subalgebra of the space of all vector fields.*

Proof. Since $f = f_t$ is a flow,

$$\begin{aligned}f_t^*(\mathcal{L}_X D) &= f_*^*(\lim_{\tau\to 0}\frac{1}{\tau}(f_\tau^*(D) - D))\\ &= \lim_{\tau\to 0}\frac{1}{\tau} f_t^*(f_\tau^*(D) - D)\\ &= \lim_{\tau\to 0}\frac{1}{\tau}(f_{t+\tau}^*(D) - f_t^*(D))\,.\end{aligned}$$

Therefore

$$f_t^*(D) - D = \int_0^t f_\tau^*(\mathcal{L}_X D)d\tau\,,$$

hence the first claim. The second claim is an immediate consequence. □

We now derive a characterization of affine vector fields by another differential equation, a global form of the Jacobi equation. Note that the flow of an affine vector field X maps geodesics to geodesics, hence X restricted to a geodesic c is a Jacobi field along c.

1.12. Proposition. *A vector field X on M is affine if and only if for all vector fields Y and Z on M*

$$D^2X(Y,Z) + R(X,Y)Z = T(Z, D_Y X) + (D_Y T)(Z,X)\,.$$

Proof. Direct calculation shows that the right hand side is equal to

$$D^2X(Y,Z) + R(X,Y)Z - (\mathcal{L}_X D)_Y Z\,.$$

Now the claim is immediate from Proposition 1.11. □

We say that M respectively D is *complete* if the maximal domain of definition of geodesics is $\mathbb{R}$.

1.13. Proposition. *If M is complete, then affine vector fields on M are complete.*

Proof. We can assume that M is connected. Let X be an affine field on M and f_t be the flow of X. Let $p \in M$ and assume that $f_t(p)$ is defined on $(-\varepsilon, \varepsilon)$. Let $q \in M$ and $c = c(s)$, $0 \le s \le 1$, be a geodesic polygon from p to q. Let

$$(v_0, s_1), \ldots, (v_{k-1}, s_k = 1) \in T_pM \times [0,1]$$

be data for c. For $t \in (-\varepsilon, \varepsilon)$, set $v_i(t) = f_{t*}(v_i)$. Since M is complete, there exists a geodesic polygon c_t starting from $f_t(p)$ and with data

$$(v_0(t), s_1), \ldots, (v_{k-1}(t), s_k)\,.$$

For each $t \in (-\varepsilon, \varepsilon)$, there is a $\delta > 0$ such that $f_\tau(c_t(s))$ is defined for all $s \in [0,1]$ and $\tau \in (-\delta, \delta)$.

By Proposition 1.7, $f_\tau(c_t(1)) = c_{t+\tau}(1)$. Since f is a flow, $f_t(q) = f_t(c(1))$ exists for all $t \in (-\varepsilon, \varepsilon)$ and is equal to $c_t(1)$. Since q is arbitrary, X is complete. □

2. Parallel Translation of Torsion and Curvature

Let M and $\tilde{M}$ be affine manifolds. Let $p \in M$, $\tilde{p} \in \tilde{M}$ and $L : T_pM \to T_{\tilde{p}}\tilde{M}$ be a linear isomorphism. Suppose that U_0 is an open neighborhood of 0 in T_pM, star shaped with respect to 0, such that

$$\exp : U_0 \to U := \exp(U_0) \quad \text{and} \quad \widetilde{\exp} : LU_0 \to \tilde{U} := \widetilde{\exp}(LU_0)$$

are diffeomorphisms. Then $f = \widetilde{\exp} \circ L \circ \exp^{-1}$ is a diffeomorphism from U to $\tilde{U}$ with $f(p) = \tilde{p}$ and $df(p) = L$.

For $q \in U$, denote by $c_q : [0,1] \to U$ the radial geodesic from p to q and set $\tilde{q} = f(q) \in \tilde{U}$ and $\tilde{c}_q = f \circ c_q$. Furthermore, denote by P_q parallel translation along c_q and $\tilde{P}_q$ parallel translation along $\tilde{c}_q$.

2.1. Theorem. *The above map $f = \widetilde{\exp} \circ L \circ \exp^{-1}$ is affine if and only if $L^*\tilde{P}_q^*\tilde{T}_{\tilde{q}} = P_q^*T_q$ and $L^*\tilde{P}_q^*\tilde{R}_{\tilde{q}} = P_q^*R_q$ for all $q \in U$.*

Proof. Necessity of the condition on the pull back of torsion and curvature follows from Proposition 1.8. Assume now that f satisfies these conditions.

Choose a basis $b_1, \ldots, b_n$ to identify $T_pM = \mathbb{R}^n$. Let $x = \exp^{-1} : U \to U_0$ and $\tilde{x} = L^{-1}\widetilde{\exp}^{-1} : \tilde{U} \to U_0$. In these coordinates, f is represented by the identity, that is, $\tilde{x} \circ f \circ x^{-1} = \mathrm{id}$. Hence using the chosen coordinates, we may think of the connections D of M and $\tilde{D}$ of $\tilde{M}$ as connections on $U_0 \subset \mathbb{R}^n$. Our aim is to show that D and $\tilde{D}$ are equal. Note that with respect to both connections, lines through 0 are geodesics. We call these lines *radial geodesics*.

Let $E_1, \ldots, E_n$ be the frame on U_0 which is D–parallel along radial geodesics and such that $E_m(0) = b_m$. Similarly, let $\tilde{E}_1, \ldots, \tilde{E}_n$ be the frame on U_0 which is $\tilde{D}$–parallel along radial geodesics and with $\tilde{E}_m(0) = \tilde{b}_m$. Let T_{ij}^l and R_{ijk}^l be the coefficients of T and R with respect to the frame $E_1, \ldots, E_n$ and $\tilde{T}_{ij}^l$ and $\tilde{R}_{ijk}^l$ be the coefficients of $\tilde{T}$ and $\tilde{R}$ with respect to the frame $\tilde{E}_1, \ldots, \tilde{E}_n$. Now $\tilde{E}_m(0) = E_m(0)$, hence our assumption on the pull backs of torsion and curvature implies that $T_{ij}^l = \tilde{T}_{ij}^l$ and $R_{ijk}^l = \tilde{R}_{ijk}^l$.

