INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

Lecture Notes

given by

DINH NHO HÀO

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Foreword

The author has worked for 4 years (1993–1996) at the University of Siegen in the group of Applied and Numerical Mathematics. His stay was supported by a DFG–scholarship and by a guest professorship from the University of Siegen. During this time, this lecture was held for graduate students.

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Comments on this lecture note should be sent directly to the author via email (hao@thevinh.ac.vn) or to hjreinhardt@numerik.math.uni-siegen.de .

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Chapter 1

Introduction

1.1 Definitions

A partial differential equation (PDE) for a function $u(x, y, \ldots)$ is a relation of the form

$$F(x, y, \ldots, u, u_x, u_y, \ldots, u_{xx}, u_{xy}, \ldots) = 0, \tag{1.1.1}$$

where F is a given function of the independent variables $x, y, \ldots$, and of the "unknown" function u and of a *finite* or *infinite* number of its derivatives. We call u a *solution* of (1.1.1) if after substitution of $u(x, y, \ldots)$ and its partial derivatives, (1.1.1) is satisfied identically in $x, y, \ldots$ in some region Ω in the space of these independent variables. For simplicity, we suppose that $x, y, \ldots$ are real and that the derivatives of u occuring in (1.1.1) are continuous functions of $x, y, \ldots$ in the real domain Ω. Several PDEs involving one or more unknown functions and their derivatives constitute a *system*.

The *order* of a PDE or of a system is the order of the highest derivative that occurs. A PDE is said to be *linear* if it is linear in the unknown functions and their derivatives, with coefficients depending on the independent variables $x, y, \ldots$. The PDE of order m is called *quasilinear* if it is linear in the derivatives of order m with coefficients that depend on $x, y, \ldots$ and the derivatives of order $< m$.

Remark 1.1.1 The dimension of the space where $x, y, \ldots$ belong to can be infinite.

Remark 1.1.2 If in (1.1.1) derivatives of u are of infinite order, then we say that (1.1.1) is *of infinite order*.

1.2 Examples

1.2.1 Equations

- $u_x + u_y = 0$ (transport)
- $u_x + g(u)_y = 0$ (conservation law, shock wave)

- $u_{xx} + u_{yy} = 0$ (the Laplace equation)
- $u_{tt} - u_{xx} = 0$ (the wave equation)
- $u_t = k\Delta u$ (the heat equation)
- $i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi + V\psi$ (Schrödinger's equation)
- $u_t + cuu_x + u_{xxx} = 0$ (the Korteweg - de Vries equation)
- $\sum_0^\infty \frac{1}{n!}\frac{d^n u}{dx^n} = g(x)$

1.2.2 Solutions

Example 1.2.1 Find all $u(x, y)$ satisfying the equation $u_{xx} = 0$. We can integrate once to get $u_x = \text{const}$ (in x). Since the solution depends on y, we have $u_x = f(y)$, where $f(y)$ is arbitrary. Do it again to get $u(x, y) = f(y)x + g(y)$.

Example 1.2.2 Find $u(x, y)$ satisfying PDE $u_{xx} + u = 0$.

Solution: $u(x, y) = f(y)\cos x + g(y)\sin x$, f and g are arbitrary.

Example 1.2.3 Find $u(x, y)$ satisfying $u_{xy} = 0$. First integrate in x, regarding y as fixed. So we get

$$u_y(x, y) = f(y).$$

Next let integrate in y, regarding x as fixed. We get the solution

$$u(x, y) = F(y) + G(x),$$

where $F' = f$.

1.3 First-order linear equations

1.3.1 The constant coefficient case

Consider the equation

$$au_x + bu_y = 0, \tag{1.3.1}$$

where a and b are constants not both zero.

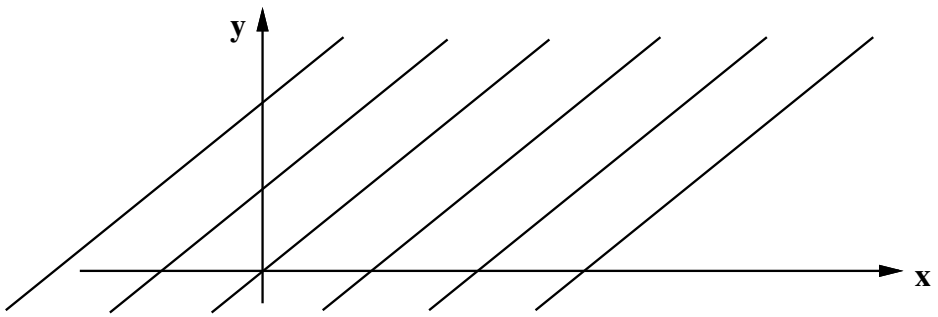
Geometric Method. The quantity $au_x + bu_y$ is the directional derivative of u in the direction of the vector $\vec{V} = (a, b) = a\vec{i} + b\vec{j}$. It must always be zero. This means that $u(x, y)$ must be constant in the direction of $\vec{V}$. The vector $(b, -a)$ is orthogonal to $\vec{V}$. The lines parallel to $\vec{V}$ satisfy the equation $bx - ay = \text{const} = c$ and along them the solution u has a constant value, say $f(c)$. Then $u(x, y) = f(c) = f(bx - ay)$. Since c is arbitrary, we have

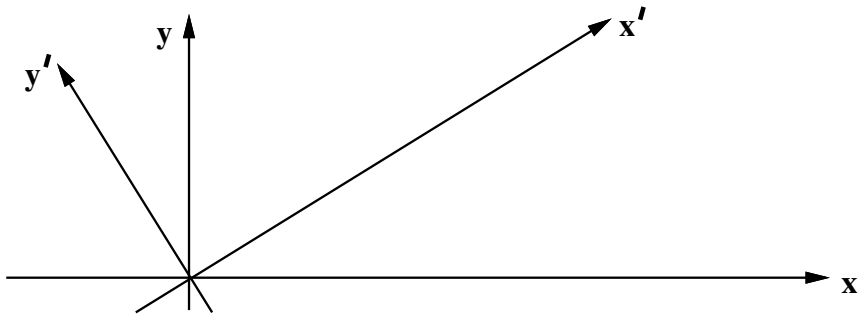
$$u(x, y) = f(c) = f(bx - ay) \tag{1.3.2}$$

for all values of x and y.

Coordinate method. Change variables to

$$x' = ax + by, \quad y' = bx - ay. \tag{1.3.3}$$





We have

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = au_{x'} + bu_{y'}$$

and

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} + \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} = bu_{x'} - au_{y'}.$$

Hence $au_x + bu_y = a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) = (a^2 + b^2)u_{x'}$. Since $a^2 + b^2 \neq 0$, we have $u_{x'} = 0$. Thus, $u = f(y') = f(bx - ay)$.

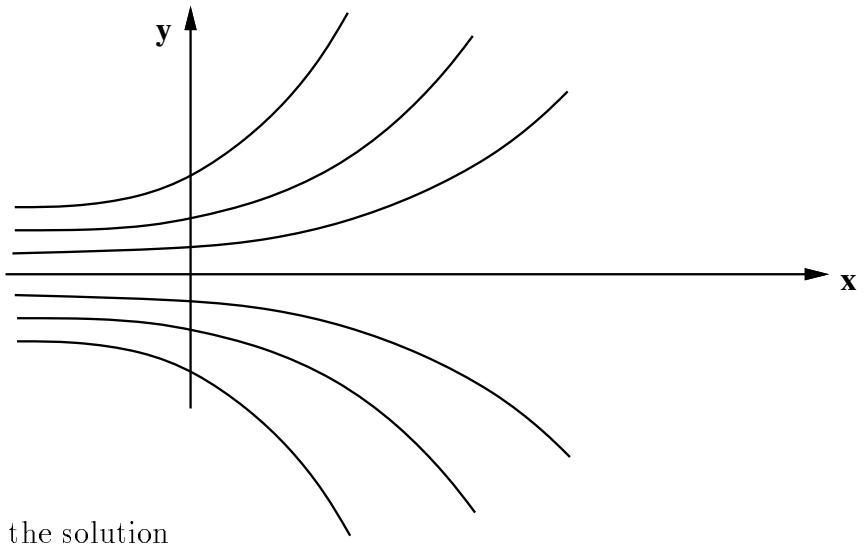
1.3.2 A variable coefficient case

The equation

$$u_x + yu_y = 0 \tag{1.3.4}$$

is linear and homogeneous but has a variable coefficient y. The PDE (1.3.4) itself asserts that the directional derivative in the direction of the vector $(1, y)$ is zero. The curves in the xy plane with $(1, y)$ as tangent vectors have slopes y. Their equations are

$$\frac{dy}{dx} = \frac{y}{1}. \tag{1.3.5}$$



This ODE has the solution

$$y = Ce^x. \tag{1.3.6}$$

These curves are called the *characteristic curves* of the PDE (1.3.4). As C is changed, the curves fill out the xy plane perfectly without intesecting. On each of the curves $u(x, y)$ is a constant because

$$\frac{d}{dx}u(x, Ce^x) = \frac{\partial u}{\partial x} + Ce^x\frac{\partial u}{\partial y} = u_x + yu_y = 0.$$

Thus, $u(x, Ce^x) = u(0, Ce^0) = u(0, C)$ is independent of x. Putting $y = Ce^x$ and $C = e^{-x}y$, we have

$$u(x, y) = u(0, e^{-x}y).$$

It follows that

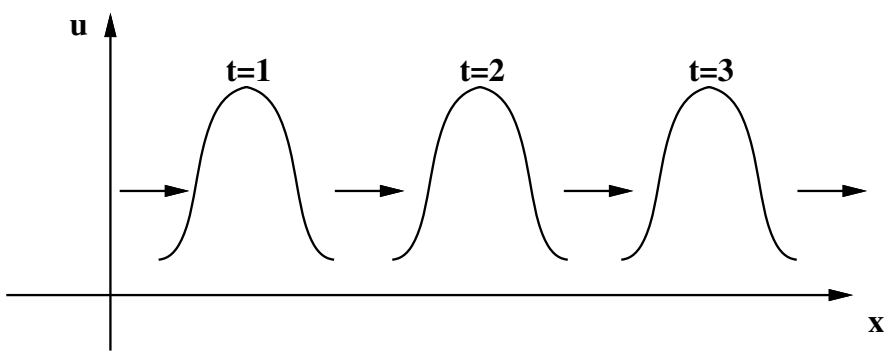
$$u(x, y) = f(e^{-x}y) \tag{1.3.7}$$

is the general solution of our PDE.

1.4 Where PDE come from?

1.4.1 Simple Transport

Consider a fluid, say water, flowing at a constant rate c along a horizontal pipe of a fixed cross section in the positive x direction. A substance, say a pollutant, is suspended in the water. Let $u(x, t)$ be its concentration in grams/centimeter at time t.



We know that the amount of pollutant in the interval $[0, b]$ at the time t is $M = \int_0^b u(x,t)dx$, in grams. At the later time $t + h$, the same molecules of pollutant have moved to the right by $c \cdot h$ centimeters. Hence,

$$M = \int_0^b u(x,t)dx = \int_{ch}^{b+ch} u(x, t+h)dx.$$

Differentiate with respect b, we get

$$u(b,t) = u(b + ch, t + h).$$

Differentiate with respect to h and putting $h = 0$, we get

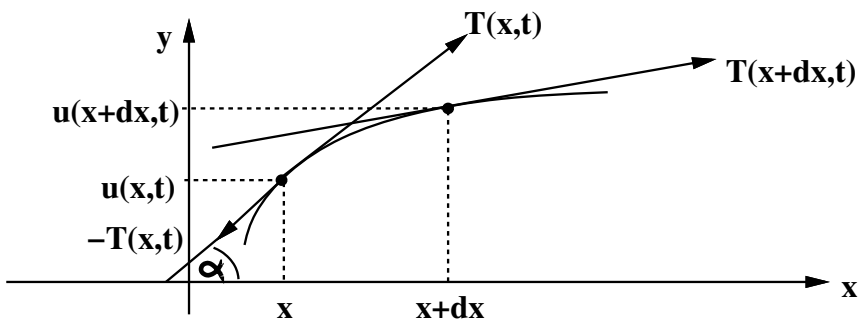
$$0 = cu_x(b,t) + u_t(b,t).$$

Since b is arbitrary, the equation describing our simple transport problem looks as follows

$$u_t + cu_x = 0. \tag{1.4.1}$$

1.4.2 Vibration Equations

Consider a flexible, elastic homogeneous string or thread of length l, which undergoes relatively small transverse vibrations. Assume that it remains in a plane. Let $u(x,t)$ be its displacement from equilibrium position at time t and position x. Because the string is perfectly flexible, the tension (force) is directed tangentially along the string. Let $T(x,t)$ be the magnitude of this tension vector. Limiting ourselves to the consideration of small vibrations of the string, we shall ignore magnitudes of order greater than $\tan\alpha = \partial u/\partial x$.



Any small section of the string (a, b), after displacement from a state of equilibrium within the range of our approximation, does not change its length

$$S = \int_a^b \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}\, dx \simeq b - a$$

and therefore, in agreement with Hook's law, T is independent of t. We shall show now it is also independent of x. This means

$$T = T_0 = \text{const}\,.$$

For the projections of the tension into the $u-$ and $x-$axes (denoted by T_u and T_x, respectively) we have

$$\begin{aligned} T_x(x) &= T(x)\cos\alpha = \frac{T(x)}{\sqrt{1+(u_x)^2}} \simeq T(x), \\ T_u(x) &= T(x)\sin\alpha = T(x)\tan\alpha = T(x)u_x, \end{aligned}$$

where α is the angle between the tangent at the curve $u(x,t)$ and the $x-$axis.

The tension, the external force and the initia (Trägheitskräfte) are acting into a small piece (x_1, x_2) of the string. The sum of the projections into $x-$axis of all these forces must be zero, since we consider only the case of transverse vibrations. Because the external force and the initia (Trägheitskräfte) are acting along the $u-$axis direction, we have

$$T_x(x_1) - T_x(x_2) = 0, \quad \text{or} \quad T(x_1) = T(x_2).$$

Since x_1 and x_2 are arbitrary, we see that T is independent of x. Thus, for all x and t, we have

$$T \equiv T_0.$$

Let $F(x,t)$ be the magnitude of the external forces acting on the string at the point x at an instant of time t and directed perpendicular to the $x-$axis. Finally, let $\rho(x)$ be the linear density of the string at the point x, so that $\rho(x)dx$ is the mass of the element of the string $(x, x+dx)$.

We now construct an equation of montion for the element $(x, x+dx)$ of the string. The tension forces $T(x+dx,t)$, $-T(x,t)$ and the external force, the sum of which, according to Newton's law, must be equal to the product of the mass of the element and its acceleration, are acting upon the element $(x, x+dx)$. Projecting this vector equation onto the u axis, on account of all that has been said, we obtain the equation

$$T_0 \sin\alpha|_{x+dx} - T_0 \sin\alpha|_x + F(x,t)dx = \rho(x)dx\frac{\partial^2 u(x,t)}{\partial t^2} \tag{1.4.2}$$

However, whithin the range of our approximation

$$\sin\alpha = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} \approx \tan\alpha = \frac{\partial u}{\partial x}$$

and therefore

$$\rho\frac{\partial^2 u(x,t)}{\partial t^2} = T_0\frac{1}{dx}\left[\frac{\partial u(x+dx,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right] + F(x,t)$$

that is,

$$\rho\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + F. \tag{1.4.3}$$

This is indeed the *equation of small transverse vibrations of a string.*

If the density ρ is constant, $\rho(x) = \rho$, the equation of vibration of a string assumes the form

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + f, \tag{1.4.4}$$

where we have set $a^2 = T_0/\rho$ (constant), $f = F/\rho$. We shall also call this equation the *one-dimensional wave equation.*

By the same way one can also derive the equation of *small transverse vibrations of a membrane* as

$$\rho\frac{\partial^2 u}{\partial t^2} = T_0\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + F := T_0\Delta u + F. \tag{1.4.5}$$

1.4.3 Diffusion Equation

We now derive the equation of heat diffusion or the diffusion of particles in a medium. Denote by $u(x,t)$ the temperature of the medium at the point $x = (x_1, x_2, x_3)$ at an instant of time t. We shall consider the medium to be isotropic and shall denote its density by $\rho(x)$, its specific heat by $c(x)$, and its coefficient of thermal conductivity at point x by $k(x)$. $F(x,t)$ denotes the intensity of heat sources at the point x at the instant of time t. We shall calculate the balance of heat in an arbitrary volume V during the time interval $(t, t+dt)$. The boundary of V is denoted by S and $\vec{n}$ is the external normal to it. In agreement with Fourier's law, an amount of heat

$$Q_1 = \int_S k\frac{\partial u}{\partial \vec{n}}dSdt = \int_S (k \text{ grad } u, \vec{n})dSdt$$

equal, by virtue of the Gauss–Ostrogradskii formula, to

$$Q_1 = \int_V \text{div}(k \text{ grad } u)dxdt$$

enters volume V through surface S.

An amount of heat

$$Q_2 = \int_V F(x,t)dxdt$$

arises in volume V as a result of heat sources. Since the temperature in the volume V during the interval of time $(t, t+dt)$ has increased in magnitude by

$$u(x, t+dt) - u(x,t) \simeq \frac{\partial u}{\partial t}dt$$

it follows that for this it is necessary to expend an amount of heat

$$Q_3 = \int_V c\rho\frac{\partial u}{\partial t}dxdt.$$

On the other hand, $Q_3 = Q_1 + Q_2$, and thus

$$\int_V \left[\text{div}(k \text{ grad } u) + F - c\rho\frac{\partial u}{\partial t}\right]dxdt = 0$$

from which, since volume V is arbitrary, we obtain the equation of heat diffusion

$$c\rho\frac{\partial u}{\partial t} = \text{div}(k \text{ grad } u) + F. \tag{1.4.6}$$

If the medium is homogeneous, that is, if c, ρ, and k are constants, Eq. (1.4.6) assumes the form

$$\frac{\partial u}{\partial t} = a^2 \Delta u + f \tag{1.4.7}$$

where

$$a^2 = \frac{k}{c\rho}, \quad f = \frac{F}{c\rho}.$$

Eq. (1.4.7) is called the *heat equation.*

1.4.4 Steady State Equation

For steady state processes $F(x,t) = F(x), u(x,t) = u(x)$, and both the wave equation (1.4.5) and the diffusion equation (1.4.6) assume the form

$$-\operatorname{div}(p \text{ grad } u) + qu = F(x). \tag{1.4.8}$$

When $p = \text{const}$ and $q = 0$, Eq. (1.4.8) is called *Poisson's equation*

$$\Delta u = -f, \quad f = \frac{F}{p} \tag{1.4.9}$$

when $f = 0$, Eq. (1.4.9) is called *Laplace's equation*

$$\Delta u = 0. \tag{1.4.10}$$

In the wave equation (1.4.5), let the external disturbance $f(x,t)$ be periodic with frequency ω and amplitude $f_0(x)$,

$$f(x,t) = f_0(x)e^{i\omega t}.$$

If we seek periodic solutions $u(x,t)$ with the same frequency and unknown amplitude $u_0(x)$, then for the function $u_0(x)$ we obtain the equation

$$\Delta u_0 + k^2 u_0 = -\frac{f_0(x)}{a^2}, \quad k^2 = \frac{\omega^2}{a^2} \tag{1.4.11}$$

called the *Helmholtz equation.*

1.4.5 Schrödinger's Equation

Let a quantum particle of mass m move in an external force field with potential $V(x)$. We shall denote by $\psi(x,t)$ the wave function of this particle, so that $|\psi(x,t)|^2 dx$ is the probability that the particle will be in a neighbourhood $d(x)$ of the point x at an instant of time t; here dx is the volume of $d(x)$. Then, the function ψ satisfies Schrödinger's equation

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi + V\psi, \tag{1.4.12}$$

where $\hbar = 1.054.10^{-27}$erg sec is Planck's constant.

1.5 Classification of Linear (Second-Order) Differential Equations

1.5.1 Differential Equations with Two Independent Variables

Let us consider a second-order differential equation with two independent variables

$$a_{11}\frac{\partial^2 u}{\partial x^2} + 2a_{12}\frac{\partial^2 u}{\partial x \partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \tag{1.5.1}$$

where we shall suppose that the coefficients a_{11}, a_{12}, and a_{22} belong to C^2 in a certain neighbourhood and do not become zero simultaneously anywhere in this neighbourhood. For definiteness we assume that $a_{11} \neq 0$ in this neighbourhood. In fact, it may happen that $a_{22} \neq 0$. But then, by reversing the roles of x and y, we obtain an equation in which $a_{11} \neq 0$. If then a_{11} and a_{22} become zero simultaneously at a point, then $a_{12} \neq 0$ in the neighbourhood of this point. In this case the change of variables $x' = x + y, y' = x - y$ leads to an equation in which both a_{11} and a_{22} are not zero.

Using the new variables

$$\xi = \xi(x, y), \quad \eta = \eta(x, y), \quad \xi \in C^2, \quad \eta \in C^2, \quad D\left(\frac{\xi, \eta}{x, y}\right) \neq 0, \tag{1.5.2}$$

we have

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x, \\
u_y &= u_\xi \xi_y + u_\eta \eta_y, \\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}, \\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}, \\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}.
\end{aligned}$$

Putting these equations in our original equations, we get a new equation of the form

$$\bar{a}_{11}u_{\xi\xi} + 2\bar{a}_{12}u_{\xi\eta} + \bar{a}_{22}u_{\eta\eta} + \bar{F} = 0, \tag{1.5.3}$$

where

$$\begin{aligned}
\bar{a}_{11} &= a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2, \\
\bar{a}_{12} &= a_{11}\xi_x\eta_x + a_{12}(\xi_x\eta_y + \eta_x\xi_y) + a_{22}\xi_y\eta_y, \\
\bar{a}_{22} &= a_{11}\eta_x^2 + 2a_{12}\eta_x\eta_y + a_{22}\eta_y^2
\end{aligned}$$

and $\bar{F}$ is independent of the second order derivatives with respect to ξ and η of u.

We shall require that the functions $\xi(x, y)$ and $\eta(x, y)$ make the coefficients $\bar{a}_{11}$ and $\bar{a}_{22}$ vanish, that is they should satisfy the equations

$$a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2 = 0, \tag{1.5.4}$$

$$a_{11}\eta_x^2 + 2a_{12}\eta_x\eta_y + a_{22}\eta_y^2 = 0. \tag{1.5.5}$$

First we prove the following lemma.

Lemma 1.5.1 1. *If $z = \varphi(x, y)$ is a particular solution of the equation*

$$a_{11}z_x^2 + 2a_{12}z_x z_y + a_{22}z_y^2 = 0, \tag{1.5.6}$$

then $\varphi(x, y) = C$ is a general integral of the ordinary differential equation

$$a_{11}dy^2 - 2a_{12}dxdy + a_{22}dx^2 = 0. \tag{1.5.7}$$

2. *Conversely, if $\varphi(x, y) = C$ is a general integral of Eq. (1.5.7), then $z = \varphi(x, y)$ is a solution of Eq. (1.5.6).*

Proof. Since the function $z = \varphi(x, y)$ is a solution of Eq. (1.5.6),

$$a_{11}\left(\frac{\varphi_x}{\varphi_y}\right)^2 - 2a_{12}\left(-\frac{\varphi_x}{\varphi_y}\right) + a_{22} \tag{1.5.8}$$

for all x, y in the neighbourhood where $z = \varphi(x, y)$ is defined and $\varphi_y(x, y) \neq 0$. Let $\varphi(x, y) = C$, Because $\varphi_y(x, y) \neq 0$, we can find a neighbourhood where $y = f(x, C)$, then

$$\frac{dy}{dx} = -\left[\frac{\varphi_x(x, y)}{\varphi_y(x, y)}\right]_{y=f(x,C)} \tag{1.5.9}$$

and

$$a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\frac{dy}{dx} + a_{22} = \left[a_{11}\left(-\frac{\varphi_x}{\varphi_y}\right)^2 - 2a_{12}\left(-\frac{\varphi_x}{\varphi_y}\right) + a_{22}\right]_{y=f(x,C)} = 0.$$

Thus, $y = f(x, C)$ is a solution of (1.5.7). Note that the last term vanishes for all x, y.

Conversely, let $\varphi(x, y) = C$ be an integral of the Eq. (1.5.7). We shall show that for any point (x, y)

$$a_{11}\varphi_x^2 + 2a_{12}\varphi_x\varphi_y + a_{22}\varphi_y^2 = 0 \tag{1.5.10}$$

Let (x_0, y_0) be given. Draw through it an integral curve for Eq. (1.5.7), for which we set $\varphi(x_0, y_0) = C_0$. Consider now the curve $y = f(x, C_0)$. It is clear $y_0 = f(x_0, C_0)$. For any point of this curve we have

$$a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\frac{dy}{dx} + a_{22} = \left[a_{11}\left(-\frac{dy}{dx}\right)^2 - 2a_{12}\left(-\frac{dy}{dx}\right) + a_{22}\right]_{y=f(x,C_0)}.$$

Putting $x = x_0$ into this equation, we get

$$a_{11}\varphi_x^2(x_0, y_0) + 2a_{12}\varphi_x\varphi_y(x_0, y_0) + a_{22}\varphi_y^2(x_0, y_0) = 0.$$

Since x_0, y_0 are arbitrary, we have just proved our lemma. □

Eq. (1.5.7) is called the *characteristic equation* of the differential equation (1.5.1) and its integral curves are called *characteristics*.

Let $\xi = \xi(x, y)$, where $\xi(x, y) = \text{const}$ is an integral of Eq. (1.5.7), we see that the coefficient of $u_{\xi\xi}$ vanishes. Analogously, set $\eta = \eta(x, y)$ where $\eta(x, y) = \text{const}$ is another integral of Eq. (1.5.7) independent of $\xi(x, y)$, we get the coefficient of $u_{\eta\eta}$ to be also zero.

Eq. (1.5.7) is equivalent to the linear equations

$$\frac{dy}{dx} = \frac{a_{12} + \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}, \tag{1.5.11}$$

$$\frac{dy}{dx} = \frac{a_{12} + \sqrt{a_{12}^2 + a_{11}a_{22}}}{a_{11}}. \tag{1.5.12}$$

According to the terminology of the theory of second-order curves, we say that Eq. (1.5.1) at a point M is of

hyperbolic type, when at this point $a_{12}^2 - a_{11}a_{22} > 0$,

elliptic type, when at this point $a_{12}^2 - a_{11}a_{22} < 0$,

parabolic type, when at this point $a_{12}^2 - a_{11}a_{22} = 0$.

Since

$$\bar{a}_{12}^2 - \bar{a}_{11}\bar{a}_{22} = (a_{12}^2 - a_{11}a_{22})(\xi_x\eta_y - \xi_y\eta_x)^2,$$

the type of Eq. (1.5.1) is invariant under our transformation (1.5.2). Note that, at different points of the domain the equation may have different type. In what follows we consider a neighbouhood G where our equation is of only one type. Through any point of G there are two characteristics which are real and different for an equation of hyperbolic type, complex and different for an equation of elliptic type, and real and coincided for an equation of parabolic type. We consider these cases separately.

1. For equations of hyperbolic type $a_{12}^2 - a_{11}a_{22} > 0$, so the right-hand sides of (1.5.11) and (1.5.12) are real and different. The general integrals $\xi(x, y) = C$ and $\eta(x, y) = C$ of these equations define two families of characteristics. Letting

$$\xi = \xi(x, y), \quad \eta = \eta(x, y) \tag{1.5.13}$$

we get Eq. (1.5.3) after dividing by the coefficient before $u_{\xi\eta}$ into the form

$$u_{\xi\eta} = F(\xi, \eta, u, u_\xi, u_\eta) \text{ with } \bar{a}_{12} \neq 0. \tag{1.5.14}$$

This is called the *canonical form* of a hyperbolic equation.

Normally we are using another canonical form. Let

$$\xi = \alpha + \beta, \quad \eta = \alpha - \beta,$$

it means

$$\alpha = \frac{\xi + \eta}{2}, \quad \beta = \frac{\xi - \eta}{2},$$

where α and β are considered as new variables. Then,

$$u_\xi = \frac{1}{2}(u_\alpha + u_\beta), \quad u_\eta = \frac{1}{2}(u_\alpha - u_\beta), \quad u_{\xi\eta} = \frac{1}{4}(u_{\alpha\alpha} - u_{\beta\beta}),$$

and Eq. (1.5.3) has the form

$$u_{\alpha\alpha} - u_{\beta\beta} = F_1.$$

2. For equations of parabolic type $a_{12}^2 - a_{11}a_{22} = 0$, therefore the equations (1.5.11) and (1.5.12) coincide and we have only a general integral for Eq. (1.5.6): $\varphi(\xi, \eta) = \text{const}$. In this case, set

$$\xi = \varphi(x, y) \quad \text{and} \quad \eta = \eta(x, y),$$

where $\eta(x, y)$ is an arbitrary function independent of φ. For this choice we have

$$\bar{a}_{11} = a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2 = (\sqrt{a_{11}}\xi_x + \sqrt{a_{22}}\xi_y)^2 = 0,$$

since $a_{12} = \sqrt{a_{11}}\sqrt{a_{22}}$. It follows

$$\begin{aligned} \bar{a}_{12} &= a_{11}\xi_x\eta_x + a_{12}(\xi_x\eta_y + \xi_y\eta_x) + a_{22}\xi_y\eta_y \\ &= (\sqrt{a_{11}}\xi_x + \sqrt{a_{22}}\xi_y)(\sqrt{a_{11}}\eta_x + \sqrt{a_{22}}\eta_y) = 0. \end{aligned}$$

Thus,

$$u_{\eta\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta) \quad \text{with} \quad \Phi = -\frac{\bar{F}}{\bar{a}_{22}} \quad (\bar{a}_{22} \neq 0)$$

is the canonical form of a parabolic equation. If the right hand side of this equation does not depend on u_ξ then we have an ordinary differential equation.

3. For the case of elliptic equations $a_{12}^2 - a_{11}a_{22} < 0$, and we see that the equation (1.5.11) is complex conjugate to (1.5.12). If

$$\varphi(x, y) = C$$

is a complex integral of Eq. (1.5.11), then

$$\varphi^*(x, y) = C,$$

is a complex integral of Eq. (1.5.12) where φ^* is the complex function conjugate to φ. Let

$$\xi = \varphi(x, y) \quad \eta = \varphi^*(x, y).$$

In order to work with real functions, let further

$$\alpha = \frac{\varphi + \varphi^*}{2}, \quad \beta = \frac{\varphi - \varphi^*}{2i}.$$

It is clear that α and β are real, and

$$\xi = \alpha + i\beta, \quad \eta = \alpha - i\beta.$$

We note that since we are working with complex variables, we have to suppose that all coefficients $a_{11}, a_{12}, \ldots$ are analytic. In this case

$$\begin{aligned} &a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2 \\ &\quad = (a_{11}\alpha_x^2 + 2a_{12}\alpha_x\alpha_y + a_{22}\alpha_y^2) - (a_{11}\beta_x^2 + 2a_{12}\beta_x\beta_y + a_{22}\beta_y^2) \\ &\quad +2i(a_{11}\alpha_x\beta_x + a_{12}(\alpha_x\beta_y + \alpha_y\beta_x) + a_{22}\alpha_y\beta_y), \end{aligned}$$

therefore

$$\bar{a}_{11} = \bar{a}_{22} \quad \text{and} \quad \bar{a}_{12} = 0.$$

Now we have

$$u_{\alpha\alpha} + u_{\beta\beta} = \Phi(\alpha, \beta, u, u_\alpha, u_\beta) \quad \text{with} \quad \Phi = -\frac{\bar{F}}{\bar{a}_{22}} \quad (\bar{a}_{22} \neq 0).$$

We summarize the results of this paragraph as follows.

- $a_{12}^2 - a_{11}a_{22} > 0$ (hyperbolic type): $u_{xx} - u_{yy} = \Phi$ or $u_{xy} = \Phi$;
- $a_{12}^2 - a_{11}a_{22} < 0$ (elliptic type): $u_{xx} + u_{yy} = \Phi$;
- $a_{12}^2 - a_{11}a_{22} = 0$ (parabolic type): $u_{xx} = \Phi$.

Example 1.5.2 Tricomi's equation

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

belongs to the mixed type: when $y < 0$ it is of hyperbolic type, but when $y > 0$ it is of elliptic type, and when $y = 0$ it is of parabolic type.

Tricomi's equation is of interest in gas dynamics. With $y < 0$ the equations of the characteristics assume the form

$$y' = \pm\frac{1}{\sqrt{-y}}.$$

Therefore the curves

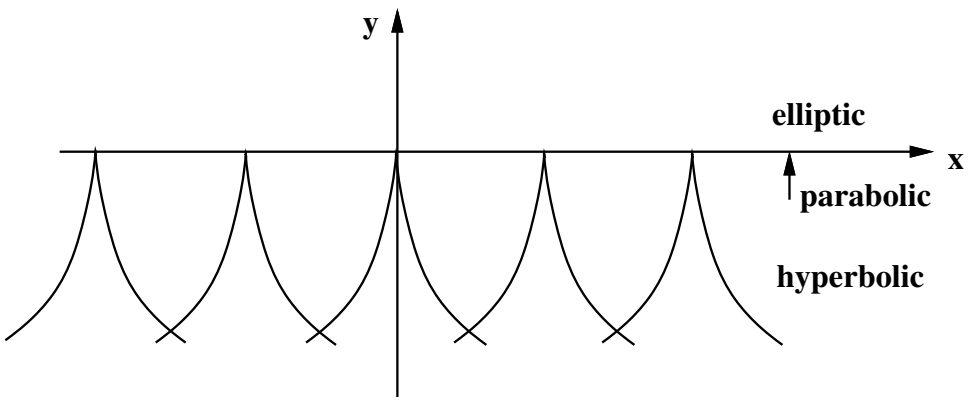
$$\frac{3}{2}x + \sqrt{-y^3} = C_1, \quad \frac{3}{2}x - \sqrt{-y^3} = C_2$$

are characteristics of Tricomi's equation. The transform

$$\xi = \frac{3}{2}x + \sqrt{-y^3}, \quad \eta = \frac{3}{2}x - \sqrt{-y^3}$$

reduces Tricomi's equation to the canonical form

$$\frac{\partial^2 \tilde{u}}{\partial\xi\partial\eta} - \frac{1}{6(\xi-\eta)}\left(\frac{\partial\tilde{u}}{\partial\xi} - \frac{\partial\tilde{u}}{\partial\eta}\right) = 0, \quad \xi > \eta.$$



If $y > 0$, then $\varphi = \frac{3}{2}x - i\sqrt{y^3}$ and with the change of variables

$$\xi = \frac{3}{2}x, \quad \eta = -\sqrt{y^3}$$

Tricomi's equation reduces to

$$\frac{\partial^2 \tilde{u}}{\partial\xi^2} + \frac{\partial^2 \tilde{u}}{\partial\eta^2} + \frac{1}{3\eta}\frac{\partial\tilde{u}}{\partial\eta} = 0, \quad \eta < 0.$$

1.5.2 Differential Equations with Several Independent Variables.

We consider now the linear differential equation with real coefficients

$$\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}u_{x_ix_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu + f = 0, \quad (a_{ij} = a_{ji}), \tag{1.5.15}$$

where a, b, c and f are functions of variables $x_1, x_2, \ldots, x_n$. Making the change of variables

$$\xi_k = \xi_k(x_1, x_2, \ldots, x_n) \quad (k = 1, \ldots, n),$$

we have

$$\begin{aligned} u_{x_i} &= \sum_{k=1}^{n} u_{\xi_k}\alpha_{ik}, \\ u_{x_ix_j} &= \sum_{k=1}^{n}\sum_{l=1}^{n} u_{\xi_k\xi_l}\alpha_{ik}\alpha_{jl} + \sum_{k=1}^{n} u_{\xi_k}(\xi_k)_{x_ix_j}, \end{aligned}$$

where $\alpha_{ik} := \partial\xi_k/\partial x_i$.

