Lecture Notes in Mathematics 1873

Roger Mansuy · Marc Yor

Random Times and Enlargements of Filtrations in a Brownian Setting

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Dedicated to the memory of J.L. Doob (1910 – 2004)
and P.-A. Meyer (1934 – 2003)

In the neighborhood of the second author's family home...

Preface

These notes represent accurately the contents of the six lectures we gave in the Statistics Department of Columbia University, between the 10th and the 25th of November 2004.

The audience was a mix of faculty, most of whom were fine “connoisseurs” of stochastic calculus, excursion theory, and so on, and graduate students who were basically acquainted with Brownian motion.

Our aim in teaching this course was two-fold:

- on one hand, to give the audience some familiarity with the theory and main examples of enlargements of filtrations, either of the initial or the progressive kinds;
- on the other hand, to update the relevant Chapters[1] of Part II [Yor97b] of the Zürich volumes, precisely, those which were devoted to martingale and filtration problems, i.e. **Chapters** 12 to 17 in Part II.

Each lecture was followed by an exercises session.
Here is the detailed organization of these lecture notes:

- as a set of **Preliminaries**, the basic operations of stochastic calculus and of the (Strasbourg) general theory of processes are recalled; no doubt that this is too sketchy, only a first aid tool kit is being presented, and the reader will want to read much more, e.g. [Del72] and the last volume, by Dellacherie, Maisonneuve and Meyer, of Probabilités et Potentiel [DMM92];
- in **Chapter** 1, the transformation of martingales in a “small” filtration into semimartingales in a bigger filtration is being studied; an important number of, by now, classical examples, drawn more or less from Jeulin's monograph [Jeu80] or Jeulin-Yor [JY85], are presented, and then collected in an appendix at the end of the chapter: this appendix consists in two

[1] An updated, revisited version of **Chapters** 1 to 11, corresponding to Part I [Yor92a], is being published in the Springer Universitext collection under the title *Aspects of Brownian motion* [MY05a].

tables, the first one for progressive enlargements, the second one for initial enlargements; we tried to gather there some most important examples, which often come up in the discussion of various Brownian path decompositions and their applications. This presentation is close to the effort made in the *Récapitulatif* in [JY85] pp. 305-313;

- in **Chapter** 2, we examine what remains of a number of classical results in martingale theory when, instead of dealing with a stopping time, one works up to a general random time;
- the main topic of **Chapter** 3 consists in the comparison of $\mathbb{E}[X|\mathcal{F}_\gamma]$ and $X_\gamma := \mathbb{E}[X|\mathcal{F}_t]_{|t=\gamma}$ where, for the simplicity of our exposition, γ is the last zero before 1 of an underlying Brownian motion, and X is a generic integrable random variable. Note how easily one may be confusing the two quantities, which indeed are identical when γ is replaced by a stopping time. Moreover, in our set-up with γ, one of these quantities is equal to 0 if and only if the other one is, and this remark leads naturally to the description of all martingales which vanish on the (random) set of the Brownian zeroes;
- **Chapter** 4 discusses the predictable and chaotic representation properties (abbreviated respectively as PRP and CRP) for a given martingale with respect to a filtration. Although the CRP is rarer than the PRP, a much better understanding of the CRP, and many examples, have been obtained since the unexpected discovery by Émery [Éme89] that Azéma's martingale enjoys the CRP. In particular, we introduce in this chapter the Dunkl martingales, which also enjoy the CRP.
- the two next **Chapters** 5 and 6 are devoted to questions of filtrations. They are tightly knit with the preceding chapters, e.g. in **Chapter** 5, Azéma's martingale plays a central role, and in **Chapter** 6, ends of predictable sets are being discussed in the framework of the Brownian filtration. In more details, the deep roots of **Chapter** 5 are to be found in excursion theory where, traditionally, a level, e.g. level 0, is being singled out from the start, and excursions away from this level are studied. It was then natural to consider how quantities and concepts related to a given level based excursion theory vary with that level. Two different suggestions for this kind of study were made, the first one by D. Williams, with following studies by J. Walsh and C. Rogers, the second one by J. Azéma, which provoked answers from Y. Hu. Both set-ups are being examined in **Chapter** 5.

  Chapter 6 develops our present understanding of the Brownian filtration, or rather, of some fundamental properties which are necessary for a given filtration to be generated by a Brownian motion. The results are due mainly to B. Tsirel'son, and collaborators, between 1996 and 2000 (roughly). In particular, it was established during this period that:
  - the filtration of a N-legged Brownian spider ($N \geq 3$) is not strongly Brownian.

  - there exist probability measures Q equivalent to Wiener measure, such that under Q, the natural filtration of the coordinate process is not strongly Brownian.

  Tsirel'son original "hands on" method of attack of these questions later developed into the search of "invariants of filtration", e.g. the notions of standard filtration, cosy filtration,..., which were studied by M. Emery and co-workers, and Tsirel'son himself, and which we briefly present at the end of **Chapter** 6.

- Each chapter ends with some exercises, which complement the content of that chapter. A standard feature of these exercises, as well as the style of their solutions, is an illustration of general "principles", which we present in the framework of explicit examples. The solutions-presented in **Chapter** 7 – are succinctly written, but should contain sufficient details for the reader. As much as possible, the arguments in the proposed solutions are closely connected with the material found in the corresponding chapters. We also took the opportunity to include some open questions, sometimes in the form of exercises, which are then indicated with the symbol ✠.

We are both very grateful for the warm hospitality we received during our stay in Columbia University as well as the strong motivation of the audience during the sessions. Thanks to everyone involved, and special thanks to Peter Bank and Ioannis Karatzas.

Paris,
November 3, 2005

Roger Mansuy
Marc Yor

Contents

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Notation and Convention

Here is a short list of current notation and convention used in the different chapters.

- We shall always assume that the underlying probability space $(\Omega, \mathcal{F}, P)$ is separable.
- Let $\mathcal{A}(\subset \mathcal{F})$ be a σ-field, X an $\mathcal{A}$-measurable random variable and Y a random variable independent of $\mathcal{A}$. Then, for any Borel function f, the conditional expectation $\mathbb{E}\left[f(X,Y)|\mathcal{A}\right]$ shall be denoted as $\hat{\mathbb{E}}\big[f(X,\hat{Y})\big]$. In other words, the expectation concerns the hat-variables with all others remaining *frozen*.
- We sometimes make the abuse of notation:
  $X \in \mathcal{A}$, meaning that the random variable X is $\mathcal{A}$-measurable.
- The symbol γ_t (resp. δ_t) will only be used to denote the last (resp. the first) zero of a certain process, usually Brownian motion, before (resp. after) the time t. We often abbreviate γ_1 and δ_1 by γ and δ.
- In general, to a process N, we associate $\overline{N}$, its one sided supremum process; namely, for $t \geq 0$, $\overline{N}_t := \sup_{s \leq t} N_s$. However, for Brownian motion $(B_t;\ t \geq 0)$, we keep the usual notation $(S_t;\ t \geq 0)$ for its one-sided supremum.
- In this book, studies of the law of a process $(X_t;\ t \geq 0)$ often begin with: "For any bounded functional F, $\mathbb{E}\left[F(X_s;\ s \leq t)\right]\ldots$". By this sentence, we mean that F is a measurable functional on $\mathcal{C}([0,t],\mathbb{R})$ if X is assumed to be continuous, on $\mathcal{D}([0,t],\mathbb{R})$ otherwise.
- $\mathcal{F}^{\sigma(X)}$ will denote the initial enlargement of the filtration $(\mathcal{F}_t;\ t \geq 0)$ with the random variable X, that is the filtration defined by

  $$\boxed{\mathcal{F}_t^{\sigma(X)} := \bigcap_{\varepsilon>0} \left(\mathcal{F}_{t+\varepsilon} \vee \sigma(X)\right), \qquad t \geq 0}$$

  We shall sometimes use the terminology: X-initial enlargement of $(\mathcal{F}_t; t \geq 0)$.
- For $\Lambda : \Omega \to [0,\infty]$, a random time, we denote by $\mathcal{F}^{\Lambda}$ the smallest filtration which contains $(\mathcal{F}_t;\ t \geq 0)$, and makes Λ a stopping time, i.e.

