Several Complex Variables
Local Theory

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Chapter I. Analytic Hypersurfaces

We investigate the local behaviour of the set of zeros of a single convergent power series. One of our main results will be that the singular locus of such a hypersurface is an analytic subset which is nowhere dense. To prove this, one needs some insight in the ring $C\{z_1, \ldots, z_n\}$ of konvergent power series, which is based on the fundamental Weierstrass preperation and division theorem.

1. Elementary properties of analytic functions

We assume that the reader is acquainted with the notion of an analytic function of one complex variable.

1.1 Definition. *A function $f : D \longrightarrow \mathbb{C}$ on an open subset $D \subset \mathbb{C}^n$ is analytic (=holomorphic), if it is continuous and if it is analytic in each of its n variables.* defan

Remark. By a non-trivial result of Hartogs the assumption of the continuity of f can be omitted in 1.1.

Sum and product of two analytic functions are analytic, constant functions are analytic and $1/f$ is analytic if f is an analytic function without zeros.

Notation.

$$\mathcal{O}(D) = \{\, f : D \longrightarrow \mathbb{C}; \quad f \text{ analytic} \,\}.$$

This is an algebra over the field of complex numbers.

We will prove that analytic functions can be expanded locally into power series.

For the convenience of the reader we recall some basic facts about absolutely convergent series, where the index set is an arbitrary countable set.

Let S be a countable set and

$$a : S \longrightarrow \mathbb{C}, \quad s \longmapsto a_s,$$

a family of complex numbers, parametrized by S. This familiy is called *summable*, if there exists a positive number C such that for every *finite* subset $T \subset S$

$$\sum_{s \in T} |a| < C.$$

If the family is summable one can define the value

$$A = \sum_{s \in S} a_s$$

One chooses an ordering of the elements of S,

$$S = \{s_1, s_2, \ldots\}.$$

The summability implies that the series

$$a_{s_1} + a_{s_2} + \ldots$$

is absolutely convergent. A standard result from elementary calculus says that this value does not depend on the choice of the ordering. Therefore one can define

$$\sum_{s \in S} a_s := a_{s_1} + a_{s_2} + \ldots$$

Instead of

$$(a_s)_{s \in S} \text{ is summable}$$

one says

$$\text{The series } \sum_{s \in S} a_s \text{ is absolutely convergent.}$$

Let

$$f_s : X \longrightarrow \mathbb{C}$$

be a family of functions on a topological space parametrized by S By definition the series

$$\sum_{s \in S} f_s$$

converges *normally* if for every point $a \in X$ there exist a neighbourhood U and a summable family of numbers (m_s) , such that

$$|f_s(x)| \leq |m_s| \text{ for } x \in U$$

Normal convergence implies absolute and locally uniform convergence (for every ordering of the index set).

Let $r = (r_1, \ldots, r_n)$ by a vector of positive real numbers and $a \in \mathbb{C}^n$ a point. The polydisc with center a and multiradius r is

$$\begin{aligned} U_r(a) &= \left\{ z \in \mathbb{C}^n; \quad |z_j - a_j| < r_j \text{ for } j = 1, \ldots, n \right\}, \\ &= U_{r_1}(a_1) \times \ldots \times U_{r_n}(a_n). \end{aligned}$$

We use the notations

$\mathbb{N}$ = set of natural numbers (without 0),
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Occasionally the elements of $\mathbb{N}_0^n$ are called *multiindices.*

For $a \in \mathbb{C}^n$ und $\nu \in \mathbb{N}_0^n$ we define

$$a^\nu := a_1^{\nu_1} \dots a_n^{\nu_n}.$$

1.2 Proposition. *Let $f : U_r(a) \longrightarrow \mathbb{C}$ be an analytic function on a polydisc.* entwik
There exists a uniquely determined family

$$a : \mathbb{N}_0^n \longrightarrow \mathbb{C}, \quad \nu \longmapsto a_\nu,$$

such that the series

$$\sum_{\nu \in \mathbb{N}_0^n} a_\nu (z-a)^\nu$$

converges normally in $U_r(a)$ and represents the function f

Proof. We choose

$$\rho = (\rho_1, \dots, \rho_n),\ 0 < \rho_j < r \quad (1 \le j \le n).$$

It is sufficient to prove 1.2 in $U_\rho(a)$ instead of $U_r(a)$. Applying Cauchy's formula to the last variable we obtain

$$f(z) = \oint\limits_{|\zeta_n - a_n| = \rho_n} \frac{f(z_1, \dots, z_{n-1}, \zeta_n) d\zeta_n}{\zeta_n - z_n}$$

Applying Caucy's formula succesively to the variables $z_{n-1}, \dots, z_1$ we obtain the generalized Cauchy formula

$$f(z) = \oint\limits_{|\zeta_n - a_n| = \rho_n} \cdots \oint\limits_{|\zeta_1 - a_1| = \rho_1} \frac{f(\zeta_1, \dots, \zeta_n) d\zeta_1 \dots d\zeta_n}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)}$$

The proof of 1.2 is now the same as in the case $n = 1$ because one can expand the integrand into a geometric series,

$$\frac{1}{1-a_1} \cdots \frac{1}{1-a_n} = \sum_{\nu \in \mathbb{N}_0^n} a^\nu.$$

One obtains this formula by multiplying n one-variable geometric series term by term (applying Cauchy's product rule).

The uniqueness of the expansion 1.2 is proved as in the case $n = 1$. □

As in the case $n = 1$ the generalized Cauchy formula implies the following stability rules:

1.3 Proposition. *Let $f_1, f_2, f_3, \ldots$ be a sequence of analytic functions on an open subset $D \subset \mathbb{C}^n$, which converges locally uniform. The the limit function f is analytic too. The sequence of partial derivatives* lokstab

$$\partial f_1/\partial z_j, \partial f_2/\partial z_j, \ldots \quad (1 \leq i \leq n)$$

converges locally uniform

$$\partial f_m/\partial z_j \longrightarrow \partial f/\partial z_j \qquad (m \longrightarrow \infty, \quad 1 \leq j \leq n).$$

Additional Remark. *If $\sum_{s\in S} f_s$ is a series of analytic functions which converges normally then the same is true for the series of partial derivatives $\sum\limits_{s\in S} \partial f_s/\partial z_j$.*

From 1.2 and 1.3 follows

1.4 Proposition. *The partial derivatives of analytic functions are analytic too. Partial derivatives with respect to different variables commute.* partc

Let

$$U \subset \mathbb{C}^n, \quad V \subset \mathbb{C}^m$$

be open subsets. A map

$$f : U \longrightarrow V$$

is called analytic if each of the m components is an analytic funtion.

1.5 Remark. *A map* totd

$$f : U \longrightarrow V, \quad U \subset \mathbb{C}^n, \; V \subset \mathbb{C}^m \text{ open}$$

is analytic if and only if it is totally differentiable in the following sense: For every point $a \in U$ there exists a $\mathbb{C}$-linear map

$$A : \mathbb{C}^n \longrightarrow \mathbb{C}^m$$

with the property:

$$f(z) - f(a) = A(z-a) + r(z), \quad \frac{||\, r(z)\,||}{||z-a||} \longrightarrow 0 \text{ for } z \longrightarrow a.$$

Here $||\cdot||$ denotes one of the standard norms on $\mathbb{C}^n$ bzw. $\mathbb{C}^m$, for example $||z|| = \max_{1\leq j\leq n} |z_j|$.

Additional Remark. *The linear map A is unique, its matrix is the Jacobian*

$$J(f,a) = \begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_m} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial z_1} & \cdots & \frac{\partial f_m}{\partial z_n} \end{pmatrix}_{z=a}$$

Proof. It is easy to see that a totally differentiable function is totally differentiable in each of its veriable, hence analytic in each of its variable and hence analytic by our definition 1.1. The converse statement follows from the fact that analytic functions admit expansions into power series (1.2). □

A $\mathbb{C}$-linear map is also $\mathbb{R}$-linear. This means that a function which is complex totally differentiable is also totally differentiable in the sense of real analysis. From the chain rule of the real analysis and from the fact that the composition of $\mathbb{C}$-linear maps is $\mathbb{C}$-linear. we obtain the following chain rule.

1.6 Remark. *Let* kett

$$U \xrightarrow{f} V \xrightarrow{g} W, \qquad U \subset \mathbb{C}^n, \quad V \subset \mathbb{C}^m, \quad W \subset \mathbb{C}^p \text{ open},$$

be analytic mappings The composition

$$g \circ f \ : U \to W$$

is analytic too and we have

$$J(g \circ f, a) = J(g, f(a)) \cdot J(f, a).$$

A similar argument gives the theorem of invertible functions:

1.7 Remark. *Let* impli

$$f : U \longrightarrow V, \quad U, V \subset \mathbb{C}^n \text{ open},$$

be an analytic map and $a \in U$ a point such the the Jacobian $J(f,a)$ is invertible. Then there exist open neighbourhoods

$$a \in U_0 \subset U, \quad f(a) \in V_0 \subset V$$

such that the restriction of f defines a bijective map

$$f_0 : U_0 \longrightarrow V_0$$

and such that the inverse map

$$f_0^{-1} : V_0 \longrightarrow U_0$$

is analytic too. One has

$$J(f_0, a)^{-1} = J(f_0^{-1}, f(a)).$$

The theorem of inverse functions is connected with the notion of a *smooth subset* of a compex vector space.

Smooth Sets

We start we a basic definiton.

1.8 Definition. *A subset $X \subset \mathbb{C}^n$ is called **smooth** in a point $a \in X$ if there exist open subsets* glat

$$a \in U \subset \mathbb{C}^n \text{ and } 0 \in V \subset \mathbb{C}^n$$

and a biholomorphic map

$$\varphi : U \xrightarrow{\sim} V, \quad \varphi(a) = 0$$

such that

$$\varphi(U \cap A) = V \cap H,$$

where $H \subset \mathbb{C}^n$ is a suitable sub-vectorspace.

It is of course possible to assume $H = \left\{ z \in \mathbb{C}^n; \quad z_{m+1} = \ldots = z_n = 0 \right\}$.

1.9 Remark and Definition. *The dimension of the subspace H in 1.8 is uniquely determined by (X, a). It is called the **(complex) dimension** of X in a,* defdim

$$\dim_a A := \dim_{\mathbb{C}} H.$$

Proof. The following has to be shown: Let $U, V \subset \mathbb{C}^n$ be open neighbourhoods of 0 and

$$\varphi : U \longrightarrow V, \quad \varphi(0) = 0,$$

a biholomorphic mapping Assume

$$\varphi(U \cap H) = V \cap H',$$

where H, H' are sub-vectorspaces of dimension m, m'. The claim is that m and m' agree. Restricting φ we obtain a bijective map

$$\varphi_0 : U \cap H \longrightarrow V \cap H'.$$

This map is biholomorphic if we choose basises and identify H with $\mathbb{C}^m$ and H' with $\mathbb{C}^{m'}$. The chain rule 1.6 implies that the Jacobian $J(\varphi_0, 0)$ is invertible. Only square matrices can be invertible. □

Smooth sets are often defined as zero sets of analytic functions:

1.10 Remark. *Let* glgla

$$\varphi : U \longrightarrow \mathbb{C}^m, \quad U \subset \mathbb{C}^n \text{ open},$$

be a holomorphic map and

$$X = \{ z \in U; \quad \varphi(z) = 0 \}.$$

its zero locus. Let $a \in X$ be a point such that the rank of $J(\varphi, a)$ is m (which is the maximal possible rank). Then X ia smooth at a and

$$\dim_a X = n - m.$$

Proof. Let

$$l : \mathbb{C}^{n-m} \longrightarrow \mathbb{C}^n$$

be a $\mathbb{C}$-linear map. We consider the analytic map

$$U \times \mathbb{C}^{n-m} \xrightarrow{\Phi} \mathbb{C}^n, \quad (z, w) \longmapsto (\varphi(z), l(w)).$$

The Jacobian of Φ in $(a, 0)$ is invertible for a suitable choice of l (because every $m \times n-$matrix of rank m is part of an invertible $n \times n-$matrix. Now the proof follows from the theorem of invertible functions 1.7.

The statement of 1.10 is false if one omits the condition about the rank of the Jacobian. If the rank is smaller then m, the point a can be (but must not be) a singularity. One of the main problems of the local theory is to get some insight in the set of singular points.

2. The Weierstrass preperation theorem

We introduce the ring of convergent power series:

2.1 Definition. *A convergent power series in n variables is a map* kopot

$$a : \mathbb{N}_0^n \longrightarrow \mathbb{C}, \quad \nu \longmapsto a_\nu,$$

such that the series

$$\sum_\nu a_\nu z^\nu$$

converges absolutely in a suitable polydisc $U_r(0) \subset \mathbb{C}^n$.

As in the case $n = 1$ it is quite clear that a convergent power series converges normal in the polydisc $U_r(a)$ and defines an analytic function. Usually one uses the notation

$$\sum a_\nu z^\nu$$

instead of a. The set of all convergent power series

$$\mathcal{O}_n := \mathbb{C}\{z_1, \ldots, z_n\}$$

is a ring:

$$\sum a_\nu z^\nu + \sum b_\nu z^\nu = \sum (a_\nu + b_\nu) z^\nu,$$
$$\Big(\sum a_\nu z^\nu\Big)\Big(\sum b_\nu z^\nu\Big) = \sum_{\nu+\mu=n} \Big(\sum a_\nu + b_\mu\Big) z^n.$$

Let $0 \in U \subset \mathbb{C}^n$ be an open neighbourhood. We obtain a natural map

$$\mathcal{O}(U) \longrightarrow \mathbb{C}\{z_1, \ldots, z_n\}.$$

taking the power series of an arbitrary $f \in \mathcal{O}(U)$ in the origin. It is quite clear that this is a homomorphism.

We can consider the polyniomal ring $\mathbb{C}[z_1, \ldots, z_n]$ in n variables over $\mathbb{C}$ as a subring of $\mathbb{C}\{z_1, \ldots, z_n\}$. It consists of all power series with the property that only finitely many coefficients a_ν do not vanish. The degree–n–part of a power series $P = \sum a_\nu z^\nu$ is the homogenous polynomial of degree k

$$P_k := \sum_{\nu_1 + \cdots + \nu_n = k} a_\nu z^\nu.$$

In any polydisc around 0, where P converges absolutely, we have the identity of functions

$$P(z) = \sum_{k=0}^{\infty} P_k(z).$$

This justifies the notation $P = \sum_k P_k$.

2.2 Definition. *The order* $\operatorname{ord}(P)$ *of a non zero power series* $P \in \mathcal{O}_n$ *is the smallest* k *such that the degree–*k*–part* P_k *does not vanish.* orD

If P_k is the lowest homogenous part of P and Q_l the lowest homegenous part of another power series Q then $P_k Q_l$ is the lowest homegenous part of PQ. Using the well-known fact that the polynomial ring is an integral domain we obtain:

2.3 Remark. *The ring of convergent power series* $\mathcal{O}_n$ *is an integral domain. If* P, Q *are two non-zero elements of* $\mathcal{O}_n$*, then* inT

$$\operatorname{ord}(PQ) = \operatorname{ord}(P) + \operatorname{ord}(Q).$$

Let $0 \in U \subset \mathbb{C}^n$ be a *connected* neighbourhoud of the origin. As in the variable case one can show that the the homomorphism $U \to \mathcal{O}_n$ is injective.

2.4 Remark. *Let $U \subset \mathbb{C}^n$ be an open connected subset. The ring of analytic functions on U is an integral domain.* UinT

We mentioned already that polynomials are special power series and constants can be considered as special polynomials. Hence we have natural inclusions

$$\mathbb{C} \subset \mathbb{C}[z_1, \ldots, z_n] \subset \mathbb{C}\{z_1, \ldots, z_n\}.$$

There is another interesting inclusion. We consider the subset of $\mathcal{O}_n$ consisting of all power series which do not depend on z_n. It is quite clear that this subset is a subring which is naturally isomorphic with $\mathcal{O}_{n-1}$. This means that we have a natural injective ring homomorphism $\mathcal{O}_{n-1} \hookrightarrow \mathcal{O}_n$ whose precise definition is left to the reader. In the following we will identify an element of $\mathcal{O}_{n-1}$ with its image in $\mathcal{O}_n$ as long as no amiguity arises. This means that we consider $\mathcal{O}_{n-1}$ as a subring of $\mathcal{O}_n$. We introduce another important subring of $\mathcal{O}_n$. It consists of all elements which can be written as polynomials in z_n with coefficients from $\mathcal{O}_{n-1}$,

$$P = P_d z_n^d + P_{d-1} z_n^{d-1} + \cdots + P_0, \qquad P_j \in \mathbb{C}\{z_1, \ldots, z_{n-1}\} \quad (0 \leq j \leq d).$$

(Here P_j of course does ot mean the degree–j–part of P.)

In this context we expect that the reader is acquainted with the abstract notion of the polynomial ring $R[X_1, \ldots, X_n]$ in n indeterminates over an arbitrary ring R with unity $1 = 1_R$. Our ring $\mathcal{O}_n[z_n]$ can be identified with the ring of polynomials over $\mathcal{O}_{n-1}$ in one indeterminate and $\mathbb{C}[z_1, \ldots, z_n]$ can be identified with the ring of polynomials over $\mathbb{C}$ in n indeterminates.

We will not use deep commutative algebra in this text, but we will use some basic facts which can be found in most text books about algebra. Let's formulate some of those basic facts.

Let R denote an arbitrary commutative ring with unit element. An element $a \in R$ is called a *unit*, if the equation $ax = 1$ is solvable in r or which means the same if $Ra = R$. The set of units forms a group

$$R^* := \{\, a \in R;\ a \text{ unit} \,\}$$

under multiplication. A ring is called *local*, if the set $\mathfrak{m} = R - R^*$ of non-units is an additive group. Then it is automatically an ideal. This ideal contains every ideal of R whch is different from R, because such an ideal never contains a unit. Hence $\mathfrak{m}$ is a maximal ideal (the unique maximal ideal) and $R/\mathfrak{m}$ is a field, which is called the residue field of R.

Examples.

1) Every nonzero element of a field K is a unit. Hence a field is a local ring, the maximal ideal is $\mathfrak{m} = \{0\}$. The residue field is K itsself.

2) Let R be an intgral domain. Every unit of the polynomial ring $R[X_1, \ldots, X_n]$ is constant, i.e. contained in R,

$$R[X_1, \ldots, X_n]^* = R^*.$$

Because the sum of two non-constant polynomials can be constant, polynomial rings give examples of rings, which are not local.

3) The ring $\mathbb{Z}$ of integers is not local. The units are ± 1.

The situation for rings of power series is quite different:

2.5 Remark. *A power series $P = \sum a_\nu z^\nu \in \mathcal{O}_n$ is a unit in $\mathcal{O}_n$ if and only if the constant coefficient* poloc

$$P(0) := a_0$$

is different from zero. As a consequence, $\mathcal{O}_n$ is a local ring with maximal ideal

$$\mathfrak{m}_n := \{\, P \in \mathcal{O}_n; \quad P(0) \neq 0 \,\}.$$

The map $P \mapsto P(0)$ defines an isomorphism of the residue field to $\mathbb{C}$.

Weierstrass found an division algorithm in the ring of power series analogous to the Euclidean algorithm in a polynnomial ring. We recall this Euclidean algorithm.