Let $c(t) = tv, 0 \le t \le 1$, be a radial geodesic in U_0. Then $J(t) = tb_m$, $0 \le t \le 1$, is a Jacobi field along c — with respect to both connections — corresponding to

the variation by the radial geodesics $c_s(t) = t(v + sb_m)$. Hence J satisfies the Jacobi equation

$$J'' + R(J, \dot c)\dot c = T(\dot c, J') + T'(\dot c, J)$$

with respect to D and the corresponding Jacobi equation for $\tilde D$. Therefore the coefficients α^l_m of J with respect to the frame $E_1, \dots, E_n$ satisfy the system of differential equations

$$\ddot\alpha^l_m + R^l_{ijk}\alpha^i_m \dot c^j \dot c^k = T^l_{ij}\dot c^i \dot\alpha^j_m + \dot T^l_{ij}\dot c^i \alpha^j_m$$

with initial condition

$$\alpha^l_m(0) = 0\,, \quad \dot\alpha^l_m(0) = \delta^l_m\,.$$

The corresponding statement holds for the coefficients $\tilde\alpha^l_m$ of J with respect to the frame $\tilde E_1, \dots, \tilde E_n$. Therefore $\tilde\alpha^l_m = \alpha^l_m$. Now the constant vector fields $b_1, \dots, b_n$ on U_0 are a frame on U_0 and hence $E_m = \tilde E_m$, $1 \le m \le n$.

Next we consider the Christoffel symbols Γ^k_{ij} of D and $\tilde\Gamma^k_{ij}$ of $\tilde D$ with respect to the frame $E_1, \dots, E_n$ on U_0,

$$D_{E_i}E_j = \Gamma^k_{ij}E_k \quad \text{and} \quad \tilde D_{E_i}E_j = \tilde\Gamma^k_{ij}E_k$$

Now if c is a curve in an open subset U_0 of an affine manifold and X, Y are vector fields on U_0 such that Y is parallel along c, then $D_t D_X Y = R(\dot c, X)Y$. This we apply to the vector fields E_m along a radial geodesic c as above to get a system of differential equations for the Christoffel symbols as functions of the parameter t of c,

$$\dot\Gamma^k_{ij}E_k = D_t(\Gamma^k_{ij}E_k) = D_t D_{E_i}E_j = R(\dot c, E_i)E_j = v^l R^k_{lij}E_k\,.$$

Now the the Christoffel symbols $\tilde\Gamma^k_{ij}$ satisfy the same system of differential equations. Since the covariant derivatives of $E_1, \dots, E_n$ with respect to D and $\tilde D$ vanish in 0,

$$\Gamma^k_{ij}(0) = \tilde\Gamma^k_{ij}(0) = 0\,,$$

we conclude that the Christoffel symbols of the two connections are equal. Hence $D = \tilde D$. □

2.2. LEMMA. *Let M be an affine manifold. Then for any two points $p, q \in M$,*
1) if $c : [0,1] \to M$ is a continuous curve from p to q, then c is homotopic to a geodesic polygon from p to q;
2) if two geodesic polygons $c_0, c_1 : [0,1] \to M$ from p to q are homotopic as continuous curves, then there is a homotopy $c_s : [0,1] \to M$, $0 \le s \le 1$, from c_0 to c_1 admitting a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0,1]$ such that $c_s|[t_{i-1}, t_i]$ is a geodesic for all $s \in [0,1]$, $1 \le i \le k$.

We refrain from discussing the straightforward but tedious proof of these rather obvious assertions.

Let M and $\tilde M$ be affine manifolds and suppose in addition that $\tilde M$ is complete. Let $p \in M$, $\tilde p \in \tilde M$ be points and $L : T_pM \to T_{\tilde p}\tilde M$ be a linear isomorphism. For a geodesic polygon $c \in \Pi_p$, let $(v_0, t_1), \dots, (v_{k-1}, t_k = 1)$ be data in

$T_pM \times [0,1]$. Since $\tilde{M}$ is complete, there is a (unique) polygon $\tilde{c} \in \Pi_{\tilde{p}}$ with data $(Lv_0, t_1), \dots, (Lv_{k-1}, t_k = 1)$ and $\tilde{c}$ is independent of the choice of data for c. For $c \in \Pi_p$ and the corresponding $\tilde{c} \in \Pi_{\tilde{p}}$, we denote by P_c parallel translation along c and $\tilde{P}_c$ parallel translation along $\tilde{c}$.

2.3. THEOREM. *Suppose that M is simply connected (and $\tilde{M}$ is complete) and that $L^*\tilde{P}_c^*\tilde{T}_{\tilde{q}} = P_c^*T_q$ and $L^*\tilde{P}_c^*\tilde{R}_{\tilde{q}} = P_c^*R_q$ for all $c \in \Pi_p$, where $q = c(1)$ and $\tilde{q} = \tilde{c}(1)$. Then there is a unique affine map $f : M \to \tilde{M}$ with $f(p) = \tilde{p}$ and $f_{*p} = L$. Moreover, if M is complete and $\tilde{M}$ is connected, then f is a covering map.*

Proof. For $q \in M$, denote by $\Pi_{pq} \subset \Pi_p$ the space of geodesic polygons $c : [0,1] \to M$ from p to q, endowed with the compact–open topology. By Lemma 1.3, Π_{pq} is not empty. For $c \in \Pi_{pq}$ and the corresponding $\tilde{c} \in \Pi_{\tilde{p}}$ as above set $\tilde{q} = f(q) = \tilde{c}(1)$. The main point of the proof is to show that f is in fact well defined, that is, $\tilde{q}$ is independent of the choice of c.

Now M is simply connected and hence by Lemma 2.2, Π_{pq} is connected. Hence it suffices to show that for any geodesic polygon $c_0 \in \Pi_{pq}$, $\tilde{c}_1(1) = \tilde{c}_0(1)$ for any polygon $c_1 \in \Pi_{pq}$ sufficiently close to c_0.

For a given subdivision $0 = t_0 < t_1 < \cdots < t_k$ of $[0,1]$, denote by P_i and $\tilde{P}_i$ parallel translation along $c_0|[t_{i-1}, t_i]$ and $\tilde{c}_0|[t_{i-1}, t_i]$ respectively and set

$$\tilde{L}_i = \tilde{P}_i \cdot L_{i-1} \cdot P_i^{-1},$$

where $L_0 = L$. Then our assumption on the parallel translation of torsion and curvature implies that the triples $(p_i = c_0(i), \tilde{p}_i = \tilde{c}_0(i), L_i)$ satisfy the assumptions of Theorem 2.1. Hence if the subdivision is chosen fine enough, then for any $i \le k-2$ there are open neighborhoods U_i of $c_0([t_i, t_{i+2}])$, $\tilde{U}_i$ of $\tilde{c}_0([t_i, t_{i+2}])$ and affine diffeomorphisms $f_i : U_i \to \tilde{U}_i$ with $f_i(c_0(t_i)) = \tilde{c}_0(t_i)$ and $df_i(c_0(t_i)) = L_i$. For $i \ge 2$ we choose neighborhoods V_i of $c_0(t_i)$ such that for any point in V_i, there is a piecewise smooth curve from $c_0(t_{i-1})$ to $c_1(t_i)$ which is contained in the intersection $U_{i-2} \cap U_{i-1}$.