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$$\boxed{\mathcal{F}_t^{\Lambda} := \bigcap_{\varepsilon > 0} \left(\mathcal{F}_{t+\varepsilon} \vee \sigma(\Lambda \wedge (t+\varepsilon)) \right), \qquad t \geq 0}$$

We shall sometimes use the terminology: Λ-progressive enlargement of $(\mathcal{F}_t;\ t \geq 0)$.

- All martingales considered in this volume are assumed to be càdlàg (i.e. right-continuous and left-limited); in a number of cases, they are even assumed to be continuous, but this will always be specified.
- $\mathbf{e}$ (resp. $\mathcal{N}$) will often denote a standard exponentially distributed variable (resp. a standard normal variable).
- The symbol $\hookrightarrow$ (resp. $\not\hookrightarrow$) denotes immersion (resp. non-immersion) between two filtrations $(\mathcal{F}_t;\ t \geq 0)$ and $(\mathcal{G}_t;\ t \geq 0)$ such that $\mathcal{F}_t \subseteq \mathcal{G}_t$ for every t; $(\mathcal{F}_t;\ t \geq 0)$ is said to be immersed in $(\mathcal{G}_t;\ t \geq 0)$ if all $(\mathcal{F}_t;\ t \geq 0)$-martingales are $(\mathcal{G}_t;\ t \geq 0)$-martingales. This notion will be studied in Chapter 5, but we already note that the more general situation when some (perhaps all...) $(\mathcal{F}_t;\ t \geq 0)$-martingales are $(\mathcal{G}_t;\ t \geq 0)$-semimartingales will be a recurrent subject of study in these lecture notes.

0

Preliminaries

Throughout these preliminaries, we are working with an underlying filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0), \mathbb{P})$. We insist that most of the notions introduced below are relative to this filtered probability space.

0.1 Doob's Maximal Identity

The following lemmas are variants of Doob's optional stopping theorem.

Lemma 0.1 *Let N be a $\mathbb{R}_+$-valued continuous local martingale with $N_0 = 1$, and $N_t \underset{t\to\infty}{\longrightarrow} 0$.*
Denote $\overline{N}_t = \sup_{s\leq t} N_s$ and $\overline{N}^t = \sup_{s\geq t} N_s$.
Then $\overline{N}_\infty \overset{(law)}{=} 1/U$ where U denotes a uniformly distributed variable.
More generally, for every finite stopping time T such that $N_T > 0$ a.s., $N_T/\overline{N}^T$ is a uniform variable independent of $\mathcal{F}_T$.

Proof

- Define, for $a > 1$, $T_a = \inf\{t \geq 0,\ N_t = a\}$.
  Then: $\quad 1 = \mathbb{E}\left[N_0\right] = \mathbb{E}\left[N_{T_a}\right] = aP(\overline{N}_\infty \geq a)(= aP(T_a < \infty))$
  Thus, $\quad P(\overline{N}_\infty \geq a) = \frac{1}{a}$ i.e. $P\left(1/\overline{N}_\infty \leq \frac{1}{a}\right) = \frac{1}{a}$
- For any finite stopping time T such that $N_T > 0$ a.s., consider the local martingale constructed from N by shifting time from T, namely $(N_{u+T}/N_T;\ u \geq 0)$. We can apply the first step of this proof to this local martingale whose supremum is $\overline{N}^T/N_T$. The result follows easily.

■

The next lemma completes, in some sense, Lemma 0.1. $(N_t;\ t \geq 0)$ is now replaced by a general continuous semi-martingale $(X_t;\ t \geq 0)$, which is not necessarily positive.

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Lemma 0.2 *Let $h : \mathbb{R} \to \mathbb{R}$ be a locally integrable function and set $H(x) = \int_0^x dy h(y)$.*
Then $H(\overline{X}_t) - h(\overline{X}_t)(\overline{X}_t - X_t) = \int_0^t h(\overline{X}_s) dX_s$; hence, if $(X_t;\ t \geq 0)$ is a local martingale, so is $(H(\overline{X}_t) - h(\overline{X}_t)(\overline{X}_t - X_t),\ t \geq 0)$.

Proof
This result is easily obtained when h is regular thanks to Itô formula, and the essential fact that $d\overline{X}_t$ is carried by $\{t;\ \overline{X}_t = X_t\}$. The general result follows from a monotone class argument.

■

Comment 0.1 *For $h(x) = 1_{x \leq a}$ and $(X_t;\ t \geq 0)$ a continuous local martingale, Lemma 0.2 yields that*

$$(a 1_{\overline{X}_t \geq a} + X_t 1_{\overline{X}_t \leq a};\ t \geq 0)$$

is a local martingale, from which the result of Lemma 0.1 follows.

Example 0.1 *(Doob's inequality in L^p for positive submartingales)*
We consider $(\Sigma_t;\ t \geq 0)$ a positive continuous submartingale.
Taking $F(x) = x^p$ with $p > 1$, Lemma 0.2 implies that $\overline{\Sigma}_t^p - p\overline{\Sigma}_t^{p-1}(\overline{\Sigma}_t - \Sigma_t)$ is a local submartingale.
Up to a localization argument, we obtain

$$\begin{aligned} \mathbb{E}\left[\overline{\Sigma}_t^p\right] &\leq \frac{p}{p-1} \mathbb{E}\left[\overline{\Sigma}_t^{p-1} \Sigma_t\right] \\ &\leq \frac{p}{p-1} \mathbb{E}\left[\overline{\Sigma}_t^p\right]^{(p-1)/p} \mathbb{E}\left[\Sigma_t^p\right]^{1/p} \qquad \textit{(Hölder)} \end{aligned}$$

Thus

$$\|\overline{\Sigma}_t\|_p \leq \frac{p}{p-1} \|\Sigma_t\|_p$$

0.2 Balayage Formula

The result of Lemma 0.2 may be understood in a more general framework. Let $(k_u;\ u \geq 0)$ be a locally bounded, predictable process, $(Y_u;\ u \geq 0)$ a continuous semi-martingale starting at 0.
Denote by γ_t and δ_t respectively the last zero of Y before t and the first zero of Y after t, namely:

$$\begin{aligned} \gamma_t &= \sup\{u \leq t;\ Y_u = 0\} \\ \delta_t &= \inf\{u \geq t;\ Y_u = 0\} \end{aligned}$$

Then

Lemma 0.3 *(see e.g. [RY05] Chapter VI pp 260-263)*
The process $(k_{\gamma_t} Y_t;\ t \geq 0)$ *is a semimartingale, more precisely:*

$$k_{\gamma_t} Y_t = \int_0^t k_{\gamma_u} dY_u$$

Proof
First, assume that k is a simple predictable process, i.e. there exists a stopping time T such that $k_u = 1_{[0,T]}(u)$ for any $u \geq 0$.
Then, there is the following sequence of easy identities:

$$\begin{aligned} k_{\gamma_t} Y_t &= 1_{\gamma_t \leq T} Y_t = 1_{t \leq \delta_T} Y_t = Y_{t \wedge \delta_T} \\ &= \int_0^t 1_{u \leq \delta_T} dY_u = \int_0^t k_{\gamma_u} dY_u \end{aligned}$$

A monotone class argument yields to the conclusion for a general predictable process k.

■

Example 0.2 *Taking* $k_u = h(\overline{X}_u)$ *and* $Y_u = \overline{X}_u - X_u$*, we recover partly the result discussed in Lemma 0.2.*

Now we consider the particular case $(k_u = h(L_u);\ u \geq 0)$ where L denotes the local time of X at 0, and $(Y_u = |X_u|;\ u \geq 0)$. Lemma 0.3, combined with Tanaka's formula, leads to

$$h(L_t)|X_t| = \int_0^t h(L_s)\text{sgn}(X_s) dX_s + \int_0^t h(L_s) dL_s$$

That is, assuming now that $(X_t;\ t \geq 0)$ is a local martingale.