The Euclidean algorithm for polynomials

let R be an integral domain and

a) $P \in R[X]$ an arbitrary polynomial,
b) $Q \in R[X]$ a normalized polynomial, i.e. the highest coefficient is 1.

Then there exist a unique decomposition

$$P = AQ + B.$$

where $A, B \in R[X]$ are polynomials and

$$\deg(B) < d.$$

This includes the case $B = 0$, if one defines $\deg(0) = -\infty$. The proof of this result is trivial (induction on the degree of P).

We have to introduce the fundamental notion of a Weierstrass polynomial. This notion can be defined for every local ring R. Let P be a normalized polynomial of degree d over R. This polynomial is called a Weierstrass polynomial if its image in $R/\mathfrak{m}[X]$ is X^d. For the ring of our interest $R = \mathcal{O}_n$ this means

2.6 Definition. *A polynomial* weiP

$$P \in \mathcal{O}_{n-1}[z] = \mathbb{C}\{z_1, \ldots, z_{n-1}\}[z_n]$$

is called a Weierstrass polynomial of degree d, if it is of the form

$$P = z_n^d + P_{d-1} z_n^{d-1} + \ldots + P_0$$

where all the coefficients besides the highest one (which is 1) vanish in the origin,

$$P_0(0) = \ldots = P_{d-1}(0) = 0.$$

Weierstrass proved two fundamental theorems, the division theorem and the preperation theorem which usually play together.

2.7 Weierstrass division theorem. *Let $Q \in \mathcal{O}_{n-1}[z]$ be a Weierstrass polynomial of degree d. Every power series $P \in \mathcal{O}_n$ admits a unique decomposition* weidiv

$$P = AQ + B \quad \text{where} \quad A \in \mathcal{O}_n, \quad B \in \mathcal{O}_{n-1}[z_n], \quad \deg_{z_n}(B) < d.$$

Here $\deg_{z_n}(B)$ means the degree of the polynomial B over the ring $\mathcal{O}_{n-1}$ (again taking $-\infty$ if $B = 0$).

We will prove this theorem a little later. The division theorem resembles the Euclidean algorithm. But there is a difference. In the Euclidean algorithm we divide through arbitrary normalized polynomials. In the division theorem we are restriced to divide through Weierstrass polynomials and this notion seems to be very restricted. But here the preperation theorem enters. It says that the class of Weierstrass polynomials is of high generality. To formulate the preperation theorem we need the notion of a *z_n-general* power series.

2.8 Definition. *A power series* znallg

$$P = \sum a_\nu z^\nu \in \mathbb{C}\{z_1, \ldots, z_n\}$$

is called ***z_n-general****, if the power series $P(0, \ldots, 0, z_n)$ does not vanish. It is called z_n-general of order d if*

$$P(0, \ldots, 0, z_n) = b_d z_n^d + b_{d+1} z_n^{d+1} + \ldots \text{ where } b_d \neq 0.$$

A Weierstrass polynomial is of course z_n-general and its degree and order agree. If P is a z_n-general power series and U is a unit in $\mathcal{O}_n$ then PU is also z_n-general of the same order.

2.9 Weierstrass preperation theorem. *Let $P \in \mathcal{O}_n$ be a z_n-general power series. There exists a unique decomposition* vorS

$$P = UQ,$$

where U is a unit in $\mathcal{O}_n$ and Q a Weierstrass polynomial.

We come to the proof of the two Weierstrass theorems:

Proof of the division theorem, I. existence. We choose a positive number $r > 0$ such that Q converges in the polydisc

$$|z_j| < 2r, \quad j = 1, \ldots, n.$$

We have

$$Q(0, \ldots, 0, z_n) = z_n^d \neq 0 \text{ for } |z_n| = r.$$

A continuity argument shows the existence of a small positive number $\varepsilon < r$ such that

$$Q(z_1, \ldots, z_n) \neq 0 \quad \text{for} \quad |z_n| = r \quad \text{and} \quad |z_j| < \varepsilon,\ 1 \leq j \leq n-1.$$

We define analytic functions A, B on the polydisc $|z_j| < \varepsilon$, $1 \leq j \leq n-1$, $|z_n| < r$ by

$$A(z_1, \ldots, z_n) = \frac{1}{2\pi i} \oint\limits_{|\zeta|=r} \frac{P(z_1, \ldots, z_{n-1}, \zeta)}{Q(z_1, \ldots, z_{n-1}, \zeta)} \frac{d\zeta}{\zeta - z_n}$$

and

$$B = P - AQ.$$

By means of Cauchy's formula for P as function of z_n we obatin for B the representation

$$B(z_1, \ldots, z_n) =$$
$$\frac{1}{2\pi i} \oint\limits_{|\zeta|=r} \frac{P(z_1, \ldots, z_{n-1}, \zeta)}{Q(z_1, \ldots, z_{n-1}, \zeta)} \left[\frac{Q(z_1, \ldots, z_{n-1}, \zeta) - Q(z_1, \ldots, z_n)}{\zeta - z_n} \right] d\zeta$$

The variable z_n only occurs inside the big brackets. The dominator is a polynomial in z_n (for fixed ζ) over the ring $\mathcal{O}_{n-1}$. This polynomial vanishes for $z_n = \zeta$. This implies that the quotient is a polynomial of degree $< d$. Replacing A, B by their power series expansion in 0 we see that $P = AQ + B$ defines a decomposition as claimed in the division theorem.

II. Uniqueness. We have to show

$$AQ + B = 0 \Longrightarrow B = 0.$$

Let $(a_1, \ldots, a_{n-1})$ be a point which lies in a sufficently small neighbourhoud of the origin. Then the d zeros (counted with multiplicity)

$$z_n \longmapsto Q(a_1, \ldots, a_{n-1}, z_n)$$

are arbitrarily close to 0. This is an easy consequence of the fact that zhe polynomial $Q(a_1, \ldots, a_{n-1}, z_n)$ is normalized and close to z_n^d. (This so-called continuity principle for the zeros of noemalized polynomials is easy to prove and by the way a very special case of the deeper 2.10.) We obtain that the polynomial

$$z_n \longmapsto B(a_1, \ldots, a_{n-1}, z_n)$$

has eat least d zeros. But its degree is smaller then d which implies $B = 0$. This completes the proof of the division theorem □

Proof of the preperation theorem. We need the following principle of "continuity of zeros"

2.10 Lemma. *Let* conZ

$$f : U_r(0) \longrightarrow \mathbb{C} \quad \big(r = (r_1, \ldots, r_n)\big),$$

be a holomorphic function on a polydisc around 0. *We assume that the power series of f at the origin is z_n-general of order d. Then there exixts a polydisc*

$$U_\rho(0) \subset U_r(0),$$

such that the function

$$g(z) = f(a_1, \ldots, a_{n-1}, z)$$

has precisely d zeros (counted with multiplicity) in the circle $|z_n| < \rho_n$ if

$$|a_i| < \rho_i \quad \text{for} \quad 0 \leq i < n.$$

Proof. By assumption the function

$$z_n \longmapsto f(0, \ldots, 0, z_n)$$

is not identically zero. The zeros of this functions are isolated. Therefore we can find a number $0 < \rho_n < r$ such that

$$f(0, \ldots, 0, z_n) \neq 0 \quad \text{for} \quad |z_n| = \rho_n.$$

A continuity argument shows the existence of numbers $\rho_1, \ldots, \rho_{n-1}$, such that

$$f(z_1, \ldots, z_{n-1}, z_n) \neq 0 \quad \text{for} \quad |z_n| = \rho_n \quad \text{and} \quad |z_j| < \rho_j \ (1 \leq j \leq n-1).$$

A well-known criterion of the theory of analytic functions in one complex variable says that the integral

$$\sigma_0(a) = \frac{1}{2\pi i} \oint\limits_{|\zeta|=\rho_n} \frac{\partial f(a,\zeta)/\partial\zeta}{f(a,\zeta)} d\zeta$$

gives the number of zeros in $|z_n| < \rho_n$. But the function σ_0 is continuous and hence constant,

$$\sigma_0(a) = \sigma_0(0) = d. \qquad \square$$

Proof of the preperation theorem, continued. The idea is to contruct a Weierstrass polynomial q with the same zeros as P in a small neighbourhood of 0. For this purpose we choose numbers with the following property (use 2.10).

1) P converges for $|z_j| < 2r$, $1 \le j \le n$.
2) $P(z) \neq 0$ for $|z_n| = r$, $|z_j| < \varepsilon$ $(1 \le j \le n-1)$.
3) For fixed $(z_1, \ldots, z_{n-1})$, $|z_j| < \varepsilon$ $(1 \le j \le n-1)$ the function $z_n \mapsto P(z)$ has precisely d zeros inside the circle $|z_n| < r$.

We denote the zeros by

$$t_1(z), \ldots, t_d(z) \ (z = (z_1, \ldots, z_{n-1})).$$

The ordering is artificial, hence we cannot expect the the functions t_j are continuous. But we state

Claim. Let $\varphi \in \mathbb{C}[X_1, \ldots, X_d]$ be any symmetric polynomial (which does not change if one permutes the variables) then

$$\varphi(t_1(z), \ldots, t_d(z))$$

is holomprhic in z.

Proof of claim. It is well-known from elementary algebra that every symmetric polynomial can be written as polynomial in the special symmetric polynomials

$$\sum_{j=1}^{n} X_j^k, \quad 1 \le k \le d.$$

Therefore we only have to prove that

$$\sigma_k(z) = \sum_{j=1}^{d} t_j(z)^k$$

are analytic. But this follows from the well-known formula

$$\sigma_k(z) = \frac{1}{2\pi i} \oint\limits_{|\zeta|=\varrho_n} \frac{\zeta^k \partial Q(z,\zeta)/\partial\zeta}{\partial\zeta} d\zeta.$$

(The case $k = 0$ already occured during the proof of 2.10.) This completes the proof of the claim. We apply the claim to the elementary symmetric polynomials and obtain that

$$Q(z, z_n) = \prod_{j=1}^{d} (z_n - t_j(z)) \quad (z = (z_1, \ldots, z_{n-1}))$$

defines a holomorphic function in a neighbourhood of the origin. We denote its power series around 0 with the same letter. From $t_1(0) = \ldots = t_d(0) = 0$ follows that Q is a Weierstrass polynomial. We want to prove that P is the product of Q and a unit U. For this purpose we apply the division theorem and obtain

$$P = AQ + B, \quad B \in \mathcal{O}_{n-1}[z_n], \quad \deg_{z_n}(B) < d.$$

It is quite clear that B vanishes because all zeros of Q in a small neighbourhood of the origin are zeros of B too. A similar argument shows that A has no zeros in a full neighbourhood of the origin. This means that the power series of A is a unit. The proof of existence part of the preperation theorem is complete.

The proof of the uniqueness of the decomposition is easy: Assume that Q_1, Q_2 are Weierstrass polynomials which only differ by a unit. Then they must have the same degree d and the difference $Q_1 - Q_2$ has degree less than d. But $Q_1 - Q_2$ is zero if Q_1 is 0. This gives $Q_1 - Q_2 = 0$. □

Let $P = \sum a_\nu z^\nu \in \mathbb{C}\{z_1, \ldots, z_n\}$ be a power series and A a complex $n \times n$-matrix. Considering P as a holomorphic function in a small neighbourhood of the origin we can define the power series $P(Az)$. The following remark shows that after a suitable change of coordinates every power series $P \neq 0$ can be made z_n-general.

2.11 Remark. *Let $P \in \mathbb{C}\{z_1, \ldots, z_n\}$ be a power series which is different from 0. The set of all matrices $A \in \mathrm{GL}(n, \mathbb{C})$ such that the power series* offdi

$$Q(z) = P(Az)$$

is z_n-general, is open and dense in $\mathrm{GL}(n, \mathbb{C})$.

The proof is simple and can be omitted. □

Finally we mention that for $A \in GL(n, \mathbb{Z})$ the map

$$P(z) \longmapsto Q(z) = P(Az)$$

defines an automorphism of the ring $\mathbb{C}\{z_1, \ldots, z_n\}$.

This shows that many general results for power series only have to be proved for z_n-general power series and —by the preperation theorem— then only for Weierstrass polynomials. This principle gives a link between $\mathcal{O}_{n-1}[z_n]$ and $\mathcal{O}_n$ and opens a possibility for inductions on n.

Appendix: Power series and germs of analytic functions

We usually consider power series around the origin. But there is no need for it. One can also consider the ring $\mathbb{C}\{z_1 - a_1, \ldots, z_n - a_n\}$ for an arbitrary point a. The notion of "germ of an analytic function" is natural way to look at this ring:

Let $D \subset \mathbb{C}^n$ be an open set and $a \in D$ a distinguished point. We consider pairs (U, f), where $a \in U \subset D$ is an open neighbourhood of a and $f : U \to \mathbb{C}$ is an analytic function. Two pairs (U, f), $(V, g$ are called equivalent,

$$(U, f) \sim (V, g),$$

if there exists an open neighbourhood $a \in W \subset U \cap V$, such that $f|W = g|W$. We denote the equivalence classe of a (U, f) by

$$f_a = [U, f] = \{\, (V, g), \quad (V, g) \sim (U, f) \,\}$$

and call f_a the germ of f in a. The set of all germs in a is denoted by $\mathcal{O}_{D,a}$, This set is eqipped with a structure as $\mathbb{C}$-algebra in an obvious manner. For example the sum of two germs is defined as

$$[U, f] + [V, g] = [U \cap V, f|(U \cap V) + g|(U \cap V)]$$

independently of the choice of the representatives.

Let $f_a \in \mathcal{O}_{D,a}$ be a germ represented by the pair (U, f). Let

$$f(z) = \sum a_\nu (z - a)^\nu.$$

be the power series expansion of f around a. The assignment

$$\mathcal{O}_{D,a} \longrightarrow \mathcal{O}_n, \quad f_a \longmapsto \sum a_\nu z^\nu,$$

is an isomophism of algebras. Usually f_a will be identified with the associated power series and the symbollically notation

$$f_a = \sum_\nu a_\nu (z - a)^\nu.$$

of this power series is used. In this sense we use the suggestive notation

$$\mathcal{O}_{D,a} = \mathbb{C}\{z_1 - a_1, \ldots, z_n - a_n\}.$$

3. First applications of the Weierstrass theorems

We want to compare the rings $\mathcal{O}_{n-1}[z_n]$ and $\mathcal{O}_n$. For this we need

3.1 Lemma. *Let* auPol

$$Q \in \mathbb{C}\{z_1, \ldots, z_{n-1}\}[z_n]$$

be a Weierstrass polynomial and $U \in \mathbb{C}\{z_1, \ldots z_n\}$ *a power series such that* $P = UQ$ *is a polynomial in* z_n*,*

$$P \in \mathbb{C}\{z_1, \ldots, z_{n-1}\}[z_n].$$

Then U *is a polynmial in* z_n *too.*

Proof. From the Euclidean algorithm in the polynomial ring $\mathcal{O}_{n-1}[z_n]$ we obtain

$$P = AQ + B, \qquad A, B \in \mathcal{O}_{n-1}[z_n], \quad \deg_{z_n} B < \deg_{z_n} Q.$$

From the uniqueness statement in the division theorem we obtain $A = U$ und $B = 0$. □

3.2 Proposition. *Let* $Q \in \mathcal{O}_{n-1}[z_n]$ *be a Weierstrass polynomial. The natural homomorphism* ringIs

$$\mathcal{O}_{n-1}[z_n]/(Q) \longrightarrow \mathcal{O}_n/(Q)$$

is a ring isomorphism.

Proof. Injectivity follows from 3.1, surjectivity from the division theorem. □

Our next goal is to prove that the ring of power series is a UFD-ring (ring with unique prime factorization). First we recall some facts about prime decomposition.

An element $a \neq 0$ of an integral domain R is called a *prime element*, if it is not a unit and if

$$a|bc \Longrightarrow a|b \text{ or } a|c.$$

This means that the principal ideal Ra is a prime ideal.

An integral domain R *is called an UFD-domain if every non-unit* $a \neq 0$ *can be written as a product of finitely many prime elements.*

Such a decomposition is uniqe in the following sense: If

$$a = p_1 \cdot \ldots \cdot p_n = q_1 \cdot \ldots \cdot q_m$$

are two decompositions into primes then $m = n$ and there exists q permutation σ of the digits $1 \ldots n$ such that

$$q_i = \varepsilon_i p_{\sigma(i)} \text{ with units } \varepsilon_i.$$

An immediate consequence of 3.2 is

3.3 Lemma. *A Weierstrass polynomial $Q \in \mathcal{O}_{n-1}[z_n]$ is prime in $\mathcal{O}_{n-1}[z_n]$ if and only if it is prime in the bigger ring $\mathcal{O}_n$.* PrM

This is not true for other —even normalized— polynomials. For example $1+z$ is prime in $\mathbb{C}[z]$ but a unit in $\mathbb{C}\{z\}$.

3.4 Remark. *Let $A, B \in \mathcal{O}_{n-1}[z_n]$ be two polynomials whose product is a Weierstrass polynomial. Then A and B are constant multiples of Weierstrass polynomials.* zwW

The proof follows from the fact that a polynomial of degeree n over $\mathcal{O}_{n-1}$ is a Weierstrass polynomial, if and only if $P(0,\ldots,0,z_n) = z_n^d$.

We come to an important application of the Weierstrass theorems:

3.5 Theorem. *The ring of power series $\mathcal{O}_n = \mathbb{C}\{z_1,\ldots,z_n\}$ is a UFD–ring.* zPe

Proof. Induction by n.

begin of induction: The case $n = 0$ is trivial.
induction step. Assume that the theorem is proved for $n-1$ instead of n. We want to write an arbitrary element $P \in \mathcal{O}_n$, $P \neq 0$, $P(0) = 0$, as product of prime elements. It is no restriction to assume that P is z_n-general (2.11) and using the preperation theorem that $P \in \mathcal{O}_{n-1}[z_n]$ is a Weierstrass polynomial. By induction hypothesis $\mathcal{O}_{n-1}$ is UFD. A well-known lemma of Gauss says that the polynomial ring over a UFD-ring is UFD again*). Therefore P can be written as product od prime elements from $\mathcal{O}_{n-1}[z_n]$,

$$P = P_1 \ldots P_m.$$

From 3.3 and 3.4 follows that the elements $P_1,\ldots,P_m$ are prime in $\mathcal{O}_n$. □

3.6 Theorem. *The ring of power series $\mathcal{O}_n = \mathbb{C}\{z_1,\ldots,z_n\}$ is noetherian, i.e. all ideals are finitely generated.* noeT

Proof, Induction by n:

Begin of induction. n = 0. This case is trivial.
Induction step . Assume that the theorem is proved for $n-1$ instead of n Let $\mathfrak{a} \subset \mathcal{O}_n$ be an ideal. We want to show that it is finitely generated and can assume that it is different from zero. We can assume that $\mathfrak{a}$ contains a z_n-general element and then —by the preperation thoerem— that it contains a Weierstrass polynomial. We consider the image $\bar{\mathfrak{a}}$ of $\mathfrak{a}$ under the natural homomorphism

$$\mathcal{O}_n \longrightarrow \mathcal{O}_m/(Q), \quad P \longmapsto \bar{P}.$$

*) More preciselytThe lemma of Gauss states that a non-constant polynomial $P \in R[X]$ over a UFD-ring is a prime element if and only if its coeeficients are coprime and if P is irreducible over the quotient field K of R. By the way the polynomial ring in one variable over a field is a Euclidean ring

It is sufficient to show that $\bar{\mathfrak{a}}$ is finitely generated, because $\mathfrak{a}$ will be generated by P and inverse images of generators of $\mathfrak{a}$. By 3.2 the ring $\mathcal{O}_n/(Q)$ is a homomorphic image of the ring $\mathcal{O}_{n-1}[z_n]$. The ring $\mathcal{O}_{n-1}$ is noetherian by induction hypothesis. The *Hilbert basis theorem* says that the polynmial ring over a noetherian ring is noetherian. The homomorphic image of a noetherian ring is noetherian by trivial reasons. We obtain that $\mathcal{O}_n/(P)$ is noetherian. □

Questions of divisibility in the ring of power series are closely related to questions about the sets of zeros of power series. This was shown already during thr proof of the preperation theorem and will be worked out now in more detail. We need some more notions and results from elementary algebra:

The discriminant

Let R be an integral domain and

$$P(X) = X^d + a_{d-1}X^{d-1} + \dots a_0$$

a normalized polynomial over R. Then the discriminant $d_P \in R$ can be defined as follows: One chooses a field K which contains R as a subring and such that P splits into a product of linear factots over K,

$$P(X) = (X - \alpha_1)\dots(X - \alpha_n) \quad (\alpha_i \in K).$$

Then one defines

$$d_P = \prod_{i<j}(\alpha_i - \alpha_j)^2.$$

It is true that d_P is contained in R and not only in K. There is actually a much stronger result which states:

There exists a universal polynmial (independent of RandP) over the ring of integers

$$\Delta \in \mathbb{Z}[X_1, \dots, X_n],$$

such that

$$d_P = \Delta(a_0, \dots, a_{n-1}).$$

An element $a \in R$ of an integral domain R is called *square free*, if it is different from zero if it is not divisible by a square of a non-zero non-unit. (By this definition units are square free). IF R is a UFD-ring than $a \neq 0$ is square free if and only if the ring $R/(a)$ containes no nilpotent elments different from 0. (An element is called nilpotent if some power is zero). The following result is related to the mentioned lemma of Gauss.