Now let $c_1 \in \Pi_{pq}$ be a polygon with $c_1([t_i, t_{i+2}]) \subset U_i$ and $c(t_i) \in V_i$. (It is not required that $c_1|[t_{i-1}, t_i]$ is a geodesic, c_1 might have a different set of breaks.) We show that $\tilde{c}_1(1) = \tilde{c}_0(1)$.

For $i \ge 1$, let $\sigma_i : [t_i, t_{i+1}] \to U_i \cap U_{i-1}$ be a piecewise smooth curve from $c_0(t_i)$ to $c_1(t_{i+1})$, where we choose $\sigma_{k-1} := c_0|[t_{k-1}, 1]$. Let $\tilde{\sigma}_i = f_i \circ \sigma_i$. Denote by $Q_i, \tilde{Q}_i, R_i$ and $\tilde{R}_i$ parallel translation along $c_1|[t_{i-1}, t_i]$, $\tilde{c}_1|[t_{i-1}, t_i]$, σ_i and $\tilde{\sigma}_i$ respectively. Set $K_i = \tilde{Q}_i \circ K_{i-1} \circ Q_i^{-1}$, where $K_0 = L$.

Since the curves $c_0|[0, t_1]$, $c_1|[0, t_2]$ and σ_1 are contained in U_0 and f_0 is affine with $f_0(p) = \tilde{p}$ and $df_0(p) = L$, we have

$$\tilde{c}_1(t_2) = f_0(c_1(t_2)) = f_0(\sigma_1(t_2)) = \tilde{\sigma}_1(t_2),$$

$$L_1 = \tilde{P}_1 \circ df_0(p) \circ P_1^{-1} = df_0(c_0(t_1))$$

and similarly $K_1 = df_0(c_1(t_1))$, $K_2 = df_0(c_1(t_2))$ and $K_2 = \tilde{R}_1 \circ L_1 \circ R_1^{-1}$. Inductively we assume for some $i \leq k-2$ that

$$\tilde{\sigma}_i(t_{i+1}) = \tilde{c}_i(t_{i+1}) \quad \text{and} \quad K_{i+1} = \tilde{R}_i \circ K_i \circ R_i^{-1}.$$

Since the four curves $c_0|[t_i, t_{i+1}]$, $c_1|[t_{i+1}, t_{i+2}]$, σ_i and σ_{i+1} are in U_i and since f_i is affine with $f_i(c_0(t_i)) = \tilde{c}_0(t_i)$ and $df_i(c_0(t_i)) = L_i$, we have $K_{i+1} = df_i(c_1(t_{i+1}))$ and hence $\tilde{c}_0 = f_i \circ c_0$ on $[t_i, t_{i+1}]$ and $\tilde{c}_1 = f_i \circ c_1$ on $[t_{i+1}, t_{i+2}]$. In particular, $\tilde{\sigma}_{i+1}(t_{i+2}) = \tilde{c}_1(t_{i+2})$. Again since f_i is affine we have $L_{i+1} = df_i(c_0(t_{i+1}))$ and

$$K_{i+2} = df_i(c_1(t_{i+2})) = \tilde{R}_{i+1} \circ K_{i+1} \circ R_{i+1}^{-1}.$$

We conclude that induction applies and hence that $\tilde{c}_1(1) = \tilde{c}_0(1)$. Therefore f is well defined.

Locally about each point q, map f is given as $\widetilde{\exp}_{\tilde{q}} \circ df(q) \circ \exp_q^{-1}$, where $\tilde{q} = f(q)$. Hence f is affine by Theorem 2.1. To prove the last assertion in the theorem, we assume first that $\tilde{M}$ is simply connected and complete. Then the roles of M and $\tilde{M}$ can be reversed: The first part of the theorem applies to L^{-1} and we obtain an affine map $\tilde{f} : \tilde{M} \to M$ with $\tilde{f}(\tilde{p}) = p$, $d\tilde{f}(\tilde{p}) = L^{-1}$. But then $\tilde{f} \circ f$ is an affine map from M to M with $f(p) = p$ and $df(p) = \mathrm{id}$. By uniqueness, $\tilde{f} \circ f$ is the identity of M. Similarly, $f \circ \tilde{f}$ is the identity of $\tilde{M}$ and hence f is a diffeomorphism. Now the general case follows by passing to the universal covering space of M with the induced connection. □

2.4. Corollary. *Let M and $\tilde{M}$ be connected affine manifolds and $f : M \to \tilde{M}$ be an affine map. If M is complete, then $\tilde{M}$ is complete and f is a covering map.*

Proof. The first claim is obvious, and hence the second claim follows immediately from Theorem 2.3 □

3. Homogeneous Structures: Uniqueness

Let M be an affine manifold with parallel torsion and curvature. Then for any two points $p, q \in M$ and piecewise smooth curve c connecting p to q we have $L^* T_q = T_p$ and $L^* R_q = R_p$. This reduces the condition on parallel translation of torsion and curvature considered in Theorem 2.3.

3.1. Theorem. *Let M, $\tilde{M}$ be simply connected, complete affine manifolds with parallel torsion and curvature. Suppose there are points $p \in M$, $\tilde{p} \in \tilde{M}$ and a linear isomorphism $L : T_pM \to T_{\tilde{p}}\tilde{M}$ with $L^*(\tilde{T}_{\tilde{p}}) = T_p$ and $L^*(\tilde{R}_{\tilde{p}}) = R_p$. Then there is an affine diffeomorphism $f : M \to \tilde{M}$.*

Proof. Since M and $\tilde{M}$ have parallel torsion and curvature, the assumption on the pull back of torsion and curvature in p implies that the assumption on parallel translation of torsion and curvature in Theorem 2.3 is satisfied. Now the corresponding affine map f is a covering. Since $\tilde{M}$ is simply connected, f is a diffeomorphism. □

3.2. THEOREM. *Let M be a simply connected, complete affine manifold with parallel torsion and curvature. Then M if affinely homogeneous. More precisely, for any two points $p, q \in M$ and piecewise smooth curve c from p to q, there is an affine automorphism $f : M \to M$ with $f(p) = q$ and $df(p) = P_c$, where P_c denotes parallel translation along c.*

Let M be a connected affine manifold. Then an affine map $f : M \to M$ is called a *transvection* if there is a point $p \in M$ and a piecewise smooth curve c from p to $f(p)$ such that $df(p) = P_c$. If the latter condition is satisfied for some point $p \in M$, it holds for any point. For if $q \in M$ is any other point and σ is any piecewise smooth curve from p to q, then

$$df(q) = P_{f\circ\sigma} P_c P_\sigma = P_{(f\circ\sigma)*c*\sigma} \tag{3.3}$$

by Proposition 1.6. In particular, the transvections of M constitute a normal subgroup $\mathfrak{T}(M) \subset \mathfrak{A}(M)$. Now Theorem 3.2 says that (under the assumptions made there) parallel translation along any curve is realized by a (unique) transvection. Hence $\mathfrak{T}(M)$ is transitive on M and the stabilizer of $\mathfrak{T}(M)$ at a given point is the holonomy group of that point.