Lemma 0.4 *Let* $H(l) = \int_0^l dy h(y)$*; then,* $(H(L_t) - h(L_t)|X_t|;\ t \geq 0)$ *is a local martingale.*

In order to show the proximity between all the martingales presented in these preliminaries, it is of some interest to recall Lévy's identity for a Brownian motion B, its supremum S and its local time L.

Proposition 0.5 *(Lévy's equivalence)*

$$(S_t - B_t, S_t;\ t \geq 0) \stackrel{(law)}{=} (|B_t|, L_t;\ t \geq 0)$$

As is well-known, this can be seen as a consequence of Skorohod's lemma, namely:

Lemma 0.6 *Given a continuous function* y*, the equation in* (z, l)

$$z(t) = -y(t) + l(t); \qquad y(0) = 0$$

where it is assumed that both z *and* l *are continuous,* $z \geq 0$*,* l *is an increasing process such that the measure* dl_t *has its support in* $\{t;\ z(t) = 0\}$*, admits a unique solution* (z^*, l^*) *given by* $l^*(t) = \sup_{s \leq t} y(s)$ *and* $z^*(t) = l^*(t) - y(t)$*.*

To prove Lévy's equivalence, it now suffices to compare the identities:

$$S_t - B_t = -B_t + S_t$$
$$|B_t| = -\left(-\int_0^t \operatorname{sgn}(B_s)dB_s\right) + L_t$$

0.3 Predictable Compensators

Definition 0.1 *Let A be an integrable, increasing, right-continuous process. There exists a unique predictable, increasing, right-continuous process $A^{(p)}$ such that, for any positive predictable process k:*

$$\mathbb{E}\left[\int_0^\infty k_u dA_u\right] = \mathbb{E}\left[\int_0^\infty k_u dA_u^{(p)}\right]$$

We shall say that $A^{(p)}$ is the predictable compensator of A.

Example 0.3 *As an easy example, we consider N a Poisson process with parameter λ; it is well-known that $(N_t - \lambda t;\ t \geq 0)$ is a martingale. Accordingly, the predictable compensator of N is $(\lambda t;\ t \geq 0)$.*
More generally, if $(X_t;\ t \geq 0)$ is a Lévy process, with $\nu(dx)$ its Lévy measure, then for every $f : \mathbb{R} \to \mathbb{R}_+$, bounded, with compact support away from 0, the process $(\sum_{s \leq t} f(\Delta X_s),\ t \geq 0)$ admits as its predictable compensator

$$t < \nu, f >= t \int_{\mathbb{R}} \nu(dx) f(x)$$

We will compute many examples in the following chapters ([Kni91] gives some examples of computations of compensators). In particular, predictable compensators will play an essential role in the study of progressive enlargements of filtrations.

In the Strasbourg terminology, $(A_t^{(p)};\ t \geq 0)$ is the "dual predictable projection" of $(A_t;\ t \geq 0)$. Note that if $(A_t;\ t \geq 0)$ is $(\mathcal{F}_t;\ t \geq 0)$-adapted, then $(A_t - A_t^{(p)}; t \geq 0)$ is a $(\mathcal{F}_t;\ t \geq 0)$-martingale. This easily follows from Definition 0.1.
See [Del72] for the beginning of the story... At the other end, see, e.g. [AJK+96], [AJKY98]...

0.4 σ-fields Associated with a Random Time Λ

To a general random time Λ, we associate the following σ-fields:

$$\mathcal{F}_{\Lambda+} = \sigma\{z_\Lambda,\ z \text{ any } (\mathcal{F}_t;\ t \geq 0) \text{ progressively measurable process}\}$$
$$\mathcal{F}_{\Lambda} = \sigma\{z_\Lambda,\ z \text{ any } (\mathcal{F}_t;\ t \geq 0) \text{ optional process}\}$$
$$\mathcal{F}_{\Lambda-} = \sigma\{z_\Lambda,\ z \text{ any } (\mathcal{F}_t;\ t \geq 0) \text{ predictable process}\}$$

These σ-fields- which were introduced by Chung [Chu72]- will play an important role in the sequel when we discuss enlargements of filtrations.
An application of the monotone class theorem shows that every, say $(\mathcal{F}_\Lambda)$-measurable variable, is of the form z_Λ, for $(z_t;\ t \geq 0)$ a $(\mathcal{F}_t;\ t \geq 0)$ predictable process. The same remark applies to every $\mathcal{F}_\Lambda$, resp. $\mathcal{F}_{\Lambda+}$, measurable random variable.

0.5 Integration by Parts Formulae

Lemma 0.7 *If X and Y are two semi-martingales, then*

$$\forall t \geq 0, \qquad X_t Y_t = \int_0^t X_{s-} dY_s + \int_0^t Y_{s-} dX_s + [X,Y]_t \tag{0.1}$$

In particular, assuming that $(M_t;\ t \geq 0)$ is a bounded martingale and $(A_t;\ t \geq 0)$ is an increasing process such that $\mathbb{E}[A_\infty] < \infty$,

- *if, moreover, $(A_t;\ t \geq 0)$ is an optional increasing process, then*

$$\mathbb{E}[M_\infty A_\infty] = \mathbb{E}\left[\int_0^\infty M_s dA_s\right] \tag{I_o}$$

- *if, moreover, $(A_t;\ t \geq 0)$ is a predictable increasing process, then*

$$\mathbb{E}[M_\infty A_\infty] = \mathbb{E}\left[\int_0^\infty M_{s-} dA_s\right] \tag{I_p}$$

We leave the proofs of (I_o) and (I_p), as consequences of (0.1), to the reader.
The integration by parts formulae (I_o) and (I_p) will be invoked many times in the sequel.

0.6 H^1 and BMO Spaces

Definition 0.2 *The $\mathbf{H^1}$ space is the set of martingales N such that*

$$\mathbb{E}\left[\sup_t |N_t|\right] < \infty \text{ or equivalently } \mathbb{E}\left[\sqrt{[N,N]_\infty}\right] < \infty$$

Fitted with these equivalent norms, $\mathbf{H^1}$ is a Banach space.

Definition 0.3 *The **BMO** space (Bounded Mean Oscillation) is the set of square integrable martingales N such that*

$$esssup_{T \in \mathcal{T}} \mathbb{E}\left[(N_\infty - N_{T-})^2 | \mathcal{F}_T\right] \leq C < \infty \tag{0.2}$$

for some constant C (depending on N), where $\mathcal{T}$ denotes the set of stopping times and $(\mathcal{F}_t;\ t \geq 0)$ the underlying filtration.
*$||N||_{BMO}$ may be defined as the smallest $\sqrt{C}$ for C in (0.2). Fitted with this norm, **BMO** is a Banach space.*

Remark 0.1 *If N is **BMO**, then ΔN is bounded, and therefore N is locally bounded.*
*Yet, $(B_{t\wedge 1},\ t \geq 0)$ is unbounded, but belongs to **BMO** (see Exercise 7).*

An important result in martingale theory, which plays the role of the $L^1 - L^\infty$ duality in measure theory is the following:

Theorem 0.8 *[GS72][Mey73](or [Mey75], [Mey77])*
*The dual space of $\mathbf{H^1}$ is **BMO**.*

0.7 Exercises

Exercise 1 *Let $c : \mathbb{R}^+ \to \mathbb{R}^n$ a curve of class $\mathcal{C}^1$. Show that $(Z_t := c(S_t) - (S_t - B_t)c'(S_t);\ t \geq 0)$, where B and S still denote respectively a standard Brownian motion and its unilateral supremum, is a continuous local martingale. In particular, if the derivative c' is bounded, Z is a true martingale.*

Exercise 2 *Let $Y_t = M_t + V_t$, $t \geq 0$, denote the canonical Doob-Meyer decomposition of an adapted continuous semimartingale with $V_0 = 0$.*
Let $(W_t;\ t \geq 0)$ denote a continuous process with bounded variation, $W_0 = 0$. Under which necessary and sufficient conditions on $(Y_t;\ t \geq 0)$ and $(W_t;\ t \geq 0)$, is the process $(F(W_t) - f(W_t)Y_t;\ t \geq 0)$ a local martingale, for any bounded Borel f?