Putting these expressions in (1.5.15) we get

$$\sum_{k=1}^{n}\sum_{l=1}^{n} \bar{a}_{kl}u_{\xi_k\xi_l} + \sum_{k=1}^{n} \bar{b}_k u_{\xi_k} + cu + f = 0$$

where

$$\begin{aligned} \bar{a}_{kl} &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\alpha_{ik}\alpha_{jl}, \\ \bar{b}_k &= \sum_{i=1}^{n} b_i\alpha_{ik} + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\xi_k)_{x_ix_j}. \end{aligned}$$

We consider now the quadratic form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a^0_{ij}y_iy_j, \tag{1.5.16}$$

where $a^0_{ij} := a_{ij}(x^0_1, \ldots, x^0_n)$. With the nonsingular linear transformation

$$y_i = \sum_{k=1}^{n} \alpha_{ik}\eta_k, \quad det(\alpha_{ik}) \neq 0,$$

we get the following new quadratic form

$$\sum_{k=1}^{n}\sum_{l=1}^{n} \bar{a}^0_{kl}\eta_k\eta_l,$$

with

$$\bar{a}^0_{kl} = \sum_{i=1}^{n}\sum_{j=1}^{n} a^0_{ij}\alpha_{ik}\alpha_{jl}.$$

Thus, the coefficients of the principal part of Eq. (1.5.15) are transformed into the coefficients of a quadratic form with the aid of a nonsingular linear transformation. It is well known that there exists a singular linear transformation which makes the coefficients of the matrix (a^0_{ij}) of a quadratic form into a diagonal matrix, in which

$$|\bar{a^0_{ii}}| = 1, \text{ or } 0, \quad \text{and} \quad \bar{a^0_{ij}} = 0 \quad (i \neq j; i = 1, 1, \ldots, n)$$

Moreover, by the Sylvester theorem about the inetia of quadratic forms, the number of coefficients $\bar{a^0_{ii}} \neq 0$ equals the rank of the martrix (a^0_{ij}) and the number of negative coefficients is invariant. We say that it is of the *canonical form.*

We say that Eq. (1.5.15) at the point x^0 is of *elliptic type*, if all n coefficients $\bar{a^0_{ii}} \neq 0$ and they have the same sign; of *hyperbolic type*, if all $\bar{a^0_{ii}} \neq 0$ and $n-1$ coefficients $\bar{a^0_{ii}}$ have the same sign and only one has other sign; of *ultrahyperbolic type*, when m coefficients $\bar{a^0_{ii}}$ are equal, and $n-m$ others with other sign are equal $(m > 1, n-m > 1)$; of *parabolic type*, if at least one of the coefficients $\bar{a^0_{ii}}$ vanishes.

We take the new independent variables ξ_i such that at the point x^0

$$\alpha_{ik} = \frac{\partial \xi_k}{\partial x_i} = \alpha^0_{ik}$$

where the α^0_{ik} are the coefficients of the transformation which makes the quadratic form (1.5.16) into the canonical form. We have

$$\begin{aligned}
u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n} + \Phi = 0, &\qquad \text{(elliptic type)},\\
u_{x_1x_1} = \sum_{i=2}^{n} u_{x_ix_x} + \Phi &\qquad \text{(hyperbolic type)},\\
\sum_{i=1}^{m} u_{x_ix_i} = \sum_{i=m+1}^{n} u_{x_ix_i} + \Phi &\qquad \text{(ultrahyperbolic type)},\\
\sum_{i=1}^{n-m} (\pm u_{x_ix_i}) + \Phi = 0, \quad (m > 0) &\qquad \text{(parabolic type)}.
\end{aligned}$$

Is it possible to use a single transformation to reduce (1.5.15) to the canonical form in a sufficiently small neighborhood of each point? To make the reduction for any equation, it is necessary that the number of conditions

$$\begin{aligned}
\bar{a}_{ik} = 0, &\qquad l \neq k, \quad l, k = 1, 2, \ldots, n,\\
\bar{a}_{lk} = \epsilon_l \bar{a}_{11} &\qquad l = 2, 3, \ldots, n; \bar{a}_{11} \neq 0,
\end{aligned}$$

where $\epsilon_l = 0, \pm 1$ should not be greater than the number of the unknown functions $\xi_l, l = 1, 2, n \in \mathbb{N}$:

$$\frac{n(n-1)}{2} + n - 1 \leq n, \quad \text{that is} \quad n \leq 2.$$

For $n = 2$ we have already proved that it is possible to use only one transformation to reduce (1.5.15) to the canonical form in a sufficiently small neighborhood of each point.

Chapter 2

Characteristic Manifolds and the Cauchy problem

2.1 Notation

Let $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$. A vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ whose components are non–negative integers α_k is called a *multi–index.* The letters $x, y, \ldots, \xi, \eta$ will be used for vectors, and $\alpha, \beta, \gamma, \ldots$ for multi–indices. Components are always indicated by adding a subscript to the vector–symbol: ξ has components ξ_k. We set

$$0 = (0, 0, \ldots, 0) \ , \quad 1 = (1, 1, \ldots, 1) \ . \tag{2.1.1}$$

For an $\alpha \in \mathbb{Z}^n$ set

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \ ; \quad \alpha! = \alpha_1! \alpha_2! \ldots \alpha_n! \ , \tag{2.1.2}$$

and for $x \in \mathbb{R}^n$, $\alpha \in \mathbb{Z}^n$

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \ . \tag{2.1.3}$$

By C_α we generally denote a coefficient depending on n nonnegative integers $\alpha_1, \alpha_2, \ldots, \alpha_n$:

$$C_\alpha = C_{\alpha_1 \ldots \alpha_n} \ . \tag{2.1.4}$$

The C_α may be real numbers, or vectors in a space $\mathbb{R}^m$. The general m-th degree polynomial in $x_1, \ldots, x_1$ is then of the form

$$P(x) = \sum_{|\alpha| \le m} C_\alpha x^\alpha \tag{2.1.5}$$

with $\alpha \in \mathbb{Z}^n$, $x \in \mathbb{R}^n$, $C_\alpha \in \mathbb{R}$.

Setting $D_k = \partial / \partial x_k$, we introduce the "gradient vector" $D = (D_1, \ldots, D_n)$, and define the gradient of a function $u(x_1, \ldots, x_n)$ as the vector

$$Du = (D_1 u, \ldots, D_n u) \ . \tag{2.1.6}$$

The general partial differential operator of order m is then denoted by

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n} = \frac{\partial^m}{\partial_{x_1}^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \tag{2.1.7}$$

where $|\alpha| = m$.

2.2 The Cauchy problem

Consider the mth–order linear differential equation for a function $u(x) = u(x_1, \ldots, x_n)$

$$Lu := \sum_{|\alpha| \leq m} A_\alpha(x) D^\alpha u = B(x) \; . \tag{2.2.1}$$

The same formula describes the general mth–order *system* of N differential equations in N unknowns if we interprete u and B as column vectors with N components and A_α as $N \times N$ square matrices. Similarly the general mth–order quasi–linear equation (respectively system of such equations) is

$$Lu := \sum_{|\alpha| \leq m} A_\alpha D^\alpha u + C = 0 \; , \tag{2.2.2}$$

where now the A und C are functions of the independent variables x_k and of the derivatives $D^\beta u$ of the unknown u of orders $|\beta| \leq m-1$. More general nonlinear equations or systems

$$F(x, D^\alpha u) = 0 \tag{2.2.3}$$

can be reduced formally to quasi–linear ones by applying a first–order differential operator to (2.2.3). On the other hand, an mth–order quasi–linear system (2.2.2) can be reduced to a (large) first–order one, by introducing all derivatives $D^\beta u$ with $|\beta| \leq m-1$ as new variables, and making use of suitable compatibility conditions for the $D^\beta u$.

The *Cauchy problem* consists of finding a solution u of (2.2.2) or (2.2.1) having prescribed *Cauchy data* on a hyper–surface $S \subset \mathbb{R}^n$ given by

$$\phi(x_1, x_2, \ldots, x_n) = 0 \; . \tag{2.2.4}$$

Here ϕ shall have m continuous derivatives and the surface should be regular in the sense that

$$D\phi = (\phi_{x_1}, \ldots, \phi_{x_n}) \neq 0 \; . \tag{2.2.5}$$

The *Cauchy data* on S for an mth–order equation consist of the derivatives of u of order less than or equal to $m-1$. They cannot be given arbitrarily but have to satisfy the compatibility conditions valid on S (instead normal derivatives of order less than m can be given independently from each other). We are to find a solution u near S which has these Cauchy data on S. We call S *noncharacteristic* if we can get all $D^\alpha u$ for $|\alpha| = m$ on S from the linear algebraic system or equations consisting of the compatibility condition for the data and the eq. (2.2.2) taken on S. We call S *characteristic* if at each point x of S the surface S is not noncharacteristic.

To get an algebraic criterion for characteristic surfaces we first consider the special case where the hyper–surface S is the coordinate plane $x_n = 0$. The Cauchy data then consist of the $D^\beta u$ with $|\beta| < m$ taken for $x_n = 0$. Singling out the "normal" derivatives on S of orders $\leq m-1$:

$$\left(\frac{\partial u^k}{\partial x_n^k} = \right) D_n^k u = \psi_k(x_1, \ldots, x_{n-1}) \text{ for } k = 0, \ldots, m-1 \text{ and } x_n = 0 \tag{2.2.6}$$

we have on S

$$D^\beta u = D_1^{\beta_1} D_2^{\beta_2} \ldots D_{n-1}^{\beta_n - 1} \psi_{\beta_n} \tag{2.2.7}$$

provided that $\beta_n < m$. In particular for $|\beta| \leq m-1$ we have here the compatibility conditions expressing all Cauchy data in terms of normal derivatives on S. Let α^* denote the multi–index

$$\alpha^* = (0, \ldots, 0, m) \; . \tag{2.2.8}$$

In the eq (2.2.1) or (2.2.2) taken on S it is only the term with $\alpha = \alpha^*$ that is not expressible by (2.2.7) in terms of $\psi_0, \ldots, \psi_{m-1}$ and hence in terms of the Cauchy data. All others contain derivatives $D^\alpha u$ with $\alpha_n \leq m-1$. Thus $D^{\alpha^*}u$, and hence all $D^\alpha u$ with $|\alpha| \leq m$, are determined uniquely on S, if we can solve the differential equation for the term $D^{\alpha^*}u$. This is always possible in a unique way if and only if the matrix A_{α^*} is nondegenerated, i. e., $\det(A_{\alpha^*}) \neq 0$. For a single scalar differential equation this condition reduces to $A_{\alpha^*} \neq 0$. In the linear case the condition

$$\det(A_{\alpha^*}) \neq 0 \tag{2.2.9}$$

does not depend on the Cauchy data on S; in the quasi–linear case, where the A_α depend on the $D^\alpha \beta$ with $|\beta| \leq m-1$ and on x, one has to know the ψ_k in order to decide if S is characteristic.

Since the condition (2.2.9) involves coefficients of mth–order derivatives, we define the *principal part* L_{pr} of L (both in (2.2.2) and (2.2.1)) as

$$L_{pr} := \sum_{|\alpha|=m} A_\alpha D^\alpha \; . \tag{2.2.10}$$

The *"symbol"* of this differential operator is the matrix form ("characteristic matrix" of L):

$$\Lambda(\zeta) = \sum_{|\alpha|=m} A_\alpha \zeta^\alpha \; . \tag{2.2.11}$$

The $N \times N$ matrix $\Lambda(\zeta)$ has elements that are mth–degree forms in the components of the vector $\zeta = (\zeta_1, \ldots, \zeta_n)$. In particular, the multiplier of $D_n^m \left(= \frac{\partial^m}{\partial x_n^m}\right)$ in L_{pr} is $A_{\alpha^*} = \Lambda(\eta)$, where

$$\eta = (0, 0, \ldots, 0, 1) = D\phi \tag{2.2.12}$$

is the unit normal to the surface $\phi = x_n = 0$. The condition for the plane $\phi = x_n = 0$ to be noncharacteristic is then

$$Q(D\phi) \neq 0 \; , \tag{2.2.13}$$

where $Q = Q(\zeta)$ is the *characteristic form* defined by

$$Q(\zeta) := \det(\Lambda(\zeta)) \tag{2.2.14}$$

for any vector ζ. (In the case of a scalar equation ($N = 1$) the characteristic form $Q(\zeta)$ coincides with the polynomial $\Lambda(\zeta)$). We shall see that (2.2.13) is the condition for a surface $\phi = 0$ to be noncharacteristic.

Consider a general S given by (2.2.4), by assumption (2.2.5) we can always suppose that in a neighborhood of a given point of S, the condition $\phi_{x_n} \neq 0$ holds. Thus, the transformation

$$y_i = \begin{cases} x_i & \text{for } i = 1, \ldots, n-1 \\ \phi(x_1, \ldots, x_n) & \text{for } i = n \end{cases} \tag{2.2.15}$$

is then locally regular and invertible. By the chain rule,

$$\frac{\partial u}{\partial x_i} = \sum_k C_{ik} \frac{\partial u}{\partial y_k} \; , \tag{2.2.16}$$

where the

$$C_{ik} = \frac{\partial y_k}{\partial x_i} \tag{2.2.17}$$

are functions of x or of y. Denoting by C the matrix of the C_{ik} and introducing the gradient operator d with respect to y with components

$$d_i = \frac{\partial}{\partial y_i} , \tag{2.2.18}$$

we can write (2.2.16) symbolically, as

$$D = Cd , \tag{2.2.19}$$

taking D and d to be column vectors. Generally, for $|\alpha| = m$

$$D^\alpha = (Cd)^\alpha + R_\alpha , \tag{2.2.20}$$

where R_α is a linear differential operator involving only derivatives of order $\leq m-1$. Then the principal part of L in (2.2.2) or (2.2.1) transformed to y–coordinates is given by

$$L_{pr} = \sum_{|\alpha|=m} A_\alpha (Cd)^\alpha = \ell_{pr} \tag{2.2.21}$$

and its symbol, the characteristic matrix of ℓ, by

$$\lambda(\eta) = \sum_{|\alpha|=m} A_\alpha (C\eta)^\alpha . \tag{2.2.22}$$

Since the mapping (2.2.15) is regular, S is noncharacteristic for L if the plane $y_n = 0$ is noncharacteristic with respect to the operator L transformed to y–coordinates, i. e. if

$$\det(\lambda(\eta)) = \det\left(\sum_{|\alpha|=m} A_\alpha (C\eta)^\alpha\right) \neq 0 \tag{2.2.23}$$

for $\eta = (0, 0, \ldots, 0, 1)$. But then

$$\begin{aligned} \zeta \;=\; C\eta \;&=\; \left(\frac{\partial y_n}{\partial x_1}, \ldots, \frac{\partial y_n}{\partial x_1}\right) \;=\; (\phi_{x_1}, \ldots, \phi_{x_1}) \\ &=\; D\phi . \end{aligned}$$

Thus, the condition for noncharacteristic behavior of S can again be written as (2.2.13).

If u in (2.2.2) stands for a vector with N components, the condition for S to be a characteristic surface is

$$Q(D\phi) = \det\left(\sum_{|\alpha|=m} A_\alpha (D\phi)^\alpha\right) = 0 . \tag{2.2.24}$$

Example 2.2.1 *For the wave equation*

$$u_{tt} = c^2 (u_{x_1 x_1} + u_{x_2 x_2}) \tag{2.2.25}$$

for $u = u(x_1, x_2, t)$, a characteristic surface $t = \psi(x_1, x_2)$ satisfies equation

$$1 = c^2 \left(\psi_{x_1}^2 + \psi_{x_2}^2\right) . \tag{2.2.26}$$

2.3 Real Analytic Functions and the Cauchy–Kowalevski Theorem

2.3.1 Real Analytic Functions

2.3.1.1 Multiple infinite series

We say that the multiple infinite series

$$\sum_{\alpha} c_\alpha \ , \ \alpha \in \mathbb{Z}^n \tag{2.3.1}$$

converges, if

$$\sum_{\alpha} |c_\alpha| < \infty \ . \tag{2.3.2}$$

The sum of a convergent series does not depend on the order of summation.

Example 2.3.1 *For* $x \in \mathbb{R}^n$, $\alpha \in \mathbb{Z}^n$

$$\sum_{\alpha} x^\alpha = \prod_{i=1}^{n} \left(\sum_{\alpha_i=0}^{\infty} x_i^{\alpha_i} \right) = \frac{1}{(1-x_1)(1-x_2)\dots(1-x_n)} = \frac{1}{(1-x)^1} \tag{2.3.3}$$

provided $|x_i| < 1$ *for all* i.

Example 2.3.2 *For* $x \in \mathbb{R}^n$, $\alpha \in \mathbb{Z}^n$,

$$\begin{aligned} \sum_{\alpha} \frac{|\alpha|!}{\alpha!} x^\alpha &= \sum_{j=0}^{\infty} \sum_{|\alpha|=j} \frac{|\alpha|!}{\alpha!} x^\alpha \\ &= \sum_{j=0}^{\infty} (x_1 + \dots + x_n)^j = \frac{1}{1-(x_1+\dots+x_n)} \end{aligned}$$

provided that $|x_1| + \dots + |x_n| < 1$.

Let $c_\alpha(x)$ be continuous scalar functions and defined in a set $S \subset \mathbb{R}^n$; if

$$|c_\alpha(x)| \le c_\alpha \ , \quad \forall \alpha \in \mathbb{Z}^n \ , \ x \in S$$

and

$$\sum_{\alpha} c_\alpha < \infty,$$

then

$$\sum_{\alpha} c_\alpha(x)$$

converges uniformly for $x \in S$ and represents a continuous function. If S is open and $c_\alpha \in C^j(S)$, $\forall \alpha$, and if the series

$$\sum_{\alpha} D^\beta c_\alpha(x)$$

converges uniformly for $x \in S$ and each β with $|\beta| \leq j$, then $\sum_\alpha c_\alpha(x) \in C^j(S)$ and

$$D^\beta \sum_\alpha c_\alpha(x) = \sum_\alpha D^\beta c_\alpha(x) \ , \quad x \in S, |\beta| \leq j \ .$$

Of particular importance are the power series

$$\sum_\alpha c_\alpha x^\alpha \ , \quad x \in \mathbb{R}^n \ , \ \alpha \in Z^n \ , \ c_\alpha \in \mathbb{R} \ . \tag{2.3.4}$$

Assume that the series converges for a certain z:

$$\mu \sum_\alpha |c_\alpha| \, |z^\alpha| < \infty \ . \tag{2.3.5}$$

Then (2.3.4) converges uniformly for all x with

$$|x_i| \leq |z_i| \quad \text{for all } i \ . \tag{2.3.6}$$

Hence,

$$f(x) = \sum_\alpha c_\alpha x^\alpha \tag{2.3.7}$$

defines a continuous function f in the set (2.3.6).

Problem: All series obtained by formal differentiation of (2.3.7) converge in the interior of (2.3.6), and even uniformly in every compact subset of the interior.

2.3.1.2 Real analytic functions

Definition 2.3.3 *Let $f(x)$ be a function whose domain is an open set Ω in $\mathbb{R}^n$ and whose range lies in $\mathbb{R}$. For $y \in \Omega$ we call f* real analytic *at y if there exist $c_\alpha \in \mathbb{R}$ and a neighborhood U of y (all depending on y) such that*

$$f(x) = \sum_\alpha c_\alpha (x-y)^\alpha \ , \quad \forall x \in U \ . \tag{2.3.8}$$

We say f is real analytic *in Ω ($f \in C^\omega(\Omega)$), if f is real analytic at each $y \in \Omega$. A vector $f(x) = (f_1(x), \ldots, f_m(x))$ defined in Ω is called* real analytic, *if each of its components is real analytic.*

Theorem 2.3.4 *If $f = (f_1, \ldots, f_m) \in C^\omega(\Omega)$, then $f \in C^\infty(\Omega)$. Moreover, for any $y \in \Omega$ there exists a neighborhood U of y and positive numbers M, r such that for all $x \in U$.*

$$\begin{aligned} f(x) &= \sum_\alpha \frac{1}{\alpha!} (D^\alpha f(x))(x-y)^\alpha \\ \left| D^\beta f_k(x) \right| &\leq M |\beta|! r^{-|\beta|} \ , \quad \forall \beta \in Z^n \ , \ \forall k \ . \end{aligned} \tag{2.3.9}$$

Theorem 2.3.5 *Let $f \in C^\omega(\Omega)$ where Ω is a connected open set in $\mathbb{R}$. Let $z \in \Omega$. Then f is uniquely determined in Ω if we know the $D^\alpha f(z)$, $\forall \alpha \in \mathbb{Z}^n$. In particular f is uniquely determined in Ω by its values in any non–empty open subset of Ω.*

Proof: Let $f_1, f_2 \in C^\omega(\Omega)$ and let $D^\alpha f_1(z) = D^\alpha f_2(z)$ for all $z \in \mathbb{Z}^n$. Write $g = f_1 - f_2$ and decompose Ω into

$$\begin{aligned} \Omega_1 &= \{x \mid x \in \Omega \, ; D^\alpha g(x) = 0 \quad \forall \alpha \in \mathbb{Z}^n\} \\ \Omega_2 &= \{x \mid x \in \Omega \, ; D^\alpha g(x) \neq 0 \quad \text{for some } \alpha \in \mathbb{Z}^n\} \ . \end{aligned}$$

Then Ω_2 is open by continuity of g. The set Ω_1 also is open; for if $y \in \Omega_1$, we have $g(x) = 0$ in a neighborhood of y by (2.3.9). Since $z \in \Omega_1$ and Ω is connected, Ω_2 must be empty. $\square$

2.3.1.3 Analytic and real analytic functions

Let $x \in \mathbb{C}^n$, $\alpha \in \mathbb{Z}^n$, $\sum \subset \mathbb{C}^n$, $f : \sum \to \mathbb{C}^m$. We call f analytic in $\sum$ ($f \in C^a(\sum)$), if for each $y \in \sum$ we can represent f in the form (2.3.8) in a neighborhood of y. Here $c_\alpha \in \mathbb{C}^m$. It can be easily seen that

$$c_\alpha = \frac{1}{\alpha!} D^\alpha f(y) \; . \tag{2.3.10}$$

Theorem 2.3.6 *A function $f(x)$ with range in $\mathbb{C}^m$ and domain in $\mathbb{C}^n$ is analytic in the open set $\sum$ if f is differentiable with respect to the independent variables.*

Theorem 2.3.7 *If $f \in C^\omega(\Omega)$ with $\Omega \subset \mathbb{R}^n$, then for every compact subset S of Ω there exists a neighborhood $\sum$ of S in $\mathbb{C}^n$ and a function $F \in C^a(\sum)$ such that $F(x) = f(x)$ for all $x \in S$.*

2.3.2 The Cauchy–Kowalevski theorem

The theorem concerns the existence of a real analytic solution of the Cauchy problem for the case of real analytic data und equations. We restrict ourselves to quasi–linear systems of type (2.2.2) since more general nonlinear systems can be reduced to quasi–linear ones by differentiation. We assume that S is real analytic in a neighborhood of one of its points x^0, that is near x^0 the surface S is given by an equation $\phi(x) = 0$, where ϕ is real analytic at x^0 and $D\phi \neq 0$ at x^0, say, $D_n\phi \neq 0$. On S we prescribe compatible Cauchy data $D^\beta u$ for $|\beta| < m$ which shall be real analytic at x^0. The coefficients A_α, C shall be real analytic functions of their arguments x, $D^\beta u$ at the point x^0, $D^\beta x^0$. We assume, moreover, S to be non–characteristic at x^0 (and hence in a neighborhood of x^0) in the sense that $Q(D\phi) \neq 0$. Then the Cauchy–Kowalevski theorem asserts that there exists a unique solution of (2.2.2) which is real analytic at x^0.

First of all, we can transform x^0 into the origin and S locally by an analytic transformation into a neighborhood of the origin in the plane $x_n = 0$. Then by introducing derivatives of orders less than m as new dependent variables one reduces the system to one of first order. We make use here of the fact that the set of real analytic functions is closed under differentiation and composition. One arrives at a first order system in which the coefficient matrix of the term with $\partial u/\partial x_n$ is non–generate because S is non–characteristic. Hence we can solve for $\partial u/\partial x_n$, obtaining a system in the *standard form*

$$\frac{\partial U}{\partial x_n} = \sum_{i=1}^{n-1} a^i(x, U) \frac{\partial U}{\partial x_i} + b(x, U) \tag{2.3.11}$$

where the $a^i(x, U)$ is a square matrix (a^i_{jk}), and $b(x, U)$ is a column vector with components b_i, U is the new vector of u and its derivatives of order less than m. On $x_n = 0$, near 0, we have prescribed initial values $U = f(x_1, \ldots, x_{n-1})$. Here we can assume that $f = 0$, introducing $U - f$ as the new unknown functions. We can add x_n as an additional dependent variable U^* or component of U satisfying the equation $\partial u^*/\partial x_n = 1$, and the initial condition $U^* = 0$. This has the effect that the a^i and b in (2.3.11) do not depend on x_n. In fact, we can write

$$\begin{aligned} a^i(x,U) &= a^i(x_1,x_2,\ldots,x_n,U) \\ &= \overline{a^i}(x_1,x_2,\ldots,x_{n-1},\overline{U}) \ , \\ b(x,U) &= b(x_1,x_2,\ldots,x_n,U) \\ &= \overline{b}(x_1,x_2,\ldots,x_n,\overline{U}) \end{aligned}$$

where $\overline{U} = (U, U^*)$. Writing (2.3.11) componentwise, we have to prove the following version of the Cauchy–Kowalevski theorem:

Theorem 2.3.8 (Cauchy–Kowalevski) *Let the a^i_{jk} and b_j be real analytic functions of $Z = (x_1, x_2, \ldots, x_{n-1}, u_1, \ldots, u_N)$ at the origin of $\mathbb{R}^{N+n-1}$. Then the system of differential equations*

$$\frac{\partial u_j}{\partial x_n} = \sum_{i=1}^{n-1}\sum_{k=1}^{N} a^i_{jk}(z)\frac{\partial u_k}{\partial x_i} + b_j(z) \ , \quad \textit{for } j = 1,\ldots,N \tag{2.3.12}$$

with initial conditions

$$u_j = 0 \quad \textit{for } x_n = 0 \ ; \quad j = 1,\ldots,N \tag{2.3.13}$$

has a unique (among the real analytic u_j) system of solutions $u_j(x_1,\ldots,x_n)$ that is real analytic at the origin.

Proof: See, e.g., John's book [10] or Petrovskii's book [17].

Remark: The theorem of Cauchy and Kowalewski is *local* in character, and applies only to analytic solutions of analytic Cauchy problems. It does not guarantee *global* existence of solutions; it does not exclude the possibility that other non–analytic solutions exist for the same Cauchy problem, nor the possibility that an analytic solution becomes *non–analytic* in some distance away from the initial surface.

2.4 The Uniqueness theorem of Holmgren

2.4.1 The Lagrange–Green Identity

Let Ω be a domain in $\mathbb{R}^n$ with the boundary $\partial\Omega$ which is sufficiently regular. Denote d/dn the differentiation in the direction of the exterior unit normal $\zeta = (\zeta_1,\ldots,\zeta_n)$ of $\partial\Omega$. The Gauss–Ostrogradskii formula says that

$$\int_\Omega D_k u(x)\,dx = \int_{\partial\Omega} u(\xi)\frac{dx_k}{dn}dS = \int_{\partial\Omega} u(\xi)\zeta_k dS \ . \tag{2.4.1}$$

If $\partial\Omega$ is sufficiently regular then this formula is applicable to all $u \in C^1(\overline{\Omega})$. The theorem can be generalized to $u \in C^1(\Omega)\cap C^0(\overline{\Omega})$ by approximating Ω from the interior. More generally, we have the formula for integration by parts,

$$\int_\Omega v^T D_k u\,dx = \int_{\partial\Omega} v^T u\zeta_k dS - \int_\Omega \left(D_k v^T\right) u\,dx \ , \tag{2.4.2}$$

where u, v are column vectors belonging to $C^1(\Omega)$ with T denoting transposition.

Let now L be a linear differential operator

$$Lu = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha u \ . \tag{2.4.3}$$

Let u, v be column vectors and a_α be square matrices in $C^m(\overline{\Omega})$. Then by repeated application of (2.4.2) it follows that

$$\begin{aligned} &\int_\Omega v^T \sum_{|a|\leq m} a_\alpha(x) D^\alpha u \, dx \\ &\quad = \int_\Omega \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha \left(v^T a_\alpha(x) \right) u \, dx \\ &\qquad + \int_{\partial\Omega} M(v, u, \zeta) dS \ . \end{aligned} \tag{2.4.4}$$

Here M in the surface integral is linear in the ζ_k with coefficients which are bilinear in the derivatives of v and u, the total number of differentiations in each term being at most $m-1$. The expression M is not uniquely determined but depends on the order of performing the integration by parts. This is the Lagrange–Green identity for L which we also write in the form

$$\int_\Omega v^T Lu \, dx = \int_\Omega (L^* v)^T u \, dx + \int_{\partial\Omega} M(u, v, \zeta) dS \ , \tag{2.4.5}$$

where L^* is the (formally) adjoint operator to L, defined by

$$L^* v := \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha \left(a_\alpha(x)^T v \right) \ . \tag{2.4.6}$$

For the Laplace operator $L = \Delta$, and scalar functions u and v we have

$$\begin{aligned} \int_\Omega v \Delta u \, dx &= \int_{\partial\Omega} \sum_i v u_{x_i} \zeta_i \, dS - \int_\Omega \sum u_{x_i} v_{x_i} \, dx \\ &= \int_{\partial\Omega} v \frac{du}{dn} \, dS - \int_\Omega \sum_i u_{x_i} v_{x_i} \, dx \ . \end{aligned}$$

Integrating once more by parts we obtain

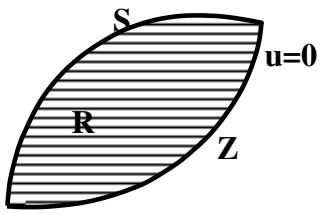
$$\int_\Omega v \Delta u \, dx = \int_\Omega u \Delta v \, dx + \int_{\partial\Omega} \left(v \frac{du}{dn} - u \frac{dv}{dn} \right) dS \ . \tag{2.4.7}$$

2.4.2 The Uniqueness theorem of Holmgren

The Cauchy–Kowalevski theorem does not exclude the possibility that *nonanalytic* solutions of the same Cauchy problem might exist, furthermore it works only for analytic Cauchy data. However, uniqueness can be proved for the Cauchy problem for a *linear* equation with analytic coefficients and for data (not necessarily analytic) prescribed in an analytic noncharacteristic surface S. The proof (due to Holmgren) uses of the Cauchy–Kowalevski theorem and the Lagrange–Green identity.

Let u be a solution of a first order system

$$Lu = \sum_{k=1}^n a^k(x) \frac{\partial u}{\partial x_k} + b(x) u = 0 \tag{2.4.8}$$



in a "lens–shaped" region R bounded by two hypersurfaces S and Z.

Here $x \in \mathbb{R}^n$, $u \in \mathbb{R}^n$ and a^k, b are $N \times N$ matrices. Assume that u has Cauchy data $u = 0$ on Z and that S is non–characteristic; that is, the matrix

$$A = \sum_{k=1}^{n} a^k(x)\zeta_k \tag{2.4.9}$$

is non–degenerate for $x \in S$, and ζ is the unit normal of S at x. Let v be a solution of the *adjoint* equation

$$L^*v = -\sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left((a^k)^T v\right) + b^T v = 0 \quad \text{for } x \in \mathbb{R} \tag{2.4.10}$$

(T for transposition) with Cauchy data

$$v = w(x) \quad \text{for } x \in S\ . \tag{2.4.11}$$

Applying the Lagrange–Green identity we find that

$$\int_S w^T A u\, ds = 0\ . \tag{2.4.12}$$

Let now Γ be the set of functions w on S for which the Cauchy problem (2.4.10)–(2.4.11) has a solution v. If Γ is dense in $C^0(S)$ we conclude that (2.4.12) holds for every $w \in C^0(S)$. But then $Au = 0$ on S, and hence also, since A is non–degenerate, $u = 0$ on S. For if $Au \neq 0$ for some $z \in S$, then also $Au \neq 0$ for all x in a neighborhood ω of z on S. We can find a continuous non–negative scalar function $\phi(x)$ on S with support in ω and with $\phi(z) > 0$. Then

$$\int_S \phi(Au)^T(Au)\, ds > 0$$

for $w = \phi(Au)$ contrary to (2.4.12). Now in the case where the matrices a^k and b are *real analytic,* and S and w are real analytic, the Cauchy–Kowalevski theorem guarantees the existence of a solution v of $L^*v = 0$ with $v = w$ on S in a *sufficiently small* neighborhood of S, and so we cannot be sure that this neighborhood includes all of R. To bridge the gap between S and Z and to conclude that $u = 0$ throughout R, we have to cover all of R by an analytic family of non–characteristic surfaces S_λ.