3.4. THEOREM. *Let M be a simply connected affine manifold with connection D. Then M is affinely homogeneous with respect to D if there exists a complete connection $\bar{D}$ on M with parallel torsion and curvature such that the difference $S = D - \bar{D}$ is $\bar{D}$–parallel.*

Proof. By Theorem 3.2 the group $\bar{\mathfrak{T}}(M)$ of transvections of $\bar{D}$ is transitive on M. Now $\bar{\mathfrak{T}}(M)$ preserves S since S is $\bar{D}$–parallel. Hence $\bar{\mathfrak{T}}(M)$ preserves $D = \bar{D} + S$ and therefore all elements of $\bar{\mathfrak{T}}(M)$ are affine with respect to D. □

For the assertion opposite to the one in Theorem 3.4 we refer to Remark 4.13 below.

The conditions in Theorem 3.4 can be restated in a different way. For this we note first that torsion and curvature tensors of D and $\bar{D}$ are related by

$$T(X,Y) = \bar{T}(X,Y) + S_X Y - S_Y X\,; \tag{3.5}$$

$$R(X,Y)Z = \bar{R}(X,Y)Z + (\bar{D}_X S)_Y Z - (\bar{D}_Y S)_X Z + [S_X, S_Y]\cdot Z\,. \tag{3.6}$$

Hence if S is $\bar{D}$–parallel, then $\bar{T}$ and $\bar{R}$ are $\bar{D}$–parallel if and only if T and R are.

3.7. PROPOSITION. *Let D, $\bar{D}$ be connections on a manifold M and $S = D - \bar{D}$. Then S, $\bar{T}$ and $\bar{R}$ are parallel with respect to $\bar{D}$ if and only if*

$$\begin{aligned}(D_X S)_Y &= [S_X, S_Y] - S_{S_X Y}\,;\\ (D_X T)_Y &= [S_X, T_Y] - T_{S_X Y}\,;\\ (D_X R)_{YZ} &= [S_X, R_{YZ}] - R_{S_X YZ} - R_{Y S_X Z}\,.\end{aligned}$$

Clearly a simply connected complete affine manifold with $T = 0$ and $R = 0$ is isomorphic to $\mathbb{R}^n$, $n = \dim M$, with the canonical connection. We now discuss two important examples where the condition $T = 0$ respectively $R = 0$ is relaxed and where Theorem 3.2 applies.

3.8. *Example.* Let M be a simply connected complete affine manifold with $DT = 0$ and $R = 0$. Since M is simply connected with $R = 0$, parallel translation is independent of the path and defines a global trivialization of TM by parallel vector fields. Since $DT = 0$, the Lie bracket of any two parallel vector fields is parallel. Hence the vector space $\mathfrak{g}$ of parallel vector fields on M together with the Lie bracket (of vector fields) is a Lie algebra. It follows that $\mathfrak{T}(M)$ operates simply transitively on M. Hence the choice of a point $p \in M$ identifies $\mathfrak{T}(M)$ and M through the map $f \mapsto f(p)$. In this way M inherits a group structure, turning M into a Lie group such that $\mathfrak{g}$ is the Lie algebra of left invariant vector fields on M.

Vice versa, if $\mathfrak{G}$ is a Lie group, then requiring left invariant vector fields to be parallel defines a connection on M with $DT = 0$ and $R = 0$.

3.9. *Example.* Assume now that M is a simply connected complete affine manifold with $T = 0$ and $DR = 0$. Then for any point $p \in M$ the linear map $L = -\mathrm{id} : T_pM \to T_pM$ satisfies the assumptions of Theorem 3.2 (since $T = 0$ and R has an odd number of arguments). Hence for any point $p \in M$ there is an affine automorphism s_p of M with $s_p(p) = p$ and $ds_p(p) = -\mathrm{id} : T_pM \to T_pM$. By definition, this means that M is an *affinely symmetric space*. Vice versa, if M is an affinely symmetric space, then $T = 0$ and $DR = 0$: for every point $p \in M$ and all vectors $u, v \in T_pM$ we have

$$-T_p(u,v) = ds_p \cdot T_p(u,v) = T_p(ds_p \cdot u, ds_p \cdot v) = T_p(-u,-v) = T_p(u,v)\,,$$

hence $T = 0$. Since DR has an even number of arguments, the same argument applies and shows that $DR = 0$.

4. Homogeneous Structures: Existence

We now consider the homogeneous case more closely. We start with a few general remarks on smooth actions of Lie groups. Let $\mathfrak{G}$ be a Lie group acting smoothly on a manifold M. The Lie algebra of left invariant vector fields of $\mathfrak{G}$ is denoted $\mathfrak{g}$. As usual, we identify $\mathfrak{g} = T_e\mathfrak{G}$. For any $X \in \mathfrak{g}$, the 1–parameter group $g_t = \exp(tX)$, $t \in \mathbb{R}$, of $\mathfrak{G}$ generates a vector field X^* on M,

$$X^*(p) = \partial_t(g_t(p))|_{t=0} = \partial_t(e^{tX}p)|_{t=0}\,. \tag{4.1}$$

The map $X \mapsto X^*$ is linear, but changes the sign of the Lie bracket,

$$[X,Y]^* = -[X^*,Y^*]\,. \tag{4.2}$$

We fix a point $p \in M$ and let $\mathfrak{K} = \{g \in \mathfrak{G} \mid g(p) = p\}$ be the stabilizer of p. The Lie algebra of $\mathfrak{K}$ is denoted $\mathfrak{k}$. We have

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid X^*(p) = 0\}\,. \tag{4.3}$$

We assume now that the action of $\mathfrak{G}$ on M is transitive. Then the smooth map $\pi : \mathfrak{G} \to M$, $\pi(g) = gp$, is a submersion. The fiber of π over $q = gp \in M$ is $g\mathfrak{K}$ and with respect to the right action of $\mathfrak{K}$ on $\mathfrak{G}$, π is a $\mathfrak{K}$–principal bundle over M. (We are not going to use the theory of principal bundles here.) The differential of π at the neutral element $e \in \mathfrak{G}$ is denoted π_*. The kernel of π_* is $\mathfrak{k}$. For any $k \in \mathfrak{K}$

$$k_{*p} \circ \pi_* = \pi_* \circ \mathrm{Ad}_k\,, \tag{4.4}$$

where k_{*p} denotes the differential of k at p, viewed as a diffeomorphism of M.

We now let M be a simply connected, complete affine manifold with parallel torsion and curvature. By Theorem 2.3, the Lie group $\mathfrak{G} = \mathfrak{A}(M)$ of affine automorphisms of M acts transitively on M. We fix a point $p \in M$ and denote by $\mathfrak{K}$ the stabilizer of p.