Exercise 3 *Taking N a Poisson process with parameter λ, compute the predictable compensators of $(N_t^2,\ t \geq 0)$ and $(\exp N_t;\ t \geq 0)$.*

Exercise 4 *Denote by ${}^{(o)}A$ the optional projection of the increasing process A, i.e. the optional process $(\alpha_t;\ t \geq 0)$ which satisfies*

$$\alpha_T = \mathbb{E}\left[A_T | \mathcal{F}_T\right], \qquad \text{on } T < \infty$$

for every $(\mathcal{F}_t;\ t \geq 0)$ bounded stopping time T.
Prove that $({}^{(o)}A_t;\ t \geq 0)$ is a submartingale, whose increasing process, in its Doob-Meyer decomposition is $A^{(p)}$.

Exercise 5 *Let $(X_t;\ t \geq 0)$ denote a Lévy process, which admits $\phi_t(\cdot)$ as the density of the law of X_t.*

a) Show the following local absolute continuity relationship between the law of the bridge of length t from x to y and the law of X starting at x

$$P^t_{x \to y | \mathcal{F}_s} = \frac{\phi_{t-s}(y - X_s)}{\phi_t(y - x)} . P_{x | \mathcal{F}_s}$$

b) Compute the predictable compensator of $(\sum_{s \leq t} f(\Delta X_s);\ t \geq 0)$ under the law $P^t_{x \to y}$

Exercise 6 *Let $(M_t;\ t \geq 0)$ be a local martingale and $(A_t;\ t \geq 0)$ an increasing process.*

a) *Show that, if moreover $(A_t;\ t \geq 0)$ is an optional increasing process, then the process $(M_t A_t - \int_0^t M_s dA_s;\ t \geq 0)$ is a local martingale.*
b) *Show that, if moreover $(A_t;\ t \geq 0)$ is a predictable increasing process, then the process $(M_t A_t - \int_0^t M_{s-} dA_s;\ t \geq 0)$ is a local martingale.*
c) *Deduce Yoeurp's lemma [Yoe76]:*
If $(M_t;\ t \geq 0)$ is a local martingale and $(A_t;\ t \geq 0)$ a predictable process with finite variation, then, for any $t \geq 0$, $[M, A]_t = \int_0^t (\Delta A_s) dM_s$ and $[M, A]$ is a local martingale.

Exercise 7

a) *Show that $(\mathbb{E}[B_1|\mathcal{F}_t],\ t \geq 0)$ and $(\mathbb{E}[|B_1||\mathcal{F}_t],\ t \geq 0)$ are in BMO.*
b) *Let X be a random variable such that $(\mathbb{E}[X|\mathcal{F}_t],\ t \geq 0)$ is in BMO. Show that X admits some exponential moments.*
Deduce that, for any $n > 2$, $(\mathbb{E}[B_1^n|\mathcal{F}_t],\ t \geq 0)$ is not in BMO.
c) *Show that $(\mathbb{E}\left[B_1^2|\mathcal{F}_t\right],\ t \geq 0)$ is not in BMO.*

1

Enlargements of Filtrations

The beginnings of the theory of enlargements of filtrations may be traced back as follows:

- K. Itô [Itô78] writes that it would be nice to give some meaning to $\int_0^t B_1 dB_s$, and that this may be done in enlarging the natural filtration of B with the variable B_1, and showing that $(B_t;\ t \geq 0)$ remains a semimartingale in the enlarged filtration;
- separately, and independently from each other, P.A. Meyer and D. Williams (circa 1977) ask what becomes of a $(\mathcal{F}_t;\ t \geq 0)$-martingale $(M_t;\ t \geq 0)$ when considered in $(\mathcal{F}_t^{\Lambda};\ t \geq 0)$, the smallest filtration which contains $(\mathcal{F}_t;\ t \geq 0)$ and makes Λ a stopping time, for a given random time Λ.

In this chapter, in particular, we shall see how both questions have been developed in subsequent years.
The reader will not fail to find some parenthood between the enlargement formulae and Girsanov's theorem. This parenthood has been closely studied in [Yoe85].

In these lecture notes, we do not deal with other methods- than enlargements of filtrations- of integrating anticipative integrands, such as the Skorokhod integral, for which we refer the interested reader to [Hit75] [NP91], [Nua93] and references therein.

1.1 Some General Problems of Enlargements

Consider two filtrations $(\mathcal{F}_t;\ t \geq 0)$ and $(\mathcal{G}_t;\ t \geq 0)$ such that

$$\forall t \geq 0, \qquad \mathcal{F}_t \subseteq \mathcal{G}_t$$

In this chapter, we shall deal with two special cases:

R. Mansuy and M. Yor: *Random Times and Enlargements of Filtrations in a Brownian Setting*, Lect. Notes Math. **1873**, 11–29 (2006)
www.springerlink.com

Progressive enlargement: $\mathcal{G}_t := \bigcap_{\varepsilon>0} (\mathcal{F}_{t+\varepsilon} \vee \sigma(\Lambda \wedge (t+\varepsilon)))$ with Λ a given random time (i.e. $\mathcal{G}$ is the smallest filtration which contains $(\mathcal{F}_t;\ t \geq 0)$ and makes Λ a stopping time). We denote this filtration as $(\mathcal{F}_t^\Lambda;\ t \geq 0)$.

Initial enlargement: $\mathcal{G}_t := \bigcap_{\varepsilon>0} (\mathcal{F}_{t+\varepsilon} \vee \mathcal{H}_0)$ with $\mathcal{H}_0$ a given σ-field. When $\mathcal{H}_0 = \sigma(X)$, we denote this filtration by $(\mathcal{F}_t^{\sigma(X)};\ t \geq 0)$.

In both set-ups, we intend to solve the following
Problem
Under which conditions on Λ in the first instance or $\mathcal{H}_0$ in the second one, do all $(\mathcal{F}_t;\ t \geq 0)$-martingales remain $(\mathcal{G}_t;\ t \geq 0)$-semi-martingales? In the affirmative, what is the corresponding $(\mathcal{G}_t;\ t \geq 0)$ canonical decomposition of a generic $(\mathcal{F}_t;\ t \geq 0)$ semi-martingale?
In many interesting cases, only certain $(\mathcal{F}_t; t \geq 0)$ martingales are $(\mathcal{G}_t;\ t \geq 0)$ semi-martingales. It is of interest to determine exactly which are these $(\mathcal{F}_t;\ t \geq 0)$ martingales.
From now on, we shall often make the following additional assumptions:

Assumption 1.1

(CA)	***(C)***	*All* $(\mathcal{F}_t;\ t \geq 0)$*-martingales are continuous.*
	(A)	Λ *avoids the* $(\mathcal{F}_t;\ t \geq 0)$ *stopping times, i.e.:* *for any* $(\mathcal{F}_t;\ t \geq 0)$ *stopping time* T, $P(\Lambda = T) = 0$

To begin with, we shall only assume that (**C**) is satisfied.

Comment 1.1 *It is noteworthy that, concerning the problem of progressive enlargement, the set-up is rather restrictive, i.e. very few studies with pairs* $((\mathcal{F}_t, \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t);\ t \geq 0)$*, other than with* $\mathcal{H}_t = \sigma(\Lambda \wedge t)$ *have been developed since 1980. An exception is when* $\mathcal{H}_t = \sigma(J_t)$*, for* $J_t = \inf_{s \geq t} X_s$*, for certain* $(\mathcal{F}_t;\ t \geq 0)$*-adapted processes* $(X_t;\ t \geq 0)$*; see Subsection 1.2.2.*

1.2 Progressive Enlargement

In this study, it is crucial to determine A^Λ the $(\mathcal{F}_t;\ t \geq 0)$ predictable compensator of $(1_{\Lambda \leq t};\ t \geq 0)$, i.e. the predictable increasing process such that

$$\mathbb{E}\left[k_\Lambda\right] = \mathbb{E}\left[\int_0^\infty k_u dA_u^\Lambda\right]$$

for all bounded predictable processes $(k_u;\ u \geq 0)$, as explicitly as possible.
We denote by $(\mathcal{F}_t^\Lambda;\ t \geq 0)$ the smallest filtration which makes Λ a stopping time and contains $(\mathcal{F}_t;\ t \geq 0)$.