Definition 2.4.1 *A family of hypersurfaces S_λ in $\mathbb{R}^n$ with parameter λ ranging over an open interval $\Lambda = (a, b)$ forms an* analytic field, *if the S_λ can be transformed bi–analytically into the cross sections of a cylinder whose base is the unit ball Ω in $\mathbb{R}^{n-1}$. This means that there exists a $1-1$ mapping $F : \Omega * \Lambda \to \mathbb{R}^n$, where $x = F(y)$ is real analytic in $\Omega \times \Lambda$ and has non–vanishing Jacobian; the S_λ for $\lambda \in \Lambda$ are the sets*

$$S_\lambda = \left\{x \;\middle|\; x = F(y);\ (y_1, \ldots, y_{n-1}) \in \Omega\ ,\ y_n = \lambda\right\}\ . \tag{2.4.13}$$

Our conditions imply that the set

$$\sum = \cup_{\lambda\in\Lambda} S_\lambda \ , \tag{2.4.14}$$

called the *support* of the field, is open, and that the transformation $x = F(y)$ has a real analytic inverse $y = G(x)$ mapping $\sum$ onto $\Omega \times \Lambda$. In particular $\lambda(x) = G_n(x)$ is real analytic in $\sum$.

Uniqueness theorem Let the S_λ for $\lambda \in \Lambda = (a, b)$ form an analytic field in $\mathbb{R}^n$ with support $\sum$. Consider the mth order linear system

$$Lu = \sum_{|\alpha|\le m} A_\alpha(x) D^\alpha u = 0 \tag{2.4.15}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^N$, and the coefficient matrices $A_\alpha(x)$ are real analytic in $\sum$. Introduce the sets

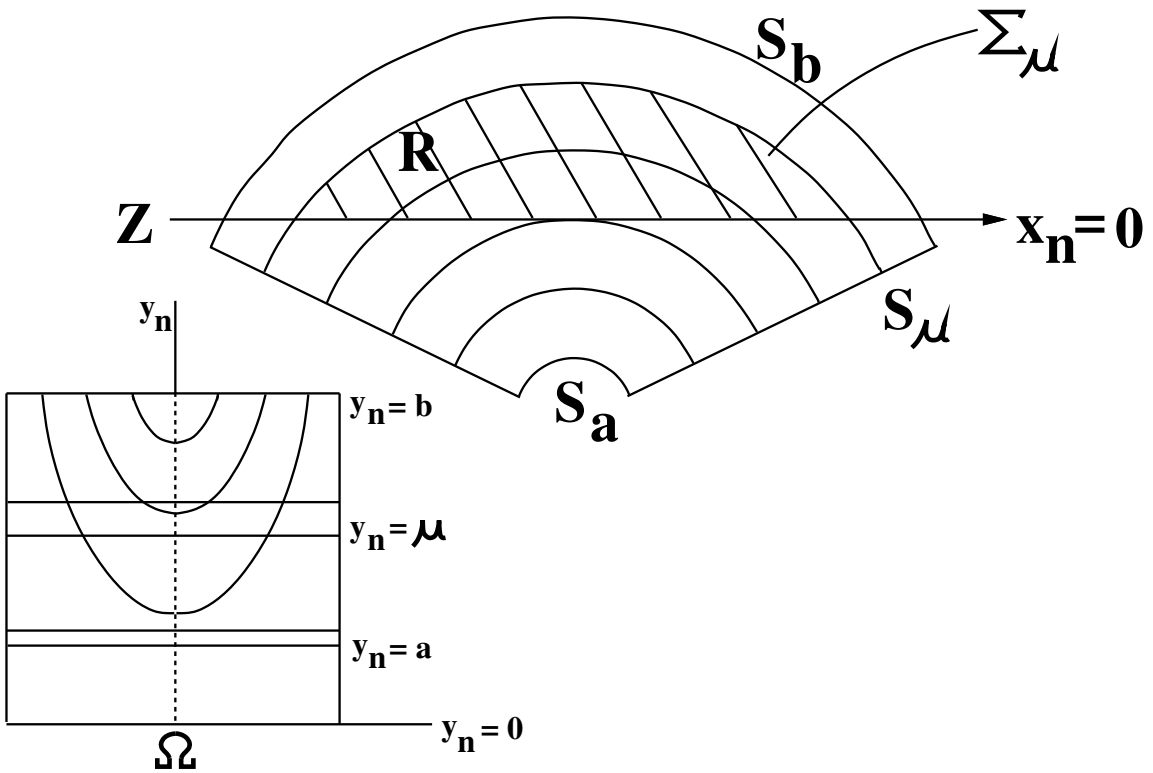
$$R = \left\{ x \;\middle|\; x \in \sum,\ x_n \ge 0 \right\} \tag{2.4.16}$$

$$Z = \left\{ x \;\middle|\; x \in \sum,\ x_n = 0 \right\} \tag{2.4.17}$$

and for $\mu \in \Lambda$

$$\sum_\mu = \left\{ x \;\middle|\; x \in S_\lambda \text{ for some } \lambda \text{ with } a < \lambda \le \mu \right\} \ . \tag{2.4.18}$$

We assume that Z and all S_λ are non–characteristic, with respect to L, and that $\sum_\mu \cap R$ for any $\mu \in \Lambda$ is a **closed** subset of the open set $\sum$. Let u be a solution of (2.4.15) of class $C^m(R)$ and having vanishing Cauchy data on Z. Then $u = 0$ in R.



Chapter 3

Hyperbolic Equations

In this chapter we shall consider the following equation

$$u_{tt} = a^2 u_{xx} + f(x,t)$$

for $x \in \mathbb{R}^1$, $t \in R^+$.

3.1 Boundary and initial conditions

Consider the equation of small transverse vibrations of a string

$$u_{tt} = a^2 u_{xx} + f(x,t) \tag{3.1.1}$$

for $0 \le x \le \ell$. If the ends of the string are fixed, say, on the plane, then *the boundary conditions*

$$u(0,t) = 0 \,, \quad u(\ell,t) = 0 \tag{3.1.2}$$

must be satisfied. Furthermore, the *initial conditions*, i. e. the form and the velocity $\partial u/\partial t$ at the moment when the process begins, say at t_0, are given:

$$u(x,t_0) = \varphi(x) \,, \tag{3.1.3}$$

$$u_t(x,t_0) = \psi(x) \,. \tag{3.1.4}$$

These conditions must coincide with the conditions (3.1.2) at the ends of $[0,\ell]$. Here φ and ψ are given functions. Later we shall prove that that these conditions fully describe the solution of the equation (3.1.1).

If the ends of the string are moving with a given law, then the boundary conditions have the form

$$u(0,t) = \mu_1(t) \,, \quad u(\ell,t) = \mu_2(t) \,, \tag{3.1.2$'$}$$

where $\mu_1(t)$ and $\mu_2(t)$ are given.

The boundary conditions can have another forms. For example, if one end of the string is fixed, say for $x = 0$, and the other end is free, then we have

$$u(0,t) = 0 \,; \tag{3.1.5}$$

and at the free end the elasticity tension

$$T(\ell,t) = k\,\frac{\partial u}{\partial x}\bigg|_{x=\ell}$$

is zero (no external force, $k(x)$ is the Young modulus at the point x). Thus,

$$u_x(\ell,t) = 0\ . \tag{3.1.6}$$

If the end $x=0$ is moving with a certain law $\mu(t)$, meanwhile at $x=\ell$ a given force $\overline{\nu}(t)$ is acting then

$$u(0,t) = \mu(t)\ , \quad u_x(\ell,t) = \nu(t)\ , \quad \nu(t) = \frac{\overline{\nu}}{k}$$

Another type of boundary conditions is

$$u_x(\ell,t) = -h[u(\ell,t) - \theta(t)] \tag{3.1.7}$$

(elastische Befestigung).

Thus, we have the following types of boundary conditions

Boundary condition of	first kind	$u(0,t) = \mu(t)$
———————	second kind	$u_x(0,t) = \nu(t)$
———————	third kind	$u_x(0,t) = h[u(0,t) - \theta(t)]$.

There are also other kinds, e. g.,

$$u_x(\ell,t) = \frac{1}{k}F[u(\ell,t)]\,,\ldots$$

We do not enter in this direction, but refer to the book by Tikhonov and Samarskii [22].

If the time interval is small, then the boundary conditions have not so much influence upon the vibration and so we can consider the problem as the limit case for the initial value problem in an unbounded domain:

Find a solution of the equation

$$u_{tt} = a^2 u_{xx} + f(x,t)\ , \quad -\infty < x < \infty\ ,\ t > 0\ ,$$

with the initial deformation

$$u(x,t)\Big|_{t=0} = \varphi(x)\ , \quad -\infty < x < \infty\ ,$$

and the initial velocity

$$u_t(x,t)\Big|_{t=0} = \psi(x)\ , \quad -\infty < x < \infty\ .$$

This is in fact the Cauchy problem for a hyperbolic equation.

If one wants to study the behaviour of the string near to one end assuming that the influence of the boundary condition at the other end is small, then we have, e. g.,

$$\begin{aligned} u(0,t) &= \mu(t)\ , \quad t \geq 0\ , \\ u(x,0) &= \varphi(x)\ , \quad 0 \leq x < \infty\ , \\ u_t(x,0) &= \psi(x)\ , \quad 0 \leq x < \infty\ . \end{aligned}$$

For a concrete vibration process we have to find appropriate boundary conditions for it.

3.2 The uniqueness

Theorem 3.2.1 *The differential equation*

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) + f(x,t)\,, \quad 0 < x < \ell\,,\ t > 0\,, \tag{3.2.1}$$

has no more than one solution which satisfies the initial and boundary value conditions

$$\begin{aligned} u(x,0) &= \varphi(x)\,, \\ u_t(x,0) &= \psi(x)\,, \\ u(0,t) &= \mu_1(t)\,, \\ u(\ell,t) &= \mu_2(t)\,. \end{aligned} \tag{3.2.2}$$

Here, we assume that the function $u(x,t)$ and its first and second derivatives are continuous in the interval $0 \le x \le \ell$ for $t \ge 0$ and $\rho(x) > 0\,,\ k(x) > 0$ are continuous, given functions.

Proof: Suppose that there exist two solutions $u_1(x,t)\,,\ u_2(x,t)$ of our problem, then its difference

$$v(x,t) = u_1(x,t) - u_2(x,t)$$

satisfies the homogeneous equation

$$\rho\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left(k\frac{\partial v}{\partial x}\right) \tag{3.2.3}$$

and the homogeneous conditions

$$\left.\begin{aligned} v(x,0) &= 0\,, \\ v_t(x,0) &= 0\,, \\ v(0,t) &= 0\,, \\ v(\ell,t) &= 0\,. \end{aligned}\right\} \tag{3.2.4}$$

We shall prove that with the conditions (3.2.4) v is identically zero.

For this purpose we consider the function

$$E(t) := \frac{1}{2}\int_0^\ell \left\{k(v_x)^2 + \rho(v_t)^2\right\} dx \tag{3.2.5}$$

and prove that it is independent of t.

The function $E(t)$ has a physical meaning; it is the total energy of the string at the moment t. Since v is twice continuously differentiable, we can differentiate $E(t)$ and get

$$\frac{dE(t)}{dt} = \int_0^\ell \left(kv_x v_{xt} + \rho v_t v_{tt}\right) dx \tag{3.2.6}$$

An integration by part in the second term of the right–hand side gives

$$\begin{aligned} \int_0^\ell kv_x v_{xt}\,dx &= kv_x v_t\Big|_0^\ell - \int_0^\ell v_t(kv_x)_x\,dx \\ &= -\int_0^\ell v_t(kv_x)_x\,dx\,. \end{aligned} \tag{3.2.7}$$

(Since $v(0,t)=0$ it follows that $v_t(0,t)=0$ and $v(\ell,t)=0$ implies $v_t(\ell,t)=0$). Thus,

$$\begin{aligned}\frac{dE(t)}{dt} &= \int_0^\ell \left[\rho v_t v_{tt} - v_t (k v_x)_x\right] dx \\ &= \int_0^\ell \left[\rho v_{tt} - (k v_x)_x\right] v_t \, dx = 0 \ .\end{aligned}$$

It means, $E(t)=\text{const}$. Furthermore, since $v(x,0)=0$, $v_t(x,0)=0$ we have

$$E(t)=\text{const}=E(0)=\frac{1}{2}\int_0^\ell \left[k(v_x)^2+\rho(v_t)^2\right]\Big|_{t=0} dx = 0 \ . \tag{3.2.8}$$

On the other hand, the functions $\rho(x)$ and $k(x)$ are positive, and it follows from (3.2.6), (3.2.8) that

$$v_x(x,t)\equiv 0 \ , \quad v_t(x,t)\equiv 0 \ .$$

Hence

$$v(x,t)=\text{const}=v(x,0)\equiv 0 \ .$$

□

The same result remains valid for the second boundary problem, where

$$u_x(0,t)=\mu_1(t) \text{ and } u_x(\ell,t)=\mu_2(t) \ ,$$

because in this case $v_x(0,t)=v_x(\ell,t)=0$.

The proof for the third boundary value problem

$$\begin{aligned} u_x(0,t)-h_1 u(0,t) &= \mu_1(t) \ , \quad h_1\ge 0 \ , \\ u_x(\ell,t)+h_2 u(0,t) &= \mu_2(t) \ , \quad h_2\ge 0 \ , \end{aligned}$$

is a little bit different. In this case

$$\begin{aligned} v_x(0,t)-h_1 v(0,t) &= 0 \ , \\ v_x(\ell,t)+h_2 v(0,t) &= 0 \ , \end{aligned}$$

and in (3.2.7) we obtain

$$k v_x v_t\Big|_0^\ell = -\frac{k}{2}\frac{\partial}{\partial t}\left[h_2 v^2(\ell,t)+h_1 v^2(0,t)\right] \ .$$

Integrating $\frac{dE}{dt}$ from 0 to t gives

$$\begin{aligned} E(t)-E(0) &= \int_0^t \int_0^\ell v_t \left[\rho v_{tt} - (k v_x)_x\right] dx\, dt \\ &- \frac{k}{2}\left\{h_2\left[v^2(\ell,t)-v^2(\ell,0)\right]+h_1\left[v^2(0,t)-v^2(0,0)\right]\right\} \ . \end{aligned}$$

Taking into account that

$$E(0)=\frac{1}{2}\int_0^\ell \left\{k v_x^2+\rho v_t^2\right\}\Big|_{t=0} dx = 0$$

and $v(\ell,0)=v(0,0)=0$, we get

$$E(t)=-\frac{k}{2}\left[h_2 v^2(\ell,t)+h_1 v^2(0,t)\right]\le 0 \ .$$

On the other hand $E(t)\ge 0$, it follows that $E(t)\equiv 0$ and so $v(x,t)\equiv 0$.

3.3 The method of wave propagation (Wellenausbreitungsmethode)

3.3.1 The D'Alembert method

We consider the problem

$$u_{tt} - a^2 u_{xx} = 0\ , \quad -\infty < x < \infty\ ,\ t > 0\ , \tag{3.3.1}$$

$$u(x,0) = \varphi(x)\ , \quad u_t(x,0) = \psi(x)\ , \quad -\infty < x < \infty\ . \tag{3.3.2}$$

First, we transform the equation (3.3.1) into the canonical form, in which there is the mixed derivative. The characteristic equation

$$dx^2 - a^2 dt^2 = 0$$

is equivalent to

$$dx - a\,dt = 0\ , \quad dx + a\,dt = 0\ ,$$

the integrals of which are

$$x - at = C_1\ , \quad x + at = C_2\ .$$

Introducing the new variables

$$\xi = x + at\ , \quad \eta = x - at$$

we get

$$\begin{array}{lll} u_x = u_\xi + u_\eta & , & u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \\ u_t = a(u_\xi - u_\eta) & , & u_{tt} = (u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\xi})\,a^2\ . \end{array}$$

Thus, the equation (3.3.1) is transformed into

$$u_{\xi\eta} = 0\ . \tag{3.3.3}$$

It is clear that every solution of Eq. (3.3.3) has the form

$$u_\eta(\xi,\eta) = f^*(\eta)\ ,$$

where $f^*(\eta)$ is an arbitrary function only of η.

Integrating this equation we obtain

$$u(\xi,\eta) = \int f^*(\eta) d_\eta + f_1(\xi) = f_1(\xi) + f_2(\eta)\ , \tag{3.3.4}$$

where f_1 depends only on ξ and f_2 depends only on η. Conversely, if f_1 and f_2 are arbitrary differentiable functions, then the function $u(\xi,\eta)$ defined by (3.3.4) is a solution of (3.3.3). It follows that

$$u(x,t) = f_1(x+at) + f_2(x-at) \tag{3.3.5}$$

is a general solution of (3.3.3).

Assume that there exists a solution of our Cauchy problem (3.3.1), (3.3.2), then it is defined by (3.3.5). The functions f_1 and f_2 are defined by

$$u(x,0) = f_1(x) + f_2(x) = \varphi(x) \tag{3.3.6}$$

$$u_t(x,0) = af_1'(x) - a_2 f_2'(x) = \psi(x)\ . \tag{3.3.7}$$

From (3.3.7) we have

$$f_1(x) - f_2(x) = \frac{1}{a}\int_{x_0}^{x} \psi(\alpha)d\alpha + c\ ,$$

where x_0 and c are some constants. The last equation and (3.3.6) yield

$$\left.\begin{aligned} f_1(x) &= \frac{1}{2}\varphi(x) + \frac{1}{2a}\int_{x_0}^{x} \psi(\alpha)d\alpha + \frac{c}{2} \\ f_2(x) &= \frac{1}{2}\varphi(x) - \frac{1}{2a}\int_{x_0}^{x} \psi(\alpha)d\alpha - \frac{c}{2}\ . \end{aligned}\right\} \tag{3.3.8}$$

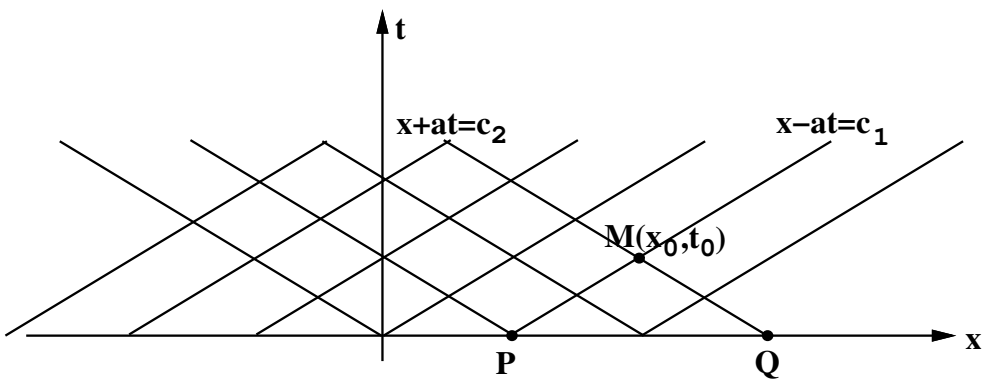
It follows that

$$u(x,t) = \frac{\varphi(x+at)+\varphi(x-at)}{2} + \frac{1}{2}\left\{\int_{x_0}^{x+at} \psi(\alpha)d\alpha - \int_{x_o}^{x-at} \psi(\alpha)d\alpha\right\}$$

or

$$\boxed{u(x,t) = \frac{\varphi(x+at)+\varphi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\alpha)d\alpha} \tag{3.3.9}$$

The formula (3.3.9) is called D'Alembert's formula. It proves the uniqueness of the solution of the problem (3.3.1), (3.3.2). It is clear also that if φ is twice differentiable and ψ is once differentiable then (3.3.9) is a solution of (3.3.1)–(3.3.2). Thus the method of D'Alembert proves not only the uniqueness of the solution but also its existence.



Let (x_0, t_0) be a point in the plane (x,t), $t \geq 0$. There are two characteristics crossing this point $M = (x_0, t_0)$. Denoting the intersections of these characteristics with the line $x = 0$ by P and Q, respectively, we see that in order to determine $u(x_0, t_0)$ we need only φ and ψ in $[P, Q]$. The triangle MPQ is called the *characteristic triangle* of the point M.

3.3.2 The stability of the solution

The solution of Eq. (3.3.1) is uniquely determined by the condition (3.3.2). We shall prove that it depends continuously on the Cauchy data. Namely, we prove the following result.

Theorem 3.3.1 *For every time interval $[0, t_0]$, and $\varepsilon > 0$ there always exists a number $\delta = \delta(\varepsilon, t_0)$ such that if*

$$|\varphi| < \delta\ , \quad |\psi| < \delta\ ,$$

then

$$|u(x,t)| < \varepsilon \quad (0 \leq t \leq t_0)\ .$$

Remark 3.3.2 Let $u_i(x,t)$ be the solutions of (3.3.1) with the Cauchy data φ_i and ψ_i, $i = 1,2$. The theorem says that, for every time interval $[0,t_0]$, and $\varepsilon > 0$ there always exists a number $\delta = \delta(\varepsilon, t_0)$ such that if

$$|\varphi_1 - \varphi_2| < \delta \ , \quad |\psi_1 - \psi_2| < \delta \ ,$$

then

$$|u_1(x,t) - u_2(x,t)| < \varepsilon \quad (0 \le t \le t_0) \ .$$

Proof: From the D'Alembert formula (3.3.9) we have

$$\begin{aligned} |u(x,t)| &\le \frac{|\varphi(x+at)| + |\varphi(x-at)|}{2} + \frac{1}{2a}\int_{x-at}^{x+at} |\psi(\alpha)| d\alpha \\ &< \delta + \frac{1}{2a}\delta . 2at \ \le \ \delta(1+t_0) \ . \end{aligned}$$

Thus, if we take $\delta = \frac{\varepsilon}{1+t_0}$, then $|u(x,t)| < \varepsilon$. □

A boundary value problem is called **well–posed**, if

1. It has a solution (Existence)
2. This solution is unique (Uniqueness)
3. The solution depends continuously on the initial and boundary conditions (Stability).

If one of these three condtions is not met, we say that our problem is **ill–posed**.

For our Cauchy problem we know that

$$u(x,t) = \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\alpha) d\alpha$$

is a unique solution, if $\varphi \in C^2$, $\psi \in C^1$, and this solution is stable in L_∞–norm. Thus, our problem is well–posed in this class. However, if $\varphi \notin C^2$, $\psi \notin C^1$, then the D'Alembert formula gives no solution to our Cauchy problem. Anyway, we have proved that the function $u(x,t)$ designed by (3.3.9) is stable for any φ and ψ, thus if $\varphi \notin C^2$, $\psi \notin C^1$ then we can approximate them by $\varphi^\varepsilon \in C^2$, $\psi^\varepsilon \in C^1$ and receive a solution u^ε designed by (3.3.9) with respect to these data, such that $|u - u^\varepsilon| < \varepsilon$. Letting $\varepsilon \to 0$, we have $u^\varepsilon \to u$. We say that u is a generalized solution of our Cauchy problem with respect to non-smooth Cauchy data φ and ψ.

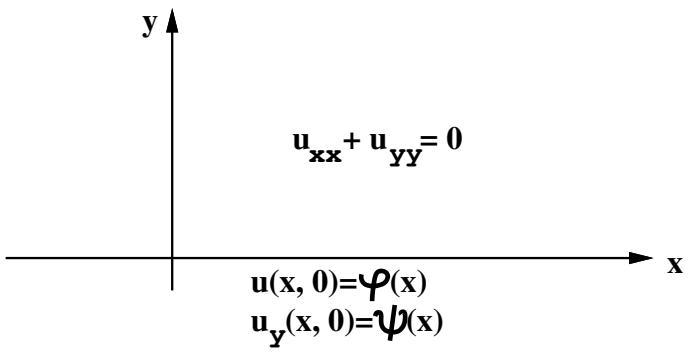
Another example of ill–posedness

The Cauchy problem for the Laplace equation has the form

$$\begin{aligned} &u_{xx} + u_{yy} = 0 \ , \quad -\infty < x < \infty \ , \ y > 0 \\ &u(x,0) = \varphi(x) \ , \quad u_y(x,0) = \psi(x) \ , \quad -\infty < x < \infty \ . \end{aligned}$$

The functions

$$u^{(1)}(x,y) = 0 \quad , \quad u^{(2)}(x,y) = \tfrac{1}{\lambda}\sin(\lambda x)\cosh(\lambda y)$$



satisfy the Laplace equation; furthermore

$$u^{(1)}(x,0) = 0 \quad , \quad u^{(2)}(x,0) = \varphi(x) = \frac{1}{\lambda}\sin(\lambda x)$$
$$u_y^{(1)}(x,0) = 0 \quad , \quad u_y^{(2)}(x,0) = \psi(0) = 0 \ .$$

If $\lambda \to 0$, then $|u^{(1)}(x,0) - u^{(2)}(x,0)| \to 0$; however, for $y > 0$

$$\left|u^{(1)}(x,y) - u^{(2)}(x,y)\right| = \frac{1}{\lambda}|\sin\lambda x|\,|\cosh(xy)| \not\to 0 \ .$$

Thus, the Cauchy problem for the Laplace equation is ill–posed.

3.3.3 The reflection method

Consider the problem in the half axis $x > 0$:

$$\begin{aligned} &u_{tt} - a^2 u_{xx} = 0 \quad , && 0 < x < \infty \ , \ t > 0 \ , \\ &u(0,t) = \mu(t) \quad , && (\text{or} \ \ u_x(0,t) = \nu(t) \ ) \ , \\ &u(x,0) = \varphi(x) \quad , && 0 < x < \infty \ , \\ &u_t(x,0) = \psi(x) \quad , && 0 < x < \infty \ . \end{aligned}$$

For simplicity, first, suppose that

$$u(0,t) = 0 \qquad (\text{or}, \ \ u_x(0,t) = 0) \ ,$$

Lemma 3.3.3 *Consider the problem (3.3.1), (3.3.2).*

1. *If the Cauchy data φ and ψ are odd functions with respect to some point x_0, then the corresponding solution $u(x,t)$ is equal to zero at this point.*

2. *If the Cauchy data φ and ψ are even functions with respect to x_0, then the derivative of u with respect to x is equal to zero at this point.*

Proof 1. We can suppose that $x_0 = 0$, and thus

$$\varphi(x) = -\varphi(-x) \ , \quad \psi(x) = -\psi(-x) \ .$$

For $x = 0$ we have

$$u(0,t) = \frac{\varphi(at) + \varphi(-at)}{2} + \frac{1}{2a}\int_{-at}^{at} \psi(\alpha)d\alpha = 0 \ .$$

2. If $\varphi(x) = \varphi(-x)\,,\ \psi(x) = \psi(-x)$ then

$$u_x(0,t) = \frac{\varphi'(at) + \varphi'(-at)}{2} + \frac{1}{2a}\big[\psi(at) - \psi(-at)\big] = 0\ .$$

□

Now we consider the problem

$$\begin{array}{lll} u_{tt} = a^2 u_{xx} & , & 0 < x < \infty\,,\ t > 0\ , \\ u(x,0) = \varphi(x) & , & 0 < x < \infty\ , \\ u_t(x,0) = \psi(x) & , & 0 < x < \infty\ , \\ u(0,t) = 0 & , & t > 0\ . \end{array}$$

Let

$$\begin{aligned} \Phi(x) &= \begin{cases} \varphi(x) & x > 0 \\ -\varphi(-x) & x < 0 \end{cases} \\ \Psi(x) &= \begin{cases} \psi(x) & x > 0 \\ -\psi(-x) & x < 0 \end{cases} \end{aligned}$$

be odd extensions of φ and ψ, respectively. It is clear that the function

$$u(x,t) = \frac{\Phi(x+at) + \Phi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \Psi(\alpha)d\alpha$$

is well definded. Furthermore, taking Lemma 3.3.3 into account , we see that

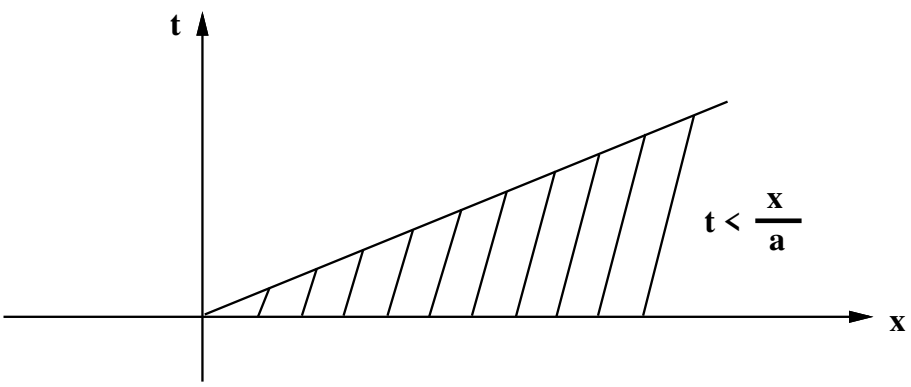
$$u(0,t) = 0\ .$$

Besides, for $x > 0$

$$\left.\begin{array}{rcccl} u(x,0) & = & \Phi(x) & = & \varphi(x)\ , \\ u_t(x,0) & = & \Psi(x) & = & \psi(x)\ . \end{array}\right\}$$

Thus $u(x,t)$ is a solution of our problem. We can write the solution in the following way

$$u(x,t) = \begin{cases} \dfrac{\varphi(x+at) + \varphi(x-at)}{2} + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at} \psi(\alpha)d\alpha \text{ for } t < \dfrac{x}{a}\,,\ x > 0 \\ \dfrac{\varphi(x+at) + \varphi(at-x)}{2} + \dfrac{1}{2a}\displaystyle\int_{at-x}^{x+at} \psi(\alpha)d\alpha\,, \quad t > \dfrac{x}{a}\,,\ x > 0 \end{cases}$$



In the domain $t < \frac{x}{a}$ the boundary condition has no influence on the solution and it coincides with the solution of (3.3.1) in $(-\infty, \infty)$.

We consider now the case

$$u_x(0,t) = 0 \ .$$

With this purpose we take the even extension of φ and ψ:

$$\Phi(x) = \begin{cases} \varphi(x) & x > 0 \\ \varphi(-x) & x < 0 \end{cases} \ , \quad \Psi(x) = \begin{cases} \psi(x) \ , & x > 0 \\ \psi(-x) \ , & x < 0 \end{cases} \ .$$

As a solution of our equation (3.3.1) we have

$$u(x,t) = \frac{\Phi(x+at) + \Phi(x-at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \Psi(\alpha) d\alpha$$

or

$$u(x,t) = \begin{cases} \dfrac{\varphi(x+at) + \varphi(x-at)}{2} + \dfrac{1}{2a} \displaystyle\int_{x-at}^{x+at} \psi(\alpha) d\alpha \ , & t < \dfrac{x}{a} \\ \dfrac{\varphi(x+at) + \varphi(at-x)}{2} + \dfrac{1}{2a} \left\{ \displaystyle\int_0^{x+at} \psi(\alpha) d\alpha + \int_0^{at-x} \psi(\alpha) d\alpha \right\} \ , & t > \dfrac{x}{a} \end{cases} \ .$$

It is clear that $u(x,t)$ satisfies the equation, the initial conditions and the boundary condition $u_x(0,t) = 0$.

It remains to consider non–homogeneous cases

$$u(0,t) = \mu(t) \neq 0 \ ,$$

and

$$u_x(0,t) = \nu(t) \neq 0 \ .$$

Let us consider the case of homogeneous Cauchy data

$$\overline{u}(x,0) = 0 \ , \quad \overline{u}_t(x,0) = 0 \ , \quad \overline{u}(0,t) = \mu(t) \ , \ t > 0 \ .$$

The general solution to this problem has the form

$$\overline{u}(x,t) = f_1(x+at) + f_2(x-at) \ .$$

Since $\overline{u}(x,0) = 0$, $\overline{u}_t(x,0) = 0$, we see that $f_1(s) = -f_2(s) = c$ for $s \geq 0$, where c is a constant.

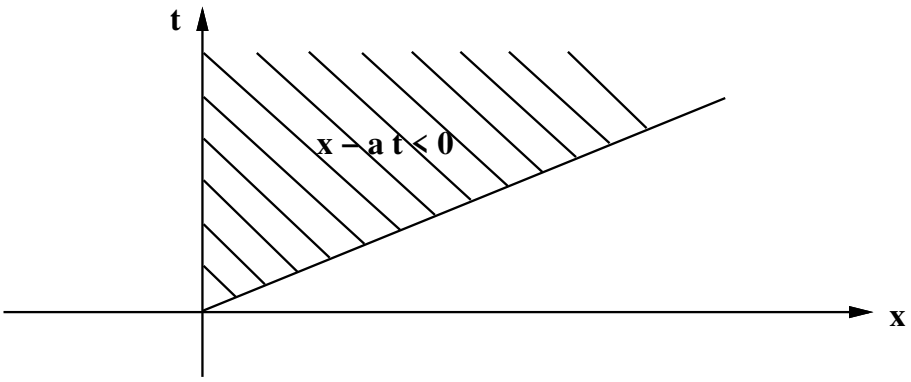
From the boundary condition

$$\begin{aligned} \overline{u}(0,t) \ = \ \mu(t) \ &= \ f_1(at) + f_2(-at) \\ &= \ f_2(-at) + c \ , \ t > 0 \ . \end{aligned}$$

Thus, $f_2(s) = \mu\left(\frac{-s}{a}\right) + c$, $s < 0$. Therefore, if $x - at < 0$, say $t > \frac{x}{a}$, we have

$$\overline{u}(x,t) = \mu\left(-\frac{x-at}{a}\right) + c = \mu\left(\frac{-x}{a} + t\right) + c \ .$$

Putting $x = 0$, we get $c = 0$.