Let $u \in T_pM$ and c be the geodesic through p with $\dot c(0) = u$. Let $P_t : T_pM \to T_{c(t)}M$ be parallel translation along c from $c(0)$ to $c(t)$. Then P_t is a linear isomorphism and since T and R are parallel we have $P_t^* T_{c(t)} = T_p$ and $P_t^* R_{c(t)} = R_p$. Hence by Theorem 3.2, there is $g_t \in \mathfrak{G}$ with $g_t(p) = c(t)$ and $dg_t(p) = P_t$, a *transvection along* c.

Now c is a geodesic and hence $P_t v = \dot c(t)$. Hence the geodesic c is invariant under P_t, that is, $P_t(c(s)) = c(s+t)$ for all $s \in \mathbb{R}$. Since g_t is affine, it maps parallel vector fields along c to parallel vector fields along c. By our choice of $dg_t(p)$, g_t simply reparameterizes these parallel vector fields. Hence $dg_t(c(s))$ is parallel translation along c from $c(s)$ to $c(s+t)$. Therefore $g_{s+t} = g_s \circ g_t$ for all $s, t \in \mathbb{R}$. Hence the transvections g_t, $t \in \mathbb{R}$, along c constitute a 1–parameter subgroup of $\mathfrak{G}$. The corresponding generator $X \in \mathfrak{g}$ or vector field X^* of M is called the *infinitesimal transvection* along c.

Let X^* be the infinitesimal transvection along c as above. Then $X^*(p) = u$ and, more generally, $X^*(c(t)) = \dot c(t)$. Let $v \in T_pM$ and $\sigma = \sigma(s)$ be a curve through p with $\dot\sigma(0) = v$. Then

$$\begin{aligned} D_v X^* &= D_s \partial_t \{g_t(\sigma(s))\}|_{s=t=0} \\ &= T_p(v,u) + D_t \sigma_s \{g_t(\sigma(s))\}|_{t=s=0} \\ &= T_p(v,u) + D_t\{dg_t \cdot v\}|_{t=0} \\ &= T_p(v,u) + D_t(P_t \cdot u) = T_p(v,u)\,. \end{aligned} \tag{4.5}$$

Recall that an affine vector field is determined by its value and covariant derivative at a point. Now the above equation for $DX^*(p)$ is linear in $X^*(p) = u$ and therefore the set of infinitesimal transvections X^* along geodesics through p forms

a vector subspace $\mathfrak{p}$ of $\mathfrak{g}$. In fact, the above computation gives

$$\mathfrak{p} = \{X \in \mathfrak{g} \mid D_v X^*(p) = T_p(v, X^*(p)) \text{ for all } v \in T_pM\}\,. \tag{4.6}$$

Since any vector $u \in T_pM$ determines a geodesic through p and thus a 1–parameter group of transvections, we obtain a decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}\,. \tag{4.7}$$

If g is an affine transformation of M and g_t, $t \in \mathbb{R}$, is the 1–parameter group of transvections along a geodesic c through p, then gg_tg^{-1}, $t \in \mathbb{R}$, is the 1–parameter group of transvections along the geodesic $gc = g \circ c$ through gp. In particular, for any k in the stabilizer $\mathfrak{K}$ of p, kg_tk^{-1}, $t \in \mathbb{R}$, is the 1–parameter group of transvections along the geodesic kc through $kp = p$. Hence for any $k \in \mathfrak{K}$,

$$\mathrm{Ad}_k(\mathfrak{p}) \subset \mathfrak{p}\,. \tag{4.8}$$

Recall that an action of a Lie group $\mathfrak{G}$ on a manifold M is called *reductive* if for one — and hence any — point $p \in M$, there is an $\mathrm{Ad}_{\mathfrak{K}}$–invariant splitting $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of the stabilizer $\mathfrak{K}$ of p in $\mathfrak{G}$. Thus our discussion shows that the action of $\mathfrak{A}(M)$ on M is reductive: the choice of point p leads to a natural $\mathrm{Ad}_{\mathfrak{K}}$–invariant splitting $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the Lie algebra of $\mathfrak{G}$.

We show next that all the information on the connection is encoded in the Lie algebra structure of $\mathfrak{g}$ and the splitting. However, we do this in somewhat greater generality.

Let $\mathfrak{G}$ be a Lie group, $\mathfrak{K} \subset \mathfrak{G}$ a closed subgroup and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ an $\mathrm{Ad}_{\mathfrak{K}}$–invariant splitting. Let $M = \mathfrak{G}/\mathfrak{K}$ be the orbit space and $p = [\mathfrak{K}] \in M$ be the chosen, distinguished point in M. As above, we let $\pi : \mathfrak{G} \to M$ be the map $\pi(g) = gp$. Then $\pi_* : \mathfrak{p} \to T_pM$ is an isomorphism which we use to identify $\mathfrak{p} \cong T_pM$. This simplifies some of the statements and formulas below. For $X \in \mathfrak{g}$, we let $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$ be the decomposition into the components belonging to $\mathfrak{k}$ and $\mathfrak{p}$ respectively. Then

$$\pi_* X = \pi_* X_{\mathfrak{p}} = X^*(p)\,. \tag{4.9}$$

4.10. Theorem. *There is a unique $\mathfrak{G}$–invariant connection D on M such that for any $X \in \mathfrak{p}$, $c(t) = \exp(tX)p$, $t \in \mathbb{R}$, is the geodesic through p with initial velocity $\pi_*(X) = X^*(p)$ and such that $\exp(tX)$, $t \in \mathbb{R}$, is a 1–parameter group of transvections along c. Moreover,*

1. *the restriction to $T_pM \cong \mathfrak{p}$ identifies $\mathfrak{G}$–invariant tensor fields on M with $\mathrm{Ad}_{\mathfrak{K}}$–invariant tensors on $\mathfrak{p}$ and any $\mathfrak{G}$–invariant tensor field on M is parallel with respect to D.*
2. *connection D has parallel torsion and curvature; for $X, Y, Z \in \mathfrak{p} \cong T_pM$,*
$$T_p(X,Y) = -[X,Y]_{\mathfrak{p}}\,, \quad R_p(X,Y)Z = -[[X,Y]_{\mathfrak{k}}, Z]\,.$$
3. *any automorphism φ of $\mathfrak{G}$ leaving $\mathfrak{K}$ and $\mathfrak{p}$ invariant induces an affine transformation f of M, and f leaves those $\mathfrak{G}$–invariant tensor fields on M invariant whose restriction to $T_pM \cong \mathfrak{p}$ is invariant under $\mathrm{Ad}_{\mathfrak{K}}$ and φ.*

Proof. Let $W \subset \mathfrak{p}$ be an open neighborhood of 0, star shaped with respect to 0 and such that

$$W \ni X \mapsto e^X p$$

is a diffeomorphism onto an open neighborhood U of p in M. Choose a basis $b_1, \dots, b_n$ of T_pM. For $g \in \mathfrak{G}$ and $q = gp$, define smooth vector fields V_i on $gU = g(U)$ by

$$V_i(ge^X p) = g_* e^X_* b_i \,.$$

Then $V_1, \dots, V_n$ is a frame of TM over gU. If Y is a smooth vector field on gU, then over gU, $Y = \varphi^i V_i$ with smooth functions $\varphi^1, \dots, \varphi^n$. For $u \in T_qM$, set

$$D_u X := u(\varphi^i) V_i(q).$$

In other words, we define the vector fields V_i to be parallel at q.