3.7 Proposition. *Let P be a normalized polynomial from the polynomial ring $R[X]$ over an UFD-ring R. The following three statements are equivalent:* copD

1) *P is a square free element from $R[X]$.*
2) *P und P' have besides units from R no common divisor in $R[X]$-*
3) *$d_P \neq 0$.*

We are able to formulate and prove the announced relation between divisibility and sets of zeros:

3.8 Proposition. *Let $P, Q \in \mathcal{O}_n$, $Q \neq 0$, be two power series. We assume that there exists a neighbourhood of the origin in which both series converge and such that every zero of Q in this neighbourhood is also a zero of P. Then there exist a natural number n such that P^n is divisible by Q,* divI

$$P^n = AQ, \quad A \in \mathcal{O}_n.$$

If Q is square free, one can take $n = 1$, i.e. then P is divisible by Q.

Proof. Because of the existence of the prime decomposition we can assume that Q is square free. By our standard procedure we can assume that Q is a Weierstrass polynomial. From the divison theorem we obtain

$$P = AQ + B, \quad B \in \mathcal{O}_{n-1}[z_n], \quad \deg_{z_n} B < d.$$

By assumption we know in a small neigbourhood of the origin

$$Q(z) = 0 \Longrightarrow B(z) = 0.$$

Now we make use of the fact that Q is a square free element of $\mathcal{O}_n$. From 3.2 one can deduce that Q is square free in $\mathcal{O}_{n-1}[z_n]$. From 3.7 we know that the discrimant of Q is different from 0. Now we consider the polynomial

$$Q_a(z) = Q(a_1, \ldots, a_{n-1})(z) \in \mathbb{C}[z]$$

for fixed sufficiently small $a = (a_1, \ldots, a_{n-1})$. The discriminant d_{Q_a} can be obtained from d_Q by spezializing $z_1 = a_1, \ldots, z_{n-1} = a_{n-1}$. This follows for example from the existen of the universal polynomial Δ. Therefore there exists a dense subset M of a small neighbourhood of 0 such that d_{Q_a} is different from 0 for $a \in M$. This means that Q_a is a square free element from $\mathbb{C}[z]$. because $\mathbb{C}$ is algebraically closed this means nothing else that Q_a has no multiple zeros. Hence Q_a has d pairwise distinct zeros (for $a \in M$). As we pointed out several times the d zeros are arbitrarily small if a is sufficiently small. We obtain that $z \mapsto B(a, z)$ has d pairwise distinct zeros if a lies in a dense subset of a sufficiently small neighbourgood of the origin. It follows that B_a vanishes for theese a. By a continuity argument we obtain $B = 0$. □

4. Hypersurfaces

Hypersurfaces are special cases of analytic sets. Analytic sets are locally sets of common zeros if a finite number of holomorphic numbers:

4.1 Definition. *A subset $A \subset \mathbb{C}^n$ is called* ***analytic*** *if for every point $a \in A$ there exists an open neighbourhood $U = U(a)$ and a holomorphic map* anM

$$f : U \longrightarrow \mathbb{C}^m \quad (m = m(a) \text{ suitable})$$

such that

$$A \cap U = \{\, z \in U(a); \quad f(z) = 0 \,\}.$$

Analytic sets are topologically reasonable sets. For example they are locally compact which follows from

4.2 Remark. *For every analytic subset $A \subset \mathbb{C}^n$ there exists an open subset $D \subset \mathbb{C}^n$ such that A is contained in D and closed in D.* lokAb

For D one can take the union of all the $U(a)$ which occur in 4.1.

4.3 Definition. *A subset $A \subset \mathbb{C}^n$ is called a* ***hypersurface****, if every point $a \in A$ admits an open connected neighbourhood and a holomorphic function $f : U \longrightarrow \mathbb{C}$ which does not vanish identically and such that* hypeR

$$A \cap U = \{\, z \in U; \quad f(z) = 0 \,\}.$$

Let $A \subset \mathbb{C}^n$ be an analytic set. A point $a \in A$ is called a *singular point* of A if A is not smooth in a. The *singular locus* S of A is the set of all singular points of A. It is clear that S is a closed subset of A. The complement $X - S$ sometimes is called the regular locus of A. One of the main resuts of this volume will be:

1) *The singular locus is a thin subset of A.*
2) *The singular locus is an analytic set.*

In this section we will prove this fundamental case of a hypersurface. We start with a sufficient criterion for smoothness. A special case of 1.10 states:

4.4 Lemma. *Let* hinR

$$f : D \longrightarrow \mathbb{C}, \quad D \subset \mathbb{C}^n \text{ open},$$

be an analytic function which does not vanish identically on a non-empty open subset and let

$$A = \{\, z \in D; \quad f(z) = 0 \,\}$$

be the hypersurface defined by f. Assume that a is a point of A such that

$$df(a) := (\partial f/\partial z_1(a), \ldots, \partial f/\partial z_n(a)) \neq (0, \ldots, 0).$$

Then a is a smooth point of A. The dimension of A in a is $n - 1$.

The converse of this criterion is false. Take for example $D = \mathbb{C}$ and $f(z) = z^2$. The zero set is one point, hence smooth (of dimension 0) in this point. But $f'(0) = 0$. The point is that $f(z)$ is not reduced. The "correct" function to define the origin as analytic set is to take $f(z) = z$.

4.5 Definition. *A holomorphic function* reD

$$f : D \longrightarrow \mathbb{C} \quad (D \subset \mathbb{C}^n \text{ open})$$

is called ***reduced*** *in a point $a \in D$ if the power series of f in a is a square free element of $\mathbb{C}\{z_1 - a_1, \ldots z_n - a_n\}$.*

If a is a non-zero element of an UFC-domain one can define its "square free part" b. This is a square free element which divides a and such that a divides a suitable power of a. The square free part is determined up to a unit of R. The defnition of b is obvious fron the prime decomposition of a. For example the square free part of $z_1^2 z_2^3$ is $z_1 z_2$. If we want to investigate local properties of a hypersurface A around a given point $a \in A$ we can assume that the defining equation $f(z) = 0$ in a small neighbourhood of a is given by a function f which is reduced at a. This has the big advatage that now the converse of 4.4 is true.

4.6 Lemma. *Let* notW

$$f : D \longrightarrow \mathbb{C}, \quad D \subset \mathbb{C}^n \text{ open},$$

be an analytic function which does not vanish identically on a non-empty open subset and let

$$A = \{\, z \in D; \quad f(z) = 0 \,\}$$

be the hypersurface defined by f. Assume that f is reduced in the point $a \in A$. Then a is a smooth point of A ***if and only if***

$$df(a) := (\partial f/\partial z_1(a), \ldots, \partial f/\partial z_n(a)) \neq (0, \ldots, 0)$$

and the dimension of A in a is $n - 1$.

Corollary. *The dimension of a hypersurface in any smooth point is $n - 1$.*

Poof. Assume that a is a smooth point. Let

$$\varphi : D \xrightarrow{\sim} D', \quad b = \varphi(a),$$

be a biholomorphic map. We can replace A by $\varphi(A)$ and f by $g = f \circ \varphi^{-1}$ without changing the statement. Therefore we can assume that $a = 0$ and that A is a linear space. We can assume that A is contained in the hypersurface "$z_n = 0$ ". From 1.3.8 we can conclude that the power series P of f in the origin divides the power series z_n. The latter is a prime and we obtain

$$P = U \cdot z_n, \quad U \text{ a unit in } \mathcal{O}_n.$$

The statement $(dP)(a) \neq 0$ is an obvious consequence. □

Lemma 4.6 indicates that the singular locus is an analytic set. But there is still a problem. As we pointed any hypersurfece can be defined locally around a given point as zero set of a holomorphic function which is reduced in a. To prove that the singular locus is an analytic set needs more, namlely thet f is reduced in all points of a complete neighbourhood of a. So what we still have to prove is

4.7 Proposition. *Let f be a holomorphic function on an open set $U \subset \mathbb{C}^n$, The set of all points $a \in U$ in which f is reduced is an open set.* redO

For the prove we need the following two remarks:

4.8 Remark. *Let $P \in \mathcal{O}_{n-1}[z_n]$ be a normalized polynomial, which is square free in the ring $\mathcal{O}_{n-1}[z_n]$. Then P is square free in the bigger ring $\mathcal{O}_n$.* squFr

We already used this result for Weierstrass polynomials where it is a consequence of 3.2. For the general case we use the preperation thoerem

$$P = UQ, \quad U \text{ unit in } \mathcal{O}_n, \quad Q \text{ Weierstrass polynomial.}$$

We know that U is a polynomial in z_n (1.3.1). This implies that Q is square free in te ring $\mathcal{O}_{n-1}[z_n]$ and therefore in $\mathcal{O}_n$. But U is a unit in $\mathcal{O}_n$ ist.Therefore P is square free in $\mathcal{O}_n$. □

The same argument shows

4.9 Remark. *Let $P \in \mathcal{O}_{n-1}[z_n]$ be a normalized polynomial which is prime in the ring $\mathcal{O}_{n-1}[z_n]$. The P eather is a unit in $\mathcal{O}_n$ or it is a prime in $\mathcal{O}_n$* UoE

Proof of 4.7. Let $a \in D$ be a point in which f is reduced. We can assume $a = 0$ and that the power series $P = f_0$ is a Weierstrass polynomial. We consider the power series of f in all points b in a small plydisc around 0.

$$f_b \in \mathbb{C}\{z_1 - b_1, \ldots, z_n - b_n\}$$

This power series is still a normalized polynomial in $\mathbb{C}\{z_1 - b_1, \ldots, z_{n-1} - b_{n-1}\}[z_n - b_n]$ but usually not a Weierstrass polynomial. By asumption P is square free (in $\mathcal{O}_n$ but the also in $\mathcal{O}_{n-1}[z_n]$ because it is a Weierstrass polynomial). Therefore the discriminant does not vanish. This (and the universal fomula for the discrimant) shows that the discriminant of P_b does dot vanish if b is cloes to 0. This means that P_b is square free in the polynomial ring and square free in $\mathcal{O}_a$ by 4.8. □

The proposition 4.7 is the basic result of this section. Together with 4.6 it shows that the singular locus S of a hypersurface A is a an analytic set. We want to show that S is thin in A, i.e. that it contains no nonempty open subset of A. For this it is sufficient to prove that in any neigboourhood of a given point

$a \in A$ smooth points exist. Wecan assume that A is defined by one equation $f(z) = 0$ and that f is reduced inits domain of definition. If all points of A are singular we have

$$f(z) = 0 \Longrightarrow \partial f/\partial z_1 = \ldots = \partial f/\partial z_n = 0.$$

From 3.8 follows that the power series of f in any point, for example in a divides the power series of the partial derivatives. But one has

If P, $P(0) = 0$, is a power series which divides the partial derivative $\partial f/\partial z_j$ for some j then f is independent of z_j

This is quite clear in the case $n = 1$ and folows then for arbitrary n. Now we obtain the desired result:

4.10 Theorem. *Let $A \subset \mathbb{C}^n$ be an analytic hypersurface. The singular locus S is a closed thin and analytic subset of A.* siHy

We give another application of the basic result 4.7.

4.11 Riemann extension theorem. *Let $A \subset U$ be a thin closed analytic subset of an open subset $U \subset \mathbb{C}^n$. Every bounded holomorphic function $f : U - A \longrightarrow \mathbb{C}$ extends holomorphically to the whole U.* riemE

Corollary. *The complement $U - A$ is connected.*

Proof. We can assume that A is contained in the zero set of one holomorphic function g which does not vanish on any non empty open subset. Furthermore we can assume that g_a is reduced for all $a \in U$.

The function fg extends to a continuous function on the whole U (values 0 at A), because f is bounded and g vanishes along A. It is easy to see that the partial derivatives of the function

$$f \cdot g^2 = (f \cdot g)g$$

exist (they are 0 along A). W use now the well-known theorem of the theory of functions of one complex variable that a differentiable complex function (on an open domain in $\mathbb{C}$) is analytic. We obtain that $f \cdot g^2$ is analytic on U. Now we can apply two times 3.8 to prove that f can be extended analytically to the whole U.

Chapter II. Analytic algebras

An anlytic algebra is a factor algebra of the ring of power series $\mathbb{C}\{z_1, \ldots, z_n\}$ by an ideal $\mathfrak{a}$. This means that the ideal theory of the ring of power series and the theory of analytic algebras is the same. Analytic algebras reflect local properties od analytic sets.

1. Noether normalization

An ideal

$$\mathfrak{a} \subset \mathcal{O}_n = \mathbb{C}\{z_1, \ldots, z_n\}$$

is called z_n-general if $\mathfrak{a}$ contains a z_n-general element. by the preperation theorem $\mathfrak{a}$ then contains a Weierstrass polynomial. Any ideal $\mathfrak{a}$ which is different from 0 can be made z_n-genral by means of a linear transformation of coordinates. Therefore it is very often possible to reduce the investigation of ideals to z_n-general ones. Our main technique will be induction in n. For this purpose we have to intersect an ideal

$$\mathfrak{a} \subset \mathcal{O}_n = \mathbb{C}\{z_1, \ldots, z_n\}$$

with the ring $\mathcal{O}_{n-1}$ using the natural inclusion

$$\begin{array}{ccc} \mathbb{C}\{z_1, \ldots z_{n-1}\} & \hookrightarrow & \mathbb{C}\{z_1, \ldots, z_n\} \\ \| & & \| \\ \mathcal{O}_{n-1} & & \mathcal{O}_n \end{array}$$

Let

$$\mathfrak{b} = \mathcal{O}_{n-1} \cap \mathfrak{a}$$

denote the intersection. We consider the quotient rings and obtain an embedding of algebras

$$\mathcal{O}_{n-1}/\mathfrak{b} \hookrightarrow \mathcal{O}_n/\mathfrak{a}.$$

We will investigate this extension of algebras in terms of notions from commutative algebra. Let's recall some simple facts about commutative algebra. Our main reference will be [Bo].

In the following all rings A are assumed to be commutative with unit-element 1_A. All homomorphisms of rings are assumed to map the unit elmenet into the unit element. All modules M over a ring A are assumed to be unital,

i.e. $1_A m = m$ for all $m \in A$. A module is called od finite type if there exist elements $m_1, \ldots, m_n$, such that

$$M = Am_1 + \ldots + Am_n$$

If

$$\varphi : A \longrightarrow B$$

is a ring homomorphism we can consider B as an A-module by means of

$$ab := \varphi(a)b$$

An element $b \in B$ is called *integral* over A, if every element $b \in B$ satisfies an equation

$$b^d + a_{d-1}b^{d-1} + \ldots + a_0 = 0 \quad (a_i \in A \text{ for } 0 \leq i < d).$$

1.1 Remark. *Let $A \to B$ be a ring homomorphism, A noetherian and B of finite type as A-module. Then B is integral over A.* ganzE

In this context we mention also

1.2 Remark. *Let $A \to B \to C$ be two ring homomorphisms, B of finite type as A-module and C of finite type as B-module. Then C is of finite type as A-module.* endtra

An important application of the Weierstrass theorems is:

1.3 Lemma. *Let $\mathfrak{a} \subset \mathcal{O}_n$ ba a z_n-general ideal. Then $\mathcal{O}_n/\mathfrak{a}$ is a $\mathcal{O}_{n-1}/\mathfrak{b}$ module of finite type with respect to the natural inclusion* natE

$$\mathcal{O}_{n-1}/\mathfrak{b} \hookrightarrow \mathcal{O}_n/\mathfrak{a} \quad (\mathfrak{b} = \mathcal{O}_{n-1} \cap \mathfrak{a}).$$

Additional remark. *If $\mathfrak{a}$ contains a Weierstrass polynomial of degree d, then $\mathcal{O}_n/\mathfrak{a}$ is generated as $\mathcal{O}_{n-1}/\mathfrak{b}$-module by the images of the powers*

$$1, z_n, \ldots, z_n^{d-1}.$$

The proof is an immediate consequence of the division theorem. □

1.4 Definition. *An analytic algebra A is a (commutaive and associative algebra with unit) such that for suitable n there exists a surjective homomorphism of algebras* anAL

$$\mathcal{O}_n \to A$$

whose kernel is contained in the maximal ideal $\mathfrak{m} \subset \mathcal{O}_n$.

This means that A is isomorphic to a quotient algebra of a ring of power series by a proper ideal. The image $\mathfrak{m} \subset A$ of the maximal ideal is a proper ideal of A. Otherwise there would exist an element $P \in \mathfrak{m}_n$ whose image is 1. But then $P-1$ wuld be in the kernel, which cannot be the case because the kernel cannot contain 1. Under a homomorphism units are mapped to units. Therefore $\mathfrak{m}$ is the biggest proper ideal in A. We see that A is a local ring with maximal ideal $\mathfrak{m}$. It is quite clear that the composition of the two natural homomorphisms

$$\mathbb{C} \longrightarrow A \longrightarrow A/\mathfrak{m}, \quad CA \longmapsto C1_A \mod \mathfrak{m},$$

is an isomorphism. Taking its invers we obtain a canonical surjective homomorphism

$$A \longrightarrow \mathbb{C}$$

with kernel $\mathfrak{m}$.

By a homorphism of analytic algebras we mean a $\mathbb{C}$-linear ring homomorphism (which maps the unit element to the unit element).