If $x - at \geq 0$, say $t < \frac{x}{a}$, then

$$\overline{u}(x,t) = f_1(x+at) + f_2(x-at) = 0 \ .$$

Now it is easily seen that the solution of our boundary value problem is

$$u(x,t) = \begin{cases} \dfrac{\varphi(x+at)+\varphi(x-at)}{2} + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at} \psi(\alpha)d\alpha \ , & t < \dfrac{x}{a} \\ \mu\left(t - \frac{x}{a}\right) + \dfrac{\varphi(x+at)+\varphi(at-x)}{2} + \dfrac{1}{2a}\displaystyle\int_{at-x}^{x+at} \psi(\alpha)d\alpha \ , & t > \dfrac{x}{a} \ . \end{cases}$$

3.4 The Fourier method

3.4.1 Free vibration of a string

Consider the problem

$$u_{tt} = a^2 u_{xx} \ , \quad 0 < x < \ell \ , \ t > 0 \ , \tag{3.4.1}$$
$$u(0,t) = u(\ell,t) = 0 \ , \quad t > 0 \ , \tag{3.4.2}$$
$$u(x,0) = \varphi(x) \ , \quad u_t(x,0) = \psi(x) \ , \quad 0 \leq x \leq \ell \ . \tag{3.4.3}$$

The equation (3.4.1) is linear and homogeneous, and therefore the sum of its special solutions is again a solution. We shall try to find the solution of our problem (3.4.1)–(3.4.3) via a sum of special solutions with appropriate coefficients. With this purpose we consider the auxiliary problem: Find a solution of the problem

$$u_{tt} = a^2 u_{xx} \ , \quad 0 < x < \ell \ , \ t > 0 \ , \tag{3.4.1}$$

which is not identically vanishing and satisfying the homogeneous boundary conditions

$$u(0,t) = 0 \ , \quad u(\ell,t) = 0 \ , \tag{3.4.2}$$

and can be represented in the form

$$u(x,t) = X(x)T(t) \ . \tag{3.4.4}$$

Here X depends only on x and T depends only on t.

Putting (3.4.4) into (3.4.1) we get

$$X''T = \frac{1}{a^2}T''X \ , \tag{3.4.5}$$

or, dividing by XT,

$$\frac{X''(x)}{X(x)} = \frac{1}{2a^2}\,\frac{T''(t)}{T(t)} \ . \tag{3.4.6}$$

The function (3.4.4) is a solution of (3.4.1), if (3.4.5) or (3.4.6) are satisfied. The right–hand side of (3.4.6) depends only on t, meanwhile the left–hand side depends only on x. It follows that they must be equal to a constant, say $-\lambda$:

$$\frac{X''(x)}{X(x)} = \frac{1}{a^2}\,\frac{T''(t)}{T(t)} = -\lambda \ , \tag{3.4.7}$$

here, we put the sign minus before λ only for our convenience.

From (3.4.7) we obtain two ordinary differential equations for X and T

$$X''(x) + \lambda X(x) \ = \ 0 \ , \tag{3.4.8}$$

$$T''(t) + a^2\lambda T(t) \ = \ 0 \ . \tag{3.4.9}$$

The boundary value conditions (3.4.2) give

$$\begin{aligned} u(0,t) \ &= \ X(0)T(t) \ = \ 0 \ , \\ u(\ell,t) \ &= \ X(\ell)T(t) \ = \ 0 \ . \end{aligned}$$

It follows that

$$X(0) = X(\ell) = 0 \tag{3.4.10}$$

otherwise $T(t) \equiv 0$ and $u(x,t) \equiv 0$, and u is not a non–trivial solution of our problem.

Thus, in order to find $X(x)$ we get an *eigenvalue problem:*

Find λ such that there exists a non–trivial solution of the problem

$$\left.\begin{aligned} X'' + \lambda X \ &= \ 0 \ , \\ X(0) \ = \ X(\ell) \ &= \ 0 \end{aligned}\right\} , \tag{3.4.11}$$

and find the solutions corresponding to these λ. The λ, for which a non–trivial solution exists, are called *eigenvalues*, and the corresponding solutions to them are called *eigenfunctions.*

In what follows we distinguish between three cases:

1. For $\lambda < 0$, the problem has no non–trivial solution. In fact, the general solution of (3.4.11) has the form

   $$X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} \ .$$

   This solution must satisfy the boundary conditions:

   $$\begin{aligned} X(0) \ &= \ C_1 + C_2 \ = \ 0 \ , \\ X(\ell) \ &= \ C_1 e^{\ell\sqrt{-\lambda}} + C_2 e^{-\ell\sqrt{-\lambda}} \ = \ 0 \ . \end{aligned}$$

   Thus,

   $$C_1 = -C_2 \quad \text{and} \quad C_1\left(e^{\ell\sqrt{-\lambda}} - e^{-\ell\sqrt{-\lambda}}\right) = 0 \ .$$

   Because $\lambda < 0$, $\sqrt{-\lambda}$ is real and positive, and so $e^{\ell\sqrt{-\lambda}} - e^{-\ell\sqrt{-\lambda}} \neq 0$. Hence, $C_1 = 0$, $C_2 = 0$ and $X(x) \equiv 0$.

2. For $\lambda = 0$ there exists no non–trivial solution, since the general solution is

$$X(x) = ax + b \ ,$$

and the boundary conditions then give

$$\begin{aligned} X(0) &= [ax+b]\Big|_{x=0} = b = 0 \\ X(\ell) &= a\ell = 0 \ . \end{aligned}$$

Thus, $a = 0$, $b = 0$ and $X(x) \equiv 0$.

3. For $\lambda > 0$, the general solution has the form

$$X(x) = D_1 \cos\sqrt{\lambda}x + D_2 \sin\sqrt{\lambda}x \ .$$

The boundary conditions give

$$\begin{aligned} X(0) &= D_1 \ = \ 0 \ , \\ X(\ell) &= D_2 \sin\sqrt{\lambda}\ell \ = \ 0 \ . \end{aligned}$$

Since $X(x)$ is not identically vanishing, $D_2 \neq 0$, and therefore

$$\sin\sqrt{\lambda}\ell = 0 \ , \tag{3.4.12}$$

say

$$\sqrt{\lambda} = \frac{\pi n}{\ell} \ , \quad n \in \mathbb{Z} \ .$$

Thus, a non–trivial solution is possible only for the values

$$\lambda_n = \left(\frac{\pi n}{\ell}\right)^2 \ , \quad n \in \mathbb{N} \ .$$

These eigenvalues correspond to the eigenfunctions

$$X_n(x) = D_n \sin\frac{\pi n}{\ell}x \ .$$

Thus, for the values λ which are equal to

$$\lambda_n = \left(\frac{\pi n}{\ell}\right)^2 \ , \quad n \in \mathbb{N} \ , \tag{3.4.13}$$

there exist non–trivial solutions

$$X_n(x) = \sin\frac{\pi n}{\ell}x \ , \tag{3.4.14}$$

which are uniquely defined up to a constant coefficient. For these λ, the solutions of (3.4.9) are

$$T_n(t) = A_n \cos\frac{\pi n}{\ell}at + B_n \sin\frac{\pi n}{\ell}at \ , \tag{3.4.15}$$

where A_n and B_n are to be defined. It follows that the functions

$$u_n(x,t) = X_n(x)T_n(t) = \left(A_n \cos\frac{\pi n}{\ell}at + B_n \sin\frac{\pi n}{\ell}at\right)\sin\frac{\pi n}{\ell}x \tag{3.4.16}$$

are special solutions of (3.4.1), which satisfy the boundary condition (3.4.2). Since (3.4.1) is linear and homogeneous, the function

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{\pi n}{\ell} at + B_n \sin \frac{\pi n}{\ell} at \right) \sin \frac{\pi n}{\ell} x \ , \tag{3.4.17}$$

— if it converges and one can differentiate it termwise two times with respect to x and t — is a solution of (3.4.1) and satisfies the boundary conditions (3.4.2). (We shall come back to this question in the next paragraph.) The initial conditions give

$$\begin{aligned} u(x,0) = \varphi(x) = \sum_{n=1}^{\infty} u_n(x,0) &= \sum_{n=1}^{\infty} A_n \sin \frac{\pi n}{\ell} x \ , \\ u_t(x,0) = \psi(x) = \sum_{n=1}^{\infty} \frac{\partial u_n}{\partial t}(x,0) &= \sum_{n=1}^{\infty} \frac{\pi n}{\ell} a B_n \sin \frac{\pi n}{\ell} x \ . \end{aligned} \tag{3.4.18}$$

Let φ and ψ be piecewise continuous and differentiable, then we have

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n \sin \frac{\pi n}{\ell} x \ , \quad \varphi_n = \frac{2}{\ell} \int_0^{\ell} \varphi(\xi) \sin \frac{\pi n}{\ell} \xi d\xi \tag{3.4.19}$$

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n \sin \frac{\pi n}{\ell} x \ , \quad \psi_n = \frac{2}{\ell} \int_0^{\ell} \psi(\xi) \sin \frac{\pi n}{\ell} \xi d\xi. \tag{3.4.20}$$

A comparison of these two series with (3.4.18) gives

$$A_n = \varphi_n \ , \quad B_n = \frac{\ell}{\pi n a} \psi_n \ . \tag{3.4.21}$$

Thus, the series (3.4.17) is completely defined.

3.4.2 The proof of the Fourier method

Let $L(u)$ be a linear differential operator. It means that $L(u)$, the action of L onto u, is equal to the sum of the corresponding derivatives of u with coefficients independent of u.

Lemma 3.4.1 *Let the $u_i (i = 1, 2, \ldots)$ be special solutions of a linear homogeneous differential equation $L(u) = 0$. Then the series $u = \sum_{i=1}^{\infty} C_i u_i$ is also a solution of this equation, if the derivatives in $L(u)$ can be differentiated termwise.*

We come back to our boundary value problem. First, we have to prove the continuity of the function

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{\pi n}{\ell} at + B_n \sin \frac{\pi n}{\ell} at \right) \sin \frac{\pi n}{\ell} x \ . \tag{3.4.22}$$

As

$$|u_n(x,t)| \le |A_n| + |B_n| \ ,$$

we conclude that if

$$\sum_{n=1}^{\infty} \left(|A_n| + |B_n| \right) \tag{3.4.23}$$

converges, then the series (3.4.22) uniformly converges and $u(x,t)$ is continuous.

Analogously, in order to prove the continuity of $u_t(x,t)$ we have to prove the uniform convergence of the series

$$u_t(x,t) \sim \sum_{n=1}^{\infty} \frac{\partial u_n}{\partial t} = \sum_{n=1}^{\infty} a\frac{\pi n}{\ell}\left(-A_n \sin\frac{\pi n}{\ell}at + B_n \cos\frac{\pi n}{\ell}at\right)\sin\frac{\pi n}{\ell}x \tag{3.4.24}$$

or the convergence of the majorant series

$$\frac{a\pi}{\ell}\sum_{n=1}^{\infty} n\Big(|A_n| + B_n|\Big) \ .$$

Also we have to prove the uniform convergence of the series

$$\begin{aligned} u_{xx} &\sim \sum_{n=1}^{\infty}\frac{\partial^2 u_n}{\partial x^2} = -\left(\frac{\pi}{\ell}\right)^2\sum_{n=1}^{\infty} n^2\left(A_n\cos\frac{\pi n}{\ell}at + B_n\sin\frac{\pi n}{\ell}at\right)\sin\frac{\pi n}{\ell}x \ , \\ u_{tt} &\sim \sum_{n=1}^{\infty}\frac{\partial^2 u_n}{\partial t^2} = -\left(\frac{\pi a}{\ell}\right)^2\sum_{n=1}^{\infty} n^2\left(A_n\cos\frac{\pi n}{\ell}at + B_n\sin\frac{\pi n}{\ell}at\right)\sin\frac{\pi n}{\ell}x \ . \end{aligned}$$

These correspond, up to a constant, to the majorant series

$$\sum_{n=1}^{\infty} n^2\Big(|A_n| + |B_n|\Big) \ . \tag{3.4.25}$$

Since

$$A_n = \varphi_n \ , \quad B_n = \frac{\ell}{\pi n a}\psi_n \ ,$$

where

$$\varphi_n = \frac{2}{\ell}\int_0^{\ell}\varphi(x)\sin\frac{\pi n}{\ell}x\,dx \ , \quad \psi_n = \frac{2}{\ell}\int_0^{\ell}\psi(x)\sin\frac{\pi n}{\ell}x\,dx \ ,$$

our method is proved if we can prove the convergence of the series

$$\begin{aligned} &\sum_{n=1}^{\infty} n^k|\varphi_n| \quad (k=0,1,2) \\ &\sum_{n=1}^{\infty} n^k|\psi_n| \quad (k=-1,0,1) \ . \end{aligned} \tag{3.4.26}$$

Results from the theory of Fourier series

Let F be a 2ℓ–periodic function, then we can expand F into its Fourier series

$$\begin{aligned} F(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{\pi n}{\ell}x + b_n\sin\frac{\pi n}{\ell}x\right) \ , \\ a_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} F(\xi)\cos\frac{\pi n}{\ell}\xi\,d\xi \ , \quad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} F(\xi)\sin\frac{\pi n}{\ell}\xi\,d\xi \ . \end{aligned}$$

If $F(x)$ is odd, then $a_n = 0$ and

$$\begin{aligned} F(x) &= \sum_{n=1}^{\infty} b_n\sin\frac{\pi n}{\ell}x \\ b_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} F(\xi)\sin\frac{\pi n}{\ell}\xi\,d\xi = \frac{2}{\ell}\int_0^{\ell} F(\xi)\sin\frac{\pi n}{\ell}\xi\,d\xi \ . \end{aligned}$$

If $F(x)$ is defined only in $(0,\ell)$, then we can oddly extend $F(x)$ to be defined in $(-\ell,\ell)$ and then use the above expansion.

It is known that, if $F \in C^k$ and $F^{(k)}$ is piecewise continuous then the series

$$\sum_{n=1}^{\infty} n^k \Big(|a_n| + |b_n| \Big)$$

converges. If a function $f(x)$ is defined only in $(0,\ell)$ then we can extend it oddly to be defined in $(-\ell,\ell)$, say to $F(x)$. Since $F(x)$ is continuous, $f(0)$ must be 0. Further, $f(\ell)$ must be also 0, since the $F(x)$ must be 2ℓ–periodic and continuous. The continuity of the first derivative at $x = 0$ and $x = \ell$ is automatically satisfied. Generally, one has to require

$$f^{(k)}(0) = f^{(k)}(\ell) = 0 \quad (k = 0, 2, 4, \ldots, 2n) \; .$$

From these results we conclude:

1. In order to guarentee the convergence of the series

$$\sum_{n=1}^{\infty} n^k |\varphi_n| \quad (k = 0, 1, 2) \; ,$$

the function φ must be two times continuously differentiable, furthermore the third derivative of φ must be piecewise continuous, and

$$\varphi(0) = \varphi(\ell) = 0 \; , \quad \varphi''(0) = \varphi''(\ell) = 0 \; . \tag{3.4.27}$$

2. In order to guarantee the convergence of the series

$$\sum_{n=1}^{\infty} n^k |\psi_n| \quad (k = -1, 0, 1)$$

the function ψ must be continuously differentiable, have a piecewise continuous second derivative, and

$$\psi(0) = \psi(\ell) = 0 \; . \tag{3.4.28}$$

With these conditions, the representation (3.4.17) is in fact a solution of the problem (3.4.1), (3.4.2). Note that this solution is unique.

3.4.3 Non–homogeneous equations

We consider the non–homogeneous hyperbolic equation

$$u_{tt} = a^2 u_{xx} + f(x,t) \; , \quad 0 < x < \ell \; , \; t > 0 \; , \tag{3.4.29}$$

with the initial conditions

$$u(x,0) = \varphi(x) \; , \quad u_t(x,0) = \psi(x) \; , \quad 0 \le x \le \ell \; , \tag{3.4.30}$$

and homogeneous boundary value conditions

$$u(0,t) = u(\ell,t) = 0 \; , \quad t > 0 \; . \tag{3.4.31}$$

We try to find the solution of (3.4.30)–(3.4.31) in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{\pi n}{\ell} x \ , \tag{3.4.32}$$

where $u_n(t)$ are to be determined. For this purpose we expand f, φ and ψ in the Fourier series

$$\begin{aligned} f(x,t) &= \sum_{n=1}^{\infty} f_n(t) \sin \frac{\pi n}{\ell} x \ , \quad f_n(t) = \frac{2}{\ell} \int_0^{\ell} f(\xi,t) \sin \frac{\pi n}{\ell} \xi \, d\xi \ , \\ \varphi(x) &= \sum_{n=1}^{\infty} \varphi_n \sin \frac{\pi n}{\ell} x \ , \qquad \varphi_n = \frac{2}{\ell} \int_0^{\ell} \varphi(\xi) \sin \frac{\pi n}{\ell} \xi \, d\xi \ , \\ \psi(x) &= \sum_{n=1}^{\infty} \psi_n \sin \frac{\pi n}{\ell} x \ , \qquad \psi_n = \frac{2}{\ell} \int_0^{\ell} \psi(\xi) \sin \frac{\pi n}{\ell} \xi \, d\xi \ . \end{aligned} \tag{3.4.33}$$

Plugging (3.4.32) into (3.4.29), we get

$$\sum_{n=1}^{\infty} \sin \left(\frac{\pi n}{\ell} x \right) \left(-a^2 \left(\frac{\pi n}{\ell} \right)^2 u_n(t) - \ddot{u}_n(t) + f_n(t) \right) = 0 \ .$$

Thus,

$$\ddot{u}_n(t) + \left(\frac{\pi n}{\ell} \right)^2 a^2 u_n(t) = f_n(t) \ . \tag{3.4.34}$$

On the other hand,

$$\begin{aligned} u(x,0) &= \varphi(x) = \sum_{n=1}^{\infty} u_n(0) \sin \frac{\pi n}{\ell} x = \sum_{n=1}^{\infty} \varphi_n \sin \frac{\pi n}{\ell} x \ , \\ u_t(x,0) &= \psi(x) = \sum_{n=1}^{\infty} \dot{u}_n(0) \sin \frac{\pi n}{\ell} x = \sum_{n=1}^{\infty} \psi_n \sin \frac{\pi n}{\ell} x \ . \end{aligned}$$

It follows

$$u_n(0) = \varphi_n \ , \quad \dot{u}_n(0) = \psi_n \ . \tag{3.4.35}$$

Consequently, we can find $u_n(t)$ in the form

$$u_n(t) = u_n^{(I)}(t) + u^{(II)}(t) \ ,$$

where

$$u_n^{(I)}(t) = \frac{\ell}{\pi n a} \int_0^t \sin \left(\frac{\pi n}{\ell} a(t-\tau) \right) f_n(\tau) \, d\tau \tag{3.4.36}$$

and

$$u_n^{(II)}(t) = \varphi_n \cos \left(\frac{\pi n}{\ell} a t \right) + \frac{\ell}{\pi n a} \psi_n \sin \left(\frac{\pi n}{\ell} a t \right) \ . \tag{3.4.37}$$

Thus

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \frac{\ell}{\pi n a} \int_0^t \sin \frac{\pi n}{\ell} a(t-\tau) \sin \frac{\pi n}{\ell} x \cdot f_n(\tau) \, d\tau \\ &\quad + \sum_{n=1}^{\infty} \left(\varphi_n \cos \frac{\pi n}{\ell} a t + \frac{\ell}{\pi n a} \psi_n \sin \frac{\pi n}{\ell} a t \right) \sin \frac{\pi n}{\ell} x \ , \\ &:= u^{(I)}(x,t) + u^{(II)}(x,t) \ . \end{aligned} \tag{3.4.38}$$

Taking (3.4.33) into account for $f_n(t)$, we can represent $u^{(I)}(x,t)$ in the form

$$\begin{aligned} u^{(I)}(x,t) &= \int_0^t \int_0^\ell \left\{ \frac{2}{\ell} \sum_{n=1}^{\infty} \frac{\ell}{\pi n a} \sin \frac{\pi n}{\ell} a(t-\tau) \frac{\sin \pi n}{\ell} x \cdot \sin \frac{\pi n}{\ell} \xi \right\} f(\xi,\tau)\, d\xi\, d\tau \\ &:= \int_0^t \int_0^\ell G(x,\xi,t-\tau) f(\xi,\tau)\, d\xi\, d\tau \ , \end{aligned} \tag{3.4.39}$$

where

$$G(x,\xi,t-\tau) := \frac{2}{\pi a} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{\pi n}{\ell} a(t-\tau) \sin \frac{\pi n}{\ell} x \sin \frac{\pi n}{\ell} \xi \ . \tag{3.4.40}$$

3.4.4 General first boundary value problem

Find a solution of the problem

$$u_{tt} = a^2 u_{xx} + f(x,t) \ , \quad 0 < x < \ell \ , \ t > 0 \ , \tag{3.4.41}$$

$$u(x,0) = \varphi(x) \ , \quad u_t(x,0) = \psi(x) \ , \quad 0 < x < \ell \ , \tag{3.4.42}$$

$$u(0,t) = \mu_1(t) \ , \quad u(\ell,t) = \mu_2(t) \ , \quad t > 0 \ . \tag{3.4.43}$$

To deal with this problem we shall find a differentiable function U, such that

$$U(0,t) = \mu_1(t) \ , \quad U(\ell,t) = \mu_2(t) \tag{3.4.44}$$

and find an equation of the type (3.4.41) for which $U(x,t)$ is a solution. It can be easily seen that the function

$$U(x,t) = \mu_1(t) + \frac{x}{\ell} \left[\mu_2(t) - \mu_1(t) \right] \tag{3.4.45}$$

satisfies (3.4.44) and the equation

$$\begin{aligned} U_{tt} &= a^2 U_{xx} + \ddot{\mu}_1(t) + \frac{x}{\ell} [\ddot{\mu}_2(t) - \ddot{\mu}_1(t)] \\ &= a^2 U_{xx} + \bar{f}(x,t) \ . \end{aligned} \tag{3.4.46}$$

Now consider the problem

$$\begin{aligned} v_{tt} &= a^2 v_{xx} + f - \bar{f} \ , \\ v(x,0) &= \varphi(x) - U(x,0) \ , \\ v_t(x,0) &= \psi(x) - U_t(x,0) \ , \\ v(0,t) &= 0 \ , \quad v(\ell,t) = 0 \ . \end{aligned} \tag{3.4.47}$$

It is clear that we can find v in the form of (3.4.38) and furthermore

$$u(x,t) = U(x,t) + v(x,t) \ . \tag{3.4.48}$$

3.4.5 General scheme for the Fourier method

Let $\varphi(x)$, $k(x)$, $q(x)$ be positive functions.

Consider the problem

$$L[u] := \frac{\partial}{\partial x}\left[k(x)\frac{\partial u}{\partial x}\right] - q(x)u = \rho(x)\frac{\partial^2 u}{\partial t^2}\ , \tag{3.4.49}$$

$$u(0,t) = 0\ , \quad u(\ell,t) = 0\ , \quad t>0\ , \tag{3.4.50}$$

$$u(x,0) = \varphi(x)\ , \quad u_t(x,0) = \psi(x)\ , \quad 0<x<\ell\ . \tag{3.4.51}$$

We now use the Fourier method to solve this problem. Namely, we try to find a non–trivial solution of (3.4.49) which satisfies the boundary conditions

$$u(0,t) = 0\ , \quad u(\ell,t) = 0$$

and can be represented in the form

$$u(x,t) = X(x)T(t)\ .$$

With the same argument as in § 3.4.1, we arrive at the problems

$$\frac{d}{dx}\left[k(x)\frac{dX}{dx}\right] - qX + \lambda\rho X = 0$$

and

$$T'' + \lambda T = 0\ .$$

To find $X(x)$ we get the following eigenvalue problem: *Find λ, for which the boundary value problem*

$$L[X] + \lambda\rho X = 0\ , \quad 0<x<\ell\ , \tag{3.4.52}$$

$$X(0) = 0\ , \quad X(\ell) = 0\ , \tag{3.4.53}$$

has a non–trivial solution. The λ are called *eigenvalues,* and the associated non–trivial solutions of (3.4.52), (3.4.53) are called *eigenfunctions.*

Properties of the problem (3.4.52), (3.4.53)

1. There exist countably many eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, and their associated non–trivial solutions (eigenfunctions) $X_1(x)$, $X_2(x)\ldots, X_n(x), \ldots$

2. For $q \geq 0$, all λ_n are positive.

3. The eigenfunctions are orthogonal in the interval $[0,\ell]$ with the weight $\rho(x)$ in the sense that
$$\int_0^\ell X_m(x)X_n(x)\rho(x)\,dx = 0\ , \quad m \neq n\ . \tag{3.4.54}$$

4. Any function $F(x)$ defined in $[0,\ell]$ which is two times continuously differentiable and satisfies the boundary condition $F(0) = F(\ell) = 0$ can be expanded into a uniformly and absolutely convergent series of the $X_n(x)$:
$$F(x) = \sum_{n=1}^{\infty} F_n X_n(x)\ , \quad F_n = \frac{1}{N_n}\int_0^\ell F(x)X_n(x)\rho(x)\,dx\ ,$$
$$N_n = \int_0^\ell X_n^2(x)\rho(x)\,dx\ . \tag{3.4.55}$$

The properties 1 and 4 are provided in the Theory of Integral Equations, and we do not give any proof for them here. We shall prove the properties 2 and 3. At first, we give here the Green formula. Let u and v be defined in $[a,b]$, $u,v \in C^2(a,b)$ and $u,v \in C^1[a,b]$. Then

$$\begin{aligned} uL[v] - vL[u] &= u(kv')' - v(ku')' \\ &= \Big[k(uv' - vu')\Big]' . \end{aligned}$$

Integrating the last from a to b we get

$$\int_a^b \Big(uL[v] - vL[u]\Big) = k(uv' - vu')\Big|_a^b .$$

Proof of the property 3. Let $X_m(x)$ and $X_n(x)$ be two eigenfunctions and λ_m, λ_n be their associated eigenvalues. From the boundary conditions (3.4.53) we have

$$\int_0^\ell \Big\{X_m L[X_n] - X_n L[X_m]\Big\}\, dx = 0 .$$

Taking Eq. (3.4.52) into account we obtain

$$(\lambda_n - \lambda_m) \int_0^\ell X_m(x) X_n(x) \rho(x)\, dx = 0 .$$

Since $\lambda_n \neq \lambda_m$, it follows

$$\int_0^\ell X_m(x) X_n(x) \rho(x)\, dx = 0 . \tag{3.4.56}$$

We shall prove that for every eigenvalue there exists, up to a constant factor, only one eigenfunction. In order to prove this fact, we use a result which states that the solution of a linear ordinary differential equation is uniquely defined by its value and the value of its first derivative at a point.

Let $\overline{X}$ and $\overline{\overline{X}}$ be two eigenfunctions associated with the eigenvalue λ. Since $\overline{X}(0) = \overline{\overline{X}}(0) = 0$, we have $\overline{X}'(0) \neq 0$ and $\overline{\overline{X}}'(0) \neq 0$. Set

$$X^*(x) := \frac{\overline{X}'(0)}{\overline{\overline{X}}'(0)}\, \overline{\overline{X}}(x) .$$

The function $X^*(x)$ satisfies the equation (3.4.52), furthermore

$$\begin{aligned} X^*(0) &= \frac{\overline{X}'(0)}{\overline{\overline{X}}'(0)}\, \overline{\overline{X}}(0) \;=\; 0 , \\ X^{*'}(0) &= \frac{\overline{X}'(0)}{\overline{\overline{X}}'(0)}\, \overline{\overline{X}}'(0) \;= \overline{X}'(0) . \end{aligned}$$

Thus, $X^*(x) = \overline{X}(x)$ and therefore

$$X^*(x) = \overline{X}(x) = \frac{\overline{X}'(0)}{\overline{\overline{X}}'(0)}\, \overline{\overline{X}}(x) = A\overline{\overline{X}}(x) .$$

We conclude that if $X_n(x)$ is an eigenfunction associated with the eigenvalue λ_n, then $A_n X_n(x)$ (A_n is a nonvanished constant) is also an eigenfunction. Since these eigenfunctions differ only by a constant factor, the value of this factor is not important, and it can be normalized through the equality

$$N_n = \int_0^\ell X_n^2(x)\rho(x)\,dx = 1 \ .$$

If an eigenfunction $\hat{X}_n(x)$ is not normalized, then we can normalize it by

$$X_n(x) = \frac{\hat{X}_n(x)}{\sqrt{\int_0^\ell \hat{X}_n^2(x)\rho(x)\,dx}} \ .$$

By this way, we have found an orthogonal system of eigenfunctions $X_n(x)$ of the problem (3.4.52), (3.4.53):

$$\int_0^\ell X_m(x)X_n(x)\rho(x)\,dx = \begin{cases} 0 & , \quad m \neq n \ , \\ 1 & , \quad m = n \ . \end{cases}$$

□

Proof of the property 2. Let $X_n(x)$ be the normalized eigenfunction associated to the eigenvalue λ_n:

$$L[X_n] = -\lambda_n \rho(x) X_n(x) \ .$$

Multiplying both sides of this equation by $X_n(x)$ and integrating with respect to x from 0 to ℓ, we obtain

$$\lambda_n \int_0^\ell X_n^2(x)\rho(x)\,dx = -\int_0^\ell X_n(x)L[X_n]\,dx \ .$$

Since $\int_0^\ell X_n^2(x)\rho(x)\,dx = 1$, we get

$$\lambda_n = -\int_0^\ell X_n \frac{d}{dx}\left[k(x)\frac{dX_n}{dx}\right]\,dx + \int_0^\ell q(x)X_n^2(x)\,dx \ .$$

Integrating by part gives

$$\begin{aligned} \lambda_n &= -X_n k X_n'\Big|_0^\ell + \int_0^\ell k(x)[X_n'(x)]^2\,dx + \int_0^\ell q(x)X_n^2(x)\,dx \\ &= \int_0^\ell k(x)[X_n'(x)]^2\,dx + \int_0^\ell q(x)X_n^2(x)\,dx \ . \end{aligned}$$

Since $k(x) > 0$, $q(x) \geq 0$, $X_n \in C^2(0,\ell)$, $X_n'(x)$ is not identically vanishing, we have

$$\lambda_n > 0 \ .$$

For a function $F(x)$ defined in $(0,\ell)$ we can formally expand $F(x)$ as follows

$$\begin{aligned} F(x) &= \sum_{n=1}^\infty F_n X_n(x) \ , \\ F_n &= \frac{\int_0^\ell \rho(x)F(x)X_n(x)\,dx}{\int_0^\ell \rho(x)X_n^2(x)\,dx} \ , \quad n \in \mathbb{N} \ . \end{aligned}$$

We say that F_n are the Fourier coefficients of the function F in the system $\{X_1, X_2, \ldots\}$. With the same method as in (3.4.1), we can prove that

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \sqrt{\lambda_n} t + B_n \sin \sqrt{\lambda_n} t \right) X_n(x) \ ,$$

where

$$A_n = \varphi_n \ , \quad B_n = \frac{\psi_n}{\sqrt{\lambda_n}} \ .$$

Here, φ_n and ψ_n are the Fourier coefficients of φ and ψ, respectively.

This method has been developed by Steklov. We do not pursuit it in this lecture.

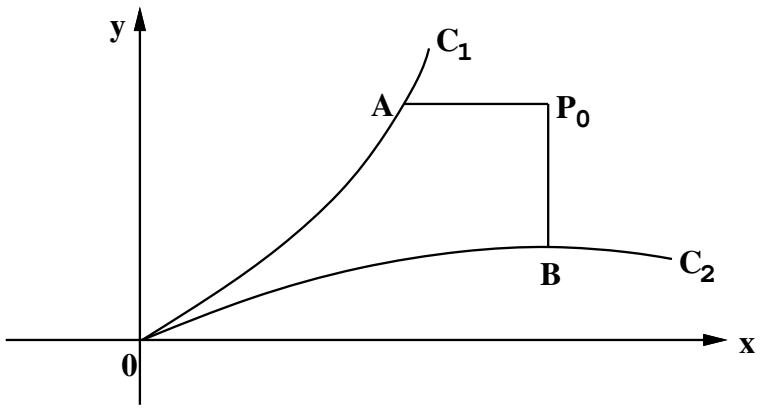
3.5 The Goursat Problem

3.5.1 Definition

Consider the equation

$$u_{xy} = f(x, y) \ . \tag{3.5.1}$$

Let C_1 and C_2 be two curves described by equations $x = \chi_1(y)$ and $x = \chi_2(y)$ and emanating from a common point, which we take to be the origin of the coordinate system. We assume that they do not intersect elsewhere. We assume further that the curves C_1 and C_2 are non–tangent to characteristics situated in the first quarter and that the functions $\chi_1(y)$ and $\chi_2(y)$ are increasing, i. e. the curves C_1 and C_2 have time–like orientation. The functions $\chi_1(y)$ and $\chi_2(y)$ have inverses $\vartheta_1(x)$ and $\vartheta_2(x)$, respectively. For definiteness suppose that $\chi_1(y) < \chi_2(y)$ for the positive values of the variable y for which these two functions exist.



Let D be a domain contained between these arcs and two characteristics P_0A and P_0B. We seek a solution $u(x, y)$ of (3.5.1) defined in the closure $\overline{D}$ of the domain D and satisfying the Dirichlet boundary conditions on the arcs C_1 and C_2:

$$u\Big|_{OA} = u(\chi_1(y), y) = \Psi_1(y) \ , \quad 0 \le y \le \beta_1 \ , \tag{3.5.2}$$

$$u\Big|_{OB} = u(\chi_2(y), y) = \Psi_2(y) \ , \quad 0 \le y \le \beta_2 \ , \tag{3.5.3}$$

where β_1 and β_2 are the y–coordinates of the points A and B. We assume that $\Psi_1(0) = \Psi_2(0) = 0$. Conditions (3.5.2), (3.5.3) can also be represented in the form

$$u(x, \vartheta_1(x)) = \varphi_1(x) \ , \quad 0 \le x \le \alpha_1 \ , \tag{3.5.4}$$

$$u(x, \vartheta_2(x)) = \varphi_2(x) \ , \quad 0 \le x \le \alpha_2 \ , \tag{3.5.5}$$

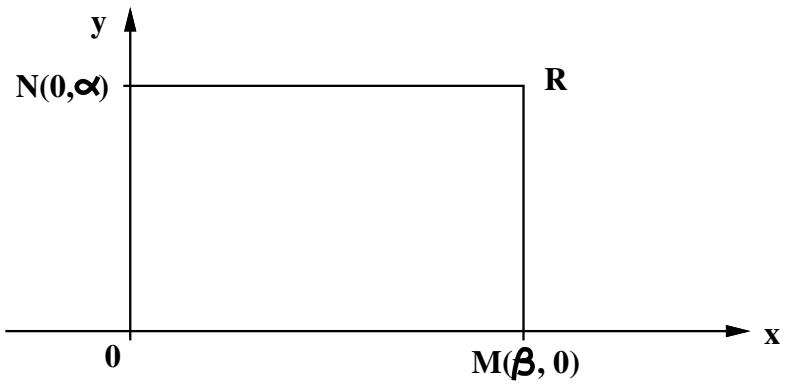
where α_1 and α_2 are the abscissae of the points A and B. This problem is called **the Goursat problem.**

If $\Psi_1(0) = \Psi_2(0) = a \neq 0$ it would be sufficient to solve the Goursat problem with the conditions $u = \Psi_1 - c$ on C_1 and $u = \Psi_2 - c$ on C_2 and then add the function $u' \equiv c$ to the derived solution.