Example 1.1 [1]*Consider* $\Lambda = \gamma_{T_a^*}$ *with* $T_a^* = \inf\{t \geq 0,\ |B_t| = a\}$
The balayage formula (Lemma 0.3) implies that $(k_{\gamma_t}|B_t| - \int_0^t k_u dL_u;\ t \geq 0)$ *is a* $(\mathcal{F}_t;\ t \geq 0)$*-martingale.*
Therefore

$$a\mathbb{E}\left[k_{\gamma_{T_a^*}}\right] = \mathbb{E}\left[\int_0^{T_a^*} k_u dL_u\right]$$

Comparing these two formulae, we obtain that

$$A_u^\Lambda = \frac{1}{a} L_{u\wedge T_a^*}$$

Example 1.2 *More generally, let* T *be a* $(\mathcal{F}_t;\ t \geq 0)$ *stopping time. We assume that* $(B_{t\wedge T};\ t \geq 0)$ *is uniformly integrable, and we consider* $\Lambda = \gamma_T$*. Then, for any bounded predictable process* $(k_u;\ u \geq 0)$*, one has*

$$\mathbb{E}\left[k_{\gamma_T}|B_T|\right] = \mathbb{E}\left[\int_0^T k_u dL_u\right]$$

again as a consequence of the balayage formula (Lemma 0.3).Denote by $(\xi_t;\ t \geq 0)$ *a* $(\mathcal{F}_t;\ t \geq 0)$*-predictable process such that:*

$$\mathbb{E}\left[|B_T||\mathcal{F}_{(\gamma_T)^-}\right] = \xi_{\gamma_T}$$

Then,

$$\mathbb{E}\left[k_{\gamma_T}\xi_{\gamma_T}\right] = \mathbb{E}\left[\int_0^T k_u dL_u\right]$$

Accordingly, $\int_0^T 1_{\xi_u=0} dL_u = 0$*, and*

$$A_t^\Lambda = \int_0^{t\wedge T} \frac{dL_u}{\xi_u}$$

Explicit computations of ξ *with e.g.* $T = t$ *constant, i.e.* $\xi_u = \sqrt{\frac{2}{\pi}(t-u)}$*, will be performed in* ***Chapter*** *4.*

1.2.1 Decomposition Formula

We are now ready to deal with the global resolution of the enlargement problem with Λ the end of a predictable set Σ, namely $\Lambda = \sup\{t \geq 0,\ (t,\omega) \in \Sigma\}$. The semi-martingale decompositions in $(\mathcal{F}_t^\Lambda;\ t \geq 0)$ shall be expressed in terms of the **Azéma supermartingale** $Z_t = P(\Lambda > t|\mathcal{F}_t)$ (we choose a càdlàg version of this supermartingale) the decomposition of which is

$$Z_t = \mu_t - A_t^\Lambda = \mathbb{E}\left[A_\infty^\Lambda|\mathcal{F}_t\right] - A_t^\Lambda$$

[1] All the computations of compensators are summarized at the end of this chapter.

Theorem 1.1 *If* $(M_t;\ t \geq 0)$ *is a* $(\mathcal{F}_t;\ t \geq 0)$ *local martingale, there exists a* $(\mathcal{F}_t^\Lambda;\ t \geq 0)$ *local martingale* $(\widetilde{M}_t;\ t \geq 0)$ *such that*[2]*:*

$$M_t = \widetilde{M}_t + \int_0^{t\wedge\Lambda} \frac{d < M, Z >_s}{Z_{s-}} + \int_\Lambda^t \frac{d < M, 1-Z >_s}{1-Z_{s-}}$$

In particular, every $(\mathcal{F}_t;\ t \geq 0)$ *local martingale remains a* $(\mathcal{F}_t^\Lambda;\ t \geq 0)$ *semi-martingale.*

We first recall the following description of $(\mathcal{F}_t^\Lambda;\ t \geq 0)$ predictable processes[3]:

Lemma 1.2 *[Jeu80] Let* J *be a* $(\mathcal{F}_t^\Lambda;\ t \geq 0)$ *predictable process.*
Then there exist J^+ *and* J^- *two* $(\mathcal{F}_t;\ t \geq 0)$ *predictable processes such that*

$$\forall u \geq 0, \qquad J_u = J_u^- 1_{[0,\Lambda]}(u) + J_u^+ 1_{(\Lambda,\infty)[}(u)$$

Proof of Theorem 1.1
We first assume that $(M_u;\ u \geq 0)$ belongs to $H^1(\mathcal{F}_t;\ t \geq 0)$ (see Section 0.6). Let $\Gamma_s \in \mathcal{F}_s^\Lambda$, and $s < t$; then,

$$\begin{aligned}
\mathbb{E}\left[1_{\Gamma_s}(M_t - M_s)\right] &= \mathbb{E}\left[\int_0^\infty J_u dM_u\right] \qquad\qquad \text{with } J_u = 1_{\Gamma_s} 1_{]s,t]}(u) \\
&= \mathbb{E}\left[\int_0^\Lambda J_u^- dM_u + \int_\Lambda^\infty J_u^+ dM_u\right] \\
&= \mathbb{E}\left[\int_0^\Lambda (J_u^- - J_u^+) dM_u\right] \qquad \text{since } \mathbb{E}\left[\int_0^\infty J_u^+ dM_u\right] = 0 \\
&= \mathbb{E}\left[\int_0^\infty \left(\int_0^v (J_u^- - J_u^+) dM_u\right) dA_v^\Lambda\right]
\end{aligned}$$

Denote $N_v = \int_0^v (J_u^- - J_u^+) dM_u$; then, we obtain by integration by parts(i.e. (I_p); in fact, it is (I_o) which is applied).

$$\begin{aligned}
\mathbb{E}\left[1_{\Gamma_s}(M_t - M_s)\right] &= \mathbb{E}\left[N_\infty A_\infty^\Lambda\right] \\
&= \mathbb{E}\left[\int_0^\infty (J_u^- - J_u^+) d < M, \mu >_u\right]
\end{aligned}$$

Introducing on the right-hand side $1_{[0,\Lambda]} + 1_{(\Lambda,\infty)}$ and replacing $1_{u\leq\Lambda}$ (resp. $1_{(\Lambda,\infty)}$) by its predictable projection Z_{u-} (resp. $(1 - Z_{u-})$), we obtain:

---

[2] Throughout, we adopt the convention that integrals such as $\int_\varrho^t ds\ \vartheta_s$ are equal to 0 for $t \leq \varrho$.

[3] The fact that Λ is the end of a predictable set (therefore a "honest" variable) is crucial for this result to be true. For a general random time Λ, see Yor [Yor78a] and Dellacherie-Meyer [DM78]. Then, there is only a weak version of Theorem 1.1 *before* Λ.

$$\mathbb{E}\left[1_{\Gamma_s}(M_t - M_s)\right] = \mathbb{E}\left[\int_0^{\Lambda} J_u^- \frac{d < \mu, M >_u}{Z_{u-}} - \int_{\Lambda}^{\infty} J_u^+ \frac{d < \mu, M >_u}{1 - Z_{u-}}\right]$$
$$= \mathbb{E}\left[\int_0^{\infty} J_u \left(\frac{d < \mu, M >_u}{Z_{u-}} 1_{u \leq \Lambda} - \frac{d < \mu, M >_u}{1 - Z_{u-}} 1_{u > \Lambda}\right)\right]$$

This proves the result for M in H^1, and, by localization, for every $(\mathcal{F}_t;\ t \geq 0)$-local martingale $(M_t;\ t \geq 0)$.