1.5 Remark. *Homomorphisms of analytic algebras are **local** in the sense that the image of the maximal ideal is contained in the maximal ideal.* loC

This gets quit clear if one looks at the residue fields. □

1.6 Noether normalization theorem. *Let A be an analytic algebra. There exists an injective homomorphism of analytic algebras* noetH

$$\mathbb{C}\{z_1, \ldots, z_d\} \hookrightarrow A \quad (d \text{ suitable})$$

such that A is a module of finite type over $\mathbb{C}\{z_1, \ldots, z_d\}$. The number d is unique.

Proof. The existence of such an embedding follows from 1.3 by repeated application in connection with 1.2. The essential point is the uniquness of d. This can be proved by analytic considerations using the projection techniques from section 3. We prefer here an argument which uses some commutative algebra, namely the concept of Krull dimension:

Let R be a ring. The Krull dimension $\dim R$ is the supremum of all $n \geq 0$ such there exists a chain of pairwise distinct prime ideals

$$\mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_n.$$

The dimension can be ∞. We use the following basic facts about the Krull dimension:

1. *Let R be a local noetherian Ring with maximal ideal $\mathfrak{m} = (a_1, \ldots, a_n)$. Then*

$$\dim R \leq n.$$

2. *Theorem of Cohen Seidenberg: Let $A \to B$ an injective ring homomorphism such that B is as A-module of finite type. Then*

$$\dim A = \dim B.$$

From 1. follows that the dimension of $\mathbb{C}\{z_1, \ldots z_d\}$ is at most d. Using the chain of prime ideals

$$(0) \subset (z_1) \subset (z_1, z_2) \subset \ldots \subset (z_1, \ldots, z_d)$$

one sees that the dimension is precisely d. From the theorem of Cohen Seidenberg we now deduce that the number d in 1.6 is the Krull dimension of A.

2. The two alternatives for prime ideals

The Noether normalization admits a refinement if the starting ideal $\mathfrak{a}$ is a prime ideal. Racall that an ideal $\mathfrak{p} \subset R$ in a ring R is called a prime ideal if the factor ring is an integral domain.

So let $\mathfrak{P} \subset \mathcal{O}_n$ be a prime ideal. and $\mathfrak{p} = \mathcal{O}_{n-1} \cap \mathfrak{p}$. We have an injective homomorphism

$$\mathcal{O}_{n-1}/\mathfrak{p} \hookrightarrow \mathcal{O}_n/\mathfrak{P}$$

which shows that $\mathfrak{p}$ is also a prime ideal. Let K resp. L be the field of quotients of $\mathcal{O}_{n-1}/\mathfrak{p}$ resp. $\mathcal{O}_n/\mathfrak{p}$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{n-1}/\mathfrak{p} & \hookrightarrow & \mathcal{O}_n/\mathfrak{P} \\ \cap & & \cap \\ K & \hookrightarrow & L\,. \end{array}$$

We distinguish to cases which behave completely different:

2.1 Theorem, the first alternative. *Let $\mathfrak{P} \subset \mathcal{O}_n = \mathbb{C}\{z_1, \ldots, z_n\}$ be a z_n-general prime ideal. Assume* erstA

$$\mathfrak{P} \cap \mathcal{O}_{n-1} = \{0\}.$$

Then $\mathfrak{P}$ is a principal ideal (i.e. generated by one element).

Proof. Let $Q \in \mathfrak{P}$ be a z_n-general element. One of the prime divisors of Q must be contained in $\mathfrak{P}$. It is z_n-general too. Hence we can assume that Q is prime. We will show that Q generates $\mathfrak{P}$. By the preperation theorem we can assume

that Q is a Weierstrass polynomial. Let $P \in \mathfrak{p}$ be an arbitrary element. From 1.3 applied to the ideal $\mathfrak{a} = (Q)$ we get an equation

$$P^k + A_{k-1}P^{k-1} + \ldots + A_0 \equiv \mod (Q), \quad A_i \in \mathcal{O}_{n-1} \ (0 \leq i < k).$$

The equation shows that A_0 is contained in $\mathfrak{p}$, hence in $\mathfrak{p}$. By assumption this ideal is 0 and we obtain $A_0 = 0$. We see

$$P \cdot (P^{k-1} + \ldots + A_1) \equiv 0 \bmod Q.$$

But (Q) is a prime ideal and we get

$$\text{eather} \qquad P \in (Q) \qquad \text{or} \qquad P^{k-1} + \ldots + A_1 \equiv \mod (Q).$$

Repeated application of this argument shows $P \in (Q)$ in any case.

2.2 Theorem, the second alternative. *Let* gleiQ

$$\mathfrak{P} \subset \mathcal{O}_n = \mathbb{C}\{z_1, \ldots, z_n\}$$

be a prime ideal which is not a principal ideal. After a suitable linear transformation of the coordiantes we can obtain:

a) $\mathfrak{P}$ *is* z_n*-general.*

b) *The rings*

$$\mathcal{O}_{n-1}/\mathfrak{p} \hookrightarrow \mathcal{O}_n/\mathfrak{p} \quad (\mathfrak{p} = \mathcal{O}_{n-1} \cap \mathfrak{P})$$

have the same field of quotients $K = L$.

"After a suitable linear transformation of the coordiantes" means that we allow to replace $\mathfrak{P}$ by its image under the automorphism

$$\mathcal{O}_n \to \mathcal{O}_n, \quad P(z) \mapsto P(Az),$$

for suitable $A \in \mathrm{GL}(n, \mathbb{C})$.

Proof of 2.2. We may assume that $\mathfrak{P}$ is already z_n-general. From 2.1 we know that

$$\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_{n-1}$$

is different from 0. After a linear transformation of the variables $(z_1, \ldots, z_{n-1}$ we can assume that $\mathfrak{p}$ is z_{n-1}-general. The ideal $\mathfrak{P}$ remains z_n-general. Now we consider

$$\mathfrak{q} = \mathfrak{p} \cap \mathcal{O}_{n-2} = \mathfrak{p} \cap \mathcal{O}_{n-2}.$$

The extension

$$\mathcal{O}_{n-2}/\mathfrak{q} \subset \mathcal{O}_n/\mathfrak{P}$$

is of finite type. We denote the fields of fractions by $K \subset L$. This is a finite algebraic extension and we have $L = K[\bar{z}_{n-1}, \bar{z}_n]$. The bar indicates that we have to take cosets mod mod P. From elementary algebra we will use

Theorem of primitive element. *Let $K \subset L$ be a finite algebraic extension of fields of characteristic zero, which is generated by two elements, $L = K[a, b]$. Then for all $x \in K$ but a finite number of exceptions one has*

$$L = K[a + xb].$$

As a consequence every finite algebraic extension of fields of characteristic zero is generated by one element. This is the usual formulation of this theorem. The above variant is contained in the standard proofs.

We obtain that

$$L = K[\bar{z}_{n-1} + a\bar{z}_n].$$

for almost all $a \in \mathbb{C}$. We consider now the following (invertible) linear transformation of variables,

$$w_{n-1} = z_{n-1} + az_n, \qquad w_j = z_j \text{ for } j \neq n-1.$$

We have to take care that $\mathfrak{P}$ remains general in the new coordinates, which now means w_n-general. This possible because we have infinitely many possibilities for a.

Thus we have proved that we can assume without loss of generality $L = K[\bar{z}_{n-1}]$. But then the quotient fields of $\mathcal{O}_{n-1}/\mathfrak{p}$ and $\mathcal{O}_n/\mathfrak{p}$ agree. □

3. Geometric realization of analytic ideals

Let $\mathfrak{a} \subset \mathcal{O}_n$ be a proper ideal of power series. The ring $\mathcal{O}_n$ being noetherian we can choose a finite system of generators $\mathfrak{a} = (P_1, \ldots, P_m)$. The generators converge in a common polydisc U around 0 and in this polydisc the analytic set

$$X := \{\, z \in U; \quad P_1(z) = \cdots = P_m(z) = 0 \,\}$$

is well defined. We call X a *geometric realization* of $\mathfrak{a}$. This realization depends on the choice of the generators and of U. But it is clear that two geometric realizations X, Y agree in a small neighbourhood of the origin. This means that for all local questions around the origin the geometric realization behaves as if it were unique.

The technique of the last two sections was to consider the intersection $\mathfrak{b} = \mathfrak{a} \cap \mathcal{O}_{n-1}$. Let X resp. Y be geometric realizations of $\mathfrak{a}$ resp. $\mathfrak{b}$. We consider the projection (cancellation of the last variable)

$$\mathbb{C}^n \longrightarrow \mathbb{C}^{n-1}, \quad (z_1, \ldots, z_n) \longmapsto (z_1, \ldots, z_{n-1}).$$

The generators of $\mathfrak{b}$ can be expressed by means of the generators of $\mathfrak{a}$. Therefore a point $a \in X$ which is sufficiently close to the origin will be mapped to a point of Y. If we replace X by its intersection with a small polydisc around Y, we obtain a map

$$X \longrightarrow Y$$

induced by the projection. We call this map the geometric realization of the pair $(\mathfrak{a}, \mathfrak{b} = \mathfrak{a} \cap \mathcal{O}_{n-1})$. Again this realization is uniquely determined in an obvious local sense around 0.

3.1 Remark. *Let $\mathfrak{a}$ be a z_n-general ideal in $\mathcal{O}_n$ and $\mathfrak{b} = \mathfrak{a} \cap \mathcal{O}_{n-1}$. There exists a geometric realization $f : X \to Y$ of $(\mathfrak{a}, \mathfrak{b})$ such that the inverse image of $0 \in Y$ consists of only one point, namely $0 \in X$. Furthermore for every neighbourhood $0 \in U \subset X$ there exists a neighbourhood $0 \in V \subset Y$ such that* einP

$$f^{-1}(V) \subset U.$$

Proof. There exists a Weierstrass polynomial $P \in \mathfrak{a}$. Close to the origin the inverse image is contained in the set of zeros of $P(0, \ldots, 0, z_n) = 0$. But $P(0, \ldots, 0, z_n) = z_n^d$ implies that 0 is the only solution. The rest comes from the frequently used argument of "continuity of zeros" of a Weierstrass polynomial. □

We want to mention here an important result, which we cannot prove at the moment but which is always behind the scenes and motivates our constructions:

The geometric realization $X \to Y$ (under the assumtion that $\mathfrak{a}$ is z_n-general) can be chosen such that it is surjective and proper and such that the fibres are finite.

We consider now the case that $\mathfrak{P} \subset \mathcal{O}_n$ is a prime ideal of the second alternative, i.e. it is z_n-general and $\mathcal{O}_n/\mathfrak{P}$ and $\mathcal{O}_{n-1}/\mathfrak{p}$ ($\mathfrak{p} := \mathfrak{P} \cap \mathcal{O}_{n-1}$) have the same field of fractions. We consider a geometric realization $f : X \to Y$ of the pair $(\mathfrak{P}, \mathfrak{p})$. We may assume that X is closed in the polydisc $U_{(\varrho_1, \ldots, \varrho_n)}(0)$ and Y is closed in $U_{(\varrho_1, \ldots, \varrho_{n-1})}(0)$.

3.2 Proposition. *Let $\mathfrak{P} \subset \mathcal{O}_n$ be a z_n-general prime ideal and $\mathfrak{p} := \mathfrak{P} \cap \mathcal{O}_{n-1}$. We assume that the fieds of fractions of $\mathcal{O}_n/\mathfrak{P}$ and $\mathcal{O}_{n-1}/\mathfrak{p}$ agree (second alternative). There exists a geometric realization $f : X \to Y$ of the pair $(\mathfrak{P}, \mathfrak{p})$ such the following holds:* bimeR

There exists a power series $A \in \mathcal{O}_{n-1}$ which is not contained in $\mathfrak{p}$ and which converges in a polydisc containing Y. Let be

$$S := \{\, z \in Y; \quad A(z) = 0 \,\} \quad \text{and} \quad T := f^{-1}(S).$$

The restriction

$$f_0 : X - T \longrightarrow Y - S$$

of f is topological.

Proof. We make use of the fact that the two fields of fractions agree. Expressing the coset of z_n as a fraction we obtain:

There exist power series $A, B \in \mathcal{O}_{n-1}$ with the properties

$$A \notin \mathrm{p} \qquad Az_n - B \in \mathrm{P}.$$

We can choose our realization $X \to Y$ such that A and B both converge in a polydisc containing Y. Especially the sets S and T are defined now. All points $z \in X$ satisfy

$$z_n A(z_1, \ldots, z_{n-1}) = B(z_1, \ldots, z_{n-1}).$$

This means

$$z_n = \frac{B(z_1, \ldots, z_{n-1})}{A(z_1, \ldots, z_{n-1})}$$

if z is not contained in T. So we have proved the injectivitiy of the map $f_0 : X - T \to Y - S$.

It remains to show that f_0 is surjective for properly choosen X and Y. To do this we choose X and Y as closed subsets of polydiscs. This is of course possible,

$$Y \subset U_{(\varrho_1, \ldots, \varrho_{n-1})}(0), \quad X \subset U_{(\varrho_1, \ldots, \varrho_n)}(0) \qquad \text{(both closed)}.$$

We assume furthermore $\varrho_1 = \cdots = \varrho_{n-1}$ and write

$$r := \varrho_1 = \cdots = \varrho_{n-1}, \quad \varepsilon := \varrho_n.$$

We define

$$g(z_1, \ldots, z_{n-1}) := (z_1, \ldots, z_n), \quad z_n := \frac{B(z_1, \ldots, z_{n-1})}{A(z_1, \ldots, z_{n-1})}.$$

What we need is $g(z) \in X$ for $z \in Y - S$. In a first step we show:

3.3 Lemma. *Let $P \in \mathfrak{P} \cap \mathcal{O}_{n-1}[z_n]$. There exists a r', $0 < r' \leq r$, such that* teL

$$P(g(z)) = 0 \quad \text{for all} \quad z \in Y - S, \ ||z|| < r'.$$

($||\cdot||$ denotes the maximum norm.) *Proof.* We choose r' small enough such that the coefficients of P converge in the polydisc with multiradis $(r', \ldots, r')$. Let d be the degree of P. Then $A^d P$ can be written as polynomial in Az_n with coefficients from $\mathcal{O}_{n-1}$. By means of $Az_n = (Az_n - B) + B$ we can rearrange P as polynomial in $Az_n - B$,

$$P = \sum_{j=0}^{d} (Az_n - B)^d P_j \quad (P_j \in \mathcal{O}_{n-1}).$$

We want to show $P(g(z)) = 0$ which is equivalent to $P_0(z) = 0$. But this is clear because $P_0 \in \mathfrak{P} \cap \mathcal{O}_{n-1} = \mathfrak{p}$. This completes the proof of the Lemma. □

We continue the proof of 3.2 and claim:

There exists r', $0 < r' \leq r$, *such that*

$$|z_n| < \varepsilon \quad \text{for} \quad ||(z_1, \ldots, z_{n-1})|| < r'.$$

One applies the Lemma 3.3 to a Weierstrass polynomial Q contained in $\mathfrak{P}$ and uses tha standard argument of "continuity of roots".

The set X can be defined by a finite number of equations $P_1(z) = \cdots = P_m(z)$, $P_j \in \mathfrak{P}$, which converge in the polydisc of multiradius $(r, \ldots, r, \varepsilon)$. By means of the division theorem ($P_j = A_j Q + B_j$) and the above lemma 3.3 we obtain $P_j(g(z)) = 0$ and hence $g(z) \in X$ for $||z|| < r'$ and suitable $r' \leq r$. If we replace Y resp. X by their intersections with the polydiscs of multiradius $(r', \ldots, r')$ resp. $(r', \ldots, r', \varepsilon)$ we obtain that f_0 is surjective and then that f_0 is bijective. The above formula for z_n shows that the inverse of f_0^{-1} is continous. □

Lemma 3.2 should be interpreted as a result which states that the realization $X \to Y$ in case of the second alternative is close to a biholomorphic map. One could say that f is *bimeromorphic*. But there is a big problem upto now. In principle it could be that S equals the whole Y. The *Rückert Nullstellensatz* will show that this is not the case. This Nullstellensatz will be the goal of the next section.

4. The Nullstellensatz

A pointed analytic set (X, a) is an analytic set with a distinguished point a. We are interested in local properties of X at a and can asssume for this purpose that $a = 0$ is the origin. We associate an ideal $\mathfrak{A} \subset \mathcal{O}_n$. A power series $P \in \mathcal{O}_n$ belongs to $\mathfrak{A}$ if there eists a small polydisc U around 0 such that P converges in U and such that p vanishes on $X \cap U$. The ideal $\mathfrak{A}$ is called the *vanishing ideal* of $(X, 0)$. It is a proper ideal, i.e. contained in the maximal ideal $\mathfrak{m}_n$. It is clear that the vanishing ideal $\mathfrak{A}$ of the realization only depends on $\mathfrak{a}$ and that $\mathfrak{a} \subset \mathfrak{A}$. We call $\mathfrak{A}$ the *saturation* of $\mathfrak{a}$.

The radical of an ideal

Let R be a ring. The radical $\operatorname{rad} \mathfrak{a}$ of an ideal $\mathfrak{a}$ is the set of all elements $a \in R$ such that a suitable power a^n, $n \geq 1$ is contained in $\mathfrak{a}$. It is easy to prove that $\operatorname{rad} \mathfrak{a}$ is an ideal which contains $\mathfrak{a}$. Fortheremore $\operatorname{rad} \operatorname{rad} \mathfrak{a} = \operatorname{rad} \mathfrak{a}$. An ideal is called *radical ideal* is it coincides wit its radical. This equivalent with the property that $R/\mathfrak{a}$ is a reduced ring, i.e. a ring which contains no nilpotent

elements differen form 0. Let R be a UFD-domain. A principal ideal Ra, $a \neq 0$ is a radical ideal if and only if a is square free. We are able to state and prove a fundamental result of local compex analysis:

4.1 The Rückert Nullstellensatz. *The saturation $\mathfrak{A}$ of a proper ideal $\mathfrak{a} \subset \mathcal{O}_n$ is the radical of $\mathfrak{a}$,* RNS

$$\mathfrak{A} = \operatorname{rad} \mathfrak{a}.$$

Proof. We want to reduce the nullstellensatz to prime ideals $\mathfrak{a}$. Prime ideals are of course radical ideals. The easiest way to do this reduction is to use a little commutative algebra, namely:

Every proper radical ideal in a noetherian ring is the intersection of finitely many prime ideals.

We use this and write the radical of our given ideal as intersection of prime ideals:

$$\operatorname{rad} \mathfrak{a} = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_m.$$

The saturation $\mathfrak{A}$ of $\mathfrak{a}$ is contained in the intersection of the of the saturations of the prime ideals. If we assume the nullstellensatz for prime ideals we obtain

$$\mathfrak{A} \subset \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_m = \operatorname{rad} \mathfrak{a}$$

This implies $\mathfrak{A} = \operatorname{rad} \mathfrak{a}$ becaus the converse inclusion is trivial.

Now we can assume that $\mathfrak{P} := \mathfrak{a}$ is a prime ideal. We have to distinguish the two alternatives:

1. *Alternative.* The ideal $\mathfrak{P}$ is principal, $\mathfrak{P} = (P)$. The element P is a prime element in $\mathcal{O}_n$. In this case the nullstellensatz is a consequence of the theory of hypersurfaces (I.3.8).