The problem formulated above is the Goursat problem in a narrow sense. In a wider sense, in the Goursat problem the function $\chi_1(y)$ appearing in the equation of the curve C_1 is

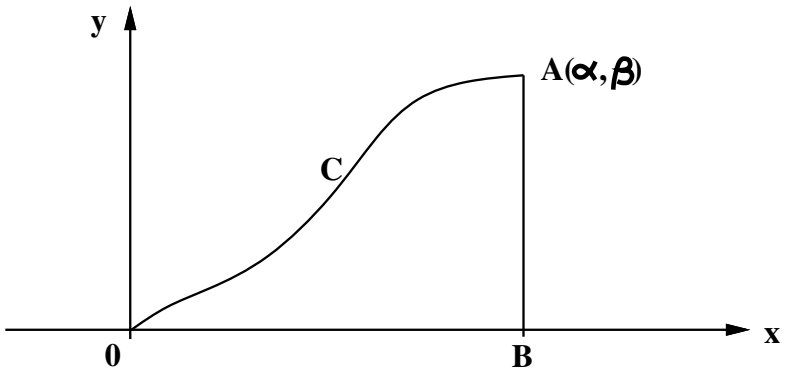
monotonically non–decreasing and the curve C_2 is represented by an equation of the form $y = \vartheta(x)$, where the function $\vartheta(x)$ is monotonically non–decreasing.

If C_1 is the interval $[0, \alpha]$ of the axis Oy and C_2 is the interval $[0, \beta]$ of the axis Oy, then we have **the Darboux problem:**



Find the solution of (3.5.1) in OMRN, where $M = (\beta, 0)$, $R = (\beta, \alpha)$, $N = (0, \alpha)$ and

$$\begin{aligned} u(x,0) &= \varphi(x), \quad 0 \le x \le \beta, \\ u(0,y) &= \psi(y), \quad 0 \le y \le \beta. \end{aligned}$$



Let the arc OA of a curve C with the equation $y = \vartheta(x)$, beginning at the origin of the coordinate system, belong to the class C^1 and $\vartheta'(x) > 0$. Let D be the domain contained between the arc OA, the axis Ox and the characteristic AB perpendicular to Ox. Let α and β be the coordinates of the point A. If we seek the solution of (3.5.1) in the domain D which satisfies the boundary conditions

$$\begin{aligned} &u(x,0) = \varphi(x), \quad 0 \le x \le \alpha, \\ &u(\chi(y), y) = \psi(x), \quad 0 \le y \le \beta, \end{aligned}$$

then we have **the Picard problem.**

Remark: The Darboux problem and the Picard problem are special cases of the Goursat problem.

Remark: The Goursat problem has many applications in gas dynamics, etc.

We restrict ourselves to the Darboux problem in this introductory lecture.

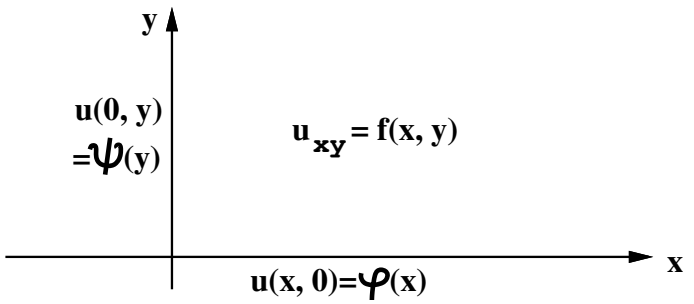
3.5.2 The Darboux problem. The method of successive approximations

We consider the Darboux problem

$$u_{xy} = f(x,y)\,, \quad x \geq 0\,,\ y \geq 0\,, \tag{3.5.1}$$

$$u(x,0) = \varphi(x)\,, \quad x \geq 0\,, \tag{3.5.6}$$

$$u(0,y) = \psi(x)\,, \quad y \geq 0\,. \tag{3.5.7}$$



Let φ and ψ be differentiable and $\varphi(0) = \psi(0)$. Integrating (3.5.1) first with respect to x, and then w.r.t. y, we get

$$\begin{aligned} u_y(x,y) &= u_y(0,y) + \int_0^x f(\xi,y)\,d\xi\,, \\ u(x,y) &= u(x,0) + u(0,y) - u(0,0) + \int_0^y d\eta \int_0^x f(\xi,\eta)\,d\xi\,. \end{aligned}$$

Thus,

$$u(x,y) = \varphi(x) + \psi(y) - \varphi(0) + \int_0^y \int_0^x f(\xi,\eta)\,d\xi\,d\eta\,. \tag{3.5.8}$$

It is clear that $u(x,y)$ defined by (3.5.8) is a solution of the Darboux problem (3.5.1), (3.5.6), (3.5.7). Furthermore it is unique.

We consider now a more general hyperbolic equation

$$u_{xy} = a(x,y)u_x + b(x,y)u_y + c(x,y)u + f(x,y) \tag{3.5.9}$$

with the boundary conditions (3.5.6), (3.5.7). We assume also that $\varphi(0) = \psi(0)$, and the coefficients a, b, and c are continuous functions of x and y.

From (3.5.9) we see that a solution $u(x,y)$ of (3.5.9) satisfies the integral equation

$$\begin{aligned} u(x,y) = {} & \int_0^y \int_0^x \left[a(\xi,\eta)u_\xi + b(\xi,\eta)u_\eta + c(\xi,\eta)u\right] d\xi\,d\eta \\ & + \varphi(x) + \psi(y) - \varphi(0) + \int_0^y \int_0^x f(\xi,\eta)\,d\xi\,d\eta\,. \end{aligned} \tag{3.5.10}$$

The solution of (3.5.10) will be constructed via the method of successive approximations. As the zero-th approximation we take the function

$$u_0(x,y) = 0\,.$$

The equation (3.5.10) gives then the following expressions for the successive approximations

$$\begin{aligned} u_1(x,y) &= \varphi(x)+\psi(y)-\varphi(0)+\int_0^y\int_0^x f(\xi,\eta)\,d\xi\,d\eta\ , \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ u_n(x,y) &= u_1(x,y)+\int_0^y\int_0^x \Big[a(\xi,\eta)\frac{\partial u_{n-1}}{\partial\xi}+b(x,\eta)\frac{\partial u_{n-1}}{\partial\eta}+ \\ &\qquad +c(\xi,\eta)u_{n-1}\Big]\,d\xi\,d\eta\ . \end{aligned} \tag{3.5.11}$$

Note also that

$$\begin{aligned} \frac{\partial u_n}{\partial x} &= \frac{\partial u_1}{\partial x}+\int_0^y\left[a(x,\eta)\frac{\partial u_{n-1}}{\partial x}+b(x,\eta)\frac{\partial u_{n-1}}{\partial\eta}+c(x,\eta)u_{n-1}\right]\,d\eta\ , \\ \frac{\partial u_n}{\partial y} &= \frac{\partial u_1}{\partial y}+\int_0^x\left[a(\xi,y)\frac{\partial u_{n-1}}{\partial\xi}+b(\xi,y)\frac{\partial u_{n-1}}{\partial y}+c(\xi,\eta)u_{n-1}\right]\,d\xi\ . \end{aligned} \tag{3.5.12}$$

In order to prove the uniform convergence of the series

$$\{u_n(x,y)\}\ ,\quad \left\{\frac{\partial u_n}{\partial x}(x,y)\right\}\ ,\quad \left\{\frac{\partial u_n}{\partial y}(x,y)\right\}$$

we consider the differences

$$\begin{aligned} z_n(x,y) &= u_{n+1}(x,y)-u_n(x,y) \\ &= \int_0^y\int_0^x\left[a(\xi,\eta)\frac{\partial z_{n-1}}{\partial\xi}+b(\xi,\eta)\frac{\partial z_{n-1}}{\partial\eta}+c(\xi,\eta)z_{n-1}(\xi,\eta)\right]\,d\xi\,d\eta\ , \\ \frac{\partial z_n(x,y)}{\partial x} &= \frac{\partial u_{n+1}(x,y)}{\partial x}-\frac{\partial u_n(x,y)}{\partial x} \\ &= \int_0^y\left[a(x,\eta)\frac{\partial z_{n-1}}{\partial x}+b(x,\eta)\frac{\partial z_{n-1}}{\partial\eta}+c(x,\eta)z_{n-1}(x,\eta)\right]\,d\eta\ , \\ \frac{z_n(x,y)}{\partial y} &= \frac{\partial u_{n+1}(x,y)}{\partial y}-\frac{\partial u_n(x,y)}{\partial x} \\ &= \int_0^x\left[a(\xi,y)\frac{\partial z_{n-1}}{\partial\xi}+b(\xi,\eta)\frac{\partial z_{n-1}}{\partial y}+c(\xi,\eta)z_{n-1}(\xi,y)\right]\,d\xi\ . \end{aligned}$$

Assume that $|a(x,y)|,\ |b(x,y)|,\ |c(x,y)|\le M$ and

$$|z_0|<H\ ,\quad \left|\frac{\partial z_0}{\partial x}\right|<H\ ,\quad \left|\frac{\partial z_0}{\partial y}\right|<H\ ,$$

where $x\in[0,L]\,,\ y\in[0,L],\ L$ is a given positive number. We want to estimate $z_n\,,\ \frac{\partial z_n}{\partial x}\,,\ \frac{\partial z_n}{\partial y}$ above. First we note that

$$\begin{aligned} |z_1| &< 3HMxy < 3HM\frac{(x+y)^2}{2!}\ , \\ \left|\frac{\partial z_1}{\partial x}\right| &< 3HMy < 3HM(x+y)\ , \\ \left|\frac{\partial z_1}{\partial y}\right| &< 3HMx < 3HM(x+y)\ . \end{aligned}$$

We shall prove by induction that

$$\begin{aligned} |z_n| &< 3HM^nK^{n-1}\frac{(x+y)^{n+1}}{(n+1)!}\,, \\ \left|\frac{\partial z_n}{\partial x}\right| &< 3HM^nK^{n-1}\frac{(x+y)^n}{n!}\,, \\ \left|\frac{\partial z_n}{\partial y}\right| &< 3HM^nK^{n-1}\frac{(x+y)^n}{n!}\,, \end{aligned} \tag{3.5.13}$$

where $K = 2L + 2$.

For $n = 1$, our estimates are proved. Suppose that they are proved for n, we want to show that they are also valid for $n + 1$.

In fact, from our definition for z_n we have

$$\begin{aligned} |z_{n+1}| &= \int_0^y\int_0^x \left[a(\xi,\eta)\frac{\partial z_n}{\partial \xi} + b(\xi,\eta)\frac{\partial z_n}{\partial \eta} + c(\xi,\eta)z_n(\xi,\eta)\right]\,d\xi\,d\eta \\ &\le 3HM^{n+1}K^{n-1}\int_0^y\int_0^x \left[2\frac{(\xi+\eta)^n}{n!} + \frac{(\xi+\eta)^{n+1}}{(n+1)!}\right]\,d\xi\,d\eta\,. \end{aligned}$$

Noting that

$$\begin{aligned} \int_0^y\int_0^x \frac{(\xi+\eta)^k}{k!}\,d\xi\,d\eta &= \int_0^y\left(\frac{(x+\eta)^{k+1}}{(k+1)!} - \frac{\eta^{k+1}}{(k+1)!}\right)\,d\eta \\ = \frac{(x+y)^{k+2}}{(k+2)!} - \frac{x^{k+2}}{(k+2)!} - \frac{y^{k+2}}{(k+2)!} &\le \frac{(x+y)^{k+2}}{(k+2)!}\,, \end{aligned}$$

we have

$$\begin{aligned} |z_{n+1}| &\le 3HM^{n+1}K^{n-1}\left(2\frac{(x+y)^{n+2}}{(n+2)!} + \frac{(x+y)^{n+3}}{(n+3)!}\right) \\ &= 3HM^{n+1}K^{n-1}\frac{(x+y)^{n+2}}{(n+2)!}\left(2 + \frac{x+y}{n+3}\right) \\ &< 3HM^{n+1}K^n\frac{(x+y)^{n+2}}{(n+2)!}\,. \end{aligned}$$

Analogously,

$$\begin{aligned} \left|\frac{\partial z_{n+1}}{\partial x}\right| &\le 3HM^{n+1}K^{n-1}\frac{(x+y)^{n+1}}{(n+1)!}\left(\frac{x+y}{n+2} + 2\right)\,, \\ &< 3HM^{n+1}K^n\frac{(x+y)^{n+1}}{(n+1)!}\,, \end{aligned}$$

and

$$\begin{aligned} \left|\frac{\partial z_{n+1}}{\partial y}\right| &\le 3HM^{n+1}K^{n-1}\frac{(x+y)^{n+1}}{(n+1)!}\left(\frac{x+y}{n+2} + 2\right)\,, \\ &< 3HM^{n+1}K^n\frac{(x+y)^{n+1}}{(n+1)!} \end{aligned}$$

Thus the estimates in (3.5.13) are proved.

From (3.5.13) we see that

$$\begin{aligned}
|z_n| &< 3HM^nK^{n-1}\frac{(x+y)^{n+1}}{(n+1)!} < \frac{3H}{K^2M}\frac{(2KLM)^{n+1}}{(n+1)!}\,,\\
\left|\frac{\partial z_n}{\partial x}\right| &< 3HM^nK^{n-1}\frac{(x+y)^{n}}{n!} < \frac{3H}{K}\frac{(2KLM)^{n}}{n!}\,,\\
\left|\frac{\partial z_n}{\partial y}\right| &< 3HM^nK^{n-1}\frac{(x+y)^{n}}{n!} < \frac{3H}{K}\frac{(2KLM)^{n}}{n!}\,.
\end{aligned}$$

It follows

$$\begin{aligned}
\sum_{n=0}^{\infty}|z_n| &< \sum_{n=0}^{\infty}\frac{3H}{K^2M}\frac{(2KLM)^{n+1}}{(n+1)!} = \frac{3H}{K^2M}\left(e^{2KLM}-1\right)\,,\\
\sum_{n=0}^{\infty}\left|\frac{\partial z_n}{\partial x}\right| &< \sum_{n=0}^{\infty}\frac{3H}{K}\frac{(2KLM)^{n}}{n!} = \frac{3H}{K}e^{2KLM}\,,\\
\sum_{n=0}^{\infty}\left|\frac{\partial z_n}{\partial y}\right| &< \sum_{n=0}^{\infty}\frac{3H}{K}\frac{(2KLM)^{n}}{n!} = \frac{3H}{K}e^{2KLM}\,.
\end{aligned}$$

We thus conclude that the series

$$\begin{aligned}
u_n &= u_0+z_1+\cdots+z_n\,,\\
\frac{\partial u_n}{\partial x} &= \frac{\partial u_0}{\partial x}+\frac{\partial z_1}{\partial x}+\cdots+\frac{\partial z_n}{\partial x}\,,\\
\frac{\partial u_n}{\partial y} &= \frac{\partial u_0}{\partial y}+\frac{\partial z_1}{\partial y}+\cdots+\frac{\partial z_n}{\partial y}
\end{aligned}$$

converges uniformly as $n\to\infty$. We denote their limit functions by

$$\begin{aligned}
u(x,y) &:= \lim_{n\to\infty}u_n(x,y)\,,\\
v(x,y) &:= \lim_{n\to\infty}\frac{\partial u_n}{\partial x}(x,y)\,,\\
w(x,y) &:= \lim_{n\to\infty}\frac{\partial u_n}{\partial y}(x,y)\,.
\end{aligned}$$

Taking the limit in the integral equations (3.5.11), (3.5.12), we get

$$\begin{aligned}
u(x,y) &= u_1(x,y)+\int_0^y\int_0^x\Big[a(\xi,\eta)v+b(\xi,\eta)w+c(\xi,\eta)u\Big]\,d\xi d\eta\,,\\
v(x,y) &= \frac{\partial u_1}{\partial x}(x,y)+\int_0^y\Big[a(x,\eta)v+b(x,\eta)w+c(x,\eta)u\Big]\,d\eta\,,\\
w(x,y) &= \frac{\partial u_1}{\partial y}(x,y)+\int_0^y\Big[a(\xi,y)v+b(\xi,y)w+c(\xi,y)u\Big]\,d\xi\,.
\end{aligned}$$

From the first equation it follows that

$$\begin{aligned}
v &= u_x\,,\\
w &= u_y\,,
\end{aligned}$$

and therefore

$$\begin{aligned} u(x,y) &= \varphi(x)+\psi(y)-\varphi(0)+\int_0^y\int_0^x f(\xi,\eta)\,d\xi\,d\eta \\ &\quad+\int_0^y\int_0^x \Big[a(\xi,\eta)u_\xi+b(\xi,\eta)u_\eta+c(\xi,\eta)u\Big]\,d\xi\,d\eta\ , \end{aligned} \tag{3.5.10}$$

which satisfies (3.5.9) and clearly the boundary conditions. Thus, there exists a solution of the Darboux problem (3.5.1), (3.5.6), (3.5.7).

We shall prove that the solution is unique. In fact, suppose that there are two solutions $u_1(x,y)$ and $u_2(x,y)$. Let

$$U(x,y)=u_1(x,y)-u_2(x,y)\ .$$

The function U satisfies the homogeneous integral equation

$$U(x,y)=\int_0^y\int_0^x (aU_x+bU_y+cU)\,d\xi\,d\eta\ .$$

Let H_1 be the majorant of $|U|, |U_x|$ and $|U_y|$ for $x\in[0,L]$, $y\in[0,L]$. Following the successive procedure we get

$$|U|<3H_1M^{n+1}K^n\frac{(x+y)^{n+2}}{(n+2)!}<\frac{3H_1}{K^2M}\frac{(2KLM)^{n+2}}{(n+2)!}$$

for any n. It follows

$$U(x,y)\equiv 0\ ,\quad \text{or}\quad u_1(x,y)\equiv u_2(x,y)\ .$$

Thus, the solution of the Darboux problem is unique.

If a, b, and c are constant, then we make the transformation

$$u=ve^{\lambda x+\mu y}\ ,$$

for which

$$\begin{aligned} u_x &= v_xe^{\lambda x+\mu y}+\lambda ve^{\lambda x+\mu y}\ , \\ u_y &= v_ye^{\lambda x+\mu y}+\mu ve^{\lambda x+\mu y}\ , \\ u_{xy} &= (v_{xy}+\mu v_x+\lambda v_y+\lambda\mu v)\,e^{\lambda x+\mu y}\ . \end{aligned}$$

Thus

$$\begin{aligned} &v_{xy}+\mu v_x+\lambda v_y+\lambda\mu v= \\ &\quad=a(v_x+\lambda v)+b(v_y+\mu v)+cv+fe^{-\lambda x-\mu y}\ . \end{aligned}$$

It follows

$$\begin{aligned} v_{xy}=&(a-\mu)v_x+(b-\lambda)v_y+ \\ &+(a\lambda+b\mu+c-\lambda\mu)v+fe^{-\lambda x-\mu y}\ . \end{aligned}$$

Letting $\mu=a\ ,\ \lambda=b$, we get

$$v_{xy}=(ab+c)v+fe^{-(bx+ay)}\ . \tag{3.5.14}$$

If $ab+c=0$, then we can find the solution explicitly in the form (3.5.8), if $ab+c\neq 0$ then we shall use the method of the next paragraph.

3.6 Solution of general linear hyperbolic equations

3.6.1 The Green formula

Let

$$L[u] = u_{xx} - u_{yy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u \ , \tag{3.6.1}$$

where $a(x,y)\,,\ b(x,y)\,,\ c(x,y)$ are differentiable functions. Let

$$L^*[v] = v_{xx} - v_{yy} - (av)_x - (bv)_y + cv \ , \tag{3.6.2}$$

$$H = \left\{ \begin{array}{l} vu_x - v_x u + avu = (vu)_x - (2v_x - av)u = \\ -(vu)_x + (2u_x + au)v \ , \end{array} \right. \tag{3.6.3}$$

$$K = \left\{ \begin{array}{l} (uv)_y - (2u_y - bu)v = \\ -vu_y + vu + bvu = -(vu)_y + (2v_y + bv)u \ , \end{array} \right. \tag{3.6.4}$$

$$L[u] = u_{xx} - u_{yy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u.$$

Since

$$L^*[v] = v_{xx} - v_{yy} - (av)_x - (bv)_y + cv,$$

we have

$$vL[u] - uL^*[v] = \frac{\partial H}{\partial x} + \frac{\partial K}{\partial y}.$$

Here we take

$$H = vu_x - v_y u + avu,$$

and

$$K = -vu_y + v_y u + bvu.$$

The Green formula says that

$$\begin{aligned} \iint\limits_G \Big(vL[u] - uL^*[v]\Big)\, d\xi\, d\eta = \\ = \iint\limits_G \Big(\frac{\partial H}{\partial x} + \frac{\partial K}{\partial y}\Big)\, d\xi\, d\eta \\ = -\int_S \Big(H\cos(n,x) + K\cos(n,y)\Big)\, ds \\ = \int_S \Big(H d\eta - K d\xi\Big) \ . \end{aligned} \tag{3.6.5}$$

Here, n is the inward normal vector to S, that is

$$dx = \cos(n,y)\, ds \ , \quad dy = -\cos(n,x)\, ds \ ,$$

assuming that S is traced out anticlockwise so as to keep the interested domain area always on the left, and ds is taken to be positive.

3.6.2 Riemann's method

We consider the problem of finding the solution of the linear hyperbolic equation

$$L[u] = u_{xx} - u_{yy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = -f(x,y) \ , \tag{3.6.6}$$

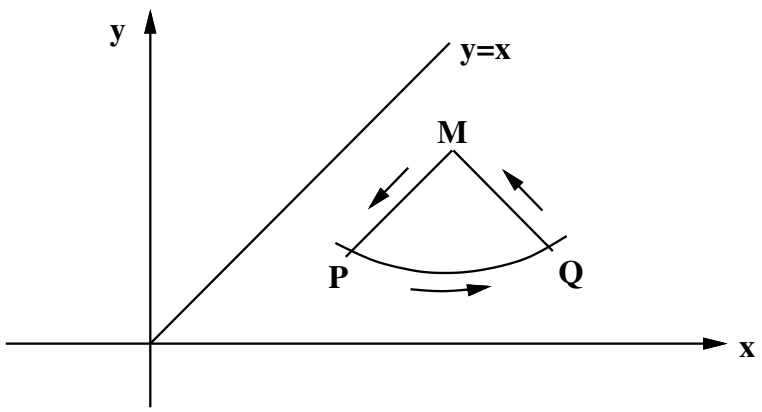
which satisfies the Cauchy data on a curve C,

$$\begin{aligned} u|_C &= \varphi(x) \ , \\ u_n|_C &= \psi(x) \ . \end{aligned} \tag{3.6.7}$$

Here, u_n is the normal derivative of u with respect to the curve C. The conditions posed on C are as follows: C is described by

$$y = f(x) \ ,$$

where $f(x)$ is a differentiable function. Further, every characteristic $y - x = \text{const}$, $y + x = \text{const}$ intersects the curve C at most one time (for this, it is necessary that $|f'(x)| \leq 1$).



Let P and Q be two points on C. Through P and Q we draw two characteristics such that they intersect each other at a point M. Consider the domain bounded by the arc PQ of the curve C and by QM and MP. From the Green formula we have

$$\begin{aligned} &\iint\limits_{MPQ} \Big(vL[u] \ - \ uL^*[v] \Big) \, d\xi \, d\eta = \\ &\qquad = \int_Q^M (H \, d\eta - K \, d\xi) + \int_M^P (H \, d\eta - K \, d\xi) + \int_P^Q (H \, d\eta - K \, d\xi) \ . \end{aligned} \tag{3.6.8}$$

On QM and MP we have

$$\begin{aligned} d\xi = -d\eta = -\frac{ds}{\sqrt{2}} \qquad &\text{on } QM \ , \\ d\xi = d\eta = -\frac{ds}{\sqrt{2}} \qquad &\text{on } MP \ , \end{aligned}$$

where s is the curve element along $\vec{QM}$ and $\vec{MP}$. Thus

$$\begin{aligned} \int_Q^M (H \, d\eta - K \, d\xi) \ = \ &\int_Q^M \Big[\Big((vu)_\xi - (2v_\xi - av)u \Big) \, d\eta \\ &- \Big(- (vu)_\eta + (2v_\eta + bv)u \Big) \, d\xi \Big] \end{aligned}$$

$$
\begin{aligned}
&= \int_Q^M \left[(vu)_\xi\, d\eta + (vu)_\eta\, d\xi\right] - \int_Q^M 2\,(v_\xi u\, d\eta + v_\eta u\, d\xi) \\
&\quad + \int_Q^M (avu\, d\eta - bvu\, d\xi) \\
&= -\int_Q^M d(uv) + \int_Q^M \left(2\frac{\partial v}{\partial s} - \frac{a+b}{\sqrt{2}}v\right) u\, ds \\
&= -(uv)_M + (uv)_Q + \int_Q^M \left(2\frac{\partial v}{\partial s} - \frac{a+b}{\sqrt{2}}v\right) u\, ds\ .
\end{aligned}
$$

Analogously,

$$
\int_M^P (H\, d\eta - K\, d\xi) = -(uv)_M + (uv)_P + \int_Q^M \left(2\frac{\partial v}{\partial s} - \frac{b-a}{\sqrt{2}}v\right) u\, ds\ .
$$

The last two equalities and (3.6.8) yield

$$
\begin{aligned}
(uv)_M &= \frac{(uv)_P + (uv)_Q}{2} + \int_P^M \left(\frac{\partial v}{\partial s} - \frac{b-a}{2\sqrt{2}}v\right) u\, ds \\
&\quad + \int_Q^M \left(\frac{\partial v}{\partial s} - \frac{b+a}{2\sqrt{2}}v\right) u\, ds \qquad (3.6.9) \\
&\quad + \frac{1}{2}\int_P^Q (H\, d\eta - K\, d\xi) - \frac{1}{2}\iint\limits_{MPQ} \Big(vL(u) - uL^*(v)\Big) d\xi\, d\eta\ .
\end{aligned}
$$

Let u be a solution of the problem (3.6.6), (3.6.7), whence v be a solution of the following problem depending on M as a parameter,

$$
L^*(v) = v_{\xi\xi} - v_{\eta\eta} - (av)_\xi - (bv)_\eta + cv = 0 \quad \text{in } \Delta MPQ \qquad (3.6.10)
$$

$$
\left.
\begin{aligned}
\frac{\partial v}{\partial s} &= \frac{b-a}{2\sqrt{2}}v \quad \text{on the characteristic } MP\ , \\
\frac{\partial v}{\partial s} &= \frac{b+a}{2\sqrt{2}}v \quad \text{on the characteristic } MQ\ , \\
v(M) &= 1\ .
\end{aligned}
\right\} \qquad (3.6.11)
$$

The conditions of (3.6.11) yield

$$
\begin{aligned}
v &= \exp\left(\int_{s_0}^s \frac{b-a}{2\sqrt{2}}\, ds\right) \quad \text{on } MP\ , \\
v &= \exp\left(\int_{s_0}^s \frac{b+a}{2\sqrt{2}}\, ds\right) \quad \text{on } MQ\ ,
\end{aligned}
$$

where s_0 is the value of s at the point M.

Thus, we obtain the Darboux problem for v, which is uniquely determined in ΔMPQ. The function v is called **the Riemann function**.

Hence, from (3.6.6) and (3.6.9) we have

$$
\begin{aligned}
u(M) &= \frac{(uv)_P + (uv)_Q}{2} \\
&\quad + \frac{1}{2}\int_Q^P \left[v(u_\xi\, d\eta + u_\eta\, d\xi) - u(v_\xi\, d\eta + v_\eta\, d\xi) + uv(ad\eta - bd\xi)\right] \qquad (3.6.12) \\
&\quad + \frac{1}{2}\iint\limits_{MPQ} v f\, d\xi\, d\eta\ .
\end{aligned}
$$

By this formula, our problem (3.6.6), (3.6.7) is solved, since v is known in ΔMPQ, and, on C, u and $\frac{du}{dn}$ are given:

$$\begin{aligned} u|_C &= \varphi(x)\,, \\ u_x &= u_x\cos(x,s) + u_n\cos(x,n) = \frac{\varphi'(x) - \psi(x)f'(x)\sqrt{1+f'^2(x)}}{1+f'^2(x)}\,, \\ u_y &= u_s\cos(y,s) + u_n\cos(y,n) = \frac{\varphi'(x)f'(x) + \psi(x)\sqrt{1+f'^2(x)}}{1+f'^2(x)}\,. \end{aligned}$$

The formula (3.6.12) ensures the existence and uniqueness of the solution of the problem (3.6.6), (3.6.7). One can show also that the function u defined by (3.6.12) satisfies in fact the conditions of the problem. We do not pursuit it in this lecture.

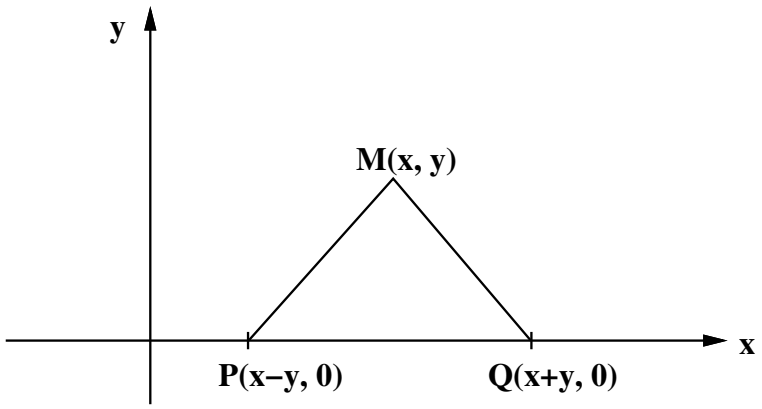
3.6.3 An application of the Riemann method

Consider the problem

$$\begin{aligned} u_{yy} &= u_{xx} + f_1(x,t)\,,\ -\infty < x < \infty\,,\ y > 0 \quad \left(y = at\,,\ f_1 = \frac{f}{a^2}\right) \\ u(x,0) &= \varphi(x)\,,\ -\infty < x < \infty\,, \\ u_y(x,0) &= \psi_1(x)\,,\ \left(\psi_1 = \frac{\psi}{a}\right)\ -\infty < x < \infty\,. \end{aligned}$$

The operator $L(u) = u_{xx} - u_{yy}$ is self-adjoint, that is $L(u) = L^*(u)$.

PQ is now an interval of the axis $y = 0$.



From (3.6.11) we see that $v \equiv 1$ in ΔMPQ. Since $d\eta = 0$ on PQ, we get

$$u(M) = \frac{u(P)+u(Q)}{2} + \frac{1}{2}\int_P^Q u_\eta\,d\xi + \frac{1}{2}\iint\limits_{MPQ} f(\xi,\eta)\,d\xi\,d\eta\,.$$

Let the coordinate of M be (x,y), we have that the coordinate of P is $(x-y,0)$ and of Q is $(x+y,0)$. It follows that

$$u(x,y) = \frac{u(x-y)+u(x+y)}{2} + \frac{1}{2}\int\limits_{x-y}^{x+y}\psi_1(\xi)\,d\xi + \frac{1}{2}\int\limits_0^y\int\limits_{x-(y-\eta)}^{x+(y-\eta)} f_1(\xi,\eta)\,d\xi\,d\eta\,.$$

Changing the variable, we get

$$u(x,t) = \frac{u(x-at)+u(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\,d\xi + \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\eta)\,d\xi\,d\tau \ .$$

Chapter 4

Parabolic Equations

In this chapter we shall consider the parabolic equation

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)$$

$$\begin{cases} k & : \quad \text{(Wärmeleitfähigkeit)} \\ c & : \quad \text{(spezifische Wärmekapazität)} \\ \rho & : \quad \text{(die Dichte)} \end{cases}$$

in a simple form. Namely, we shall consider the heat equation

$$u_t = a^2 u_{xx}, \qquad a^2 = \frac{k}{c\rho}.$$

4.1 Boundary conditions

In contrast to hyperbolic equations, for parabolic equations we need only one initial condition $u(x,t)$ for the initial time $t = t_0$.

Suppose that we consider our process for a bar $[0, l]$.

a) At one end of the bar ($x = 0$, or $x = l$) the temperature is prescribed by

$$u(0,t) = \mu(t) \qquad (\text{or } u(l,t) = \mu_l(t)),$$

where $\mu(t)$ ($\mu_l(t)$) is a given function defined on $[t_0, T]$. Here T characterizes the time interval, where our heat transfer process is considered.

b) At one end of the interval $[0, l]$, the derivative of u is given:

$$\frac{\partial u}{\partial x}(l,t) = \nu(t).$$

Normally, the heat flux is defined by

$$Q(l,t) = k\frac{\partial u}{\partial x}\Big|_{x=l}.$$

Thus

$$\frac{\partial u}{\partial x}(l,t) = \frac{Q(l,t)}{k\Big|_{x=l}} = \nu(t).$$

Note that the heat flux at $x = 0$ is

$$Q(0,t) = -k\frac{\partial u}{\partial x}\Big|_{x=0}.$$

c) At one end, one has the following relation between the solution and its derivative

$$\frac{\partial u}{\partial x}(l,t) = -\lambda[u(l,t) - \theta(t)].$$

This corresponds to the Newton law of heat transfer of the body boundary with its outside environment, the temperature $\theta(t)$ of which is given. At $x = 0$ we have another condition,

$$\frac{\partial u}{\partial x}(0,t) = \lambda[u(0,t) - \theta(t)].$$

d) If the interval $[0, l]$ is very long and the time interval $[t_0, T]$ is small, then we ignore the boundary conditions by consider the initial value problem for parabolic equations in the domain $-\infty < x < \infty,\ t \geq t_0$ with the initial condition

$$u(x,t_0) = \varphi(x) \qquad (-\infty < x < \infty),$$

where $\varphi(x)$ is a given function.

e) Similarly, if the temperature at one end of the interval is given and the other end is very far, then we can consider our parabolic equation in the domain $0 \leq x < \infty,\ t \geq t_0$ which satisfies the conditions

$$u(x,t_0) = \varphi(x) \qquad (0 < x < \infty),$$

$$u(x,0) = \mu(t) \qquad (t \geq t_0).$$

Here φ and μ are given functions.

f) We have also nonlinear boundary conditions. For example

$$k\frac{\partial u}{\partial x}(0,t) = \sigma[u^4(0,t) - \theta^4(0,t)].$$

This condition is called the Stefan-Boltzmann law.