■

Example 1.3 *Consider N a continuous, positive, local martingale with $N_0 = 1$, and*

$$N_t \underset{t \to \infty}{\longrightarrow} 0$$

Define $\Lambda = \sup\{t \geq 0,\ \overline{N}_t - N_t = 0\} = \sup\{t \geq 0,\ N_t = \overline{N}_\infty\}$. Then $Z_t = N_t / \overline{N}_t$.
Indeed,

$$P(\Lambda > t | \mathcal{F}_t) = P(\overline{N}^t > \overline{N}_t | \mathcal{F}_t) = N_t / \overline{N}_t \qquad \textit{from Lemma 0.1}$$

Remark 1.1 *Here we use the multiplicative representation $(N_t / \overline{N}_t)$ of Z instead of its additive (Doob-Meyer) decomposition $\mu_t - A_t^{\Lambda}$, which we can easily deduce, thanks to Itô formula:*

$$1 + \int_0^t \frac{1}{\overline{N}_s} dN_s - \log(\overline{N}_t) = \mu_t - A_t^{\Lambda}$$

Thus, $A_t^{\Lambda} = \log(\overline{N}_t)$, and $\mu_t = 1 + \int_0^t dN_s / \overline{N}_s$.

From now on, we will often assume that **(CA)** is satisfied. Consequently, it may be worth noticing that

Remark 1.2 *Under assumption **(C)**, assumption **(A)** is equivalent to the continuity of A^{Λ}.*
*Indeed, under **(C)**, all stopping times T are predictable; thus, if moreover **(A)** is satisfied, then for any stopping time T,*

$$0 = \mathbb{E}\left[1_{\Lambda = T}\right] = \mathbb{E}\left[A_T^{\Lambda} - A_{T-}^{\Lambda}\right],$$

hence the continuity of A^{Λ}.

We now show that under these assumptions, Example 1.3 turns out to yield the general representation of Azéma's supermartingales:

Proposition 1.3 *[NY05b] Under the assumption **(CA)**, there exists a continuous local martingale N, with $N_0 = 1$, and*

$$N_t \underset{t\to\infty}{\longrightarrow} 0$$

which satisfies

$$\forall t \geq 0, \qquad Z_t = N_t/\overline{N}_t$$

Proof
Since[4] $Z_\Lambda = 1$, the support of dA_t^Λ is included in $\{t \geq 0,\ Z_t = 1\}$ and $1 - Z_t = (1 - \mu_t) + A_t^\Lambda$.
From Skorohod's lemma (See Lemma 0.6), $A_t^\Lambda = \sup_{s\leq t}(\mu_s - 1)$.
We now consider (*) $N_t := \exp(A_t^\Lambda)Z_t$ and we show that it defines a local martingale. Indeed,

$$\begin{aligned} N_t &= 1 + \int_0^t \exp(A_u^\Lambda)dZ_u + \int_0^t \exp\,(A_u^\Lambda)Z_u dA_u^\Lambda \\ &= 1 + \int_0^t \exp(A_u^\Lambda)d\mu_u + \int_0^t \exp\,(A_u^\Lambda)(Z_u - 1)dA_u^\Lambda \\ &= 1 + \int_0^t \exp\,(A_u^\Lambda)d\mu_u \end{aligned}$$

From (*), one deduces that (**) $\overline{N}_t = \exp\,(A_t^\Lambda)$: indeed, (*) implies immediately $\overline{N}_t \leq \exp\,(A_t^\Lambda)$; conversely, if $\gamma_t = \sup\{s \leq t; 1 - Z_s = 0\}$, then $N_{\gamma_t} = \exp(A_t^\Lambda)$. Finally, since $A_\infty^\Lambda < \infty$ and $Z_t \underset{t\to\infty}{\longrightarrow} 0$, then from (*), $N_t \underset{t\to\infty}{\longrightarrow} 0$ which proves (**).

■

Another consequence of **(CA)** is the following result, due to Azéma:

Proposition 1.4 *Under **(CA)**, if Λ is the end of a predictable set, A_∞^Λ is exponentially distributed with parameter 1.*

Proof
For any positive, Borel function φ, we have, since Λ is the end of a predictable set,

$$\begin{aligned} \mathbb{E}\left[\varphi(A_\infty^\Lambda)\right] &= \mathbb{E}\left[\varphi(A_\Lambda^\Lambda)\right] \\ &= \mathbb{E}\left[\int_0^\infty dA_s^\Lambda \varphi(A_s^\Lambda)\right] \\ &= \mathbb{E}\left[\int_0^{A_\infty^\Lambda} dx \varphi(x)\right] \end{aligned}$$

[4] In fact, since Λ is the end of a predictable set, $\Lambda = \sup\{t \geq 0,\ Z_t = 1\}$

In particular, with $\varphi(x) = e^{-\lambda x}$, we obtain $\mathbb{E}\left[e^{-\lambda A^{\Lambda}_{\infty}}\right] = \frac{1}{1+\lambda}$, hence the result.

■

Remark 1.3 *This result can also be seen as a consequence of the multiplicative representation $(N_t/\overline{N}_t)$ of Z^{Λ} thanks to Lemma 0.1. More details are provided in Exercise 8 c).*

Example 1.4 *(Example 1.1 continued) $L_{T^*_a}$ is exponentially distributed with parameter a.*

1.2.2 Pitman's Theorem on $2S - B$ and Some Generalizations via Some Progressive Enlargement of Filtration

In this subsection, we give an example of a progressive enlargement which does not fit the framework of the preceding decomposition Theorem 1.1.
More precisely, we present Pitman's theorem and some of its generalizations in the light of the progressive enlargement of the filtration $(\mathcal{R}_t;\ t \geq 0)$ of a 3-dimensional Bessel process $(R_t;\ t \geq 0)$, with $J_t = \inf_{s \geq t} R_s$. Indeed, Jeulin ([Jeu79], [Jeu80]) used the fact that

$$(J_t < a) = (t < \Lambda_a)$$

with $\Lambda_a = \sup\{t;\ R_t = a\}$, and developed enlargement formulae related to all the Λ_a's simultaneously to prove Pitman's theorem in this way.

Theorem 1.5 *([Pit75], [Jeu80])*

1. *If $(B_t;\ t \geq 0)$ is a standard Brownian motion and $S_t = \sup_{s \leq t} B_s$, $t \geq 0$, then the process $(R_t := 2S_t - B_t;\ t \geq 0)$ is a 3-dimensional Bessel process.*
2. *If $(R_t;\ t \geq 0)$ is a 3-dimensional Bessel process (not necessarily starting from 0) and $J_t = \inf_{s \geq t} R_s$, $t \geq 0$, then the process $(\beta_t := 2J_t - R_t;\ t \geq 0)$ is a standard Brownian motion.*
   In fact, if $(\mathcal{R}_t;\ t \geq 0)$ denotes the natural filtration of R, then $(\beta_t;\ t \geq 0)$ is a Brownian motion with respect to $(\mathcal{R}_t \vee \sigma(J_t);\ t \geq 0)$.

Comment 1.2 *In [ST90], this result has been generalized to transient, $\mathbb{R}^*_+$-valued diffusions $(R_t;\ t \geq 0)$ such that:*

- *$(R_t;\ t \geq 0)$ satisfies the following SDE assumed to enjoy uniqueness in law*
$$R_t = r + B_t + \int_0^t ds\ c(R_s)$$
- *the scale function of R may be chosen to satisfy*

$$e(0+) = -\infty \qquad e(\infty) = 0 \qquad \frac{1}{2}e'' + ce' = 0$$

If $J_t = \inf_{s \geq t} R_s$ still denotes the future minimum and $(\mathcal{R}_t;\ t \geq 0)$ the natural filtration of R, then $\left(\frac{2}{e(J_t)} - \frac{1}{e(R_t)};\ t \geq 0\right)$ is a $(\mathcal{R}_t \vee \sigma(J_t);\ t \geq 0)$-local martingale.
With $c(x) = 1/x$, one easily recovers the second assertion of Theorem 1.5.
With $c(x) = \frac{n-1}{2x}$ (i.e. R is a n dimensional Bessel process; $n > 2$), we find that both $(2J_t^{n-2} - R_t^{n-2};\ t \geq 0)$ and $(2J_t^2 - R_t^2 - (n-4)t;\ t \geq 0)$ are $(\mathcal{R}_t \vee \sigma(J_t);\ t \geq 0)$-martingales.