We show that D is a connection as required. First of all, D is independent of the chosen basis. If $\tilde{b}_1, \dots, \tilde{b}_n$ is another basis of T_pM and $Y = \tilde{\varphi}^j \tilde{V}_j$ for the corresponding local frame $\tilde{V}_1, \dots, \tilde{V}_n$, then $\varphi^i = \tilde{\varphi}^j \alpha^i_j$, where $\tilde{b}_j = \alpha^i_j b_i$ is the change of basis. Since the α^j_i are constants,

$$u(\varphi^i) V_i(q) = u(\tilde{\varphi}^j) \alpha^i_j V_i(q) = u(\tilde{\varphi}^j) \tilde{V}_j(q) \,.$$

Second, $D_u Y$ is independent of the choice of g: If we write $q = \tilde{g}p$, then $\tilde{g} = gk$ for some $k \in \mathfrak{K}$ and

$$\tilde{g} e^X p = gke^X k^{-1} kp = g \exp(\mathrm{Ad}_k X) kp = g \exp(\mathrm{Ad}_k X) p$$

and

$$\tilde{g}_* e^X_* v = g_* k_* e^X_* k^{-1}_* k_* v = g_* (\exp(\mathrm{Ad}_k X))_* k_* v$$

for all $v \in T_pM$. Hence replacing g by gk amounts to replacing the basis $b_1, \dots, b_n$ by the new basis $k_* b_1, \dots, k_* b_n$. We already saw above that $D_u Y$ does not depend on the choice of a basis of T_pM. Hence $D_u Y$ is well defined. Now for a smooth vector field Z on gU, $D_Z Y$ is smooth. Furthermore, D is tensorial in Z and a derivation in Y. Hence D is a connection. By definition, D is $\mathfrak{G}$–invariant.

For all $X \in \mathfrak{p}$ and $v \in T_pM$ the vector field $V(t) = \exp(tX)_* v$ is parallel along the curve $c(t) = \exp(tX)p$, $t \in \mathbb{R}$. Hence c is a geodesic and e^{tX} is a transvection. This shows existence of a $\mathfrak{G}$–invariant connection such that $\exp(tX)$, $t \in \mathbb{R}$, is a 1–parameter group of transvections through p. Uniqueness of such a connection is obvious.

Since D is $\mathfrak{G}$–invariant, restriction to $T_pM \cong \mathfrak{p}$ defines an isomorphism between $\mathfrak{G}$–invariant tensor fields on M and $\mathrm{Ad}_{\mathfrak{K}}$–invariant tensor fields on $\mathfrak{p}$. Now let A be a $\mathfrak{G}$–invariant tensor field on M. Then for all $X \in \mathfrak{p}$ and $t \in \mathbb{R}$,

$$A(\exp(tX)p) = \exp(-tX)^* A(p) \,.$$

Since $\exp(tX)$, $t \in \mathbb{R}$, is parallel translation along the geodesic $c(t) = \exp(tX)p$, $t \in \mathbb{R}$, we conclude that A is parallel along c. Since this holds for any X in $\mathfrak{p}$ we get that $DA(p) = 0$. Now A and D are $\mathfrak{G}$–invariant, hence A is parallel. Since D is $\mathfrak{G}$–invariant, T and R are $\mathfrak{G}$–invariant. In particular they are parallel.

If φ is an automorphism of $\mathfrak{G}$ leaving $\mathfrak{K}$ and $\mathfrak{p}$ invariant, then φ descends to a diffeomorphism of M. Since φ leaves the data which determine D invariant, f is an affine automorphism of M. Clearly, tensor fields on M which are invariant under $\mathfrak{G}$ and f correspond to tensors in $T_pM \cong \mathfrak{p}$ which are invariant under $\mathrm{Ad}_{\mathfrak{K}}$ and φ.

For $X, Y \in \mathfrak{p}$ with $X^*(p) = u$ and $Y^*(p) = v$ we have $D_uY^* = T_p(u,v)$ as in (4.5) above. The Lie bracket of X^* and Y^* at p is also completely determined by the torsion tensor at p,

$$\begin{aligned} T_p(u,v) &= D_{X^*}Y^*(p) - D_{Y^*}X^*(p) - [X^*,Y^*](p) \\ &= 2 \cdot T_p(u,v) + [X,Y]^*(p) \end{aligned}$$

and hence

$$T_p(u,v) = -[X,Y]^*(p)\,.$$

Since T is parallel, an affine vector field X satisfies the differential equation

$$D^2X(Y,Z) + R(X,Y)Z = T(Z, D_YX)\,.$$

Hence for $X, Y, Z \in \mathfrak{p}$ with $X^*(p) = u$, $Y^*(p) = v$, $Z^*(p) = w$, we have at p

$$\begin{aligned} D_w[X^*,Y^*] &= D_w\{D_{X^*}Y^* - D_{Y^*}X^* - T(X^*,Y^*\} \\ &= D^2Y^*(w,u) - D^2X^*(w,v) \\ &= -R_p(v,w)u - T_p(T_p(w,v),u) + R_p(u,w)v + T_p(T_p((w,u),v) \\ &= R_p(u,v)w + T_p(w,T_p(u,v))\,. \end{aligned}$$

Now if we decompose $[X,Y]$ according to the splitting of $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$,

$$[X,Y] = [X,Y]_{\mathfrak{k}} + [X,Y]_{\mathfrak{p}}$$

then since $[X,Y]^*_{\mathfrak{k}}(p) = 0$,

$$D_w[X,Y]^*_{\mathfrak{k}} = [Z^*, [X,Y]^*_{\mathfrak{k}}](p) = [[X,Y]_{\mathfrak{k}}, Z]^*(p)\,.$$

Furthermore, by the definition of $\mathfrak{p}$ and what we said above,

$$D_w[X,Y]^*_{\mathfrak{p}} = T_p(Z^*, [X,Y]^*) = -T_p(w, T_p(u,v))\,.$$

Now $[X^*,Y^*] = -[X,Y]^*$ and hence

$$R_p(u,v)w = -[[X,Y]_{\mathfrak{k}}, Z]^*(p)\,.$$

In conclusion we see that the splitting $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and the Lie algebra structure of $\mathfrak{g}$ determine the affine structure of $M = \mathfrak{G}/\mathfrak{K}$ completely. □

For ease of reference we note the following additional formulas from the proof which express covariant derivatives at p in terms of T and R.