2. *Alternative.* $\mathfrak{P}$is not a principal ideal. Then we can assume that $\mathfrak{P}$ is z_n-general, that the exstension

$$\mathcal{O}_{n-1}/\mathfrak{p} \hookrightarrow \mathcal{O}_n/\mathfrak{P} \qquad (\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_{n-1}).$$

is modul-finite. and that the two rings have the same field of fractions. We make use of the geometric realization 3.2.

$$\begin{array}{lccc} f: & X & \longrightarrow & Y \\ & \cup & & \cup \\ f_0: & X - T & \xrightarrow{\sim} & Y - S\,. \end{array}$$

We indicated already in the last section that in principle S could be the whole Y before the nullstellensatz is known. But now we are in better situation. We can prove the nullstellensatz by induction on n and therefore assume:

The nullstellensatz is true for $\mathfrak{p}$.

From this we derive:

Let $P_0 \in \mathcal{O}_{n-1}$ *be a power series which converges in a small polydisc* V *around* 0 *and vanishes on* $(Y - S) \cap V$. *Then* P_0 *is contained in* $\mathfrak{p}$.

This is quite clear, because AP_0 (A as in 3.2) vanishes on $Y \cap V$. The nullstellensatz for $\mathfrak{p}$ gives $AP_0 \in \mathfrak{p}$ and get $P_0 \in \mathfrak{p}$ because $\mathfrak{p}$ is a prime ideal and A is not contained in $\mathfrak{p}$.

So in some sense the set S is neglectible. The proof of the nullstellensatz now runs as follows. We take an element P from the saturation of $\mathfrak{P}$. The claim is $P \in \mathfrak{P}$. The idea is to use an integral equation

$$P^m + P_{m-1}P^{m-1} + \ldots + P_0 \in \mathfrak{P}, \quad P_i \in \mathcal{O}_{n-1}\ (0 \leq i < m).$$

We take a minimal degree m. We distinguish two cases:

1. *case:* P_0 is contained in $\mathfrak{p}$: Then

$$P \cdot (P^{m-1} + P_{m-1}P^{m-2} + \ldots + P_1) \in \mathrm{P}.$$

Because of the minimality of M the expression in the bracket is not contained in $\mathfrak{P}$. But $\mathfrak{P}$ is a prime ideal and we obtain $P \in \mathfrak{P}$ what we wanted to show.

2. *case:* P_0 is not contained in $\mathfrak{p}$: We know that P vanishes on X in a neighbourhood of 0. We can assume that P vanishes on the whole X (use 3.1). Using the bijection $X - T \to Y - S$ we obtain that P_0 vanishes on $Y - S$. But as we have seen this implies $P_0 \in \mathfrak{p}$ which is a contradiction. This completes the proof of the nullstellensatz. □

We want to intruduce the notion "thin at" which reflects thet the set S is neglectible in Y in a certain sense.

4.2 Definition. *Let* $Y \subset X \subset \mathbb{C}^n$ *be analytic sets and* $a \in Y$ *a distinguished point. We call* Y ***thin at*** a *if the following is true:* thinA

> *If* f *is an analytic function on a neighbourhood* $a \in U \subset \mathbb{C}^n$ *which vanishes on* $(X - Y) \cap U$ *then* f *vanishes on* X *in a (possibly smaller) neighbourhood of* a.

So the essential part of the proof of the nullstellensatz was to show:

4.3 Remark. *Let* $\mathfrak{P} \subset \mathcal{O}_n$ *be a prime ideal with geometric realization* X. *Let* $P \in \mathcal{O}_n$ *be a power series which is not contained in* $\mathfrak{P}$. *Assume that* P *converges in a polydisc around* 0 *which contains* X. *Then* $Y := \{z \in X;\ P(z) = 0\}$ *is thin at* 0. thinL

Again we get an obvious problem. On should expect that the property "thin at a" extends to a full neighbourhood of a and that Y is thin in the usual topological sense in X (in this neighbourhood). At the moment we are not able to prove this. This needs the principle of *coherence* which will be our next goal. Before we have developped this basic tool we must (and can) be content with the notion "thin at". But the reader should have in mind that "thin at" is in reality the same as thin in a neighbourhood.

Chapter III. Coherence

A typical result for coherence was proposition I.4.7, which states that a holomorhic function f on an open subset $U \subset \mathbb{C}^n$, which is reduced in a point $a \in U$, is reduced in all points of whole neighbourhood of a. This result was basic for the proof that the singular locus of a hypersurface is an analytic set. The proof of this coherence result used the theory of the discriminant. Similar results on arbitrary analytic sets need deeper coherence theorems.

1. The coherence theorem of Oka

Let $U \subset \mathbb{C}^n$ be an open domain. In the appendix of I.2 we introduced the ring

$$\mathcal{O}_{U,a} = \mathbb{C}\{z_1 - a_1, \ldots, z_n - a_n\}$$

of germs of analytic functions and explained how to identify it with the ring of power series. Every holomorphic function f on an open neighbourhood of a has a germ $f_a \in \mathcal{O}_n$ which is nothing else than its power series expansion. We have a natural injection

$$\mathbb{C}\{z_1 - a_1, \ldots, z_{n-1} - a_{n-1}\} \longrightarrow \mathbb{C}\{z_1 - a_1, \ldots, z_n - a_n\}$$

and can define the ring

$$\mathbb{C}\{z_1 - a_1, \ldots, z_{n-1} - a_{n-1}\}[z_n - a_n] \subset \mathbb{C}\{z_1 - a_1, \ldots, z_n - a_n\}$$

in an obvious way. An element P of this ring is called a Weierstrass polynomial, if it is noramilized as polynomial in $z_n - a_n$ and if it has the property $P(a_1, \ldots, a_{n-1}, z_n - a_n) = (z_n - a_n)^d$, where d is the degree of P in the variable $z_n - a_n$.

Let m be anatural number. We are interested in $\mathcal{O}_{U,a}$-submodules of the free module $\mathcal{O}_{U,a}^m$. In the case $m = 1$ such a submodule is nothing else but an ideal and ideals are the modules in which we are intested. For technical reasons ist is important to allow arbitrary m. Every submodule of $\mathcal{O}_{U,a}^m$ is finitely generated because the ring of power series is noetherian.

We are not only interested in invidual modules but in *systems of modules*. This means that we assume that for every $a \in U$ a submodule

$$\mathcal{M}_a \subset \mathcal{O}_{U,a}^m$$

is given. We denote this system usually by a single letter,

$$\mathcal{M} = (\mathcal{M}_a)_{a \in U}.$$

If V is an open subset of U one defines in an obvious way the restricted system $\mathcal{M}|V := (\mathcal{M}_a)_{a \in U}$.

1.1 Definition. *A system* endEr

$$\mathcal{M} = (\mathcal{M}_a)_{a\in U}, \quad \mathcal{M}_a \subset \mathcal{O}^m_{U,a},$$

is called ***finitely generated****, if there exist finitely many vectors of holomorphic functions*

$$f^{(j)} \in \mathcal{O}(U)^m \quad for \quad 1 \leq j \leq k,$$

such that the $\mathcal{O}_a$-module $\mathcal{M}_a$ is genererated by the germs

$$(f^{(1)})_a, \ldots, (f^{(k)})_a.$$

The germs are taken of course componentwise.

1.2 Definiton. *The system $\mathcal{M} = (\mathcal{M}_a)_{a\in U}$ is called* ***coherent****, if it is locally finitely generated, which means that every point $a \in U$ admits an open neighbourhood $a \in V \subset U$ such that $\mathcal{M}|V$ is finitely generated.* coH

Let p, q be natural numbers and let

$$F = \begin{pmatrix} F_{11} & \ldots & F_{1p} \\ \vdots & & \vdots \\ F_{q1} & \ldots & F_{qp} \end{pmatrix}$$

by a matrix of holomorphic functions on U. We can consider the $\mathcal{O}(U)$-linear map

$$F : \mathcal{O}(U)^p \longrightarrow \mathcal{O}(U)^q$$

which is defined by

$$Ff := g; \quad g_i := \sum_{j=1}^{p} F_{ij} f_j \quad (1 \leq i \leq q).$$

As the notation indicates we identify the matrix and the linear map. For every point $a \in U$ we can consider the germ $F_a = ((F_{ik})_a)$ and the corresponding map

$$F_a : \mathcal{O}^p_{U,a} \to \mathcal{O}^q_{U,a}.$$

1.3 Oka's coherence theorem. *Let* OkC

$$F : \mathcal{O}(U)^p \to \mathcal{O}(U)^q \quad (U \subset \mathbb{C}^n \text{ open})$$

be an $\mathcal{O}(U)$–linear map. The system

$$\mathcal{M} = (\mathcal{M}_a)_{a\in U} \quad \mathcal{M}_a := \text{kernel}\,(F_a)$$

is coherent

The proof will be given in three steps:
1. *step, reduction to the case* $q = 1$. This will be done by induction on q. So let's assume $q > 1$ and that the theorem is proved for $q - 1$ instead of q. Let $a_0 \in U$ be a distuingished point. We want to prove that $\mathcal{M}$ is finitely generated in a neighbourhood of a. For this purpose we can replace U by a smaller neighbourhood of a_0. We consider the two projections

$$\mathcal{O}(U)^q = \mathcal{O}(U)^{q-1} \times \mathcal{O}(U) \quad \begin{matrix} \xrightarrow{\alpha} \mathcal{O}(U) \\ \xrightarrow{\beta} \mathcal{O}(U)^{q-1}. \end{matrix}$$

By the induction hypothesis, applied to

$$\alpha \circ F : \mathcal{O}(U)^p \to \mathcal{O}(U)^{q-1}$$

we can assume that there exist a finite sxstem

$$A^{(1)}, \ldots, A^{(m)} \in \mathcal{O}(U)^p,$$

such that the germs $A_a^{(1)}, \ldots, A_a^{(m)}$ generated the kernel of $(\alpha \circ F)_a$ for each point $a \in U$. Now we consider the linear map

$$G : \mathcal{O}(U)^m \longrightarrow \mathcal{O}(U)^p, \quad (f_1, \ldots, f_m) \longmapsto f_1 A^{(1)} + \ldots + f_m A^{(m)},$$

and compose it with the projection β,

$$\beta \circ A : \mathcal{O}(U)^m \to \mathcal{O}(U).$$

We assumed that the case $q = 1$ is proved and can therefore assume that ther exists a finite system

$$B^{(1)}, \ldots, B^{(l)} \in \mathcal{O}(U)^m,$$

whose germs in an arbitrary point $a \in U$ generate $(\beta \circ A)_a$. It is easy to see that the germs of

$$C^{(i)} = G(B^{(i)}) \in \mathcal{O}(U)^p \quad (1 \leq i \leq m).$$

generate the kernel of our original F_a. Thus we have show:
If Oka's theorem is true for $q = 1$ *in a given dimension* n *then it is true for all* q *in this dimension.*
2.*step.* The proof of Oka's theorem rests on *Oka's Lemma*, which is a lemma for an individual ring of power series (not a system). Before we cam formulate it, we need a notation:

$$\mathcal{O}_{n-1}[z_n : m] = \left\{ P \in \mathcal{O}_{n-1}[z_n]; \quad \deg_{z_n} P < m \right\}.$$

This is a free module over $\mathcal{O}_{n-1}$ with basis $1, z_n, \ldots, z_n^{m-1}$,

$$\mathcal{O}_{n-1}[z_n : m] \cong \mathcal{O}_{n-1}^m.$$

1.4 Oka's Lemma. *Let* okaL

$$F : \mathcal{O}_n^p \to \mathcal{O}_n$$

be a $\mathcal{O}_n$-linear map and let K be its kernel.

Assumption. *The components of the matrix F are normalized polynomials in $\mathcal{O}_{n-1}[z_n]$ of degree $< d$ (in the variable z_n).*

We consider the restriction of F

$$\mathcal{O}_{n-1}[z_n : m]^p \to \mathcal{O}_{n-1}[z_n : m+d]$$

and denote by K_m its kernel.

Claim. *The $\mathcal{O}_n$-module K is generated by K_m for $m \geq 3d$.*

Proof. In a first step we assume that the first component of the map $F = (F_1, \ldots, F_p)$ is aWeierstrass polynomial (and not only a normalized polynomial). We will prove Oka's Lemma in this case with the better bound $2d$ instead of $3d$. Let $G = (G_1, \ldots, G_p) \in K$ be an element of the kernel. The division theorem gives

$$G = F_1 A + B, \quad A \in \mathcal{O}_n^p, \quad B \in \mathcal{O}_{n-1}[z_n : d]^p.$$

We notice that the elements

$$H^{(j)} = (-F_j, 0, \ldots, 0, F_1, 0, \ldots, 0) \quad (1 < j \leq p)$$

are contained in the kernel. The trivial formula

$$F_1 A = \sum_{j=2}^{p} A_j H^{(j)} + (A_1 F_1 + \ldots + A_p F_p, 0, \ldots, 0)$$

shows that besides G also the element $H := B + (A_1 F_1 + \ldots + A_p F_p, 0, \ldots, 0)$ is contained in the kernel, i.e.

$$F_1(B_1 + A_1 F_1 + \ldots + A_p F_p) + F_2 B_2 + \ldots + F_p B_p = 0.$$

This equation shows

$$F_1(A_1 F_1 + \ldots + A_p F_p) \in \mathcal{O}_{n-1}[z_n : 2d].$$

Using again that F_1 is a Weierstrass polynomial we obtain (I.3.1)

$$A_1 F_1 + \ldots + A_p F_p \in \mathcal{O}_{n-1}[z_n : 2d].$$

Now we see that the components of H are contained in K_{2d}. The trivial formula

$$G = \sum_{j=2}^{p} A_j H^{(j)} + H$$

finally shows that G is contained in the module which generated by the $H^{(j)}$ and H, which are elements of K_{2d}.

Now we treat the general case where F_1 is not necessarily a Weierstrass polynomial. We apply the preperation theorem

$$F_1 = Q \cdot U, \quad Q \text{ Weierstrass polynomial}, \quad U \text{ unit in } \mathcal{O}_n.$$

We are interested in the solutions of the equation $F_1P_1 + F_2P_2 + \ldots + F_pP_p = 0$ or equivalently

$$Q\tilde{P}_1 + F_2P_2 + \ldots + F_pP_p = 0 \quad (\tilde{P}_1 = UP_1).$$

Bacause Q is a Weierstrass polynomial, this system is generated by solutions of z_n-degree $< 2d$. But UP_1 is of degree $< 3d$ if P_1 is of degree $< 2d$. This completes the proof of Oka's lemma.

3. *step, the proof of Oka's theorem in the case* $q = 1$*:*
The proof now is given by induction on n. As beginning of the induction can be taken the trivial case $n = 0$. We have to consider a $\mathcal{O}(U)$-linear map

$$F : \mathcal{O}(U)^p \longrightarrow \mathcal{O}(U),$$

which is geiven by a vector $(F_1, \ldots, F_P)$. We want to show that the kernel system is finitely generated in a neighbourhood of a given point and can assume that this point is the origin 0 and that U is a polydisc with center 0. After a suitable linear coordinate transformation we can assume that the power series expansions of $F_1, \ldots, F_p$ in the origin are z_n-general. By the preperation theorem we can assume that the all are Weierstrass polynomials. If we consider the power series expansions in other points $a \in U$ we still have normalized polynomials

$$(F_i)_a \in \mathbb{C}\{z_1 - a_1, \ldots, z_{n-1} - a_{n-1}\}[z_n - a_n].$$

(but usually not Weierstrass polynomials). The degree of all those polynomials is bounded by a suitable number d. We write U in the form

$$U = V \times (-r, r) \quad (V \subset \mathbb{C}^{n-1})$$

and denote by $\mathcal{O}(V)[z_n : m]$ the set of all holomorphic functions on U which are polynomials in z_n of degree $< m$ with coeffients independent of z_n. This is a free $\mathcal{O}(V)$ module,

$$\mathcal{O}(V)[z_n : m] \cong \mathcal{O}(V)^m.$$

Our given map F induces an $\mathcal{O}(V)$-linear map

$$\begin{array}{ccc} \mathcal{O}(V)[z_n : m]^p & \longrightarrow & \mathcal{O}(V)[z_n : m + d] \\ \| & & \| \\ \mathcal{O}(V)^{mp} & \longrightarrow & \mathcal{O}(V)^{m+d}. \end{array}$$

From the induction hypothesis we can assume that the kernel of this map is finitely generated. From Oka's lemma we obtain that the kernel system of F is finitely generated. Oka's theorem is proved. □

Some important properties of coherent systems

The following trivial property of coherent systems will be used frequently:

1.5 Remark. *Let $\mathcal{M},\mathcal{N}$ be two coherent systems on an open set $U \subset \mathbb{C}^n$. Assume $\mathcal{M}_{a_0} \subset \mathcal{N}_{a_0}$ for a distinguished point a_0. Then $\mathcal{M}_a \subset \mathcal{N}_a$ in a complete neighbourhood of a_0 holds.* umgC

Corollary. *$\mathcal{M}_{a_0} = \mathcal{N}_{a_0}$ implies $\mathcal{M}|V = \mathcal{N}|V$ for an open neihbourhood V of a_0.*

Another trivial observation is

1.6 Remark. *Let* bilC

$$F : \mathcal{O}(U)^m \to \mathcal{O}(U)^l \quad (U \subset \mathbb{C}^n \text{ open})$$

be an $\mathcal{O}(U)$ linear map and let

$$\mathcal{M} = (\mathcal{M}_a)_{a\in U}, \quad \mathcal{M}_a \subset \mathcal{O}^m_{U,a},$$

be a coherent system. The the image system

$$\mathcal{N} = (\mathcal{N}_a)_{a\in U}, \quad \mathcal{N}_a := F_a(\mathcal{M}_a) \subset \mathcal{O}^l_{U,a}.$$

is coherent. (The same is true already for "finitely generated" instead for "coherent".)

The next result is not trivial, it uses Oka's theorem:

1.7 Proposition. *Let $\mathcal{M},\mathcal{N}$ be two coherent systems on the open set $U \in \mathbb{C}^n$,* durC

$$\mathcal{M}_a, \mathcal{N}_a \subset \mathcal{O}^m_{U,a} \quad (a \in U).$$

The the intersection system $\mathcal{M} \cap \mathcal{N}$ which is defined by

$$(\mathcal{M} \cap \mathcal{N})_a := \mathcal{M}_a \cap \mathcal{N}_a \quad (a \in U)$$

is coherent too.

Proof. The idea is to write the intersection as a kernel. We explain the principle for individual modules $M, N \subset R^n$ of finite type over a ring R instead of a system: We can write M resp. N as image of a linear map $F : R^p \to R^m$ resp. $G : R^q \to R^m$. We denote by K the kernel of the linear map

$$R^{p+q} \longrightarrow R^m, \quad (m,n) \longmapsto F(m) - G(n).$$

The image of K under the map

$$R^{p+q} \longrightarrow R^m, \quad (m,n) \longmapsto F(m).$$

is precisely the intersection $M \cap N$. The proof of 1.7 is clear now. On "reads" M, N as coherent systems. By Oka's theorem K now stands for a coherent system amd the image $M \cap N$ is is coherent by 1.6. □

1.8 Proposition. *Let* invC

$$F : \mathcal{O}(U)^m \to \mathcal{O}(U)^l \quad (U \subset \mathbb{C}^n \text{ open})$$

be an $\mathcal{O}(U)$-linear map and let

$$\mathcal{N} = (\mathcal{N}_a)_{a\in U}, \quad \mathcal{N}_a \subset \mathcal{O}_{U,a}^l,$$

be a coherent system. The inverse image system

$$\mathcal{M} = (\mathcal{M}_a)_{a\in U}, \quad \mathcal{M}_a := F_a^{-1}(\mathcal{N}_a) \subset \mathcal{O}_{U,a}^m,$$

is koherent.