Definition 4.1.1 A function u is called a *solution* of the first boundary value problem for a parabolic equation, if

i) it is defined and continuous in the closed domain $0 \leq x \leq l,\ t_0 \leq t \leq T$,

ii) it satisfies the parabolic equation in the domain $0 < x < l,\ t_0 < t < T$,

iii) it satisfies the initial and boundary conditions

$$u(x,t_0) = \varphi(x),\ u(0,t) = \mu_1(t),\ u(l,t) = \mu_2(t),$$

where $\varphi(x)$, $\mu_1(t)$ and $\mu_2(t)$ are continuous functions, and

$$\varphi(0) = \mu_1(t_0)\ [= u(0,t_0)]$$

$$\varphi(l) = \mu_2(t_0)\ [= u(l,t_0)].$$

Definition 4.1.2 If the condition iii) in Definition 4.1.1 is replaced by

$$u(x,t_0)=\varphi(x),\ \frac{\partial u}{\partial x}(0,t)=\mu_1(t),\ \frac{\partial u}{\partial x}(l,t)=\mu_2(t),$$

then we have "the second boundary value problem".

Definition 4.1.3 If the condition iii) in Definition 4.1.1 is replaced by

$$u(x,t_0)=\varphi(x),\ \frac{\partial u}{\partial x}(0,t)=\lambda[u(0,t)-\theta_1(t)],\ \frac{\partial u}{\partial x}(l,t)=-\lambda[u(l,t)-\theta_2(t)],$$

then we have "the third boundary value problem".

We shall consider in the sequel the following questions:

i) Is the solution of our problems unique?

ii) Is there a solution?

iii) Do the solutions depend continuously on the data?

4.2 The maximum principle

In the sequel, we shall consider the equation with constant coefficients

$$v_t=a^2v_{xx}+\beta v_x+\gamma v. \tag{4.2.1}$$

If we make the transformation

$$v=\ \epsilon^{\mu x+\lambda t}u,\ \mu=-\frac{\beta}{2a^2},\ \lambda=\gamma-\frac{\beta^2}{4a^2},$$

then we have the equation

$$u_t=a^2u_{xx}. \tag{4.2.2}$$

Thus, we have to consider the equation (4.2.2).

The maximum principle: *Let a function $u(x,t)$ be defined and continuous in the closed domain $0\le t\le T$, $0\le x\le l$ and be satisfied the equation (4.2.2) in the open domain $0<t<T$, $0<x<l$. Then the function $u(x,t)$ admits its maximum (minimum) at $t=0$, or at $x=0$ or $x=l$.*

Proof: Let M be the maximum of $u(x,t)$ for $t=0$ $(0\le x\le l)$, or for $x=0,\ x=l$ $(0\le t\le T)$. Assume further that $u(x,t)$ attains its maximum at some interior point (x_0,t_0), say $0<x_0<l,\ 0<t_0<T$:

$$u(x_0,t_0)=M+\epsilon,\ \ \epsilon>0. \tag{4.2.3}$$

Since $0<x_0<l$ and $u(x,t)$ attains its maximum at (x_0,t_0), we have

$$\frac{\partial u}{\partial x}(x_0,t_0)=0,\quad \frac{\partial^2u}{\partial x^2}(x_0,t_0)\le 0. \tag{4.2.4}$$

Further, since the function $u(x_0, t)$ attains its maximum at $t_0 \in (0, T]$ we have

$$\frac{\partial u}{\partial t}(x_0, t_0) \geq 0. \tag{4.2.5}$$

(In fact, $\frac{\partial u}{\partial t}(x_0, t_0) = 0$ if $t_0 < T$, $\frac{\partial u}{\partial t}(x_0, t_0) \geq 0$ if $t_0 = T$.)

We shall find another point $(x_1, t_1) \in (0, l) \times (0, T]$ such that $\frac{\partial^2 u}{\partial x^2}(x_1, t_1) \leq 0$, $\frac{\partial u}{\partial t}(x_1, t_1) > 0$.

Consider the function

$$v(x, t) = u(x, t) + k(t_0 - t), \tag{4.2.6}$$

where k is a constant. We have

$$v(x_0, t_0) = u(x_0, t_0) = M + \epsilon$$

and

$$k(t_0 - t) \leq kT.$$

Choose $k > 0$ such that $kT < \frac{\epsilon}{2}$, that is $k < \frac{\epsilon}{2T}$; we see that the maximum of $v(x, t)$ at $t = 0$ $(0 \leq x \leq l)$, or $x = 0$ $(0 \leq t < T)$, or $x = l$ $(0 \leq t < T)$ is not greater than $M + \frac{\epsilon}{2}$. It means that

$$v(x, t) \leq M + \frac{\epsilon}{2} \ \text{(for } t = 0, \text{or } x = 0, \ x = l), \tag{4.2.7}$$

since the first member on the right-hand side of (4.2.6) is not greater than M and the second one is not greater than $\frac{\epsilon}{2}$.

Since the function $v(x, t)$ is continuous in $[0, l] \times [0, T]$, there is a point $(x_1, t_1) \in [0, l] \times [0, T]$ where $v(x, t)$ attains its maximum:

$$v(x_1, t_1) \geq v(x_0, t_0) = M + \epsilon.$$

From (4.2.7) we see that $0 < x_1 < l$ and $0 < t_1 \leq T$. It follows that

$$v_{xx}(x_1, t_1) = u_{xx}(x_0, t_0) \leq 0$$

and

$$v_t(x_1, t_1) = u_t(x_1, t_1) - k \geq 0.$$

The last inequality says that $u_t(x_1, t_1) \geq k > 0$. Thus, at (x_1, t_1) the function $u(x, t)$ does not satisfy the equation (4.2.2). This contradicts the assumption of the maximum principle. Thus, the first part of the maximum principle is proved. To prove the second part we note that the function $u_1 = -u$ satisfies (4.2.2) and all other conditions of the maximum principle in the case of "maximum". □

4.3 Applications of the maximum principle

4.3.1 The uniqueness theorem

The uniqueness theorem: *Let $u_1(x, t)$ and $u_2(x, t)$ be continuous functions defined on $0 \leq x \leq l$, $0 \leq t \leq T$, satisfying the equation*

$$u_t = a^2 u_{xx} + f(x, t), \ \ 0 < x < l, \ t > 0 \tag{4.3.1}$$

and the same initial and boundary value conditions

$$\begin{aligned} u_1(x,0) &= u_2(x,0) = \varphi(x), \quad 0 \le x \le l, \\ u_1(0,t) &= u_2(0,t) = \mu_1(t), \quad 0 \le t \le T, \\ u_1(l,t) &= u_2(l,t) = \mu_2(t), \quad 0 \le t \le T. \end{aligned}$$

Then $u_1(x,t) \equiv u_2(x,t)$.

Proof: Let

$$v(x,t) := u_1(x,t) - u_2(x,t).$$

Since u_1 and u_2 are continuous in $[0,l] \times [0,T]$, the function v is also continuous there. Further, the function v satisfies the equation $v_t = a^2 v_{xx}$ for $0 < x < l$, $t > 0$. Thus, v satisfies the conditions of the maximum principle; that is v attains its maximum for $t = 0$ or $x = 0$, or $x = l$. Since $v(x,0) = 0$, $v(0,t) = 0$ and $v(l,t) = 0$ we have $v(x,t) \equiv 0$. Thus, $u_1(x,t) \equiv u_2(x,t)$. □

4.3.2 Comparison of solutions

a) Let $u_1(x,t)$ and $u_2(x,t)$ be solutions of the heat equation with

$$\begin{aligned} u_1(x,0) &\le u_2(x,0) \\ u_1(0,t) &\le u_2(0,t), \quad u_1(l,t) \le u_2(l,t). \end{aligned}$$

Then

$$u_1(x,t) \le u_2(x,t)$$

for $(x,t) \in [0,l] \times [0,T]$.

In fact, the function $v = u_2 - u_1$ satisfies the heat equation, and $v(x,0) \ge 0$, $v(0,t) \ge 0$, $v(l,t) \ge 0$. From the maximum principle we have

$$v(x,t) \ge 0 \quad \text{for} \quad 0 < x < l,\ 0 < t \le T.$$

b) If the solutions $u(x,t)$, $\underline{u}(x,t)$ and $\overline{u}(x,t)$ of the heat equation satisfy the inequalities

$$\underline{u}(x,t) \le u(x,t) \le \overline{u}(x,t) \quad \text{for} \quad t = 0,\ x = 0 \ \text{ and } \ x = l,$$

then these inequalities remain valid for all $(x,t) \in [0,l] \times [0,T]$. This statement follows immediatly from a).

c) Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of the heat equation, which satisfy the inequality

$$|u_1(x,t) - u_2(x,t)| \le \epsilon \quad \text{for} \quad t = 0,\ x = 0,\ x = l.$$

Then

$$|u_1(x,t) - u_2(x,t)| \le \epsilon \quad \forall \ (x,t) \in [0,l] \times [0,T].$$

The proof of this statement follows immediately from b) with

$$\begin{aligned} \underline{u}(x,t) &= -\epsilon, \\ u(x,t) &= u_1(x,t) - u_2(x,t), \\ \overline{u}(x,t) &= \epsilon. \end{aligned}$$

Now we turn to the question about the stability of the solution of the heat equation: Let $u(x,t)$ be the solution of the first boundary value problem for the heat equation with the initial and boundary value conditions

$$u(x,0) = \varphi(x), \;\; u(0,t) = \mu_1(t), \;\; u(l,t) = \mu_2(t).$$

Let these functions be given approximately by $\varphi^*(x)$, $\mu_1^*(t)$ and $\mu_2^*(t)$, respectively:

$$|\varphi(x) - \varphi^*(x)| \le \epsilon, \quad |\mu_1(t) - \mu_1^*(t)| \le \epsilon, \quad |u_2(t) - \mu_2^*(t)| \le \epsilon,$$

such that there is a solution $u^*(x,t)$ of the heat equation with these data. Then

$$|u^*(x,t) - u(x,t)| \le \epsilon.$$

4.3.3 The uniqueness theorem in unbounded domains

Consider the problem

$$u_t = a^2 u_{xx}, \;\; -\infty < x < \infty, \;\; t > 0. \tag{4.3.2}$$

Uniqueness Theorem: *Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of (4.3.2) which are continuous and bounded by M:*

$$|u_1(x,t)| < M, \;\; |u_2(x,t)| < M \quad \textit{for} \quad -\infty < x < M, \;\; t \ge 0.$$

If $u_1(x,0) = u_2(x,0)$, $-\infty < x < \infty$, then

$$u_1(x,t) \equiv u_2(x,t), \;\; -\infty < x < \infty, \;\; t \ge 0.$$

Proof: Let $v(x,t) = u_1(x,t) - u_2(x,t)$. The function $v(x,t)$ is continuous, satisfies the heat equation and is bounded by $2M$:

$$|v(x,t)| \le |u_1(x,t)| + |u_2(x,t)| < 2M, \quad -\infty < x < \infty, \;\; t \ge 0.$$

Furthermore, $v(x,0) = 0$.

We shall apply the maximum principle to prove that $v(x,t) = 0$, $-\infty < x < \infty$, $t \ge 0$. Let L be a positive number. Consider the function

$$V(x,t) = \frac{4M}{L^2}\left(\frac{x^2}{2} + a^2 t\right). \tag{4.3.3}$$

The function $V(x,t)$ is continuous and satisfies the heat equation (4.3.2). Furthermore,

$$V(x,0) \ge |v(x,0)| = 0, \quad V(\pm L, t) \ge 2M \ge v(\pm L, t).$$

Applying the maximum principle in the domain $|x| \le L$, we get

$$-\frac{4M}{L^2}\left(\frac{x^2}{2} + a^2 t\right) \le v(x,t) \le \frac{4M}{L^2}\left(\frac{x^2}{2} + a^2 t\right).$$

Let (x,t) be fixed, and let L tend to infinity, we get $v(x,t) = 0$. Thus, $v(x,t) \equiv 0$. □

4.4 The Fourier method

In this section we shall apply the Fourier method to the following problem

$$u_t = a^2 u_{xx} + f(x,t), \quad 0 < x < l, \quad 0 < t \le T, \tag{4.4.1}$$

$$u(x,0) = \varphi(x), \quad 0 \le x \le l, \tag{4.4.2}$$

$$u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \quad 0 \le t \le T. \tag{4.4.3}$$

4.4.1 The homogeneous problem

Find the solution of the problem

$$u_t = a^2 u_{xx}, \quad 0 < x < l, \quad 0 < t \le T, \tag{4.4.4}$$

$$u(x,0) = \varphi(x), \quad 0 \le x \le l, \tag{4.4.5}$$

$$u(0,t) = u(l,t) = 0, \quad 0 \le t \le T. \tag{4.4.6}$$

To do this we shall find a solution of the equation

$$u_t = a^2 u_{xx},$$

which is not identically vanishing and satisfies the boundary conditions

$$u(0,t) = 0, \quad u(l,t) = 0. \tag{4.4.7}$$

Furthermore, this solution can be represented in the form

$$u(x,t) = X(x)\,T(t), \tag{4.4.8}$$

where $X(x)$ is a function depending only on x and $T(t)$ is a function depending only on t. Putting (4.4.8) into (4.4.4) and then deviding both sides by a^2XT, we get

$$\frac{1}{a^2}\frac{T'}{T} = \frac{X''}{X} = -\lambda, \tag{4.4.9}$$

where $\lambda = \text{const}$, since T'/T depends only on t and X''/X depends only on x. This leads us to consider the equations

$$X'' + \lambda X = 0, \tag{4.4.10}$$

$$T' + a^2 \lambda T = 0. \tag{4.4.11}$$

Because of (4.4.7) we have

$$X(0) = X(l) = 0. \tag{4.4.12}$$

Thus, we obtain the following eigenvalue problem for $X(x)$:

$$X'' + \lambda X = 0, \; X(0) = 0, \; X(l) = 0. \tag{4.4.13}$$

We have proved that the eigenvalues of this problem are

$$\lambda_n = \left(\frac{\pi n}{l}\right)^2, \; n = 1, 2, 3, \ldots \tag{4.4.14}$$

and the eigenfunctions associated with them are

$$X_n(x) = \sin \frac{\pi n}{l} x. \tag{4.4.15}$$

With these values λ_n the solutions of (4.4.11) have the form

$$T_n(t) = c_n \, e^{-a^2 \lambda_n t}, \tag{4.4.16}$$

where c_n will be defined later.

We see that the function

$$u_n(x,t) = X_n(x)\, T_n(t) = c_n \, e^{-a^2 \lambda_n t} \, X_n(x) \tag{4.4.17}$$

is a special solution of (4.4.4), which satisfies the homogeneous boundary conditions (4.4.6).

Consider the formal series

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{\pi n}{l}\right)^2 a^2 t} \sin\left(\frac{\pi n}{l} x\right). \tag{4.4.18}$$

The function $u(x,t)$ satisfies (4.4.6), since every element of the series does so. If $u(x,t)$ satisfies also the initial condition, then

$$\varphi(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{\pi n}{l} x\right). \tag{4.4.19}$$

Thus, c_n are the Fourier coefficients of $\varphi(x)$, when it is expanded on $(0,l)$ by

$$c_n = \varphi_n = \frac{2}{l} \int_0^l \varphi(\xi) \sin\left(\frac{\pi n}{l} \xi\right) \, d\xi. \tag{4.4.20}$$

The series (4.4.18) is now completely defined. We shall find conditions which guarantee the convergence of the series and, furthermore, which allow to differentiate (4.4.18) termwise two times w.r.t. x, and once w.r.t. t. We shall prove that the series

$$\sum_{n=1}^{\infty} \frac{\partial u_n}{\partial t} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\partial^2 u_n}{\partial x^2}$$

for $t \geq \overline{t} > 0$ ($\overline{t}$ is fixed) converge uniformly. For this, we note that

$$\begin{aligned} \left|\frac{\partial u_n}{\partial t}\right| &= \left| -c_n \left(\frac{\pi}{l}\right)^2 a^2 n^2 e^{-\left(\frac{\pi n}{l}\right)^2 a^2 t} \sin\left(\frac{\pi n}{l} x\right) \right| \\ &< |c_n| \left(\frac{\pi}{l}\right)^2 a^2 n^2 e^{-\left(\frac{\pi n}{l}\right)^2 a^2 t}. \end{aligned}$$

Let φ be bounded: $|\varphi(x)| < M$. Then

$$|c_n| = \left|\frac{2}{l}\right| \left| \int_0^l \varphi(\xi) \sin\left(\frac{\pi n}{l} \xi\right) \, d\xi \right| < 2M.$$

Hence,

$$\left|\frac{\partial u_n}{\partial t}\right| < 2M\left(\frac{\pi}{l}\right)^2 a^2 n^2 e^{-\left(\frac{\pi n}{l}\right)^2 a^2 \overline{t}},\ t \geq \overline{t},$$

and

$$\left|\frac{\partial^2 u_n}{\partial x^2}\right| < 2M\left(\frac{\pi}{l}\right)^2 n^2 e^{-\left(\frac{\pi n}{l}\right)^2 a^2 \overline{t}},\ t \geq \overline{t}.$$

Generally,

$$\left|\frac{\partial^{k+l} u}{\partial t^k \partial x^l}\right| < 2M\left(\frac{\pi}{l}\right)^{2k+l} n^{2l+l} a^{2k} e^{-\left(\frac{\pi n}{l}\right)^2 a^2 \overline{t}},\ t \geq \overline{t}.$$

Consider the majorant series $\sum\limits_{n=1}^{\infty} \alpha_n$, where

$$\alpha_n = A\, n^q e^{-\left(\frac{\pi n}{l}\right)^2 a^2 \overline{t}}. \tag{4.4.21}$$

Series of this kind converge because of the D'Alembert criterion

$$\lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^q \frac{e^{-\left(\frac{\pi}{l}\right)^2 a^2 (n^2+2n+1)\overline{t}}}{e^{-\left(\frac{\pi}{l}\right)^2 a^2 n^2 \overline{t}}}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^q e^{-\left(\frac{\pi}{l}\right)^2 a^2 (2n+1)\overline{t}} = 0.$$

Thus, the series (4.4.18) for $t \geq \overline{t} > 0$ is infinitely differentiable. Since $\overline{t}$ is arbitrary we can say that $u(x,t)$ for $t > 0$ is infinitely differentiable, furthermore, it is clear that for $t > 0$ it satisfies the equation (4.4.4).

Theorem 4.4.1 *If φ is piecewise differentiable and $\varphi(0) = \varphi(l) = 0$, then the series*

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{\pi n}{l}\right)^2 a^2 t} \sin\left(\frac{\pi n}{l} x\right) \tag{4.4.22}$$

is a continuous function for $t \geq 0$.

In fact, the inequalities

$$|u_n(x,t)| < |c_n| \quad (t \geq 0,\ 0 \leq x \leq l)$$

yield the uniform convergence of the series (4.4.18) for $t \geq 0,\ 0 \leq x \leq l$: For $t > 0$, we have proved this already; for $t = 0$, this follows directly, because $\varphi(0) = \varphi(l) = 0$ and φ is piecewise differentiable.

4.4.2 The Green function

From (4.4.22) we have

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} c_n e^{-\left(\frac{\pi n}{l}\right)^2 a^2 t} \sin\left(\frac{\pi n}{l}x\right) \\ &= \sum_{n=1}^{\infty} \left[\frac{2}{l}\int_0^l \varphi(\xi)\sin\left(\frac{\pi n}{l}\xi\right)\,d\xi\right] e^{-\left(\frac{\pi n}{l}\right)^2 a^2 t} \sin\left(\frac{\pi n}{l}x\right) \\ &= \int_0^l \left[\frac{2}{l}\sum_{n=1}^{\infty} e^{-\left(\frac{\pi n}{l}\right)^2 a^2 t} \sin\left(\frac{\pi n}{l}x\right)\sin\left(\frac{\pi n}{l}\xi\right)\right] \varphi(\xi)\,d\xi. \end{aligned}$$

We can change the order of summation and integration for $t > 0$, since the series

$$\sum_{n=1}^{\infty} e^{-\left(\frac{\pi n}{l}\right)^2 a^2 t} \sin\left(\frac{\pi n}{l}x\right)\sin\left(\frac{\pi n}{l}\xi\right) \quad \text{for} \quad t > 0$$

converges uniformly w.r.t. ξ.

Set

$$G(x,\xi,t) = \frac{2}{l}\sum_{n=1}^{\infty} e^{-\left(\frac{\pi n}{l}\right)^2 a^2 t} \sin\left(\frac{\pi n}{l}x\right)\sin\left(\frac{\pi n}{l}\xi\right). \tag{4.4.23}$$

The function $u(x,t)$ can then be written in the form

$$u(x,t) = \int_0^l G(x,\xi,t)\,\varphi(\xi)\,d\xi. \tag{4.4.24}$$

The function $G(x,\xi,t)$ is called the *Green function.* It has a physical meaning.

We shall prove that $G(x,\xi,t)$, as a function of x, represents the temperature in the bar $0 \le x \le l$ at time t, if the temperature at the initial moment $t = 0$ equals to zero and at the same moment, at the point $x = \xi$, produces a heat quantity Q (which will be determined later), while at the ends of the bar ($x = 0,\ x = l$) the temperature remains zero.

The notion "a heat quantity Q is produced at the point ξ" means that this is produced in a sufficiently small neighbourhood of ξ. The temperature change $\varphi_\epsilon(\xi)$, which produces a heat quantity at the point ξ, is equal to zero outside the interval $(\xi-\epsilon, \xi+\epsilon)$ and, inside this interval, $\varphi_\epsilon(\xi)$ is a positive, continuous and differentiable function such that

$$c\,\rho \int_{\xi-\epsilon}^{\xi+\epsilon} \varphi_\epsilon(\tau)\,d\tau = Q. \tag{4.4.25}$$

Note that the left-hand side of (4.4.25) represents the heat quantity produced by $\varphi_\epsilon(\xi)$.

The temperature in this case is

$$u_\epsilon(x,t) = \int_0^l G(x,\xi,t)\,\varphi_\epsilon(\tau)\,d\tau. \tag{4.4.26}$$

Since $G(x,\xi,t)$ for $t>0$ is a continuous function of ξ we have

$$\begin{aligned} u_\epsilon(x,t) &= \int_{\xi-\epsilon}^{\xi+\epsilon} G(x,\xi,t)\,\varphi_\epsilon(\xi)\,d\xi \;=\; G(x,\xi^*,t)\int_{\xi-\epsilon}^{\xi+\epsilon}\varphi_\epsilon(\tau)\,d\tau \\ &= G(x,\xi^*,t)\,\frac{Q}{c\,\rho}, \end{aligned} \tag{4.4.27}$$

where ξ^* is a point lying between $\xi-\epsilon$ and $\xi+\epsilon$. Letting $\epsilon\to 0$, and taking into account that $G(x,\xi,t)$ is a continuous function of ξ for $t>0$, we get

$$\begin{aligned} \lim_{\epsilon\to 0} u_\epsilon(x,t) &= \frac{Q}{c\rho}G(x,\xi,t) \\ &= \frac{Q}{c\rho}\cdot\frac{2}{l}\sum_{n=1}^{\infty} e^{-\left(\frac{\pi n}{l}\right)^2 t}\sin\frac{\pi n}{l}x\cdot\sin\frac{\pi n}{l}\xi \end{aligned} \tag{4.4.28}$$

(Here, we suppose that there exists the limit of the integral $\int_{\xi-\epsilon}^{\xi+\epsilon}\varphi_\epsilon(\tau)\,d\tau$ as $\epsilon\to 0$, and $\lim\int_{\xi-\epsilon}^{\xi+\epsilon}\varphi_\epsilon(\tau)\,d\tau=\frac{Q}{c\rho}$.) Thus, $G(x,\xi,t)$ represents the temperature influence at the instant of time into the heat pole of intensity $Q=c\rho$, which is lying at $t=0$ and $\xi\in(0,l)$.

We shall prove that for any x, ξ and $t>0$, $G(x,\xi,t)\ge 0$. Consider the function $\varphi_\epsilon(\xi)$ defined above, since $\varphi_\epsilon(\tau)>0$ for $\tau\in(\xi-\epsilon,\xi+\epsilon)$, $u_\epsilon(0,t)=u_\epsilon(l,t)=0$, from the maximum principle

$$u_\epsilon(x,t)\ge 0,\quad 0\le x\le l,\quad t>0.$$

Thus,

$$u_\epsilon(x,t)=G(x,\xi^*,t)\frac{Q}{c\rho}\ge 0\quad(t>0).$$

Letting $\epsilon\to 0$, we obtain that

$$G(x,\xi,t)\ge 0,\quad 0\le x,\,\xi\le l,\,t>0.$$

4.4.3 Boundary value problems with non-smooth initial conditions

We have considered our first boundary value problem in the class of all continuous functions in the closed domain $[0,l]\times[0,T]$. This has some restriction, for example, if $u(x,0)=u_0\neq 0$, then the solution must be discontinuous at the points $(0,0)$ and $(l,0)$. It leads to the idea to consider our problem for non-smooth initial conditions.

Theorem 4.4.2 *Let φ be a continuous function defined on $[0,l]$ and $\varphi(0)=\varphi(l)=0$. Then the solution of the equation*

$$u_t=a^2u_{xx}\quad(0<x<l,\;\;t>0), \tag{4.4.4}$$

which is continuous in $[0,l]\times[0,T]$ and satisfies the conditions

$$u(0,t) = u(l,t) = 0,\ t \in [0,T], \tag{4.4.6}$$

$$u(x,0) = \varphi(x),\ x \in [0,l], \tag{4.4.5}$$

is unique and can be represented by

$$u(x,t) = \int_0^l G(x,\xi,t)\,\varphi(\xi)\,d\xi\ . \tag{4.4.24}$$

Proof: The case where $\varphi(\xi)$ is continuous and piecewise differentiable is already proved. Consider a series of continuous, piecewise differentiable functions $\varphi_n(x)$ $(\varphi_n(0) = \varphi_n(l) = 0)$, which uniformly converges to $\varphi(x)$ (as $\varphi_n(x)$ we can take the piecewise linear function, which coincides with $\varphi(x)$ at the points $\frac{lk}{n}$ $(k = 0, 1, \ldots, n)$). For the $\varphi_n(x)$ we can determine $u_n(x,t)$ by (4.4.24), since $\varphi_n(x)$ is continuous and piecewise differentiable. The functions $u_n(x,t)$ uniformly converge to $u(x,t)$. In fact, for $\epsilon > 0$ small enough, there exists an $n(\epsilon)$ such that

$$|\varphi_{n_1}(x) - \varphi_{n_2}(x)| < \epsilon \quad (0 \le x \le l)$$

$\forall n_1,\, n_2 \ge n(\epsilon)$, since the φ_n uniformly converge to $\varphi(x)$.

It follows then by the maximum principle that

$$|u_{n_1}(x,t) - u_{n_2}(x,t)| < \epsilon \quad (0 \le x \le l,\ 0 \le t \le T)$$

if $n_1,\, n_2 \ge n(\epsilon)$. Thus, $\{u_n(x,t)\}$ uniformly converge to $u(x,t)$. Since $u_n(x,t)$ is continuous, $u(x,t)$ is so, too.

For any fixed $(x,t) \in [0,l] \times [0,T]$ we have

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = \lim_{n\to\infty} \int_0^l G(x,\xi,t)\,\varphi_n(\xi)\,d\xi = \int_0^l G(x,\xi,t)\,\varphi(\xi)\,d\xi.$$

Thus, the function $u(x,t)$ is continuous and satisfies the condition (4.4.5). Further, as we have proved in § 4.4.1 this function also satisfies (4.4.6).

The uniqueness follows immediately from the maximum principle. □

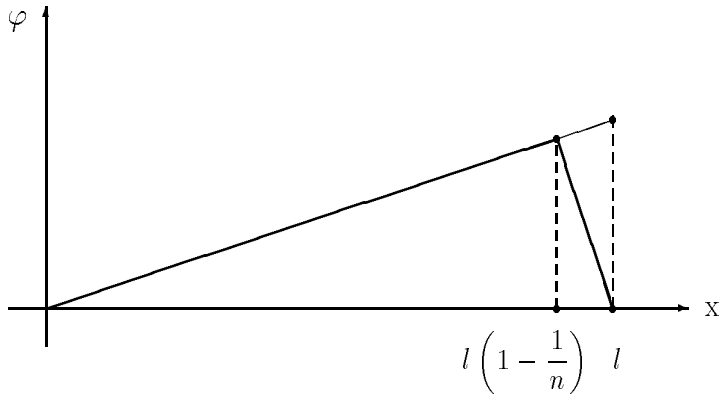
Theorem 4.4.3 *Let φ be a piecewise continuous function. Then the function $u(x,t)$ defined by (4.4.24) represents a solution of the equation (4.4.4), satisfies the condition (4.4.5), and is continuous where φ is continuous.*

Proof: First, let φ be linear:

$$\varphi(x) = c\,x. \tag{4.4.29}$$

Consider the sequence

$$\varphi_n(x) = \begin{cases} c\,x & ,\ 0 \le x \le l\left(1 - \dfrac{1}{n}\right) \\ c(n-1)(l-x), & l\left(1 - \dfrac{1}{n}\right) \le x \le l\,, \quad n \in \mathbb{N}\,. \end{cases}$$



The functions $\varphi_n(x)$ are continuous and $\varphi_n(0) = \varphi_n(l) = 0$, hence the functions $u_n(x,t)$ defined by (4.4.24) for φ_n are the solutions of (4.4.4) which satisfy (4.4.6) and

$$u_n(x,0) = \varphi_n(x).$$

Since

$$\varphi_n(x) \le \varphi_{n+1}(x) \quad (0 \le x \le l),$$

from the maximum principle,

$$u_n(x,t) \le u_{n+1}(x,t).$$

The function $U_0(x) = cx$ is a continuous solution of the heat equation. The maximum principle now gives

$$u_n(x,t) \le U_0(x),$$

as this is valid for $t = 0$, $x = 0$ and $x = l$.

The series $\{u_n(x,t)\}$ is monoton increasing and bounded above by the bounded function $U_0(x)$, so it is convergent. We have

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = \lim_{n\to\infty} \int_0^l G(x,\xi,t)\,\varphi_n(\xi)\,d\xi = \int_0^l G(x,\xi,t)\,\varphi(\xi)\,d\xi \le U_0(x),$$

because we can take the limit process under the integral. In § 4.4.1 we have proved that $u(0,t) = u(l,t) = 0$ and $u(x,t)$ satisfies (4.4.4) for $t > 0$. It remains to show that it is continuous for $t = 0$ and $0 \le x < l$. Let $x_0 < l$. We choose n such that $x_0 < l\left(1 - \frac{1}{n}\right)$. In this case $\varphi_n(x_0) = U_0(x_0)$. Noting that

$$u_n(x,t) \le u(x,t) \le U_0(x)$$

and

$$\lim_{\substack{x\to x_0\\ t\to 0}} u_n(x,t) = \lim_{x\to x_0} U_0(x) = \varphi(x_0),$$

we conclude that the limit

$$\lim_{\substack{x\to x_0\\ t\to 0}} u(x,t) = \varphi(x_0)$$

exists and is independent of how $x \to x_0$ and $t \to 0$. This implies the continuity of $u(x,t)$ at $(x_0,0)$. This function is bounded by $U_0(x)$. Thus, the theorem is proved for $\varphi(x) = cx$.

Changing x by $l - x$, we see that the theorem is also proved for $\varphi(x) = b(l-x)$.

Thus, the theorem is correct for the functions of the form $\varphi(x) = B + Ax$. Further, the theorem is correct for any continuous function $\varphi(x)$ which needs not satisfy the condition $\varphi(0) = \varphi(l) = 0$. In fact, any such function can be represented in the form

$$\varphi(x) = \left[\varphi(0) + \frac{x}{l}(\varphi(l) - \varphi(0))\right] + \psi(x),$$

where $\psi(x)$ is a continuous function which vanishes at the ends of the interval: $\psi(0) = \psi(l) = 0$. From the superposition theorem we see that the theorem is proved as far as the continuity is concerned.

Let φ be now a piecewise continuous function; we shall prove that the function $u(x,t)$ defined by (4.4.24) gives a solution to (4.4.4) and satisfies (4.4.6).

Suppose that x_0 is a point where φ is continuous. We shall prove that for any positive ϵ, there exists a $\delta(\epsilon)$ such that if $|x - x_0| < \epsilon$, $t < \delta(\epsilon)$, then $|u(x,t) - \varphi(x_0)| < \epsilon$.