4.11. Addendum. *For $X, Y, Z \in \mathfrak{p}$ with $X^*(p) = u$, $Y^*(p) = v$, $Z^*(p) = w$,*

$$D_wY^* = T_p(w,v)\,, \quad D_w[X,Y]^*_{\mathfrak{k}} = -R_p(u,v)w\,, \quad D_w[X,Y]^*_{\mathfrak{p}} = -T_p(w,T_p(u,v))\,.$$

4.12. *Remark.* For $\mathfrak{G}$ and M as in Theorem 4.10, not all $\mathfrak{G}$–invariant connections D with parallel torsion and curvature are given by a splitting $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Note that the $\mathfrak{p}$ there corresponds to the space of infinitesimal transvections along geodesics through the chosen point p. Now $\mathfrak{G}$ might be too small to contain this particular space $\mathfrak{p}$.

A typical example coccurs in the case of the upper half plane H with the hyperbolic metric. Then the group

$$\mathfrak{G} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1/\alpha \end{pmatrix} \,\middle|\, \alpha > 0 \right\} \subset Sl(2, \mathbb{R})$$

operates simply transitively and isometrically on H (by linear fractional transformations). In particular, $\mathfrak{G}$ leaves the Levi-Civita connection D of H invariant. Now D is torsionfree and of constant Gauß curvature (and in particular has parallel torsion and curvature). However, through any point $p \in H$, there is precisely one transvection in $\mathfrak{G}$, namely the one along the vertical (in the usual picture) geodesic through p.

On the other hand, since $\mathfrak{G}$ operates simply transitively on H, stabilizers of points are trivial. Hence for any given $p \in H$, the map $\pi : \mathfrak{G} \to H$, $\pi(p) = gp$, is a diffeomorphism. The pull back of the hyperbolic metric under π is a left invariant metric on $\mathfrak{G}$, hence the connection $\bar{D}$ on H corresponding to the left invariant connection on $\mathfrak{G}$ is metric and has parallel torsion and curvature (actually R=0).

4.13. *Remark.* Let M be a manifold with a transitive action by a Lie group $\mathfrak{G}$. Suppose that the action of $\mathfrak{G}$ is reductive, that is, there is an $\mathrm{Ad}_{\mathfrak{K}}$–invariant splitting $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{K}$ is the stabilizer of a point $p \in M$ in $\mathfrak{G}$. Let $\bar{D}$ be the corresponding connection on M as in Theorem 4.10. Then for any $\mathfrak{G}$–invariant connection D on M, the difference $S = D - \bar{D}$ is a $\mathfrak{G}$–invariant tensor field and hence is parallel with respect to $\bar{D}$. Vice versa, the connection $D = \bar{D} + S$ is $\mathfrak{G}$–invariant for any $\mathfrak{G}$–invariant tensorfield S of type $(2,1)$. Hence restriction of S to $T_pM \cong \mathfrak{p}$ defines an isomorphism between the space of $\mathfrak{G}$–invariant connections on M and the space of $\mathrm{Ad}_{\mathfrak{K}}$–invariant tensors on $\mathfrak{p}$.

4.14. *Remark.* Let M be a manifold with a transitive action by a Lie group $\mathfrak{G}$. Then $\mathfrak{G}$–invariant connections need not be complete. Easy examples are the non–zero real numbers $\mathbb{R}^*$ respectively non–zero complex numbers $\mathbb{C}^*$, both of them considered as Lie groups with respect to multiplication. The standard Euclidean connection on $\mathbb{R}^*$ and $\mathbb{C}^*$ is invariant under left and right multiplication by non–zero real respectively complex numbers, but is not complete.

However, the connection D in Theorem 4.10 is complete: the 1–parameter groups of $\mathfrak{G}$ are defined on $\mathbb{R}$, hence all maximal geodesics through p are defined on all of $\mathbb{R}$. Now D is $\mathfrak{G}$–invariant, hence all maximal geodesics are defined on all of $\mathbb{R}$.

5. Riemannian Homogeneous Spaces

Let M be a manifold with metric g, Levi-Civita connection D and curvature tensor R. A map $f : M \to M$ is called an *isometry* if $f^*g = g$. The group $\mathfrak{B}(M)$ of isometries of M is a Lie group when endowed with the compact–open topology. It is a closed subgroup of the group $\mathfrak{A}(M)$ of affine automorphisms of M (with respect to D).

We say that a vector field X on M is a *Killing field* if the (local) flow $f = f_t$ of X consists of (local) isometries of M. As in the case of affine fields, this can be characterized by a differential equation of first order. To that end we define the Lie derivative of metric g with respect to vector field X,

$$\begin{aligned} \mathcal{L}_X(g) = &= \lim_{t\to 0} \frac{1}{t}((f_t^* g)(Y,Z) - g(Y,Z)) \\ &= \langle D_Y X, Z\rangle + \langle Y, D_Z X\rangle \,, \end{aligned} \tag{5.1}$$

5.2. Proposition. *Vector field X is a Killing field if and only if $\mathcal{L}_X g = 0$. The space of Killing fields is a subalgebra of the Lie algebra of all vector fields.*

The proof is similar to the one of Proposition 1.11. Moreover, the equation of second order characterizing affine fields simplifies since D is torsion free.

5.3. Proposition. *If X is a Killing field, then for all vector fields Y and Z*

$$D^2X(Y,Z) + R(X,Y)Z = 0\,.$$

As for completeness, the situation is simpler than in the more general affine case. Obviously Killing fields are complete if M is complete. It is immediate from Proposition 5.3 that the space of Killing fields is a Lie algebra (with respect to the Lie bracket of vector fields). The space of complete Killing fields is a subalgebra and is anti–isomorphic to the Lie algebra $\mathfrak{b}(M)$ of $\mathfrak{B}(M)$.

We are interested in the question whether M is homogeneous. Note that unlike in the affine case, Riemannian homogeneous spaces are complete.

5.4. Theorem. *If M is simply connected and complete, then M is homogeneous if and only if M has a metric connection $\bar D$ with $\bar D$–parallel torsion and curvature.*

Proof. Suppose first that M has a metric connection $\bar D$ with $\bar D$–parallel torsion and curvature. Let $p, q \in M$ and c be a piecewise smooth curve from p to q. Then parallel translation $P_c : T_pM \to T_qM$ is an isomorphism preserving the metric. By Theorem 3.2, there is an affine automorphism $f : M \to M$ with $f(p) = q$ and $df(p) = P_c$. Now $f_p^* g_q = P_c^* g_q = g_p$, and therefore $f^*g = g$ by Proposition 1.9. Hence f is an isometry. This proves one direction, the other direction is a special case of Theorem 5.7 below. □

For any point $p \in M$, the stabilizer $\mathfrak{K} = \mathfrak{K}_p$ of p in $\mathfrak{B}(M)$ is compact. This implies that $\mathfrak{B}(M)$ acts reductively. An immediate consequence of this is that M has a metric connection $\bar D$ with $\bar D$–parallel torsion and curvature if M is

homogeneous. However, we will construct a specific complement of the Lie algebra $\mathfrak{k}$ in the Lie algebra $\mathfrak{b}(M)$ of $\mathfrak{B}(M)$.