In the special case $\mathcal{N} = 0$ this is Oka's theorem.
Proof. We explain again the algebra behind this result. Let $F : R^m \to R^l$ be a R-linar map and $N \subset R^l$ be a R-module of finite type. We assume that $F(R^m) \cap N$ is finitely generated. Then there exists a finitely generated submodule $P \subset R^m$ such that $F(P) = F(R^m) \cap N$. We also assume that the kernel K of F is finitely generated. It is easily proved that $F^{-1}(N) = P + K$ and we obtain that the inverse image is finitely generated. Thees argument works in an obvious way for coherent systems and gives a proof of 1.8. □

2. Rings of power series are Henselian

The fact that power series are henselian rings can be considered as an abstract formulation of the Weierstrass theorems. We don't need the notion of a henselian ring to formulate this result, but for sake of completeness we give the definition of this property.

A local ring R with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$ is called a ***henselian ring*** *if the following is true:*
Let $P \in R[X]$ be a normalized polynomial. We denote by p its image in $k[X]$. Assume that $a, b \in k[X]$ are two coprime normalized polynomials with the property $p = ab$. Then there exist normalized polynomials $A, B \in R[X]$ with cosets a, b such that $P = AB$.

We recall that the polynomial ring in one variable over a field is a principal ideal ring. Therefore two polynomials a, b are coprime if and only if they generate the unit ideal $k[X]$.

We consider the special case where k is algebraically closed. Then every normalized polynomial $p \in k[X]$ is a product of linear factors, if $b_1, \dots b_m$ are the pairwise distinct zeros and $d_1, \dots, d_m$ their multiplicities then

$$p(X) = \prod_{j=1}^{m} (X - b_j)^{d_j}.$$

This is a decomposition of p into m pairwise coprime factors. So the henselian property means in this case:

There exists a decomposition $P = P_1 \cdots P_m$ of P as product of normalized polynomials such that $p_j(X) = (X - b_j)^{d_j}$ where p_j denotes the image of P_j in $k[X]$.

We want to show that the ring of power series $\mathcal{O}_n = \mathbb{C}\{z_1, \ldots, z_n\}$ is Henselian. The residue field $\mathcal{O}_n/\mathfrak{m}_n$ can be identified with $\mathbb{C}$ and the projection $\mathcal{O}_n \to \mathcal{O}_n/\mathfrak{m}_n$ corresponds to the map

$$\mathbb{C}\{z_1, \ldots, z_n\} \longrightarrow \mathbb{C}, \quad P \longmapsto P(0).$$

We have to consider the polynomial ring over $\mathbb{C}\{z_1, \ldots, z_n\}$. Therefore we need a letter for the variable. To stay close to previous notations we consider $\mathcal{O}_{n-1}$ instead of $\mathcal{O}_n$ and formulate the Hensel property for this ring. Then we have the letter z_n free for the variable of the polynomial ring. After this preperation we see that the following theorem expresses precisely that the rings of power series are Henselian.

2.1 Theorem. *Let $P \in \mathcal{O}_{n-1}[z_n]$ be a normalized polynomial of degree $d > 0$ and let β be a zero with multiplicity d_β of the polynomial $z \longmapsto P(0, \ldots, 0, z)$. Then there exists a unique normalized polynomial $P^{(\beta)} \in \mathcal{O}_{n-1}[z_n]$ which divides P and such that* HENS

$$P^{(\beta)}(0, \ldots, 0, z) = (z - \beta)^{d_P}.$$

Moreover

$$P = \prod_{P(0,\ldots,0,\beta)=0} P^\beta.$$

(Here β runs through the zeros of $z \mapsto P(0, \ldots, 0, z)$.)

For the proof of this theorem we need three lemmas:

2.2 Lemma. *Let $P \in \mathcal{O}_{n-1}[z_n]$ be an **irreducible** normalized polynomial with the property $P(0) = 0$. Then P is a Weierstrass polynomial.* irnorm

Proof. By the preperation theorem we have $P = UQ$ with a Weierstrass polynomial and a unit U. We know (I.3.1) that U is a polynomial. But P is irreducible. We obtain $U = 1$ and $P = Q$. □

2.3 Lemma. *Let $P \in \mathcal{O}_{n-1}[z_n]$ be an irreducible normalized polynomial of degree $d > 0$. Then* irrnu

$$P(0, \ldots, 0, z) = (z - \beta)^d$$

with a suitable complex number β.

Proof. Let β be a zero of the polynomial $z \mapsto P(0, \ldots, 0, z)$. We rearrange P as polynomial in $z_n - \beta$ and obtain by 2.2 a Weierstrass polynomial in $\mathcal{O}_{n-1}[z_n - \beta]$. □

2.4 Lemma. *Let P, Q be two normalized polynomials in $\mathcal{O}_{n-1}[z]$. The polynomials $p(z) = P(0,\ldots,0,z)$, $q(z) = P(0,\ldots,0,z)$ are assumed to be coprime. (This means that have no common zero.) Then P and Q generate the unit ideal* einId

,

$$(P,Q) = \mathcal{O}_{n-1}[z_n].$$

Proof. The proof will use the theorem of Cohen Seidenberg: The ring polynomial in one variable over a field is a principal ideal ring. Therefore

$$(p,q) = \mathbb{C}[z].$$

We obtain that P and Q together with the maximal ideal $\mathfrak{m}_{n-1} \subset \mathcal{O}_{n-1}$ generate the unit ideal,

$$(P,Q,\mathfrak{m}_{n-1}) = \mathcal{O}_{n-1}[z_n].$$

Now we consider the natural homomorphism

$$\mathcal{O}_{n-1} \longrightarrow \mathcal{O}_{n-1}[z_n]/(P,Q).$$

This ring extension is module-finite. This follows immediately if one applies the Euclidean algorithm to one of the polynomials P, Q. The theorem of Cohen Seidenberg deals with module finite ring extensions. We give here a formulation which is not the standard one but usually a lemma during the proof:

Let A be a noetherian local ring and $A \to B$ a ring homomorphism such that B is an A-module of finite type. We assume that B is different from the zero ring ($1_B \neq 0_B$). Then there exists a proper ideal in B which contains the image of the maximal ideal of A.

(One can take the ideal which is generated by the image of the maximal ideal of A. The problem is to show that this is diferent form B.)

We continue the proof of 2.4. We want to show that P and Q generate the unit ideal. We give an indirect argument and assume that this is not the case. Then by Cohen Seidenberg we obtain that the image of $\mathfrak{m}_{n-1}$ in $\mathcal{O}_{n-1}[z_n]/(P,Q)$ does not generate the unit ideal. This means the same that $(P,Q,\mathfrak{m}_{n-1})$ is not the unit ideal, which gives a contradiction. □

Proof of theorem 2.1. Let P be a normalized polynomial of degree $d > 0$ in $\mathcal{O}_{n-1}[z_n]$. We decompose P into a product of irreducible normalized polynomials

$$P = P_1 \cdot \cdot P_m.$$

From 2.3 we obtain

$$P_i(0,\ldots,0,z) = (z-\beta_i)^{d_i} \quad (1 \leq i \leq m).$$

The numbers β_i are the zeros of the polynomial $P(0,\ldots,0,z)$. There is no need that the β_i are pairwise distinct. But we can collect the P_i for a fixed zero and multiply them together. □

We need a further little lemma from algebra:

Let R be a UFD-domain and a, b two coprime elements. The natural homomorphism

$$R/(ab) \longrightarrow R/(a) \times R/(b)$$

is injective. It is an isomorphism if a and b generate the unit ideal.

We apply this to thorem 2.1 and obtain:

2.5 Proposition. *(We use the notations of 2.1.) The natural homomorphism* hensis

$$\mathcal{O}_{n-1}[z_n]/(P) \xrightarrow{\sim} \prod_\beta \mathcal{O}_{n-1}[z_n]/(P^{(\beta)})$$

is an isomorphism.

We recall the the P^β are Weierstrass polynomials in the ring $\mathcal{O}_{n-1}[z_n - \beta]$. From the divison theorem in the form I.3.2 we obtain

$$\mathcal{O}_{n-1}[z_n]/(P^{(\beta)}) = \mathbb{C}\{z_1, \ldots, z_{n-1}, z_n - \beta\}/(P^{(\beta)}).$$

Now we can conclude from the Hensel property the following generalization of the division theorem for normalized polynomials instead of Weierstrass polynomials:

2.6 Proposition. *(We use the notations of 2.1.) The natural homomorphism* Hensis

$$\mathbb{C}\{z_1, \ldots, z_{n-1}\}[z_n]/(P) \xrightarrow{\sim} \prod_\beta \mathbb{C}\{z_1, \ldots, z_{n-1}, z_n - \beta\}/(P^{(\beta)})$$

is an isomorphism. This remains true if one replaces $P^{(\beta)}$ by the power series expansion of P in $(0, \ldots, 0, \beta)$.

The last statement uses the decomopsition $P = \prod_\gamma P^{(\gamma)}$ and the fact that all factors besides the considered $P^{(\beta)}$ do not vanish in $(0, \ldots, 0, \beta)$ and hence define units in $\mathbb{C}\{z_1, \ldots, z_{n-1}, z_n - \beta\}$.

3. A special case of Grauert's projection theorem

Grauert's projection theoremLet $U \subset \mathbb{C}^n$ be an open domain. We consider a coherent system of idealsGrauertprojection theorem

$$\mathfrak{a} = (\mathfrak{a}_a)_{a \in U}, \quad \mathfrak{a}_a \in \mathcal{O}_{U,a}.$$

3.1 Definition. *The **support** of a coherent system $\mathfrak{a}$ of ideals is the of all $a \in U$ such that $\mathfrak{a}_a$ is different from the unit ideal.* cohI

If the system is finitely generated, let's say by $f_1, \ldots, f_m$ then the support is nothing else but the set of common zeros as follows from the nullstellensatz. So we see:

3.2 Remark. *The support of a coherent system of ideals is a closed analytic subset od U.* cohab

Conversely analytic sets can be obtained at least locally as the support of coherent systems.

. Now we assume that $U = V \times \mathbb{C}$ with a polydisc $V \subset \mathbb{C}^{n-1}$. We consider the *projection*

$$\pi : U \longrightarrow V, \quad (z, z_n) \longmapsto Z.$$

It may happen that the image of an closed analytic set $X \subset U$ in V is a closed analytic set $Y \subset V$ but this must be not the case. We want to give a sufficient condition where it is the case. The idea is to consider rather coherent systems than analytic sets. So let's assume that X is the support of the coherent system $\mathfrak{a}$. We expect that in good situations Y is the support of certain coherent system on V. It's not difficult to guess what this system should be.

3.3 Defintion. *Let $V \subset \mathbb{C}^{n-1}$ be a polydisc and $\mathfrak{a}$ a coherent system of ideals on $U \times V$. We define for a point $b \in V$ the ideal* projS

$$\mathfrak{b}_b := \mathcal{O}_{V,b} \cap \bigcap_{a \in U,\ \pi(a)=b} \mathfrak{a}_a.$$

and call $\mathfrak{b} := (\mathfrak{b}_b)_{b \in V}$ the projected system.

We recall that the projection π defines a natural inclusion $\mathcal{O}_{V,b} \hookrightarrow \mathcal{U}_{U,a}$ for all a, b with $\pi(a) = b$.

Projections of analytic sets of the above kind can be very bad and similarely the projected systems can be bad and need not to be coherent. But there exist "good" projections:

3.4 Theorem. *Assume that $V \subset \mathbb{C}^{n-1}$ is a polydisc and that $\mathfrak{a}$ is a coherent system on $U = V \times \mathbb{C}$, which can be generated by finitely many functions $f_1, \ldots, f_m \in \mathcal{O}(V)[z_n]$. We assume that $P := f_1$ is a normalized polynomial. Then the projected system $\mathfrak{b}$ is coherent.* GRAU

Additional remark. *If X is the support of $\mathfrak{a}$, then $Y = \pi(X)$ is the support of $\mathfrak{b}$. Especially $\pi(X)$ is a closed analytic subset of V. The map $\pi : X \to Y$ has finite fibres.*

(The truth is that the projection $X \to V$ is a *proper analytic map with finite fibres.* The above theorem can be considered as a special case of the deep projection theorem of Grauert.)

Proof of 3.4. The proof will use Oka's coherence theorem and the Hensel property of rings of power series. The ideal $\mathfrak{a}_a$ is the unit ideal if $P(a) \neq 0$. For every $b \in V$ the number of $a \in U$ with $\pi(a) = b$ and $P(a) = b$ is finite. Therefore $\mathfrak{b}_b$ is the intersection of *finitely many* ideals:

$$\mathfrak{b}_b := \mathcal{O}_{V,b} \cap \bigcap_{\pi(a)=b,\ P(a)=0} \mathfrak{a}_a.$$

The ideal $\mathfrak{b}_b$ contains 1 if and only this is the case for all $\mathfrak{a}_a$, $\pi(a) = b$. We see that the additional remark will follow automatically from the coherence of $\mathfrak{b}$.

We want to consider the ideal

$$\mathcal{I}_b \subset \mathcal{O}_{V,b}[z_n]/(P_b),$$

which is generated by the $f_1, \ldots, f_n$ (more precisely by their images). We have to consider this ideal also as $\mathcal{O}_{V,b}$-module. It is of finite type over this ring, more precisely it is generated as module over this ring by the elements

$$f_i z_n^j \qquad (1 \leq i \leq m, \quad 0 \leq j < d).$$

This uses the Euclidean algorithm, which gives an isomorphism

$$\mathcal{O}_{V,b}^d \xrightarrow{\sim} \mathcal{O}_{V,b}[z_n]/(P_b).$$

A vector $(H_0, \ldots, H_d)$ is mapped to $\sum H_j z_n^j$. We take the inverse image of $\mathcal{I}_b$ and get a submodule

$$\mathcal{M}_b \subset \mathcal{O}_{V,b}^d.$$

From the given generators we see that the system $\mathcal{M} = (\mathcal{M}_b)_{b \in V}$ is finitely generated hence coherent on V. This system is closely related to our projected ideals $\mathfrak{b}_b$:

Claim. *The projected ideal $\mathfrak{b}_b$ is precisely the inverse image of $\mathcal{I}_b$ with respect to the natural map*

$$\mathcal{O}_{V,b} \longrightarrow \mathcal{O}_{V,b}[z_n]/(P_b).$$

We assume for a moment that the claim is proved. Then $\mathfrak{b}$ can be considered as inverse image of the coherent system $\mathcal{M}$. But Oka's coherence theorem (1.8) then implies that $\mathfrak{b}$ is coherent. So it remains to prove the claim:

Proof of the claim. In this proof the Hensel property of rings of power series will enter. We have to make further use of our normalized polynomial $P \in \mathcal{O}(V)[z_n]$,

$$P = z_n^d + P_{d-1} z_n^{d-1} + \ldots + P_0.$$

Its coefficients P_j are holomorphic functions on V. We will use the power series expansion $(P_j)_b \in \mathcal{O}_{V,b}$ for varying points $b \in V$. We have to consider the image of P in $\mathcal{O}_{V,b}[z_n]$,

$$P_b = z_n^d + (P_{d-1})_b z_n^{d-1} + \ldots + (P_0)_b \in \mathcal{O}_{V,b}[z_n].$$

We also have to use the ring $\mathcal{O}_{V,b}[z_n]/(P_b)$. The Hensel propety of rings of power series gave us important information for this ring. Applying 2.6 we obtain a natural isomorphism*)

$$\mathcal{O}_{V,b}[z_n]/(P_b) \xrightarrow{\sim} \prod_\beta \mathcal{O}_{U,(b,\beta)}/(P_b^{(\beta)}).$$

*) In 2.6 the result has been formulated only for $b = 0$ which is no loss of generality.

Here β runs over the zeros $P(b,\beta) = 0$. The elements $P_b^{(\beta)} \in \mathcal{O}_{V,b}$ come from the "Hensel decomposition"

$$P_b = \prod_\beta P_b^{(\beta)}, \qquad P_b^{(\beta)}(b, z_n) = (z_n - \beta)^{d_\beta}.$$

We determine the image of $\mathcal{I}_b$ under this isomorphism. For this we use the simple fact thet an ideal $\mathfrak{c} \subset A \times B$ in the cartesian product of two rings always is the direct product of two ideals, $\mathfrak{c} = \mathfrak{a} \times \mathfrak{b}$, where $\mathfrak{a} \subset A$ and $\mathfrak{b} \subset B$ are the projections of $\mathfrak{c}$. Using this and the definition (3.4) of $\mathfrak{a}$ we see:

The image of the ideal $\mathcal{C}_b$ in $\prod_\beta \mathcal{O}_{U,(b,\beta)}/(P_b^{(\beta)})$ is the direct product of the ideals $\bar{\mathfrak{a}}_{(b,\beta)}$, which mean the images of $\mathfrak{a}_{(b,\beta)}$ in $\mathcal{O}_{U,(b,\beta)}/(P_b^{(\beta)})$.

We have to determine the inverse image of this ideal under the natural map

$$\mathcal{O}_{V,b} \longrightarrow \prod_\beta \mathcal{O}_{U,(b,\beta)}/(P_b^{(\beta)}).$$

The claim states that this inverse image is the projection ideal $\mathfrak{b}_b$. But this inverse image is the intersection of the inverse images of $\bar{\mathfrak{a}}_{(b,\beta)}$ under

$$\mathcal{O}_{V,b} \longrightarrow \mathcal{O}_{U,(b,\beta)}/(P_b^{(\beta)}).$$

But $P_b^{(\beta)}$ is contained in $\mathfrak{a}_{(b,\beta)}$ (s. 2.6). Therefore it is the same to take the inverse image of $\mathfrak{a}_{(b,\beta)}$ under

$$\mathcal{O}_{V,b} \longrightarrow \mathcal{O}_{U,(b,\beta)}.$$

This is $\mathcal{O}_{V,b} \cap \mathfrak{a}_{(b,\beta)}$ and the intersection of all of then is $\mathfrak{b}_b$. □

4. Cartan's coherence theorem

We give three different formulations for Cartan's theorem:

4.1 Cartan's coherence theorem. *Let $\mathfrak{a} = (\mathfrak{a}_a)_{a\in U}$ be a coherent system of ideals on an open domain $U \subset \mathbb{C}^n$. Then its radical* CAR

$$\operatorname{rad} \mathfrak{a} := (\operatorname{rad} \mathfrak{a}_a)_{a\in U}$$

is coherent too.

let $X \subset U$ be a closed analytic subset. The vanishing ideal system $\mathfrak{A}_X$ is the system of ideals $\mathfrak{A}_a$, $a \in U$ which consists of all elements from $\mathcal{O}_{U,a}$, which vanish in a small neighbourhood of a on X. If a is not in X then $\mathfrak{A}_a = \mathcal{O}_{U,a}$. For this one has to use that X is closed in U. A second form of Cartan's theorem is:

4.2 Cartan's coherence theorem. *Let $X \subset U$ be a closed analytic subset of an open set $U \subset \mathbb{C}^n$. The vanishing ideal system $\mathfrak{A}$ is coherent.* CART

To see the equivalence one has to have in mind that the support of a coherent ideal system $\mathfrak{a}$ is a closed analytic set and that by the nullstellensatz the radical of $\mathfrak{a}$ is the complete vanishing ideal system $\mathfrak{A}$. One also has to use the trivial fact the every analytic set locally is the support of a coherent system. Another formulation is

4.3 Cartan's coherence theorem. *Let $\mathfrak{a}$ be coherent system of ideals. The set of all points a such that $\mathfrak{a}_a = \operatorname{rad} \mathfrak{a}_a$ is open.* CARTA

We show that 4.3 implies 4.2. Let $a \in U$ a point. The ideal $\operatorname{rad} \mathfrak{a}_a$ is finitely generated. Therefore there exists a coherent system $\mathfrak{b}$ on an open neighbourhood $a \subset V \subset U$ auch that $\mathfrak{b}_a = \operatorname{rad} \mathfrak{a}_a$ and $\mathfrak{a}_b \subset \mathfrak{b}_b \subset \operatorname{rad} \mathfrak{a}_b$. Now 4.3 implies that in a full neighbourhood $\mathfrak{b}_b = \operatorname{rad} \mathfrak{a}_b$. The conclusion 4.2 $\Rightarrow$ 4.3is also clear. One uses the fact that two coherent systems which agree in a point agree in a full neighbourhood.