Because φ is continuous at x_0, there exists an $\eta(\epsilon)$ such that

$$|\varphi(x) - \varphi(x_0)| \le \frac{\epsilon}{2} \quad \text{for} \quad |x - x_0| < \eta(\epsilon).$$

It follows that

$$\varphi(x_0) - \frac{\epsilon}{2} \le \varphi(x) \le \varphi(x_0) + \frac{\epsilon}{2} \quad \text{for} \quad |x - x_0| < \eta(\epsilon). \tag{4.4.30}$$

Let $\overline{\varphi}(x)$ and $\underline{\varphi}(x)$ be continuously differentiable functions such that

$$\begin{aligned} &\overline{\varphi}(x) = \varphi(x_0) + \frac{\epsilon}{2} \quad \text{for} \quad |x - x_0| < \eta(\epsilon), \\ &\overline{\varphi}(x) \ge \varphi(x) \qquad\quad \text{for} \quad |x - x_0| > \eta(\epsilon), \end{aligned} \tag{4.4.31}$$

$$\begin{aligned} &\underline{\varphi}(x) = \varphi(x_0) - \frac{\epsilon}{2} \quad \text{for} \quad |x - x_0| < \eta(\epsilon), \\ &\underline{\varphi}(x) \le \varphi(x) \qquad\quad \text{for} \quad |x - x_0| > \eta(\epsilon). \end{aligned} \tag{4.4.32}$$

From (4.4.30) we have

$$\underline{\varphi}(x) \le \varphi(x) \le \overline{\varphi}(x). \tag{4.4.33}$$

Consider the functions

$$\begin{aligned} \overline{u}(x,t) &= \int_0^l G(x,\xi,t)\,\overline{\varphi}(\xi)\,d\xi, \\ \underline{u}(x,t) &= \int_0^l G(x,\xi,t)\,\underline{\varphi}(\xi)\,d\xi. \end{aligned}$$

Since $\overline{\varphi}(x)$ and $\underline{\varphi}(x)$ are continuous, $\overline{u}(x,t)$ and $\underline{u}(x,t)$ are so at x_0. It follows the existence of a $\delta(\epsilon)$ such that

$$\left.\begin{aligned} |\overline{u}(x,t) - \overline{\varphi}(x)| &\le \frac{\epsilon}{2}, \\ |\underline{u}(x,t) - \underline{\varphi}(x)| &\le \frac{\epsilon}{2} \end{aligned}\right\} \quad \text{for} \quad |x - x_0| < \delta(\epsilon), \;\; t < \delta(\epsilon).$$

It means

$$\left.\begin{array}{rcl} \overline{u}(x,t) & \leq \overline{\varphi}(x) + \dfrac{\epsilon}{2} & = \varphi(x_0)+\epsilon, \\ \underline{u}(x,t) & \geq \underline{\varphi}(x) - \dfrac{\epsilon}{2} & = \varphi(x_0)-\epsilon \end{array}\right\} \quad \text{for} \quad |x-x_0| < \delta(\epsilon), \;\; t < \delta(\epsilon).$$

Since the function $G(x,\xi,t)$ is not negative, the inequalities of (4.4.33) yield

$$\underline{u}(x,t) \leq u(x,t) \leq \overline{u}(x,t). \tag{4.4.34}$$

Consequently,

$$\varphi(x_0) - \epsilon \leq u(x,t) \leq \varphi(x_0) + \epsilon \quad \text{for} \quad |x-x_0| < \delta(\epsilon), \;\; t < \delta(\epsilon).$$

That is

$$|u(x,t) - \varphi(x_0)| < \epsilon \quad \text{for} \quad |x-x_0| < \delta(\epsilon), \;\; t < \delta(\epsilon).$$

Furthermore, (4.4.34) yields also the boundedness of $u(x,t)$. □

4.4.4 The non-homogeneous heat equation

We consider now the non-homogeneous heat equation

$$u_t = a^2 u_{xx} + f(x,t), \;\; 0 < x < l, \; 0 < t \leq T, \tag{4.4.1}$$

with the initial condition

$$u(x,0) = 0, \;\; 0 \leq x \leq l, \tag{4.4.35}$$

and the boundary conditions

$$u(0,t) = u(l,t) = 0, \;\; 0 < t \leq T. \tag{4.4.6}$$

We shall find a solution of this problem in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \, \sin\left(\frac{\pi n}{l} x\right). \tag{4.4.36}$$

In order to determine $u(x,t)$, we have to find $u_n(t)$. For this purpose, we represent $f(x,t)$ in the form

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \, \sin\left(\frac{\pi n}{l} x\right),$$

where

$$f_n(t) = \frac{2}{l} \int_0^l f(\xi,t) \, \sin\left(\frac{\pi n}{l}\xi\right) \, d\xi. \tag{4.4.37}$$

Putting (4.4.36) and (4.4.37) into (4.4.1), we get

$$\sum_{n=1}^{\infty} \sin\left(\frac{\pi n}{l} x\right) \, \left\{ \left(\frac{\pi n}{l}\right)^2 a^2 u_n(t) + \dot{u}_n(t) - f_n(t) \right\} = 0.$$

It follows that

$$\dot{u}_n(t) = -a^2 \left(\frac{\pi n}{l}\right)^2 u_n(t) + f_n(t). \tag{4.4.38}$$

On the other hand

$$u(x,0) = \sum_{n=1}^{\infty} u_n(0) \sin\left(\frac{\pi n}{l}x\right) = 0,$$

and therefore

$$u_n(0) = 0. \tag{4.4.39}$$

Solving (4.4.38) with the initial condition (4.4.39) we obtain

$$u_n(t) = \int_0^t e^{-\left(\frac{\pi n}{l}\right)^2 a^2(t-\tau)} f_n(\tau)\, d\tau. \tag{4.4.40}$$

Hence,

$$u(x,t) = \sum_{n=1}^{\infty} \left[\int_0^t e^{-\left(\frac{\pi n}{l}\right)^2 a^2(t-\tau)} f_n(\tau)\, d\tau \right] \sin\left(\frac{\pi n}{l}x\right). \tag{4.4.41}$$

Taking (4.4.37) into account, we can write

$$\begin{aligned} u(x,t) &= \int_0^t \int_0^l \left\{ \frac{2}{l} \sum_{n=1}^{\infty} e^{-\left(\frac{\pi n}{l}\right)^2 a^2(t-\tau)} \sin\left(\frac{\pi n}{l}x\right) \sin\left(\frac{\pi n}{l}\xi\right) \right\} f(\xi,\tau)\, d\xi\, d\tau \\ &=: \int_0^t \int_0^l G(x,\xi,t-\tau)\, f(\xi,\tau)\, d\xi\, d\tau, \end{aligned} \tag{4.4.42}$$

where

$$G(x,\xi,t-\tau) = \frac{2}{l} \sum_{n=1}^{\infty} e^{-\left(\frac{\pi n}{l}\right)^2 a^2(t-\tau)} \sin\left(\frac{\pi n}{l}x\right) \sin\left(\frac{\pi n}{l}\xi\right) \tag{4.4.43}$$

is in fact our Green's function defined by (4.4.23).

We can give here the physical meaning of (4.4.42) similar to that of (4.4.24), however, we omit to give it here.

4.4.5 The non-homogeneous first boundary value problem

Find the solution of the problem

$$u_t = a^2 u_{xx} + f(x,t), \quad 0 < x < l,\ 0 < t \le T, \tag{4.4.1}$$

$$u(x,0) = \varphi(x), \quad 0 \le x \le l, \tag{4.4.2}$$

$$u(0,t) = \mu_1(t),\ u(l,t) = \mu_2(t), \quad 0 < t \le T. \tag{4.4.3}$$

In order to solve this problem, we change it to a homogeneous boundary value problem.

In fact, consider the function

$$U(x,t) = \mu_1(t) + \frac{x}{l}\left[\mu_2(t) - \mu_1(t)\right]. \tag{4.4.44}$$

The function U satisfies the boundary condition

$$U(0,t) = \mu_1(t) \quad \text{and} \quad U(l,t) = \mu_2(t),$$

and the equation

$$U_t = a^2 U_{xx} + f_1(x,t),$$

where $f_1(x,t) = U_t(x,t) - a^2 U_{xx}(x,t)$.

It is clear now that the function

$$v(x,t) = u(x,t) - U(x,t)$$

is a solution of the problem

$$\begin{aligned} v_t &= a v_{xx} + \overline{f}, \\ v(0,t) &= v(l,t) = 0, \\ v(x,0) &= \varphi(x) - U(x,0). \end{aligned}$$

The solution v can be found by the Fourier method.

4.5 Problems on unbounded domains

4.5.1 The Green function in unbounded domains

We have derived the Green function

$$G_l(x,\xi,t) := \frac{2}{l}\sum_{n=1}^{\infty} e^{-\left(\frac{\pi n}{l}\right)^2 a^2 t} \sin\left(\frac{\pi n}{l}x\right) \sin\left(\frac{\pi n}{l}\xi\right) \tag{4.5.1}$$

for the problem in the finite domain $(0,l)$. Here we have used the subindex l to indicate the domain $(0,l)$. We want to develope this theory when the domain is expanded to the whole $\mathbb{R}$. In doing so, we rewrite (4.5.1) in another form, such that the ends of the bar are $-\frac{l}{2}$ and $\frac{l}{2}$.

Let $x' = x - \frac{l}{2}$, $\xi' = \xi - \frac{l}{2}$. Then x' and ξ' are lying in $(-\frac{l}{2}, \frac{l}{2})$ and the Green function has the form

$$G_l(x',\xi',t) = \frac{2}{l}\sum_{n=1}^{\infty} e^{-\left(\frac{\pi n}{l}\right)^2 a^2 t} \sin\frac{\pi n}{l}\left(x' + \frac{l}{2}\right) \sin\frac{\pi n}{l}\left(\xi' + \frac{l}{2}\right). \tag{4.5.2}$$

We rewrite (4.5.2) by the following arguments.

If n is even, that is $n = 2m$, then

$$\sin\frac{2\pi m}{l}\left(x'+\frac{l}{2}\right)\cdot\sin\frac{2\pi m}{l}\left(\xi'+\frac{l}{2}\right)=\sin\left(\frac{2\pi m}{l}x'\right)\cdot\sin\left(\frac{2\pi m}{l}\xi'\right).$$

Further, if n is odd, that is $n = 2m+1$, then

$$\sin\frac{(2m+1)\pi}{l}\left(x'+\frac{l}{2}\right)\cdot\sin\frac{(2m+1)\pi}{l}\left(\xi'+\frac{l}{2}\right)=\cos\frac{(2m+1)\pi}{l}x'\cdot\cos\frac{(2m+1)\pi}{l}\xi'.$$

Thus,

$$\begin{aligned} G_l(x',\xi',t) &= \frac{2}{l}{\sum_{n=0}^{\infty}}'' e^{-\left(\frac{\pi n}{l}\right)^2 a^2 t}\,\sin\left(\frac{\pi n}{l}x'\right)\sin\left(\frac{\pi n}{l}\xi'\right) \\ &+ \frac{2}{l}{\sum_{n=1}^{\infty}}' e^{-\left(\frac{\pi n}{l}\right)^2 a^2 t}\,\cos\left(\frac{\pi n}{l}x'\right)\cos\left(\frac{\pi n}{l}\xi'\right), \end{aligned} \tag{4.5.3}$$

where $\sum''$ stands for even n, and $\sum'$ stands for odd n.

We study the limit of $G_l(x',\xi',t)$ as $l\to\infty$. We have

$$\frac{2}{l}{\sum_{n=0}^{\infty}}'' e^{-\lambda_n^2 a^2 t}\sin(\lambda_n x')\sin(\lambda_n\xi') = \frac{1}{\pi}{\sum_{n=0}^{\infty}}'' f_1(\lambda_n)\,\Delta\lambda, \tag{4.5.4}$$

where

$$f_1(\lambda) = e^{-\lambda^2 a^2 t}\sin(\lambda x')\sin(\lambda\xi'),\ \Delta\lambda=\frac{2\pi}{l},\ \lambda_n=\frac{\pi n}{l}.$$

The sum (4.5.4) is the integral sum for the function $f_1(\lambda)$ over the interval $0\le\lambda<\infty$. For $l\to\infty$, $\Delta\lambda\to 0$. Taking the limit under the integral, we obtain

$$\lim_{\Delta\lambda\to 0}\frac{1}{\pi}{\sum_{n=0}^{\infty}}'' f_1(\lambda_n)\,\Delta\lambda = \frac{1}{\pi}\int_0^{\infty} f_1(\lambda)\,d\lambda = \frac{1}{\pi}\int_0^{\infty} e^{-\lambda^2 a^2 t}\sin(\lambda x')\sin(\lambda\xi')\,d\lambda. \tag{4.5.5}$$

Remark 4.5.1 Here we have used the following result: For a continuous function defined on $[0,\infty)$, if the integral sums

$$\sum_{i=1}^{\infty} f(\lambda_i^*)(\lambda_i-\lambda_{i-1}) \quad (\lambda_i-\lambda_{i-1}=\Delta\lambda,\ \lambda_{i-1}\le\lambda_i^*\le\lambda_i)$$

for any choice of λ_i^* converge, then there exists the integral

$$\int_0^{\infty} f(\lambda)\,d\lambda.$$

Analogously,

$$\frac{2}{l}{\sum_{n=1}^{\infty}}' e^{-\lambda_n^2 a^2 t}\cos(\lambda_n x')\cos(\lambda_n\xi') = \frac{1}{\pi}{\sum_{n=1}^{\infty}}' f_2(\lambda_n)\,\Delta\lambda, \tag{4.5.6}$$

where

$$f_2(\lambda) = e^{-\lambda^2 a^2 t}\cos(\lambda x')\cos(\lambda\xi'),\ \Delta\lambda=\frac{2\pi}{l} \quad\text{and}\quad \lambda_n=\frac{\pi n}{l}.$$

As $\Delta\lambda \to 0$, we get

$$\lim_{\Delta\lambda\to 0} \frac{1}{\pi}{\sum_{n=1}^{\infty}}' f_2(\lambda_n)\,\Delta\lambda = \frac{1}{\pi}\int_0^\infty f_2(\lambda)\,d\lambda = \frac{1}{\pi}\int_0^\infty e^{-\lambda^2 a^2 t}\cos(\lambda x')\,\cos(\lambda\xi')\,d\lambda. \tag{4.5.7}$$

Finally, we have

$$\begin{aligned} G(x,\xi,t) &= \lim_{l\to\infty} G_l(x,\xi,t) \\ &= \frac{1}{\pi}\int_0^\infty e^{-\lambda^2 a^2 t}\sin(\lambda x)\,\sin(\lambda\xi)\,d\xi + \frac{1}{\pi}\int_0^\infty e^{-\lambda^2 a^2 t}\cos(\lambda x)\,\cos(\lambda\xi)\,d\xi \\ &= \frac{1}{\pi}\int_0^\infty e^{-\lambda^2 a^2 t}\cos\lambda(x-\xi)\,d\xi. \end{aligned}$$

The Green function for the unbounded domain $(-\infty,\infty)$ has thus the form

$$G(x,\xi,t) = \frac{1}{\pi}\int_0^\infty e^{-\lambda^2 a^2 t}\cos\lambda(x-\xi)\,d\lambda. \tag{4.5.8}$$

We calculate now the integral

$$I = \int_0^\infty e^{-\lambda^2\alpha}\,\cos(\lambda\beta)\,d\lambda \quad (\alpha > 0), \tag{4.5.9}$$

where the parameters α and β are given. In order to calculate this integral, we fix α and vary β so that I is a function of β: $I(\beta)$. It is clear that

$$\begin{aligned} \frac{dI}{d\beta} &= -\int_0^\infty e^{-\lambda^2\alpha}\,\lambda\sin(\lambda\beta)\,d\lambda \\ &= \sin(\lambda\beta)\frac{1}{2\alpha}e^{-\lambda^2\alpha}\Big|_0^\infty - \frac{\beta}{2\alpha}\int_0^\infty e^{-\lambda^2\alpha}\cos(\lambda\beta)\,d\lambda \\ &= -\frac{\beta}{2\alpha}I(\beta). \end{aligned}$$

Thus,

$$\frac{I'(\beta)}{I(\beta)} = -\frac{\beta}{2\alpha}.$$

Hence

$$I(\beta) = C\,e^{-\frac{\beta^2}{4\alpha}}.$$

On the other hand,

$$I(0) = \int_0^\infty e^{-\lambda^2\alpha}\,d\lambda = \frac{1}{\sqrt{\alpha}}\int_0^\infty e^{-z^2}\,dz = \frac{1}{\sqrt{\alpha}}\cdot\frac{\sqrt{\pi}}{2}.$$

Thus,

$$I(\beta)=\int_0^\infty e^{-\lambda^2\alpha}\cos(\lambda\beta)\,d\lambda=\frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{\alpha}}e^{-\frac{\beta^2}{4\alpha}}. \tag{4.5.10}$$

Applying (4.5.10) to (4.5.8), we get

$$G(x,\xi,t)=\frac{1}{2\sqrt{\pi a^2 t}}e^{-\frac{(x-\xi)^2}{4a^2t}}, \tag{4.5.11}$$

which is the Green function for the unbounded domain $(-\infty,\infty)$. This function is often called *the fundamental solution of the heat equation.*

Properties of the fundamental solution

i) The function $G(x,\xi,t-t_0)$ as a function of x,t is a solution of the heat equation. In fact,

$$\begin{aligned}
G_x &= -\frac{1}{2\sqrt{\pi}}\cdot\frac{x-\xi}{2[a^2(t-t_0)]^{3/2}}\cdot e^{-\frac{(x-\xi)^2}{4a^2(t-t_0)}},\\
G_{xx} &= \frac{1}{2\sqrt{\pi}}\left[-\frac{1}{2}\frac{1}{[a^2(t-t_0)]^{3/2}}+\frac{(x-\xi)^2}{4[a^2(t-t_0)]^{5/2}}\right]e^{-\frac{(x-\xi)^2}{4a^2(t-t_0)}},\\
G_t &= \frac{1}{2\sqrt{\pi}}\left[-\frac{a^2}{2[a^2(t-t_0)]^{3/2}}+\frac{a^2(x-\xi)^2}{4[a^2(t-t_0)]^{3/2}}\right]e^{-\frac{(x-\xi)^2}{4a^2(t-t_0)}}.
\end{aligned}$$

Thus,

$$G_t=a^2\,G_{xx}.$$

ii) $\displaystyle\int_{-\infty}^{\infty}G(x,\xi,t-t_0)\,dx=1$ for $t>t_0$.

$$\int_{-\infty}^{\infty}G(x,\xi,t-t_0)\,dx=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\frac{(x-\xi)^2}{4a^2(t-t_0)}}\frac{dx}{2\sqrt{a^2(t-t_0)}}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\alpha^2}\,d\alpha=1.$$

4.5.2 Heat conduction in the unbounded domain $(-\infty,\infty)$

Find a bounded function $u(x,t)$ $(-\infty<x<\infty,\ t\ge 0)$, which satisfies the equation

$$u_t=a^2\,u_{xx}\quad(-\infty<x<\infty,\ t>0), \tag{4.5.12}$$

and the initial condition

$$u(x,0)=\varphi(x),\ \ -\infty<x<\infty. \tag{4.5.13}$$

Since u is bounded, the solution of our problem is unique (§ 4.3.5). We shall prove that if φ is bounded, say $|\varphi|<M$, then for $t>0$ *the Poisson integral*

$$u(x,t)=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{a^2t}}e^{-\frac{(x-\xi)^2}{4a^2t}}\varphi(\xi)\,d\xi \tag{4.5.14}$$

is a solution of (4.5.12), and $\lim\limits_{\substack{x\to x_0\\ t\to 0}} u(x,t)=\varphi(x_0)$. Here x_0 is a point where φ is continuous.

We note that

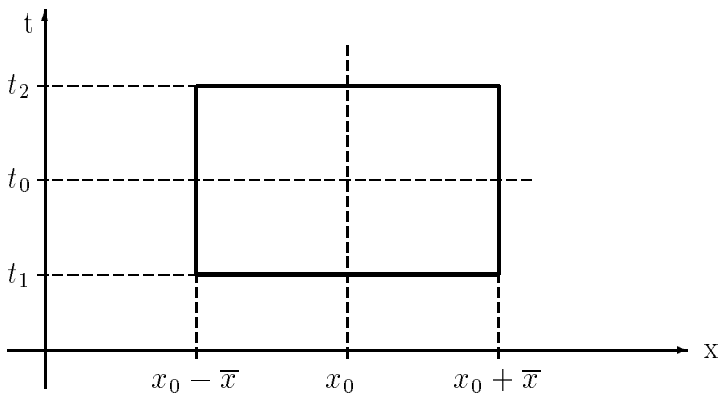
$$|u(x,t)|\le \int\limits_{-\infty}^{\infty} G(x,\xi,t)\,|\varphi(\xi)|\,d\xi < M\int\limits_{-\infty}^{\infty} G(x,\xi,t)\,d\xi = M.$$

Thus, the function $u(x,t)$ is bounded. We shall prove that for $t>0$, $u(x,t)$ satisfies the heat equation (4.5.12). In doing so, we prove that we can differentiate (4.5.14) under the integral. For example, formally differentiating both sides of (4.5.14) we get

$$\frac{\partial u}{\partial x}=\int\limits_{-\infty}^{\infty}\frac{\partial}{\partial x}\Big(G(x,\xi,t)\Big)\,\varphi(\xi)\,d\xi,$$

and it remains to prove that the integral on the right-hand side converges uniformly. It is enough to prove the differentiability of u at a point (x_0,t_0), or to prove the uniform convergence of the above integral in a neighbourhood of this point:

$$t_1\le t_0\le t_2,\quad |x-x_0|\le \overline{x}.$$



In doing so, we shall prove that there exists a positive function $F(\xi)$, which does not depend on x and t, and approximates $|G_x(x,\xi,t)\,\varphi(\xi)|$ from above:

$$|G_x(x,\xi,t)\,\varphi(\xi)|\le F(\xi), \tag{4.5.15}$$

and

$$\int\limits_{x_1}^{\infty} F(\xi)\,d\xi<\infty,\qquad \int\limits_{-\infty}^{x_1} F(\xi)\,d\xi<\infty; \tag{4.5.16}$$

x_1 is a number, for which (4.5.15) is valid.

For $|x-x_0|\le\overline{x}$ and $t_1\le t\le t_2$ we have

$$\begin{aligned}\left|\frac{\partial}{\partial x}G(x,\xi,t)\right|\cdot|\varphi(\xi)| &= \frac{1}{2\sqrt{\pi}}\,\frac{|\xi-x|}{2\sqrt{a^2t}^{\,3}}e^{-\frac{(x-\xi)^2}{4a^2t}}\,|\varphi(\xi)|\\ &\le \frac{M}{2\sqrt{\pi}}\,\frac{|\xi-x_0|+\overline{x}}{2\sqrt{a^2t_1}^{\,3}}e^{-\frac{(|\xi-x_0|-\overline{x})^2}{4a^2t_2}}\\ &=: F(\xi).\end{aligned} \tag{4.5.17}$$

Let x_1 be a number from which (4.5.17) is valid. Then

$$\begin{aligned}\int_{x_1}^{\infty} F(\xi)\,d\xi &= \int_{x_1}^{\infty} \frac{M}{2\sqrt{\pi}}\,\frac{|\xi-x_0|+\overline{x}}{2\sqrt{a^2t_1}^{\,3}} e^{-\dfrac{(|\xi-x_0|-\overline{x})^2}{4a^2t_2}}\,d\xi \\ &= \int_{x_1-\overline{x}}^{\infty} \frac{M}{2\sqrt{\pi}}\,\frac{\xi_1+2\overline{x}}{2\sqrt{a^2t_1}^{\,3}} e^{-\dfrac{\xi_1^2}{4a^2t_2}}\,d\xi \quad (\xi_1 = |\xi-x_0|-\overline{x}).\end{aligned}$$

This integral is clearly convergent. Thus,

$$\frac{\partial u}{\partial x} = \int_{-\infty}^{\infty} \frac{\partial}{\partial x} G(x,\xi,t)\,\varphi(\xi)\,d\xi. \tag{4.5.18}$$

Similarly, we can prove that for $t>0$ the function $u(x,t)$ is differentiable two times w.r.t. x and once w.r.t. t, and that u satisfies the heat equation (4.5.12).

Let x_0 be a point where $\varphi(x)$ is continuous. It remains to prove that

$$u(x,t) \longrightarrow \varphi(x_0), \quad \text{as} \quad t\to 0 \quad \text{and} \quad x\to x_0.$$

For any $\epsilon>0$ we have to show that there is a $\delta(\epsilon)$ such that

$$|u(x,t)-\varphi(x_0)|<\epsilon$$

if $|x-x_0|<\delta(\epsilon)$ and $t<\delta(\epsilon)$.

Since φ is continuous at x_0, there exists an $\eta(\epsilon)$ such that

$$|\varphi(x)-\varphi(x_0)| < \frac{\epsilon}{6} \quad \text{for} \quad |x-x_0|<\eta. \tag{4.5.19}$$

We break the integral for $u(x,t)$ into three parts as follows:

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{a^2t}} e^{-\dfrac{(x-\xi)^2}{4a^2t}}\,\varphi(\xi)\,d\xi \\ &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{x_1} \cdots + \frac{1}{2\sqrt{\pi}} \int_{x_1}^{x_2} \cdots + \frac{1}{2\sqrt{\pi}} \int_{x_2}^{\infty} \cdots \\ &:= u_1(x,t)+u_2(x,t)+u_3(x,t), \end{aligned} \tag{4.5.20}$$

where

$$x_1 = x_0-\eta \quad \text{and} \quad x_2 = x_0+\eta. \tag{4.5.21}$$

For the second integral we have

$$u_2(x,t) = \frac{\varphi(x_0)}{2\sqrt{\pi}} \int_{x_1}^{x_2} \frac{1}{\sqrt{a^2t}} e^{-\dfrac{(x-\xi)^2}{4a^2t}}\,d\xi + \frac{1}{2\sqrt{\pi}} \int_{x_1}^{x_2} \frac{1}{\sqrt{a^2t}} e^{-\dfrac{(x-\xi)^2}{4a^2t}} [\varphi(\xi)-\varphi(x_0)]\,d\xi = I_1+I_2.$$

The integral I_1 can be calculated as

$$I_1 = \frac{\varphi(x_0)}{2\sqrt{\pi}} \int_{x_1}^{x_2} \frac{e^{-\frac{(x-\xi)^2}{4a^2t}}}{\sqrt{a^2t}}\, d\xi = \frac{\varphi(x_0)}{\sqrt{\pi}} \int_{\frac{x_1-x}{2\sqrt{a^2t}}}^{\frac{x_2-x}{2\sqrt{a^2t}}} e^{-\alpha^2}\, d\alpha,$$

with $\alpha = \dfrac{\xi - x}{2\sqrt{a^2t}}$, $d\alpha = \dfrac{d\xi}{2\sqrt{a^2t}}$.

As $|x - x_0| < \eta$, $x_1 = x_0 - \eta$, $x_2 = x_0 + \eta$, we see that

$$\frac{x_1 - x}{2\sqrt{a^2t}} \longrightarrow -\infty, \quad \text{as} \quad t \to 0$$

$$\frac{x_2 - x}{2\sqrt{a^2t}} \longrightarrow +\infty, \quad \text{as} \quad t \to 0.$$

It follows that

$$\lim_{\substack{t \to 0 \\ x \to x_0}} I_1 = \varphi(x_0).$$

Thus, there is a $\delta_1(\epsilon)$ such that

$$|I_1 - \varphi(x_0| < \frac{\epsilon}{2} \quad \text{for} \quad |x - x_0| < \delta_1 \quad \text{and} \quad t < \delta_1. \tag{4.5.22}$$

We estimate I_2 as follows:

$$|I_2| \le \frac{1}{2\sqrt{\pi}} \int_{x_1}^{x_2} \frac{1}{\sqrt{a^2t}} e^{-\frac{(x-\xi)^2}{4a^2t}} |\varphi(\xi) - \varphi(x_0)|\, d\xi.$$

From (4.5.21), for $x_1 < \xi < x_2$, we see that

$$|\xi - x_0| < \eta.$$

Taking (4.5.19) and the inequality

$$\frac{1}{\sqrt{\pi}} \int_{x'}^{x''} e^{-\alpha^2}\, d\alpha < \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2}\, d\alpha = 1, \quad \forall x', x''$$

into account, we obtain

$$\begin{aligned} |I_2| &\le \frac{\epsilon}{6} \cdot \frac{1}{2\sqrt{\pi}} \int_{x_1}^{x_2} \frac{1}{\sqrt{4a^2t}}\, e^{-\frac{(x-\xi)^2}{4a^2t}}\, d\xi \\ &= \frac{\epsilon}{6} \cdot \frac{1}{\pi} \int_{\frac{x_1-x}{2\sqrt{a^2t}}}^{\frac{x_2-x}{2\sqrt{a^2t}}} e^{-\alpha^2}\, d\alpha < \frac{\epsilon}{6}. \end{aligned} \tag{4.5.23}$$

Further,

$$\begin{aligned} |u_3(x,t)| &= \frac{1}{2\sqrt{\pi}} \left| \int_{x_2}^{\infty} \frac{1}{\sqrt{a^2 t}}\, e^{-\frac{(x-\xi)^2}{4a^2 t}}\, \varphi(\xi)\, d\xi \right| \\ &< \frac{M}{\sqrt{\pi}} \int_{\frac{x_2 - x}{2\sqrt{a^2 t}}}^{\infty} e^{-\alpha^2}\, d\alpha \longrightarrow_{\substack{t \to 0 \\ x \to x_0}} 0, \end{aligned} \tag{4.5.24}$$

$$\begin{aligned} |u_1(x,t)| &= \frac{1}{2\sqrt{\pi}} \left| \int_{-\infty}^{x_1} \frac{1}{\sqrt{a^2 t}}\, e^{-\frac{(x-\xi)^2}{4a^2 t}}\, \varphi(\xi)\, d\xi \right| \\ &< \frac{M}{\sqrt{\pi}} \int_{-\infty}^{\frac{x_1 - x}{2\sqrt{a^2 t}}} e^{-\alpha^2}\, d\alpha \longrightarrow_{\substack{t \to 0 \\ x \to x_0}} 0. \end{aligned} \tag{4.5.25}$$

(Note that as $x \to x_0$, $x_2 - x > 0$ and $x_1 - x < 0$.)

Thus, there is a $\delta_2(\epsilon)$ such that

$$|u_3(x,t)| < \frac{\epsilon}{3} \quad \text{and} \quad |u_1(x,t)| < \frac{\epsilon}{3} \quad \text{for} \quad |x - x_0| < \delta_2 \quad \text{and} \quad t < \delta_2. \tag{4.5.26}$$

A combination of (4.5.22), (4.5.23) and (4.5.26) gives

$$\begin{aligned} |u(x,t) - \varphi(x_0)| &\le |u_1| + |I_1 - \varphi(x_0)| + |I_2| + |u_3| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{3}, \end{aligned} \tag{4.5.27}$$

as $|x - x_0| < \delta$ and $t < \delta$, where $\delta = \min\{\delta_1, \delta_2\}$.

Thus, we have proved that the function

$$u(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{a^2 t}} e^{-\frac{(x-\xi)^2}{4a^2 t}}\, \varphi(\xi)\, d\xi$$

is bounded and satisfies the heat equation as well as the initial condition.

If the initial condition is given at $t = t_0$, then the solution has the form

$$u(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{a^2 (t - t_0)}} e^{-\frac{(x-\xi)^2}{4a^2 (t - t_0)}}\, \varphi(\xi)\, d\xi.$$

Example: We try to find the solution of the problem (4.5.12), (4.5.13) in the case

$$u(x,0) = \varphi(x) = \begin{cases} T_1 & \text{for } x < 0, \\ T_2 & \text{for } x > 0 \end{cases}.$$

In this case

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{a^2 t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} \varphi(\xi)\, d\xi \\ &= \frac{T_2}{\sqrt{\pi}} \int\limits_{-\infty}^{0} e^{-\frac{(x-\xi)^2}{4a^2 t}} \frac{d\xi}{2\sqrt{a^2 t}} + \frac{T_1}{\sqrt{\pi}} \int\limits_{0}^{\infty} e^{-\frac{(x-\xi)^2}{4a^2 t}} \frac{d\xi}{2\sqrt{a^2 t}} \\ &= \frac{T_2}{\sqrt{\pi}} \int\limits_{-\infty}^{-\frac{x}{2\sqrt{a^2 t}}} e^{-\alpha^2}\, d\alpha + \frac{T_1}{\sqrt{\pi}} \int\limits_{-\frac{x}{2\sqrt{a^2 t}}}^{\infty} e^{-\alpha^2}\, d\alpha \\ &= \frac{T_1 + T_2}{2} + \frac{T_1 - T_2}{\sqrt{\pi}} \int\limits_{0}^{\frac{x}{2\sqrt{a^2 t}}} e^{-\alpha^2}\, d\alpha, \end{aligned} \tag{4.5.28}$$

since

$$\frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{-z} e^{-\alpha^2}\, d\alpha = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{0} e^{-\alpha^2}\, d\alpha + \frac{1}{\sqrt{\pi}} \int\limits_{0}^{z} e^{-\alpha^2}\, d\alpha = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int\limits_{0}^{z} e^{-\alpha^2}\, d\alpha$$

and

$$\frac{1}{\sqrt{\pi}} \int\limits_{-z}^{\infty} e^{-\alpha^2}\, d\alpha = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int\limits_{-z}^{0} e^{-\alpha^2}\, d\alpha = \frac{1}{2} - \frac{1}{\sqrt{\pi}} \int\limits_{0}^{z} e^{-\alpha^2}\, d\alpha.$$

If $T_2 = 0$, $T_1 = 1$, then $u(x,t) = \frac{1}{2}\left(1 + \frac{2}{\sqrt{\pi}} \int\limits_{0}^{\frac{x}{2\sqrt{a^2 t}}} e^{-\alpha^2}\, d\alpha\right)$. The function

$$\Phi(z) = \frac{2}{\sqrt{\pi}} \int\limits_{0}^{z} e^{-\alpha^2}\, d\alpha$$

is called the *error function* which has many applications in the probability theory.

4.5.3 The boundary value problem in a half-space

Consider the heat equation in the first quadrant

$$u_t = a^2\, u_{xx}, \quad x > 0, \quad t > 0, \tag{4.5.12}$$

with the initial condition

$$u(x,0) = \varphi(x), \quad x > 0, \tag{4.5.13}$$

and one of the boundary value conditions

$$u(0,t) = \mu(t),\ t \geq 0, \quad \text{(the first b.v.p.)}$$

$$\frac{\partial u}{\partial x}(0,t) = \nu(t),\ t \geq 0, \quad \text{(the second b.v.p.)}$$

$$\left(\text{or} \quad \frac{\partial u}{\partial x}(0,t) = \lambda[u(0,t) - \theta(t)], \quad \text{(the third b.v.p.)}\right)$$

In order to guarantee the uniqueness of the solution, we assume that the solution is bounded:

$$|u(x,t)| < M, \quad 0 < x < \infty,\ t \geq 0,$$

where M is a constant. It follows also

$$|\varphi(x)| < M.$$

We represent the solution of the first b.v.p. in the form

$$u(x,t) = u_1(x,t) + u_2(x,t),$$

where u_1 is the solution of (4.5.12) with the conditions

$$u_1(x,0) = \varphi(x), \quad u_1(0,t) = 0 \tag{4.5.14}$$

and u_2 is the solution of (4.5.12) with the conditions

$$u_2(x,0) = 0, \quad u_2(0,t) = \mu(t). \tag{4.5.17}$$

Below we give two lemmas about the *Poisson integral*

$$u(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{a^2 t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} \psi(\xi)\, d\xi. \tag{4.5.14}$$

Lemma 4.5.2 If ψ is a bounded and odd function,

$$\psi(x) = -\psi(-x),$$

then $u(0,t) = 0$.

Proof: Since ψ is bounded, the integral (4.5.14) converges, and

$$u(0,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{a^2 t}} e^{-\frac{\xi^2}{4a^2 t}} \psi(\xi)\, d\xi = 0.$$

□

Lemma 4.5.3 If ψ is a bounded and even function,

$$\psi(x) = \psi(-x),$$

then $\dfrac{\partial u}{\partial x}(0,t) = 0$.