Comment 1.3 *For different extensions of Pitman's theorem, we refer to [Ber91], [Ber92], [Bia94], [Tak97], [Tan90], [Tan89].*

1.3 Initial Enlargement

In order to solve the problem of initial enlargement with X, we first consider the quantity $(\lambda_t(f); t \geq 0)$ defined for any bounded Borel function $f : \mathbb{R} \to \mathbb{R}$ as the continuous version of the martingale $(\mathbb{E}\left[f(X)|\mathcal{F}_t\right]; t \geq 0)$.
Here is a first example of $(\lambda_t(f); t \geq 0)$:

Example 1.5 *We still consider a continuous local martingale N, with $N_0 = 1$, and*

$$N_t \underset{t \to \infty}{\longrightarrow} 0$$

Here are some preliminaries for the initial enlargement with the variable $X = \overline{N}_\infty$.
For any test function f,

$$\begin{aligned}
\mathbb{E}\left[f(\overline{N}_\infty)|\mathcal{F}_t\right] &= \mathbb{E}\left[f(\overline{N}_t \vee \overline{N}^t)|\mathcal{F}_t\right] \\
&= f(\overline{N}_t) P(\overline{N}_t > \overline{N}^t|\mathcal{F}_t) + \mathbb{E}\left[f(\overline{N}^t) 1_{\overline{N}^t > \overline{N}_t}|\mathcal{F}_t\right]
\end{aligned}$$

Now, Lemma 0.1 asserts that, conditionally on $\mathcal{F}_t$, $\overline{N}^t$ is distributed as $\frac{N_t}{U}$ with U a uniform variable independent of $\mathcal{F}_t$.
Therefore

$$\lambda_t(f) = f(\overline{N}_t)(1 - \frac{N_t}{\overline{N}_t}) + \int_0^{N_t/\overline{N}_t} du f\left(\frac{N_t}{u}\right)$$

Remark 1.4 *This example provides us with many martingales of the form $\varphi(N_t, \overline{N}_t)$. One may wonder whether these martingales are of the form of those studied in Lemma 0.2. The answer is positive as we now show by simple computations:*

$$\begin{aligned}
\lambda_t(f) &= \frac{f(\overline{N}_t)}{\overline{N}_t}(\overline{N}_t - N_t) + N_t \int_{\overline{N}_t}^\infty \frac{dy}{y^2} f(y) \\
&= \overline{N}_t \int_{\overline{N}_t}^\infty \frac{dy}{y^2} f(y) - (\overline{N}_t - N_t)\left(\int_{\overline{N}_t}^\infty \frac{dy}{y^2} f(y) - \frac{f(\overline{N}_t)}{\overline{N}_t}\right)
\end{aligned}$$

We obtain the form found in Lemma 0.2 with

$$H(x) = x \int_x^\infty \frac{dy}{y^2} f(y); \quad h(x) = \int_x^\infty \frac{dy}{y^2} f(y) - \frac{f(x)}{x}$$

We now start a general discussion, for which we assume that the underlying filtration is generated by a real-valued Brownian motion $(B_t;\ t \geq 0)$.
In order to state the enlargement decomposition formula, we also need to introduce the $(\mathcal{F}_t;\ t \geq 0)$ predictable process $(\dot{\lambda}_t(f); t \geq 0)$ such that:

$$\lambda_t(f) = E\left[f(X)\right] + \int_0^t \dot{\lambda}_s(f)\, dB_s \qquad (t \geq 0)$$

The process $(\dot{\lambda}_t(f);\ t \geq 0)$ exists from the representation property of Brownian martingales as stochastic integrals with respect to dB_s.
It is not difficult to show that there exists a (predictable) family of measures $(\lambda_t(dx), t \geq 0)$ such that:

$$\lambda_t(f) = \int \lambda_t(dx) f(x)$$

and we shall **assume** the existence of another predictable family $(\dot{\lambda}_t(dx);\ t \geq 0)$ of measures such that:

$$dt \text{ a.s.} \quad \dot{\lambda}_t(f) = \int \dot{\lambda}_t(dx) f(x).$$

The result is the following

Theorem 1.6 *Assume that $ds\, dP$ a.s., the measure $\dot{\lambda}_s(dx)$ is absolutely continuous with respect to $\lambda_s(dx)$ and define $\rho(x,s)$ by:*

$$\dot{\lambda}_s(dx) = \lambda_s(dx)\rho(x,s)$$

Then, for any $(\mathcal{F}_t;\ t \geq 0)$ martingale $(M_t = \int_0^t m_u dB_u;\ t \geq 0)$, there exists $(\widetilde{M}_t;\ t \geq 0)$, a $(\mathcal{F}_t^{\sigma(X)};\ t \geq 0)$ local martingale, such that

$$\begin{aligned} M_t &= \widetilde{M}_t + \int_0^t \rho(X,s) d < M, B >_s \\ &= \widetilde{M}_t + \int_0^t \rho(X,s)\, m_s\, ds \end{aligned}$$

provided that

$$\int_0^t |d < M, B >_s | \; |\rho(X,s)| < \infty \qquad a.s.$$

In particular, if $\int_0^t ds |\rho(X,s)| < \infty$ a.s., then $(B_t;\ t \geq 0)$ decomposes as:

$$B_t = \widetilde{B}_t + \int_0^t \rho(X,s) ds$$

with $\widetilde{B}$ a Brownian motion with respect to $(\mathcal{F}_t^{\sigma(X)};\ t \geq 0)$.

Proof

Let f be a test function, $\Lambda_s \in \mathcal{F}_s$ and $s < t$; then:

$$\begin{aligned}\mathbb{E}\left[1_{\Lambda_s} f(X)(M_t - M_s)\right] &= \mathbb{E}\left[1_{\Lambda_s}(\lambda_t(f)M_t - \lambda_s(f)M_s)\right] \\ &= \mathbb{E}\left[1_{\Lambda_s}\int_s^t du\; m_u \dot{\lambda}_u(f)\right] \\ &= \mathbb{E}\left[1_{\Lambda_s}\int_s^t du\; m_u \int \rho(x,u) f(x) \lambda_u(dx)\right] \\ &= \mathbb{E}\left[1_{\Lambda_s}\int_s^t du\; m_u f(X)\rho(X,u)\right]\end{aligned}$$

from the monotone class theorem; hence, the result.

■

Example 1.6 $X = B_T$ ***with a fixed time*** T

$$\lambda_t(f) = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi(T-t)}} e^{-\frac{(x-B_t)^2}{2(T-t)}} f(x),$$

$$\textit{hence: } \dot{\lambda}_t(f) = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi(T-t)}} \frac{(x-B_t)}{T-t} e^{-\frac{(x-B_t)^2}{2(T-t)}} f(x)$$

(it suffices to "differentiate $\lambda_t(f)$ with respect to B_t")

Therefore

$$\rho(x,t) = \frac{x - B_t}{T-t}$$

In particular, we obtain[5]

$$B_t = \widetilde{B}_t + \int_0^t \frac{B_T - B_s}{T-s} ds$$

with $\widetilde{B}$ a Brownian motion with respect to $(\mathcal{F}_t^{\sigma(B_T)};\ t \geq 0)$, since

$$\mathbb{E}\left[\int_0^T \frac{|B_T - B_s|}{T-s} ds\right] = \sqrt{\frac{2}{\pi}} \int_0^T \frac{ds}{\sqrt{T-s}} < \infty$$

Nonetheless, there is the following negative result

Proposition 1.7 *[JY79]*

i) A $(\mathcal{F}_t;\ t \geq 0)$-martingale $(M_t := \int_0^t m_s dB_s;\ t \geq 0)$ remains a semimartingale in $(\mathcal{F}_t^{\sigma(B_T)};\ t \geq 0)$ if, and only if $\int_0^T \frac{ds|m_s|}{\sqrt{T-s}} < \infty$

[5] For some adequate extension of this formula to Lévy processes, see [JP88]. See also [MY05b] for its relation with harnesses.

ii) There exists some deterministic function m such that:

$$\int_0^T ds\, m_s^2 < \infty \qquad\qquad \int_0^T ds \frac{|m_s|}{\sqrt{T-s}} = \infty$$

Thus there exist some $(\mathcal{F}_t;\ t \geq 0)$-martingales which do not remain semi-martingales with respect to $(\mathcal{F}_t^{\sigma(B_T)};\ t \geq 0)$.