The rest of this section is dedicated the proof of Cartan's theorem. We need some preperations:

In a first step we give a reduction. We can assume that the origin is contained U and that $\mathfrak{a}_0 = \operatorname{rad} \mathfrak{a}_0$. We have to prove that $\mathfrak{a}_a = \operatorname{rad} \mathfrak{a}_a$ in a full neighbourhood of 0. We want to show that it is enough to treat the case of a prime ideal $\mathfrak{a}_0$. For this we use again the fact that any reduced ideal is the intersection of finitely many prime ideals. Because any ideal is finiteley generated we can find (in a small neighbourhood of 0) coherent systems $\mathfrak{a}^{(1)}, \ldots, \mathfrak{a}^{(m)}$ such that

$$\mathfrak{a}_0 = \mathfrak{a}_0^{(1)} \cap \ldots \cap \mathfrak{a}_0^{(m)}.$$

From our assumption we know that the $\overset{(j)}{a}$ are reduced (in a small neighbourhood). We also know from Oka's coherence theorem that the intersection system $\mathfrak{a}^{(1)} \cap \ldots \cap \mathfrak{a}^{(m)}$ is coherent. Tis intersection system and $\mathfrak{a}$ agree in the origin and hence in a full neighbourhood,

$$\mathfrak{a}_a = \mathfrak{a}_a^{(1)} \cap \ldots \cap \mathfrak{a}_a^{(m)}.$$

Using the trivial fact that the intersection of reduced ideals is reduced we obtain that the $\mathfrak{a}_a$ are reduced.

From now on we assume that $0 \in U$ and that

$$\mathfrak{P} := \mathfrak{a}_0$$

is a prime ideal. We will show that $\mathfrak{a}_a$ is reduced in a neighbourhood of 0. We need some preperations for the proof:

An element a of a ring R is called non-zero-divisor if multiplication with a

$$R \longrightarrow R, \quad x \longmapsto ax,$$

is injective.

4.4 Lemma. *Let $\mathfrak{a}$ be a coherent system on an open set $U \subset \mathbb{C}^n$ and let $f \in \mathcal{O}(U)$ be an analytic function on U. The set of all points $a \in U$ such that the germ f_a is a non-zero-divisor in $\mathcal{O}_{U,a}$ is open* nNof

Proof. We denote the map "multiplication by a" by

$$m_f : \mathcal{O}_{U,a} \longrightarrow \mathcal{O}_{U,a}.$$

The element f_a is non-zero-divisor if and only if

$$m_f^{-1}(\mathfrak{a}_a) = \mathfrak{a}_a.$$

From Oka's coherence theorem we know that the system $\left(m_f^{-1}(\mathfrak{a}_a)\right)_{a\in U}$ is coherent. The coincidence set of two coherent systems is open. □

After this preperations the proof of Cartan's theorem runs as follows. Recall that $0 \in U$ and that $\mathfrak{P} = \mathfrak{a}_0$ is a prime ideal. We have to show that $\mathfrak{a}_a$ is reduced in a full neighbourhood of 0. We distinguish the two "alternatives".

1. *Alternative.* $\mathfrak{P} = (P)$ is a principal ideal. Te element P is a prime element, especially square free. The theory of the *discrimant* gave us that there exists a small polydisc around 0 in which P converges and such that P_a is square free in this polydisc. Coherence gives us that $\mathfrak{a}_a = (P_a)$ in a full neighbourhood. But a principal ideal generated ba a square free element is reduced. What we see that in the case of hypersurfaces the properties of the discriminant imply Cartan's theorem (s. I.4.7). □

2. *Alternative.* This case is more involved. We will have to use the special case od Grauert's projection theorem. As usual we can assume that $\mathfrak{P} = gota_0$ is z_n-general and that

$$\mathcal{O}_{n-1}/\mathfrak{p} \longrightarrow \mathcal{O}_n/\mathfrak{P} \qquad (gotp := \mathcal{O}_{n-1} \cap \mathfrak{P})$$

have the same field of fractions. The ideal $\mathfrak{P}$ is finitely generated,

$$\mathrm{P} = (Q_1, Q_2, \ldots, Q_m).$$

We can assume that $Q := Q_1$ is a Weierstrass polynomial and then by the division theorem that all Q_i are polynomials over $\mathcal{O}_{n-1}$. We can take U in the form $U = V \times (-r, r)$, where $V \subset \mathbb{C}^n$ is a polydisc around 0. We can assume that the coefficients of the Q_j converge in V and that the zeros of the polynomial $z \mapsto Q(b, z)$ for all $b \in V$ have absolute value $< r$. From the special case of Grauert's projection theorem we obtain that the system

$$\mathfrak{b}_b = \mathcal{O}_{V,b} \cap \bigcap_{a=(b,\beta),\ Q(a)=0} \mathfrak{a}_a$$

is coherent on V. Because Q is a Weierstrass polynomial we have

$$\mathrm{b}_0 = \mathfrak{p}.$$

We want to prove Cartan's theorem by induction on n. Therefore we cann assume that alle the projected ideals $\mathfrak{b}_b$ are reduced. We will make use of the natural homomorphism

$$\mathcal{O}_{V,b}/\mathfrak{b}_b \longrightarrow \prod_{a=(b,\beta),\ Q(a)=0} \mathcal{O}_{U,a}/\mathfrak{a}_a.$$

It is quite clear that that this homomorphism is an injection. In the case $a = 0$ this is the homomorphism

$$\mathcal{O}_{n-1}/\mathfrak{p} \longrightarrow \mathcal{O}_n/\mathfrak{P}.$$

Now we make use of the basic fact that the fields of fractions of both rings agree. We find elements

$$A, B \in \mathcal{O}_{n-1}, \quad A \notin \mathfrak{p}, \quad Az_n - B \in \mathfrak{P}.$$

We can assume that A and B converge in V and furthermore because of coherence

$$A_b(z_n - a) - B_b \in \mathfrak{a}_a \qquad (a = (b,\beta) \in U).$$

We have to combine this fact that $\mathcal{O}_{U,a}/\mathfrak{a}_a$ is a module of finite type*) over $\mathcal{O}_{V,b}/\mathfrak{b}_b$. More precisely it is generated by the powers

$$(z_n - a_n)^\nu, \quad 0 \leq \nu < d,$$

where d is the z_n-degree of Q. Now we consider the analytic function $f := A^d$ on U. The germ f_0 defines a non-zero element of $\mathcal{O}_{U,0}/gotP$ and hence non-zero-divisor, because this ring is an integral domain. Because of the coherence result 4.4 we can assume that the multiplication map $m_f : \mathcal{O}_{U,a}/\mathfrak{a}_a \to \mathcal{O}_{U,a}/\mathfrak{a}_a$ is injective of all a. This map is no ring homomorphism but it is good enough to test nilpotency: First we collect all points $a = (b,\beta)$ over a given b and consider

$$m_f : \prod_{a=(b,\beta),\ Q(a)=0} \mathcal{O}_{U,a}/\mathfrak{a}_a \longrightarrow \prod_{a=(b,\beta),\ Q(a)=0} \mathcal{O}_{U,a}/\mathfrak{a}_a.$$

The construction of A shows that the image of m_f is already contained in the subring

$$\mathcal{O}_{V,b}/\mathfrak{b}_b \hookrightarrow \prod_{a=(b,\beta),\ Q(a)=0} \mathcal{O}_{U,a}/\mathfrak{a}_a.$$

The proof of Cartan's theorem now can be completed as follows: Let $C \in \prod_{a=(b,\beta),\ Q(a)=0} \mathcal{O}_{U,a}/\mathfrak{a}_a$ be a nilpotent element, $C^k = 0$. Then $m_f(C^k) =$

*) This true because $Q_a \in \mathfrak{a}_a$ is a normalized polynomial, hence z_n-general, hence the product of a unit and a Weierstrass polynomial of degree $\leq d$.

$f_a C^k = 0$. But this implies $(f_a C)^k = 0$. We recall that $m_f(C) = f_a C$ is contained in the subring $\mathcal{O}_{V,b}/\mathfrak{b}_b$. But this ring is reduced (by our induction hypothesis). Hence $m_f(C) = 0$. But m_f is injective (!) and we obtain $C = 0$. Hence the ring $\prod_{a=(b,\beta),\, Q(a)=0} \mathcal{O}_{U,a}/\mathfrak{a}_a$ is free of nilpotents and the same is true for each of its factors. This completes the proof of Cartan's coherence theorem. □

Because of the importance of this theorem we formulate again the decisive consequence:

4.5 Theorem. *Every analytic set can be written locally as the set of common zeros of a finite system of analytic functions* vollV

$$f_1, \ldots, f_m : U \longrightarrow \mathbb{C} \qquad (U \subset \mathbb{C}^n \text{ open}),$$

such that the germs in any point $a \in U$ generate the ***full vanishing ideal*** *in $\mathcal{O}_{U,a}$.*

Chapter IV. The singular locus

singular locus

1. Analytic sets and analytic mappings

Time is ripe to define the notion of an analytic map between analytic sets:

1.1 Definiton. *A map* defH

$$f : X \longrightarrow Y, \qquad X \subset \mathbb{C}^n,\ Y \subset \mathbb{C}^m,$$

is called analytic (holomorphic) if every point $a \in X$ admits an open neighbourhood $a \in U \subset \mathbb{C}^n$, such that $f|X \cap U$ is the restriction of a holomorphic map $U \to \mathbb{C}^m$

A special case is the notion of an analytic function $f : X \to \mathbb{C}$. The composition of analytic maps is analytic. An analytic map $f : X \to Y$ is called *biholomorphic*, if it is bijective and if f and f^{-1} both are analytic.

We know already that every analytic set $X \subset \mathbb{C}^n$ is contained and closed in some open subset $U \subset \mathbb{C}^n$. In the following we will use this set U very often because of

1.2 Remark. *Let $U \subset \mathbb{C}^n$ be an open set and X, Y be two closed analytic subsets of U. Then $X \cap Y$ and $X \cup Y$ are analytic too.* ClAb

The proof is easy and left to the reader. We should mention that U and $\emptyset$ are special cases of closed analytic subsets of U. Another less more trivial remark is

1.3 Remark. *Let $f : X \to Y$ be an analytic map of analytic sets and let $Z \subset Y$ be an analytic subset of Y. Then $f^{-1}(Z)$ is an analytic set.* invBi

A special case says that a subset $Y \subset X$ of an analytic set which is defined as common set of zeros of a finite set of analytic functions on X is also an analytic set.

The local ring of a pointed analytic set

let (X, a) be a pointed analytic set, i.e. an analytic set X together with a distinguished point $a \in X$. We introduce the ring $\mathcal{O}_{X,a}$ as in the case of open subsets of $\mathbb{C}^n$: The elements of $\mathcal{O}_{X,a}$ are represented by analytic functions $f : U \to \mathbb{C}$, where $U \subset X$ is an open neighbourhood of a in X. Two pairs (U, f) and (V, g) define the same germ in $\mathcal{O}_{X,a}$ if there exists a neighbourhood $W \subset U \cap V$ with the property $f|W = g|W$. Again we denote by f_a the image of (U, f) in $\mathcal{O}_{X,a}$, which by the way carries a natural structure as $\mathbb{C}$-algebra. Let be $f : X \to Y$ be an analytic map of analytic set and $a \in X$ a point and $b = f(a)$. We obtain a canonical homomorphism

$$f^* = f_a^* : \mathcal{O}_{Y,b} \longrightarrow \mathcal{O}_{X,a}.$$

If $g_b \subset \mathcal{O}_{Y,b}$ is a germ which is represented by a pair (V, g) then $f^*(g)$ is represented by $g \circ f$, which is defined on some open neighbourhood oaf a in X. A special case is the case of an open subset $U \subset X$ and the natural inclusion $U \hookrightarrow X$. In this case the natural map $\mathcal{O}_{X,a} \to \mathcal{O}_{U,a}$ is an isomorphism of $a \in U$. Usually we will identify $\mathcal{O}_{X,a}$ and $\mathcal{O}_{X,a}$.

Another important case is as follows: let $U \subset \mathbb{C}^n$ be an open subset and $X \subset U$ a closed analytic subset. The natural homomorphism

$$\mathcal{O}_{U,a} \longrightarrow \mathcal{O}_{X,a}$$

is surjective for $a \in X$ and the kernel is the vanishing ideal $\mathfrak{a}$ of X in a. We obtain

$$\mathcal{O}_{X,a} = \mathbb{C}\{z_1 - a_1, \ldots, z_n - a_n\}/\mathfrak{a}$$

and especially

1.4 Remark. *Rings of germs $\mathcal{O}_{X,a}$ of analytic sets are analytic algebras.* rGrA

The Noether normalization theorem encourages to give the following definition:

1.5 Definition. *The dimension of an analytic set X ind a point a is the dimension of the ring $\mathcal{O}_{X,a}$.* defD

$$\dim_a X = \dim \mathcal{O}_{X,a}.$$

The dimension of $\mathbb{C}^n$ in any point is of course is n because —as we know— the ring $\mathcal{O}_n$ of power series has dimension n. If $A \to B$ is a surjective homomorpism of rings then clearly $\dim B \leq \dim A$. This shows:

1.6 Remark. *Let $Y \subset X$ be closed analytic sets in an open domain $U \subset \mathbb{C}^n$. Then* abSc

$$\dim_a Y \leq \dim_b X \leq n$$

for every point $a \in Y$.

The estimate $\leq n$ in this remark gives the possibility to define

1.7 Definition. *The dimension of a non-empty analytic set X is* globD

$$\dim X := \max_{a \in X} \dim_a X.$$

The following two observations give a hint that our dimension concept is correct: In Chapter I we gave already a definition of dimension in smooth points. We have of course to verify:

1.8 Remark. *Let X be an analytic set and a be a smooth point of X. Then the original geometric definition I.1.9 of $\dim_a X$ and our new one agree.* aLN

The proof is trivial. □

1.9 Lemma. *Let $X \subset \mathbb{C}^n$ be an analytic hypersurface. Then $\dim_a X = n - 1$ for all $a \in X$.* hypD

This has been proved already for smooth points. But is true for arbitrary points: For the proof one can assume that a is the origin and that the vanishing ideal of X at a is generated by a Weierstrass polynomial Q. It is clear that $(Q) \cap \mathcal{O}_{n-1} = 0$. Therefore $\mathcal{O}_n/(Q)$ is a ring extension of $\mathcal{O}_{n-1}$. This extension is module-finite. From the theorem of Cohen Seidenberg we obtain the claim. □

Appendix: Germs of analytic setsgerm

We will show in this appendix that the local theory of analytic sets and the theory of analytic algebras is less more the same. This appendix is not needed for what follows but it is useful to have it in mind:

We want to use a little commutative algebra. Recall that the product $\mathfrak{ab}$ of two ideals $\mathfrak{a}$, $\mathfrak{a}$ in a ring R is the set of all finite sums $\sum a_j b_j$, where $a_j \in \mathfrak{a}$ and $b_j \in \mathfrak{b}$. It is clear how to define the powers $\mathfrak{a}^n$ for any natural number n. We want to use the following theorem of Krull:

Let R be a noetherien local ring and $\mathfrak{m}$ be the maximal ideal of R. Then

$$\bigcap_{n>0} \mathfrak{m}^n = 0.$$

A consequence of Krulls theorem is:

1.10 Lemma. *Let* locEi

$$f, g : \mathbb{C}\{z_1, \ldots, z_n\} \to R$$

be two local homomorphisms of the ring of power series into a local ring R. The two homomorphisms agree if $f(z_j) = g(z_j)$ for $1 \leq j \leq n$.

Proof. Let m be a natural number. Every power series P can be written as $P = A + B$ where A is a polynomial in the z_j and $B \in \mathfrak{m}_n^m$. We obtain that $f(P) - g(P)$ is contained in $\mathfrak{m}^m$, where $\mathfrak{m}$ denotes the maximal ideal of R. Krulls theorem gives $f(P) = Q(P)$.

Let $P_1, \ldots, P_m \in \mathbb{C}\{z_1, \ldots, z_n\}$ be power series. They define a holomorphic map

$$U \longrightarrow \mathbb{C}^m, \quad z \longmapsto (P_1(z), \ldots, P_m(z)),$$

which is defined on some polydisc U around 0. We obtain an induced homomorphism

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{C}^n,0} & \longrightarrow & \mathcal{O}_{\mathbb{C}^m,0} \\ \| & & \| \\ \mathbb{C}\{w_1, \ldots, w_m\} & \longrightarrow & \mathbb{C}\{z_1, \ldots, z_n\}. \end{array}$$

We denote the image of an $P \in \mathbb{C}\{w_1, \ldots, w_m\}$ under this homomorphism by $P(P_1, \ldots, P_m)$. It can be proved for example by means of Taylor's formula that $P(P_1, \ldots, P_m)$ can be obtained by formal reordering. We call such a homomorphism a natural one.

1.11 Remark. *Every algebra homomorphism $\mathbb{C}\{w_1, \ldots w_m\} \to \mathbb{C}\{z_1, \ldots, z_n\}$ is a natural one.* natH

The proof follows from 1.10. But this result enters here only in a trivial way, because Krulls intersection theorem is trivial for $\mathbb{C}\{z_1, \ldots, z_n\}$. But Krulls theorem is not trivial for arbitrary analytic algebras. Using 1.10 we obtain the following result.

1.12 Proposition. *Let $f : A \to B$ be a homomorphism of analytic algebras. We assume that we have surjective homomorphism of algebras* natHO

$$\mathbb{C}\{w_1, \ldots, w_m\} \longrightarrow A, \qquad \mathbb{C}\{z_1, \ldots, z_n\} \longrightarrow B.$$

There exists a commutative diagram

$$\begin{array}{ccc} \mathbb{C}\{w_1, \ldots, w_m\} & \longrightarrow & \mathbb{C}\{z_1, \ldots, z_n\} \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

where the first row is a natural homomorphism.

We recall that we know already that homomorphisms of analytic algebras (defined as $\mathbb{C}$-linear ring homomorphisms) are automatically local. The proof is quit clear. One considers images of w_j in A and then in B. They have preimages P_j in $\mathbb{C}\{z_1, \ldots, z_n\}$ which must lie in the maximal ideal. The natural homomorphism $w_j \mapsto P_j$ has the desired property. □

We give a geometric application of this result. Let A be an analytic algebra. A pointed analytic set (X, a) together with an isomorphism of algebras $\mathfrak{O}_{X,a} \to A$ is called a *geometric realization* of A. Occasionally we will identify $\mathfrak{O}_{X,a}$ and A using this isomorphism.