Proof:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = -\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{(x-\xi)}{2\sqrt{a^2 t^3}} e^{-\frac{(x-\xi)^2}{4a^2 t}} \psi(\xi)\, d\xi \Bigg|_{x=0} = 0.$$

□

Let $\psi(x)$ be a function defined by

$$\psi(x) = \begin{cases} \varphi(x) & , \quad x > 0 \\ -\varphi(-x) & , \quad x < 0 \end{cases}.$$

Since φ is bounded (by M), ψ is bounded. It follows that the function

$$U(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{a^2 t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} \psi(\xi)\, d\xi$$

is well defined. Furthermore, since ψ is even, $U(0,t) = 0$. Thus,

$$u_1(x,t) = U(x,t) \quad \text{for} \quad x \geq 0.$$

The function $U(x,t)$ can be represented as follows:

$$\begin{aligned} U(x,t) &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{0} \frac{1}{\sqrt{a^2 t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} \psi(\xi)\, d\xi + \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{a^2 t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} \psi(\xi)\, d\xi \\ &= -\frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{a^2 t}} e^{-\frac{(x+\xi)^2}{4a^2 t}} \varphi(\xi)\, d\xi + \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{a^2 t}} e^{-\frac{(x-\xi)^2}{4a^2 t}} \varphi(\xi)\, d\xi \\ &= \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{a^2 t}} \left\{ e^{-\frac{(x-\xi)^2}{4a^2 t}} - e^{-\frac{(x+\xi)^2}{4a^2 t}} \right\} \varphi(\xi)\, d\xi. \end{aligned}$$

Thus,

$$u_1(x,t) = \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{a^2 t}} \left\{ e^{-\frac{(x-\xi)^2}{4a^2 t}} - e^{-\frac{(x+\xi)^2}{4a^2 t}} \right\} \varphi(\xi)\, d\xi. \tag{4.5.29}$$

Analogously, for the solution $\overline{u}_1(x,t)$ of the second boundary value problem with $\dfrac{\partial u_1}{\partial x}(0,t) = 0$ and $\overline{u}_1(x,0) = \varphi(x)$ we have

$$\overline{u}_1(x,t) = \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{a^2 t}} \left\{ e^{-\frac{(x-\xi)^2}{4a^2 t}} + e^{-\frac{(x+\xi)^2}{4a^2 t}} \right\} \varphi(\xi)\, d\xi. \tag{4.5.30}$$

Now we try to find $u_2(x,t)$ and $\overline{u}_2(x,t)$. In doing so we note that if we consider the equation $v_{1t} = a^2\, v_{1xx}$, for $0 < x < \infty$ and $t \geq t_0$, with the conditions $v_1(x,t_0) = T$ and $v_1(0,t) = 0$,

then from (4.5.29) we have

$$v_1(x,t) = \frac{T}{2\sqrt{\pi}} \int_0^\infty \left\{ e^{-\frac{(x-\xi)^2}{4a^2(t-t_0)}} - e^{-\frac{(x+\xi)^2}{4a^2(t-t_0)}} \right\} \cdot \frac{d\xi}{\sqrt{a^2(t-t_0)}}. \tag{4.5.31}$$

Taking the changes of variables

$$\alpha = \frac{\xi - x}{2\sqrt{a^2(t-t_0)}}, \quad \alpha_1 = \frac{\xi + \alpha}{2\sqrt{a^2(t-t_0)}},$$

we get

$$\begin{aligned} v_1(x,t) &= \frac{T}{\sqrt{\pi}} \left[\int_{-\frac{x}{2\sqrt{a^2(t-t_0)}}}^{\infty} e^{-\alpha^2}\, d\alpha - \int_{\frac{x}{2\sqrt{a^2(t-t_0)}}}^{\infty} e^{-\alpha_1^2}\, d\alpha_1 \right] \\ &= \frac{T}{\sqrt{\pi}} \int_{-\frac{x}{2\sqrt{a^2(t-t_0)}}}^{\frac{x}{2\sqrt{a^2(t-t_0)}}} e^{-\alpha^2}\, d\alpha = T\frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{a^2(t-t_0)}}} e^{-\alpha^2}\, d\alpha. \end{aligned}$$

Thus,

$$v_1(x,t) = T\ \Phi\ \left(\frac{x}{2\sqrt{a^2(t-t_0)}} \right), \tag{4.5.32}$$

where

$$\Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\alpha^2}\, d\alpha$$

is the error function.

Let $\mu(t) = \mu_0 = \text{const}$. Then the function

$$\overline{v}(x,t) = \mu_0\ \Phi\ \left(\frac{x}{2\sqrt{a^2(t-t_0)}} \right)$$

is the solution of the heat equation (4.5.12) for $t \geq t_0$ which satisfies the conditions

$$\overline{v}(x,t_0) = \mu_0, \quad \overline{v}(0,t) = 0.$$

It follows that the function

$$v(x,t) := \mu_0 - \overline{v}(x,t) = \mu_0 \left[1 - \Phi\ \left(\frac{x}{2\sqrt{a^2(t-t_0)}} \right) \right] \tag{4.5.33}$$

is the solution of (4.5.12) for $t \geq t_0$, which satisfies the conditions

$$v(x,t_0) = 0\ \ (x > 0) \quad \text{and} \quad v(0,t) = \mu_0\ \ (t > t_0).$$

We rewrite $v(x,t)$ in the form

$$v(x,t) = \mu_0\, U(x,t),$$

where

$$U(x,t) = 1 - \Phi\left(\frac{x}{2\sqrt{a^2(t-t_0)}}\right) = \frac{2}{\sqrt{\pi}} \int\limits_{\frac{x}{2\sqrt{a^2(t-t_0)}}}^{\infty} e^{-\alpha^2}\, d\alpha.$$

The function $U(x,t)$ corresponds to the case $\mu_0 = 1$.

We extend $U(x,t)$ by zero for $t < t_0$. It then corresponds to the step boundary condition

$$U(0,t) = \begin{cases} 1 & , \quad t > t_0 \\ 0 & , \quad t < t_0 \end{cases}.$$

Consider now the solution $v(x,t)$ of (4.5.12), which satisfies the conditions

$$v(x,t_0) = 0, \quad v(0,t) = \mu(t) = \begin{cases} \mu_0 & , \quad t_0 < t < t_1 \\ 0 & , \quad t > t_1 \end{cases}.$$

It can be verified that

$$v(x,t) = \mu_0[U(x,t-t_0) - U(x,t-t_1)].$$

Further, if μ has the form

$$\mu(t) = \begin{cases} \mu_0 & , \quad t_0 < t \le t_1, \\ \mu_1 & , \quad t_1 < t \le t_2, \\ & \cdots \\ \mu_{n-1} & , \quad t_{n-1} < t \le t_n, \end{cases}$$

then the solution of (4.5.12), (4.5.13) can be represented in the form

$$u(x,t) = \sum_{i=0}^{n-2} \mu_i[U(x,t-t_i) - U(x,t-t_{i+1})] + \mu_{n-1}U(x,t-t_{n-1}). \tag{4.5.34}$$

With the aid of the theorem about the mean value, we have

$$u(x,t) = \sum_{i=1}^{n-2} \mu_i \left.\frac{\partial U(x,t-\tau)}{\partial t}\right|_{\tau_i} \Delta\tau + \mu_{n-1}U(x,t-t_{n-1}), \quad t_i \le \tau_i \le t_{i+1}. \tag{4.5.35}$$

We consider the problem (4.5.12), (4.5.13) with $u(x,0) = 0$. If the function $\mu(t)$ is piecewise continuous, then we have an approximation to $u(x,t)$ of the form (4.5.34), when $\mu(t)$ is approximated by a piecewise constant function. If we refine this approximation, then we have

$$u(x,t) = \int\limits_0^t \frac{\partial U}{\partial t}(x,t-\tau)\,\mu(\tau)\,d\tau,$$

since for $x > 0$

$$\lim_{t-t_{n-1}\to 0} \mu_{n-1} U(x, t - t_{n-1}) = 0.$$

We do not go into the details, when the limit process is meaningful, and so formally take

$$u_2(x,t) = \int_0^t \frac{\partial U}{\partial t}(x, t-\tau)\, \mu(\tau)\, d\tau. \tag{4.5.36}$$

Further,

$$\frac{\partial U}{\partial t}(x,t) = \frac{\partial}{\partial t}\left(\frac{2}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{a^2 t}}}^{\infty} e^{-\alpha^2}\, d\alpha \right) = \frac{1}{2\sqrt{\pi}} \frac{a^2 x}{\sqrt{a^2 t}^3} e^{-\frac{x^2}{4a^2 t}} = -2a^2 \frac{\partial G}{\partial x}(x,0,t) = 2a^2 \frac{\partial G}{\partial \xi}\bigg|_{\xi=0}.$$

Thus,

$$u_2(x,t) = 2a^2 \int_0^t \frac{\partial G}{\partial \xi}(x,0,t-\tau)\, \mu(\tau)\, d\tau = \frac{a^2}{2\sqrt{\pi}} \int_0^t \frac{x}{\sqrt{a^2(t-\tau)}^3} e^{-\frac{x^2}{4a^2(t-\tau)}}\, \mu(\tau)\, d\tau. \tag{4.5.37}$$

We note that in the way of getting (4.5.37) we have used only the linearity of the heat equation and nothing more. Furthermore, the boundary and initial conditions on U are explicitly given

$$U(0,t) = 1, \quad t > 0,$$
$$U(x,0) = 0, \quad x > 0,$$

or

$$U(0,t) = \begin{cases} 1 & , \quad t > 0 \\ 0 & , \quad t < 0 \end{cases}.$$

If the boundary condition for a given differential equation is

$$u(0,t) = \mu(t), \quad t > 0,$$

which satisfies the homogeneous initial condition, then

$$u(x,t) = \int_0^t \frac{\partial U}{\partial t}(x, t-\tau)\, \mu(\tau)\, d\tau.$$

This is called the *Duhamel principle*, which shows that the difficulties in the boundary value problems are due to the piecewise constant boundary conditions.

By the method of odd extension of the data, we can easily find the solution of the problem

$$u_t = a^2\, u_{xx} + f(x,t), \quad (0 < x < \infty,\ t > 0),$$
$$u(0,t) = 0, \quad u(x,0) = 0,$$

in the form

$$u_3(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^{\infty} \int_0^t \frac{1}{\sqrt{a^2(t-\tau)}} \left\{ e^{-\frac{(x-\xi)^2}{4a^2 t}} - e^{-\frac{(x+\xi)^2}{4a^2 t}} \right\} f(\xi,\tau)\, d\xi\, d\tau.$$

Chapter 5

Elliptic Equations

We shall consider shortly in this chapter the Laplace equation

$$\Delta u = u_{xx} + u_{yy} = 0.$$

A solution of the Laplace equation is called *a harmonic function.*

The inhomogeneous version of the Laplace equation

$$\Delta u = f,$$

with f being a given function, is called the *Poisson equation.*

The basic mathematical problem is to solve the Laplace or the Poisson equation in a given domain Ω with a condition on the boundary $\partial\Omega$ of Ω.

$$\Delta u = f \quad \text{in} \quad \Omega$$

$$u = h \quad \text{or} \quad \frac{\partial u}{\partial n} = h \quad \text{or} \quad \frac{\partial u}{\partial n} + au = h \quad \text{on} \quad \partial\Omega.$$

5.1 The maximum principle

Let Ω be a connected bounded domain in $\mathbb{R}^2$. Let u be a harmonic function that is continuous in $\overline{\Omega} = \Omega \cup \partial\Omega$. Then the maximum and the minimum values of u are attained on $\partial\Omega$.

Proof: Consider the function

$$v(x, y) = u(x, y) + \epsilon(x^2 + y^2), \ \epsilon > 0.$$

We have

$$\Delta v = \Delta u + \epsilon\Delta(x^2 + y^2) = 0 + 4\epsilon > 0 \quad \text{in} \quad \mathbb{R}\,.$$

Thus, v has no maximum in Ω, otherwise $v_{xx} \leq 0$, $v_{yy} \leq 0$, and so $v_{xx}+v_{yy} \leq 0$. Furthermore, since v is a continuous function, it attains its maximum in the bounded closed domain $\overline{\Omega}$. As v has no maximum in Ω, v attains its maximum at some point $x_0 \in \partial\Omega$. Hence, for all $(x, y) \in \Omega$

$$u(x, y) \leq v(x, y) \leq v(x_0, y_0) = u(x_0, y_0) + \epsilon(x_0^2 + y_0^2) \leq \max_{\partial\Omega} u + \epsilon(x_0^2 + y_0^2).$$

Since Ω is bounded, $x_0^2 + y_0^2$ is bounded and since ϵ is an arbitrary positive number, we have

$$u(x,y) \leq \max_{\partial\Omega} u, \quad \forall (x,y) \in \Omega.$$

Thus, u attains its maximum at some point of the boundary $\partial\Omega$.

The proof of "minimum part" is similar. □

5.2 The uniqueness of the Dirichlet problem

$$\Delta u = f \quad \text{in} \quad \Omega,$$

$$u|_{\partial\Omega} = h.$$

Let u_1 and u_2 be two solutions of this problem. Put $v = u_1 - u_2$. Then $\Delta v = 0$ in Ω and $v\Big|_{\partial\Omega} = 0$. By the maximum principle $v \equiv 0$ in Ω. Thus $u_1(x,y) = u_2(x,y)$.

5.3 The invariance of the operator Δ

The operator Δ is invariant under translations and rotations. In fact, a *translation* in the plane is a transformation

$$x' = x + a, \quad y' = y + b.$$

The invariance under translations means that $u_{xx} + u_{yy} = u_{x'x'} + u_{y'y'}$.

A rotation in the plane through the angle α is given by

$$x' = x\cos\alpha + y\sin\alpha,$$

$$y' = -x\sin\alpha + y\cos\alpha.$$

By the chain rule we calculate

$$\begin{aligned}
u_x &= u_{x'}\cos\alpha - u_{y'}\sin\alpha, \\
u_y &= u_{x'}\sin\alpha + u_{y'}\cos\alpha, \\
u_{xx} &= (u_{x'}\cos\alpha - u_{y'}\sin\alpha)_{x'}\cos\alpha - (u_{x'}\sin\alpha + u_{y'}\cos\alpha)_{y'}\sin\alpha, \\
u_{yy} &= (u_{x'}\sin\alpha + u_{y'}\cos\alpha)_{x'}\sin\alpha + (u_{x'}\sin\alpha + u_{y'}\cos\alpha)_{y'}\cos\alpha.
\end{aligned}$$

Adding, we have

$$u_{xx} + u_{yy} = (u_{x'x'} + u_{y'y'})(\cos^2\alpha + \sin^2\alpha) + u_{x'y'} \quad \ldots = u_{x'x'} + u_{y'y'}.$$

This proves the invariance of the Laplace operator. In engineering the Laplacian Δ is a model for *isotropic* physical situations, in which there is no prefered direction.

The rotational invariance suggests that the two-dimensional Laplacian $\Delta_2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ should take a particularly simple form in *polar coordinates*. The transformation has the form

$$x = r\cos\theta, \quad y = r\sin\theta.$$

It follows that

$$r = \sqrt{x^2+y^2}, \quad \theta = \arccos\frac{x}{\sqrt{x^2+y^2}} = \arcsin\frac{y}{\sqrt{x^2+y^2}}.$$

We have

$$\begin{aligned}
&\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta,\\
&\frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta,\\
&\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta,\\
&\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r}.
\end{aligned}$$

Thus, the transformation $x = r\cos\theta$, $y = r\sin\theta$ has the Jacobian matrix

$$J = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

with the inverse

$$J^{-1} = \begin{pmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial \theta}{\partial x} \\ \dfrac{\partial r}{\partial y} & \dfrac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\dfrac{\sin\theta}{r} \\ \sin\theta & \dfrac{\cos\theta}{r} \end{pmatrix}.$$

Hence,

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r},\\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\cos\theta - \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\frac{\sin\theta}{r}\\
&= \frac{\partial^2 u}{\partial r^2}\cos^2\theta - \frac{\partial^2 u}{\partial r\,\partial\theta}\frac{\sin\theta\,\cos\theta}{r} + \frac{\partial u}{\partial \theta}\frac{\sin\theta\,\cos\theta}{r^2}\\
&\quad - \frac{\partial^2 u}{\partial r\,\partial\theta}\frac{\sin\theta\,\cos\theta}{r} + \frac{\partial u}{\partial r}\frac{\sin^2\theta}{r} + \frac{\partial^2 u}{\partial \theta^2}\frac{\sin^2\theta}{r^2} + \frac{\partial u}{\partial \theta}\frac{\sin\theta\,\cos\theta}{r^2}\\
&= \frac{\partial^2 u}{\partial r^2}\cos^2\theta + \frac{\partial^2 u}{\partial \theta^2}\frac{\sin^2\theta}{r^2} - 2\frac{\partial^2 u}{\partial r\,\partial\theta}\frac{\sin\theta\,\cos\theta}{r} + \frac{\partial u}{\partial r}\frac{\sin^2\theta}{r} + 2\frac{\partial u}{\partial \theta}\frac{\sin\theta\,\cos\theta}{r^2}.
\end{aligned}$$

$$u_{xy} = u_{yx} = u_{rr}\sin\theta\,\cos\theta - u_{\theta\theta}\frac{\sin\theta\,\cos\theta}{r^2} + u_{r\theta}\frac{\cos^2\theta - \sin^2\theta}{r} +$$

$$+ u_\theta\frac{\sin^2\theta - \cos^2\theta}{r^2} - u_r\frac{\sin\theta\,\cos\theta}{r},$$

$$u_{yy} = u_{rr}\sin^2\theta + u_{\theta\theta}\frac{\cos^2\theta}{r^2} + 2u_{r\theta}\frac{\sin\theta\,\cos\theta}{r} + u_r\frac{\cos^2\theta}{r} - 2u_\theta\frac{\sin\theta\,\cos\theta}{r^2}.$$

Thus,

$$u_{xx} + u_{yy} = u_{rr} + u_{\theta\theta}\frac{1}{r^2} + u_r\frac{1}{r} \tag{5.3.1}$$

$$= \frac{1}{r^2}\left\{r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial\theta^2}\right\}. \tag{5.3.2}$$

It is also natural to look for *special harmonic functions that themselves are rotationally invariant.* It means that we look for solutions depending only on r. Thus, by (5.3.2)

$$0 = u_{xx} + u_{yy} = \frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

if u does not depend on θ.

Hence, $r\dfrac{\partial u}{\partial r} = c_1$, and so $u = c_1\ln r + c_2$. The function $\ln r$ will play a central role later.

5.4 Poisson's formula

Consider the problem

$$u_{xx} + u_{yy} = 0, \quad x^2 + y^2 < a, \tag{5.4.1}$$

$$u = f, \quad x^2 + y^2 = a. \tag{5.4.2}$$

Here a is a given number and f is a given function. In polar coordinates (r,θ), the equation (5.4.1) has the form

$$\frac{1}{r^2}\left\{r\,\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial\theta^2}\right\} = 0. \tag{5.4.3}$$

We shall find solutions of this equation in the form

$$u(r,\theta) = R(r)\,\Theta(\theta).$$

Plugging this into (5.4.3), we get

$$\frac{\frac{d}{dr}\left(r\,\frac{dR}{dr}\right)}{\frac{R}{r}} = -\frac{\Theta''}{\Theta} = \lambda,$$

where λ is a constant. From this we get

$$\Theta'' + \lambda\Theta = 0, \tag{5.4.4}$$

$$r\,\frac{d}{dr}\left(r\,\frac{dR}{dr}\right) - \lambda R = 0. \tag{5.4.5}$$

The equation (5.4.4) gives

$$\Theta(\theta) = A\,\cos\sqrt{\lambda}\,\theta + B\,\sin\sqrt{\lambda}\,\theta.$$

As $u(r,\theta)$ is periodic in θ, we have

$$\Theta(\theta + 2\pi) = \Theta(\theta).$$

Thus, $\sqrt{\lambda} = n$ is an entire number and so

$$\Theta_n(\theta) = A_n\,\cos(n\theta) + B_n\,\sin(n\theta).$$

We shall find $R(r)$ in the form $R(r) = r^\mu$. Putting this into (5.4.5) we get

$$n^2 = \mu^2 \quad \text{or} \quad \mu = \pm n \quad (n > 0).$$

Thus,

$$R(r) = Cr^n + Dr^{-n},$$

where $n > 0$ and C and D are some constants. We see that D must be equal to zero, otherwise as $r = 0$, $R(r)$ is not bounded. Thus, special solutions of our problem have the form

$$u_n(r,\theta) = r^n\,\Big(A_n\,\cos(n\theta) + B_n\,\sin(n\theta)\Big).$$

If the series

$$u(r,\theta) = \sum_{n=0}^{\infty} r^n\Big(A_n\,\cos(n\theta) + B_n\,\sin(n\theta)\Big)$$

converges, then it represents a harmonic function. Note that the equation (5.4.3) has no meaning for $r = 0$, so we have to prove that $\Delta u_n = 0$ also for $r = 0$. In fact, the special solutions

$$r^n\cos(n\theta) \quad \text{and} \quad r^n\sin(n\theta)$$

are the real and the imaginary parts of the function

$$\rho^n\,e^{in\theta} = (\rho\,e^{i\theta})^n = (x+iy)^n,$$

which are polynomials in x and y. It is now clear that a polynomial which satisfies the equation $\Delta u = 0$ for $r > 0$, due to the continuity of the second derivatives, satisfies this equation also for $r = 0$.

In order to determine A_n and B_n, we use the boundary condition

$$u(a,\theta) = \sum_{n=0}^{\infty} a^n\Big(A_n\,\cos(n\theta) + B_n\,\sin(n\theta)\Big) = f. \tag{5.4.6}$$

Taking into account that f is a function of θ, we have

$$f(\theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}\Big(\alpha_n\,\cos(n\theta) + \beta_n\,\sin(n\theta)\Big), \tag{5.4.7}$$

where

$$\alpha_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\,d\psi, \quad \alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\,\cos(n\psi)\,d\psi, \quad \beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\,\sin(n\psi)\,d\psi.$$

A comparison of (5.4.6) with (5.4.7) gives

$$A_0 = \frac{\alpha_0}{2}, \quad A_n = \frac{\alpha_n}{a^n}, \quad B_n = \frac{\beta_n}{a^n}.$$

Thus,

$$u(r,\theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\Big(\alpha_n\cos(n\theta) + \beta_n\sin(n\theta)\Big). \tag{5.4.8}$$

We shall find conditions guaranteeing the convergence of this series.

Consider the function

$$u_n = t^n\Big(\alpha_n\cos(n\theta) + \beta_n\sin(n\theta)\Big), \quad t = \frac{r}{a} \le 1.$$

We have

$$\frac{\partial^k u_n}{\partial\theta^k} = t^n\,n^k\,\left[\alpha_n\cos\left(n\theta + k\frac{\pi}{2}\right) + \beta_n\sin\left(n\theta + k\frac{\pi}{2}\right)\right].$$

Denoting by M the maximum of $|\alpha_n|$, $|\beta_n|$, $n = 0, 1, \ldots$,

$$|\alpha_n| < M, \quad |\beta_n| < M \tag{5.4.9}$$

we have

$$\left|\frac{\partial^k u_n}{\partial\theta^k}\right| \le t^n n^k 2M.$$

For a $t_0 \le \dfrac{r_0}{a} < 1$,

$$\sum_{n=1}^{\infty} t^n n^k(|\alpha_n| + |\beta_n|) \le 2M\sum_{n=1}^{\infty} t_0^n n^k \quad (t \le t_0).$$

It follows that the series on the right-hand side is uniformly convergent for $t \le t_0 < 1$. Hence, $u(r,\varphi)$ for $r \le r_0 < a$ is infinitely differentiable w.r.t. θ. Analogously, it is also infinitely differentiable w.r.t. $r \le r_0 < a$. As $r_0 < a$ is arbitrary the function u defined by (5.4.8) satisfies (5.4.1) and for $r < a$ it is infinitely differentiable w.r.t. r and φ.

Up to this point, we have used only the conditions in (5.4.9). This conditions are satisfied, if $f(\varphi)$ is bounded by $M/2$.

In order to guarantee the continuity of the solution up to boundary (up to $r = a$), we need to suppose that f is continuous and differentiable.

Putting the expressions of the Fourier coefficients α_n, β_n into (5.4.8), we get

$$\begin{aligned} u(r,\theta) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\Big(\cos(n\psi)\cos(n\theta) + \sin(n\psi)\sin(n\theta)\Big)\right\}d\psi \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\theta-\psi)\right\}d\psi. \end{aligned} \tag{5.4.10}$$

On the other hand,

$$\begin{aligned}
\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos[n(\theta-\psi)] &= \frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty}t^n\left[e^{in(\theta-\psi)}+e^{-in(\theta-\psi)}\right]\\
&= \frac{1}{2}\left\{1+\sum_{n=1}^{\infty}\left[\left(te^{i(\theta-\psi)}\right)^n+\left(te^{-i(\theta-\psi)}\right)^n\right]\right\}\\
&= \frac{1}{2}\left[1+\frac{te^{i(\theta-\psi)}}{1-te^{i(\theta-\psi)}}+\frac{te^{-i(\theta-\psi)}}{1-te^{-i(\theta-\psi)}}\right]\\
&= \frac{1}{2}\,\frac{1-t^2}{1-2t\cos(\theta-\psi)+t^2}\quad\left(t=\frac{r}{a}<1\right).
\end{aligned}$$

Putting the last equality into (5.4.10), we get

$$u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\psi)\frac{1-\frac{r^2}{a^2}}{\frac{r^2}{a^2}-2\frac{r}{a}\cos(\theta-\psi)+1}\,d\psi.$$

Thus,

$$u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\psi)\frac{a^2-r^2}{r^2-2ar\cos(\theta-\psi)+a^2}\,d\psi. \tag{5.4.11}$$

This formula is called *Poisson's formula.*

The function

$$K(\rho,\theta,a,\psi)=\frac{a^2-r^2}{r^2-2ar\cos(\theta-\psi)+a^2}$$

is called the *Poisson kernel.* For $r<a$ it is positive, since $2ar<a^2+r^2$ for $r\neq a$.

We now prove that Poisson's formula (5.4.11) (or (5.4.10)) gives also a continuous solution to (5.4.1), (5.4.2), if f is continuous.

If f is continuous, then f is bounded, and so we have already proved that for $r<a$, the equation $\Delta u=0$ is satisfied. It remains to prove that $u(r,\theta)$ is continuous up to the boundary. Let $f_1(\theta)$, $f_2(\theta),\ldots$ be a sequence of continuous and differentiable functions, which uniformly converge to $f(\varphi)$.

For any $f_k(\theta)$ we can determine the corresponding $u_k(r,\theta)$ by the Poisson formula. Since $\{f_k(\theta)\}$ uniformly converges to $f(\theta)$, for any $\epsilon>0$ there exists a $k_0(\epsilon)>0$ such that

$$|f_k(\theta)-f_{k+l}(\theta)|<\epsilon,\ \ \forall k>k_0(\epsilon),\ l>0.$$

Thus, from the maximum principle,

$$|u_k(r,\theta)-u_{k+l}(r,\theta)|<\epsilon\quad\text{for}\quad r\le r_0<a,\ \ k>k_0(\epsilon),\ l>0,\ \epsilon>0.$$

Hence, $\{u_k\}$ uniformly converges to a function u, $u=\lim\limits_{k\to\infty}u_k(r,\theta)$. The function u is continuous in a closed domain, since all functions

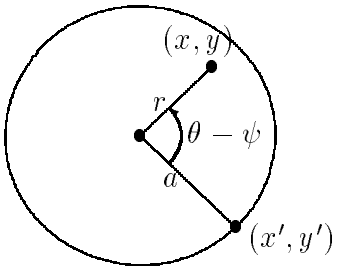
$$u_k(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{a^2-r^2}{r^2-2ar\cos(\theta-\psi)+a^2}f_k(\psi)\,d\psi$$

are continuous in closed domains. Thus,

$$u(r,\theta) = \lim_{k\to\infty} u_k(r,\theta) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \frac{a^2 - r^2}{r^2 - 2ar\cos(\theta-\psi) + a^2} f(\psi)\, d\psi, & a < r, \\ f(\theta) & , \ a = r \end{cases},$$

since $\{f_k(\theta)\}$ uniformly converges to $f(\theta)$.

Let $X = (x, y)$ have the polar coordinates (r, θ) and $X' = (x', y')$ have the the polar coordinates (a, ψ) (that means that (x', y') is lying at the boundary).



Then

$$|\vec{X} - \vec{X}'|^2 = a^2 + r^2 - 2ar\cos(\theta - \psi).$$

The arc length element on the circumference is $ds' = a\, d\psi$. Therefore, Poisson's formula takes the alternative form

$$u(X) = \frac{a^2 - |\vec{X}|^2}{2\pi a} \int\limits_{|X'|=a} \frac{u(X')}{|X - X'|^2}\, ds' \tag{5.4.12}$$

for $X \in \Omega$, where we write $u(X') = f(\theta)$.

5.5 The mean value theorem

Let u be a harmonic function in a domain $\Omega \subset \mathbb{R}^2$, continuous in $\overline{\Omega}$. Let M_0 be a point of Ω and B_a be the ball of radius a with the center at M_0 which is completely lying in Ω. Then

$$u(M_0) = \frac{1}{2\pi a} \iint\limits_{\partial B_a} u(\theta)\, d\theta. \tag{5.5.1}$$

Proof: Without loss of generality we may suppose that $M_0 = 0$. Since $B_a \subset \Omega$, the function u is harmonic in Ω, from Poisson's formula (5.4.11) we have ($X = 0$)

$$u(0) = \frac{a^2}{2\pi a} \int\limits_{|X'|=a} \frac{u(X')}{a^2}\, ds'.$$

□

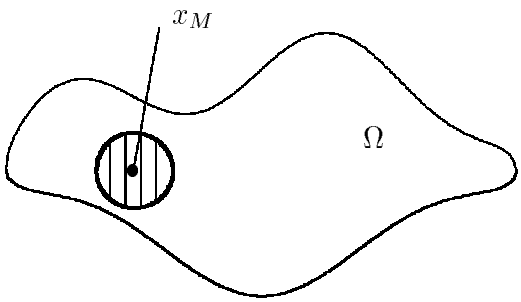
5.6 The maximum principle (a strong version)

We now prove the following: *Let Ω be a connected bounded domain in $\mathbb{R}^2$. Let u be a harmonic function in Ω, continuous in $\overline{\Omega}$. Then the maximum and the minimum values of u are attained on $\partial\Omega$ and nowhere inside unless $u \equiv$ const.*

Since u is continuous in $\overline{\Omega}$, it attains its maximum somewhere, say $x_M \in \overline{\Omega}$. We shall show that $x_M \notin \Omega$ unless $u \equiv$ const. By definition of M, we know that

$$u(x) \leq u(x_M) = M, \quad \forall x \in \Omega.$$

We draw a circle around x_M entirely contained in Ω. By the mean value theorem, $u(x_M)$ is equal to its average around the circumference.



Since the average is not greater than the maximum, we have

$$M = u(x_M) \leq M.$$

Therefore, $u(x) = M$ for all x on the circumference. This is true for any such circle. So $u(x) = M$ for all x in the diagonally shaded region. Now we repeat the argument with a different center. We can fill the whole domain up with circles. In this way, using the assumption that Ω is connected, we deduce that $u(x) \equiv M$ throughout Ω. So $u \equiv$ const.

Bibliography

[1] R. A. Adams: Sobolev Spaces, Academic Press, New York, 1975.

[2] S. Agmon: Lectures on Elliptic Boundary Value Problems. Van Nostrand, 1965.

[3] R. Courant and D. Hilbert: Methods of Mathematical Physics. Interscience Publishers, Vol. I, 1953, Vol. 2, 1962.

[4] A. Friedman: Partial Differential Equations of Parabolic Type, Prentice Hall, 1964.

[5] A. Friedman: Partial Differential Equations, Holt, Rinehart and Wiston, 1969.

[6] P. R. Garabedian: Partial Differential Equations, John Wiley & Sons, Inc., 1964.

[7] I. M. Gelfand and G. E. Shilov: Generalized Functions, Academic Press, New York, 1964.

[8] D. Gilbarg and N. . Trudinger: Elliptic Partial Differential Equations of Second Order. Springer-Verlag, Berlin, 1977.

[9] L. Hörmander: Linear Partial Differential Operators, Springer-Verlag, Berlin, 1963.

[10] F. John: Partial Differential Equations. Springer-Verlag, Berlin, 1982.

[11] M. Krzyzanski: Partial Differential Equations of Second Order, Vol. 1&2, PWN, Warszawa, 1971.

[12] O. A. Ladyzhenskaya: The Boundary Value Problems of Mathematical Physics. Springer-Verlag, New York-Berlin-Heidelberg-Tokyo 1985.

[13] O. A. Ladyženskaja, V. A. Solonnikov, N. N. Ural'ceva: Linear and Quasilinear Equations of Parabolic Type. Amer. Math. Soc. 1968.

[14] O. A. Ladyzhenskaya *et. al.*: Linear and Quasilinear Equations of Elliptic Type. Amer. Math. Soc..

[15] J.–L. Lions and E. Magenes: Non-Homogeneous Boundary Value Problems and Applications. Vol. I–III, Springer-Verlag, Berlin, 1972.

[16] J. T. Marti: Introduction to Sobolev Spaces and Finite Element Solution of Elliptic Boundary Value Problems. Academic Press, New York, 1986.

[17] I. G. Petrovskii: Lectures on Partial Differential Equations. Interscience Publishers, 1954.

[18] M. H. Protter and H. F. Weinberger: Maximum Principles in Differential Equations, Springer-Verlag, Berlin, 1984.

[19] S. L. Sobolev: Partial Differential Equations of Mathematical Physics, Pergamon, New York, 1964.

[20] W. A. Strauss: Partial Differential Equations. An Introduction. John Wiley, New York 1992.

[21] F. Treves: Basic Linear Partial Differential Equations, Academic Press, 1975.

[22] A. N. Tykhonov, A. A. Samarski: Differentialgleichungen der Mathematischen Physik. VEB, Berlin, 1959.

[23] V. S. Vladimirov: Equations of Mathematical Physics, Marcel Dekker, Inc., New York, 1971.

[24] V. S. Vladimirov: Generalized Functions in Mathematical Physics, Mir Publishers, Moscow, 1979.

[25] D. Widder: The Heat Equation, Academic Press, New York ,1975.

[26] J. Wloka: Partial Differential Equations. Cambridge Univ. Press, 1987.

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