More generally, we consider a group Lie group $\mathfrak{G}$ acting isometrically, (almost) effectively and transitively on M. We fix a point $p \in M$ and denote by $\mathfrak{K}$ the stabilizer of p in $\mathfrak{G}$. For $X, Y \in \mathfrak{g}$ we define

$$B_p(X,Y) = \operatorname{trace}(DX^*(p) \circ DY^*(p)) \,. \tag{5.5}$$

This is a bilinear form on $\mathfrak{g}$. The main point is that it is negative definite on $\mathfrak{k}$ and invariant under $\mathrm{Ad}_{\mathfrak{K}}$. Hence the space $\mathfrak{p}$ of $X \in \mathfrak{g}$ orthogonal to $\mathfrak{k}$ is a complement of $\mathfrak{k}$ and is invariant under $\mathrm{Ad}_{\mathfrak{K}}$.

5.6. Lemma. *The bilinear form (5.5) is negative definite on $\mathfrak{k}$. For all $g \in \mathfrak{G}$ and $X, Y \in \mathfrak{g}$,*

$$B_p(X,Y) = B_{gp}(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) \,.$$

In particular, B_p is invariant under $\mathrm{Ad}_{\mathfrak{K}}$.

Proof. Let $X \in \mathfrak{k}$. Then $DX^*(p)$ is a skewsymmetric endomorphism of T_pM. Hence $B_p(X,X) \leq 0$ with equality if and only if $DX^*(p) = 0$. Now X^* is a Killing field and $X^*(p) = 0$ since $X \in \mathfrak{k}$. Hence $DX^*(p) = 0$ implies $X^* = 0$. Since the action of $\mathfrak{G}$ is (almost) effective, this implies $X = 0$. As for the second claim, we note that the vector field g_*X^*, defined by $g_*X^*(q) = dg(g^{-1}(q)) \cdot X^*(g^{-1}(q))$, is equal to $(\mathrm{Ad}_g X)^*$. □

5.7. Theorem. *Let $\mathfrak{G}$ be a Lie group acting isometrically, (almost) effectively and transitively on M. Let $p \in M$ and $\mathfrak{K}$ be the stabilizer of p in $\mathfrak{G}$. Then the Lie algebra $\mathfrak{k}$ of $\mathfrak{K}$ has an $\mathrm{Ad}_{\mathfrak{K}}$–invariant complement $\mathfrak{p}$ in the Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$. In particular, M has a metric connection $\bar{D}$ with parallel torsion and curvature such that $\mathfrak{G}$ contains the $\bar{D}$–transvections.*

Proof. The first assertion is immediate from Lemma 5.6, choose $\mathfrak{p}$ as the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B_p. As for the second assertion, we let $\bar{D}$ be the connection determined by $\mathfrak{p}$, see Theorem 4.10. Now the pull back of g_p to $\mathfrak{p}$ via the identification $\mathfrak{p} \cong T_pM$ is $\mathrm{Ad}_{\mathfrak{K}}$–invariant since $\mathfrak{K}$ acts isometrically on T_pM. Hence $\bar{D}$ is metric. □

As in Proposition 3.7 we now consider the difference tensor field $D - \bar{D} = S$, where D is the Levi-Civita connection of M. We also write $S_XY = S(X,Y)$. Then the torsion tensor $\bar{T}$ and the curvature tensor $\bar{R}$ of $\bar{D}$ are given by

$$\bar{T}(X,Y) = S_YX - S_XY \tag{5.8}$$

$$\bar{R}(X,Y)Z = R(X,Y)Z - (D_XS)(Y,Z) + (D_YS)(X,Z) + [S_X, S_Y] \cdot Z \,. \tag{5.9}$$

We see that $\bar{T}$ and $\bar{R}$ are $\bar{D}$–parallel if S and R are.

5.10. Proposition. *Connection $\bar D$ is metric with $\bar D$–parallel torsion and curvature if and only if for all vector fields X, Y, Z,*

$$\begin{aligned}\langle S_X Y, Z\rangle &= -\langle Y, S_X Z\rangle\,;\\ (D_X S)(Y,Z) &= [S_X, S_Y]\cdot Z - S(S_X Y, Z)\,;\\ (D_X R)(Y,Z) &= [S_X, R(Y,Z)] - R(S_X Y, Z) - R(Y, S_X Z)\,.\end{aligned}$$

Moreover, if $\bar D$ is metric with $\bar D$–parallel torsion and curvature, then S is $\bar D$–parallel.

Proof. The first equation says that $\bar D$ is metric. The second equation holds if and only if S is parallel with respect to $\bar D$, the third equation holds if only if R is $\bar D$–parallel. This shows sufficiency of the conditions.

Vice versa, suppose that $\bar D$ is metric with $\bar D$–parallel torsion and curvature. Let $p \in M$ and P_t be $\bar D$–parallel translation along a smooth curve $c = c(t)$ through p. Then for any (small) t there is a $\bar D$–affine map f_t mapping p to $f(p)$ and with $df_t(p) = P_t$. Then f_t preserves g since parallel translation preserves g, see Proposition 1.6. Hence f_t is a local isometry and therefore preserves the Levi–Civita connection D of M. Hence f_t preserves S. Since $df_t(p) = P_t$ we get that S is $\bar D$–parallel at p. Since p was arbitrary, S is $\bar D$–parallel. □

References

[Amb56] W. Ambrose. Parallel translation of Riemannian curvature. *Ann. of Math. (2)*, 64:337–363, 1956.

[AS58] W. Ambrose and I. M. Singer. On homogeneous Riemannian manifolds. *Duke Math. J.*, 25:647–669, 1958.

[Hic59] N. Hicks. A theorem on affine connexions. *Illinois J. Math.*, 3:242–254, 1959.

[KN63] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol I.* Interscience Publishers John Wiley & Sons, Inc., New York-London, 1963.

[KN69] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. II.* Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1969.

[Kos60] Bertram Kostant. A characterization of invariant affine connections. *Nagoya Math. J*, 16:35–50, 1960.

[Nom54] Katsumi Nomizu. Invariant affine connections on homogeneous spaces. *Amer. J. Math.*, 76:33–65, 1954.

[TV83] F. Tricerri and L. Vanhecke. *Homogeneous structures on Riemannian manifolds.* Cambridge University Press, Cambridge, 1983.