1.13 Theorem. *An analytic algebra admits a geometric realization if and only if it is reduced. If $\varphi : B \to A$ is a homomorphism of analytic algebras and if (X, a) resp. (Y, b) are geometric realizations of A resp. B. Then there exist an open neighbourhood $a \in U \in X$ and a holomorphic map $f : X \to Y$ with $f(a) = b$ such that $f_a = \varphi$.* dualH

Proof. The first part follows from the nullstellensatz, the second easily from 1.12.

A consequence of 1.13 says:

1.14 Corollary. *Let (X, a) and (Y, b) be two pointed analytic sets. The algebras $\mathcal{O}_{X,a}$ and $\mathcal{O}_{Y,b}$ are isomorphic if and only if there exist open neighbourhoods $a \in U \subset X$, $b \in V \subset Y$ and a biholomorphic map $\varphi : U \to V$ with $\varphi(a) = b$.* locIS

2. Dimension of analytic sets and coherence

We want to study the local behaviour of the dimension $\dim_a X$ for varying a.

2.1 Lemma. *Let (X,a) be a pointed analytic set. There exists a neighbourhood $a \in U \subset X$ such that* dimHAL

$$\dim_b X \leq \dim_a X \quad \text{for all} \quad b \in U.$$

Proof. The proof uses Noether normalization: We can assume $X \subset \mathbb{C}^n$ and $a = 0$. The vanishing ideal $\mathfrak{a}$ of X in a can be assumed z_n-general, We denote by $\mathfrak{b} = \mathfrak{a} \cap \mathcal{O}_{n-1}$ the projected ideal. let Y be a geometric realization of Y. We can assume that the projection (cancellation of the last variable) defines a mapping $f : X \to Y$. We can assume that $\mathfrak{a}$ contains a Weierstrass polynomial $Q \in \mathcal{O}_{n-1}[z_n]$, whose coefficients converge on a polydisc which contains Y. Furthermore we can assume that for all $a \in X$ the polynomal $Q(a_1, \ldots, a_{n-1}, z_n - a_n)$ is not identical 0, hence general in $\mathbb{C}\{z_1 - a_1, \ldots, z_{n-1} - a_{n-1}\}[z_n - a_n]$. This implies that the ring homomorphism

$$f_a^* : \mathcal{O}_{Y,f(a)} \longrightarrow \mathcal{O}_{X,a}$$

is modul-finite for all $a \in X$. This homomorphism is not surjective but from Cohen Seidenberg we obtain still

$$\dim \mathcal{O}_{Y,f(a)} \geq f_a^*(\dim \mathcal{O}_{Y,f(a)}) = \mathcal{O}_{X,a}.$$

For $a = 0$ the homomorphism is injective, i.e.

$$\dim \mathcal{O}_{Y,0} = f_a^*(\dim \mathcal{O}_{Y,0}) = \mathcal{O}_{X,0}.$$

This comes from the fact that $\mathfrak{b}$ is a radical ideal and hence the full vanishing ideal. We will proof 2.1 by induction on n and can therefore assume

$$\dim_0 Y \geq \dim_b Y \qquad (b \in Y).$$

We obtain

$$\dim_0 X = \dim_0 Y \geq \dim_{f(a)} Y \geq \dim_a X,$$

which completes the proof of lemma 2.1. □

The next lemma uses coherence:

2.2 Lemma. *Let (X,a) be a pointed analytic set and $f : X \to \mathbb{C}$ an analytic function on X. We assume thet the germ f_a is a non-zero divisor in $\mathcal{O}_{X,a}$. Then there exists an open neighbourhood $a \in U \subset X$ such that the zero locus* duN

$$Y := \{\, x \in U; \quad f(x) = 0 \,\}$$

is thin in U.

Proof. We can assume that f_b is non-zero divisor in $\mathcal{O}_{X,b}$ for all $b \in X$. This implies that f does not vanish on X in any small neighbourhood of b. So in any neighbourhood of b there exist points which belong to Y but not to X. □

An analytic set $X \subset \mathbb{C}^n$ is called *integer* in a point $a \in X$, if the local ring $\mathcal{O}_{X,a}$ is integer, or which means the same, if the vanishing ideal is a prime ideal. From 2.2 follows

2.3 Lemma. *Let $Y \subset X \subset cz^n$ be two analytic sets, and $a \in Y$ a distinguished point. We assume* intDue

a) *X is integer in a.*
b) *The vanishing ideals of Y and X in a are different.*

Then there exists an open neighbourhood $a \in U \subset \mathbb{C}_n$, such that $Y \cap U$ is thin in $X \cap U$.

During the proof of the nullstellensatz we used a certain projection technique which was basic in several contexts. By means of 2.3 (which was a consequence of the coherence theoerems) we can reformulate II.3.2 as follows:

2.4 Proposition. *Let $\mathfrak{P} \subset \mathcal{O}_n$ be a z_n-general prime ideal such that the two rings* bihDu

$$\mathcal{O}_{n-1}/\mathfrak{p} \hookrightarrow \mathcal{O}_n/\mathfrak{P} \qquad (\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_{n-1})$$

have the same field of fractions. Then there exists a geometric realization $X \to Y$ of $(\mathfrak{P}, \mathfrak{p})$ with the following property: There exist closed an thin analytic subsets $S \subset Y$, $T \subset X$, such theta the restriction of f defines a biholomorphic map

$$X - T \xrightarrow{\sim} Y - S.$$

If $V \subset Y$ runs through a fundamental system of neighbourhoods of $0 \in Y$, then $U = f^{-1}(V) \subset X$ runs through a fundamental system of neighbourhoods of $0 \in X$.

An analytic set is called *pure dimensional*, if the dimension $\dim_a X$ is independent of $a \in X$. It is an important fact that analytic sets are pure dimensional around any integer point.

2.5 Proposition. *Let $X \subset \mathbb{C}^n$ be an analytic set, which is integer in $a \in X$. There exists an open neighbourhood $a \in U \subset X$ which is pure dimensional.* purD

Proof. We use induction by n. we can assume $a = 0$. Let $\mathfrak{P}$ be the vanishing ideal. We distinguish the "two alternatives".

1. *Aternative.* $\mathfrak{P}$ is a principal ideal. Then we can use the theory of hypersurfaces.

2. *Alternative.* $\mathfrak{P}$ is not a principal ideal. We use proposition 2.4. We cann assume (2.1) $\dim_a X \leq \dim_0 X$ for all $a \in X$ and by induction $\dim_b Y = \dim_0 Y$ for all $b \in Y$. Let now $a \in X$ be an arbitrary point. Because T is thin, we find in any neighbourhood of a a point $x \in X - T$. Because of 2.1 we can assume $\dim_x X \leq \dim_a X$. We obtain

$$\dim_0 X \geq \dim_a X \geq \dim_x X = \dim_{f(x)} Y = \dim_0 Y = \dim_0 X.$$

This completes the proof of 2.5.

An important result of Krull dimension theory is:

Let R be a noetherian local ring and $a \in R$ a non-zero divisor. Then

$$\dim R/(a) = \dim R - 1.$$

If $\mathfrak{a}$ is an ideal which contains a non-zero divisor we obtain

$$\dim R > \dim R/\mathfrak{a}.$$

An obvious application is

2.6 Proposition. *Let $Y \subset X$ be analytic sets and X be integer in a certain point $a \in Y$. Assume* agR

$$\dim_a Y \geq \dim_a X.$$

Then X and Y agree in a full neighbourhood of a.

We have seen that it is often useful to reduce statements about radical ideals to prime ideals. This is possible because every radical ideal is the intersection of finitely many prime ideals. We describe the geometric counterpart of this algebraic fact in more detail:

Local irreducible components

Let R be a noetherian ring. A prime ideal $\mathfrak{p}$ which contains a given ideal $\mathfrak{a}$ is called *minimal* with this property, if any prime ideal $\mathfrak{q}$, $\mathfrak{a} \subset \mathfrak{q} \subset \mathfrak{p}$, agrees with $\mathfrak{p}$. A refinement of the above statement about radical ideals is:

Let $\mathfrak{a}$ be an ideal in a noetherian ring R. There exist only finitely many minimal prime ideals containing $\mathfrak{a}$. Their intersection is $\operatorname{rad}\mathfrak{a}$. *Every prime ideal whcih contains $\mathfrak{a}$ contains one of the minimals.*

Now we consider the geometric counter part of this decomposition: Let X be a closed analytic set in an open subset $U \subset \mathbb{C}^n$. We want to study local properties of X at a given point $a \in X$ (and allow therefore to replace U by a smaller neighbourhood if neccessary). We represent the vanishing ideal $\mathfrak{a} \subset \mathcal{O}_{U,a}$ as the intersection of pairwise distinct minimal prime ideals

$$\mathfrak{a} = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_m.$$

We can assumt htat there are closed analytic sets $X_j \subset U$ with vanishing ideals $\mathfrak{p}_j$. The set

$$X_1 \cup \ldots \cup X_m \subset U$$

is analytic an its vanishing ideal at a is $\mathfrak{a}$. Hence we obtain (eventually after replacing U by smaller neighbourhood) that

$$X = X_1 \cup \ldots \cup X_m.$$

We call the X_j the local irreducible components of X at a. They are unique upto ordering and in an obvious local sense.

2.7 Lemma. *Let (X,a) be a pointed analytic set and* dimIR

$$X = X_1 \cup \ldots \cup X_m$$

be a decomposition into the local irreducible components of X at a. Then

$$\dim_a X = \max_{1\leq j\leq m} \dim_a X_j.$$

if $Y \subset X$ be an analytic subset which contains a and which is integer at a. At least after replacing X by a small neighbourhood of a the set Y is contained in one of the components X_j.

Proof. The dimension of X at a is defined by means of sequences of prime ideals in $\mathcal{O}_{X,a}$. Let $\mathfrak{a} \subset \mathcal{O}_n$ be the vanishing ideal of X at a. The chains of prime ideals in $\mathcal{O}_{X,a}$ correspond to chains

$$\mathfrak{a} \subset \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_m \subset \mathcal{O}_n.$$

The ideal $\mathfrak{p}_0$ must contain one of the minimal prime ideals containing $\mathfrak{a}$. This proofs the statement about the dimension. The last statement is also clear because the vanishing ideal of Y in a must contain one of the minimal prime ideals containing $\mathfrak{a}$. □

2.8 Lemma. *Let $X \subset U$ a closed analytic set of an open domain $U \subset \mathbb{C}^n$* notIn
and let be

$$X = X_1 \cup \ldots \cup X_m \quad (X_i \subset U \text{ closed})$$

a decomposition into irreducible components in a point $a \in X$. We consider the analytic set

$$T := \bigcup_{1\leq i<j\leq m} (X_i \cap X_j).$$

If one replaces U by a small open neighbourhood of a one has:

1. *$T \cap X_k$ is thin in X_k for $1 \leq k \leq m$.*
2. *X is not integer in any point of T.*

Proof. The finite union of thin sets is thin. Therefore for 1. we have to prove that $X_i \cap X_j \cap X_k$ is thin in X_k for any k. This follows from 2.2. To prove 2. we consider a point $b \in T$. There exist at least to different j such that $b \in X_j$. Let $\mathfrak{a}_b \subset \mathcal{O}_n$ resp. $\mathfrak{a}_b^{(j)} \subset \mathcal{O}_n$ be the vanishing ideals of X resp. X_j in b. We know from the first part of the lemma that the two ideals are different. Therefore there exist

$$P_j \in \mathfrak{a}_b^{(j)}, \quad P_j \notin \mathfrak{a}_b \qquad (b \in X_j).$$

The product of the P_j (for all j with $b \in X_j$ defines the zero element in $\mathcal{O}_{X,b}$ but the individual f_j not. This means that we found a zero divisor in $\mathcal{O}_{X,b}$.

□

3. The singular locus of an analytic set is analytic

We intoduce the notion of smooth and singular points of an analytic set. We introduced also the notion of biholomorphic mappngs between analytic sets. If a is a smooth point of an analytic set X, then there exists a biholomorphic map of an open neighbourhood $a \in U \subset X$ onto an open neighbourhood $V \subset \mathbb{C}^n$, where d is the dimension of X at a. This follows immediately from the definition of a smooth point. The converse is also true:

3.1 Remark. *Let (X, a) be a pointed analytic set. The following two statements are equivalent:* regequ

1. *The set X is smooth in a and* $\dim_a X = d$.
2. *There exists a biholomorphic map of an open neighbourhood $a \in U \subset X$ onto an open neighbourhood $V \subset \mathbb{C}^d$*

Proof. It remains to prove that 2. implies 1. We consider X as analytic set in $\mathbb{C}^n$. Taking the inverse of the biholomorphic map $U \to V$ we obtain a holomorphic map

$$F : V \xrightarrow{\sim} U \longrightarrow \mathbb{C}^n.$$

We can assume that $a = 0$. We have corresponding homomorphisms

$$\mathcal{O}_n \longrightarrow \mathcal{O}_{X,0} \xrightarrow{\sim} \mathcal{O}_{V,d} = \mathcal{O}_d$$

which are surjective. This means that we can find power series $P_1, \ldots, P_d \in \mathcal{O}_n$ with the property

$$P_j(F_1(z_1, \ldots, z_d), \ldots, F_n(Z_1, \ldots, z_d)) = z_j \quad (1 \leq j \leq d)$$

in a small neighbourhood of 0. From the chain rule follows that the rank of the Jacobian of F in the origin is d. The statement now follows from the following

3.2 Lemma. *Let $V \subset \mathbb{C}^d$ an open neighbourhood of the origin and let $F : V \to \mathbb{C}^n$ be a holomorphic map such that the rank of the Jacobian in the origin is d (which is the maximal possible value). Then there exists an open neighbourhood $0 \in W \subset V$ such that $X := F(W)$ is a everywhere smooth analytic set and that the restriction of F defines a biholomorphic map $W \to X$.* imB

Proof. One considers a linear map

$$L : \mathbb{C}^{n-d} \longrightarrow \mathbb{C}$$

The function $G(z, w) := F(z) + L(w)$ $(w \in \mathbb{C}^{n-d})$ has an invertible Jacobiam at the origin for suitable choice of L. This follows from the fact that every $(d \times n)$-matrix of rank d is part of an invertible $(n \times n$-matrix. By the theorem

of invertible functions G is biholomorphic on suitable neighbourhoods of the origin. The image of F corresponds to the set $w = 0$. □

We prepared now all tools to describe the sigular locus of an analytic set. We recall that the zero locus of an analytic map

$$F : U \longrightarrow \mathbb{C}^d, \qquad 0 \in U \subset \mathbb{C}^n \text{ open.}$$

is smooth at 0 if the rank of the Jacobian of F in the origin is d. One of the problems come from the fact that this condition is only sufficient but not necessary. (Consider the equation $z^2 = 0$.) But there are converse results. Before we can formulate and prove them we need another very simple

3.3 Lemma. *Let $P = (P_1, \ldots, P_m)$ nad $Q = (Q_1, \ldots, Q_l)$ be two systems of power series which generate the same ideal in $\mathcal{O}_n$. Then the Jacobians of P and Q at the origin have the same rank.* eqRa

The proof is left to the reader. Now we come to the announced converse criterion:

3.4 Proposition. *Let (X, a) be a pointed analytic set which is defined by analytic equations* ifaO

$$f_1(z) = \cdots = f_m(z) = 0$$

in some open neighbourhood $0 \in U \subset \mathbb{C}^n$. We assume that the germs of the f_j in a generate the whole vanishing ideal of X in $\mathcal{O}_{\mathbb{C}^n,a}$. Then a is a smooth point of X ***if and only if*** *the rank of the Jacobian of $f = (f_1, \ldots, f_m)$ in the origin is $n - d$, where $d = \dim_a X$.*

If X is smooth at a there exists a system of defining equations with this property. But the previous lemma 3.3 shows that the rank is independent of the generating system as long as it generates the whole vanishing ideal. Hence we have to assume the converse. We assume that the rank condition is fulfilled. There exist $n - d$ functions of the f_j, such that the Jacobian of this subsystem at 0 is also $n - d$. We can assume that the subsystem is $f_1, \ldots, f_{n-d}$. The zero locus of this system $\tilde{X}$ contains X. The set $\tilde{X}$ is smooth at a and of dimension d. The equation $\dim_a X = \dim_a \tilde{X}$ shows that X and $\tilde{X}$ agree near a. □

Now we are in the state to prove the main result of local complex analyis:

3.5 Theorem. *The singular locus S of an anayltic set is a thin closed analytic subset of X* HAUPT

We will apply the criterion 3.4. But there is still one difficulty. The dimension $\dim_a X$ of an analytic set can jump. So we first reduce the statement to the pure dimensional case. We fix a point $a \in X$. It is sufficient to prove the theorem for some open neighbourhood of a instead of X. Therefore we can assume that X decomposes into irreducible components in a,

$$X = X_1 \cup \ldots \cup X_m.$$

The set

$$T := \bigcup_{1 \le i < j \le m} (X_i \cap X_j).$$

is a thin closed analytic subset which is contained in the singular locus of X. Hence the singular locus of X is the union of T and the singular locuses of the X_j. The union of closed analytic sets being analytic we are reduced completely to the case that X is integer in a.

We assume now that X is integer at a and we assume that X is pure dimensional. Next we prove that in any neighbourhood of a there exist smooth points. We distinguish the "two alternatives". The first alternative is the case of a hypersurface. This case could be treated in a very early state*). The second alternative is that X is "bimeromorphic equivalentto an analytic set in $\mathbb{C}^{n-1}$. The precise sense of this is formulated in proposition 2.4. It reduces tha claim to $n-1$ instead of n and we can apply induction.

We come to the final an deepest step. We have stiil to prove that the singular locus is analytic. Here Cartan's coherence theorem is needed. We can assume that X is the zero locus of equations $F_1(z) = \cdots = F_m(z) = 0$ in some neighbourhood of the origin in $\mathbb{C}^n$ such that that the germs of the F_j generate the *full vanishing ideal in any point.* Recall that we are in the situation wher X is of pure dimension, let's say d. We know (3.4) that the singular locus now is described by the condition that the rank of the Jacobian is less than $n-d$. This means that the singular locus is the set of all points where the f_j vanish and where all subdeterminants of size $< (n-d)$ of the Jacobian vanish. This is a finte system of analytic equations. This competes the proof of 3.5. □

The insight which we obatined into the nature of the singular locus gives another charcterization of the dimension of an analytic set. Recall that we defined

$$\dim X = \max_{a \in X} \dim_a X.$$

We denote by X_{reg} the regular locus of X. Recall that the definition of the dimension of X_{reg} was rather trivial.

3.6 Remark. *Let X be an analytic set. Then* $\dim X = \dim X_{\text{reg}}$. *If $a \in X$ is an arbitrary point then* $\dim_a X = \min \dim U$, *where U runs through all open neighbourhoods of a in X.* dimRE

One can use this remark towards a new definition of the dimension of analytic sets even in singular points, which rests completely on the consideration of the regular locus. It seems that the concept of Krull dimension an be avoided. This is true but gives not at all a simplification because the deep results are not affected.

*) We used the theory of the discriminant. Instead of this one needs coherence in the general case.

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