Proof
To prove *ii)*, it suffices to take, for $1/2 < \alpha < 1$, $m_s = \frac{1}{\sqrt{T-s}\left(\log \frac{1}{T-s}\right)^{\alpha}} 1_{T>s\geq T/2}$

■

Example 1.7 $X = S_T = \sup_{s\leq T} B_s$ ***with a fixed time*** T
Using that, for $t \leq T$, $S_T = S_t \vee \sup_{t\leq u\leq T} B_u$, we obtain

$$\lambda_t(dx) = \left(\int_0^{S_t - B_t} \sqrt{\frac{2}{\pi(T-t)}} e^{-\frac{y^2}{2(T-t)}} dy \right) \varepsilon_{S_t}(dx) + \sqrt{\frac{2}{\pi(T-t)}} e^{-\frac{(x-B_t)^2}{2(T-t)}} dx 1_{x>S_t}$$

$$\dot{\lambda}_t(dx) = -\sqrt{\frac{2}{\pi(T-t)}} e^{-\frac{(S_t-B_t)^2}{2(T-t)}} \varepsilon_{S_t}(dx) + \sqrt{\frac{2}{\pi(T-t)}} \frac{x-B_t}{(T-t)} e^{-\frac{(x-B_t)^2}{2(T-t)}} dx 1_{x>S_t}$$

Therefore

$$\rho(x,t) = -\frac{1}{\sqrt{T-t}} \varphi\left(\frac{S_t - B_t}{\sqrt{T-t}}\right) 1_{S_t = x} + 1_{S_t < x} \frac{x - B_t}{T-t}$$

with $\varphi(y) = \frac{e^{-y^2/2}}{\int_0^y dr\ e^{-r^2/2}}$.
Note that this example provides a case where $\lambda_t(dx)$ is not equivalent to dx.

Example 1.8 $X = A_\infty^{(-1/2)} = \int_0^\infty du\, e^{2(B_u - \frac{1}{2}u)}$ ***(See Exercise 17 for more general perpetuities and [BS04] for similar results)***

$$\lambda_t(f) = \hat{\mathbb{E}}\left[f(A_t^{(-1/2)} + e^{2B_t - t}\ \hat{A}_\infty^{(-1/2)}) \right]$$
$$\dot{\lambda}_t(f) = \hat{\mathbb{E}}\left[2\hat{A}_\infty^{(-1/2)} f'(A_t^{(-1/2)} + e^{2B_t - t} \hat{A}_\infty^{(-1/2)}) e^{2B_t - t} \right]$$

We then use that $A_\infty^{(-1/2)}$ is distributed as a 1/2-stable variable; indeed, $A_\infty^{(-1/2)} \overset{(law)}{=} T_0 := \inf\{t \geq 0,\ \beta_t = 0\}$ where β is a Brownian motion starting at 1 thanks to the following Dubins-Schwarz representation[6]

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[6] which is also a particular case of Lamperti's relationship [Lam72]; see Exercise 17 for a discussion and applications of the general case.

$$\exp\left\{B_u - \frac{1}{2}u\right\} = \beta_{\int_0^u ds\, \exp\{2(B_s - \frac{1}{2}s)\}}$$

Therefore, we obtain:

$$\rho(x,t) = 1 - \frac{e^{2B_t - t}}{(x - A_t^{(-1/2)})}$$

Further examples are presented in the Tables at the end of this chapter after the exercises.

1.4 Further References

Motivated partly by applications in Mathematical Finance, where various filtrations related to more or less informed agents play some role, a number of papers on the subject appear regularly, e.g.: [AJKY93] [Bau03] [Bau04] [Bau02] [BM82] [FI93] [Imk96] [KH04] [LNN03] [MP03] [PK96] [Yan02]...

1.5 Exercises

Exercise 8 *Let* $(N_t; t \geq 0)$ *denote a* $\mathbb{R}^+$*-valued, continuous* $(\mathcal{F}_t;\ t \geq 0)$*-local martingale, with* $N_0 = 1$*, and*

$$N_t \underset{t\to\infty}{\longrightarrow} 0$$

Denote $\Lambda := \sup\{t;\ \overline{N}_t - N_t = 0\}$.

The aim of this exercise is to prove, using the balayage formula that:

$$A_t^{\Lambda} = \log(\overline{N}_t) \tag{1.1}$$

where $(A_t^{\Lambda}; t \geq 0)$ *denotes the* $(\mathcal{F}_t)$*-compensator of* $1_{\{\Lambda \leq t\}}$.

a) Denote $\gamma_t = \sup\{s \leq t;\ \overline{N}_s - N_s = 0\}$.
Prove that, for any bounded, $(\mathcal{F}_s;\ s \geq 0)$*-predictable process* $(k_s; s \geq 0)$*, one has*

$$k_{\gamma_t}(\overline{N}_t - N_t) = \int_0^t k_{\gamma_u} d(\overline{N}_u - N_u)$$

b) Deduce that:

$$\mathbb{E}\left[k_{\Lambda}\overline{N}_{\infty}\right] = \mathbb{E}\left[\int_0^{\infty} k_u d\overline{N}_u\right]$$

c) Deduce that (1.1) holds. Conclude that A_{∞}^{Λ} *is exponentially distributed.*

d) *Consider the same question, but do not assume that* $N_t \underset{t\to\infty}{\longrightarrow} 0$.
As is well-known, $N_\infty = \lim_{t\to\infty} N_t$ *exists a.s.*
Find a substitute for formula (1.1).
Hint*: There exists some predictable process* $(n_u;\ u \geq 0)$ *such that:*

$$n_\Lambda = \mathbb{E}\left[N_\infty | \mathcal{F}_\Lambda\right].$$

Then prove that

$$A_t^\Lambda = \int_0^t \frac{d\overline{N}_s}{\overline{N}_s - n_s}$$

Exercise 9 *Let* Λ *denote an honest time, and* $(Z_t = \mu_t - A_t;\ t \geq 0)$ *its associated Azéma supermartingale.*

a) *Using the progressive enlargement formula for* $(\mu_t;\ t \geq \Lambda)$*, show that*

$$1 - Z_{\Lambda+t} = \hat{\mu}_t + \int_0^t \frac{d < \hat{\mu} >_s}{1 - Z_{\Lambda+s}}$$

where $\hat{\mu}_t = -(\mu_{\Lambda+t} - \mu_\Lambda)$

b) *Deduce that* $< \mu >_\infty - < \mu >_\Lambda \overset{(law)}{=} T_1^{(3)}$ *where* $T_1^{(3)}$ *denotes the first hitting of 1 by a 3-dimensional Bessel process.*

Exercise 10 *(Last passage time of a transient diffusion, cf [RY05] exercise 4.16 Chapter VII)*
Let Y *be a diffusion with scale function* s *such that* $s(-\infty) = -\infty$ *and* $s(\infty) = 0$ *and speed measure* m *(see [BS02b] p.13). Consider the last passage time at a given level* a*, namely the random time* $\mathcal{L}_a = \sup\{t \geq 0,\ Y_t = a\}$.

a) *Compute* $A^{\mathcal{L}_a}$*, the predictable compensator of* $\mathcal{L}_a$.
b) *We recall now (see e.g. [BS02b]) that*
- *the transition semi-group of* Y *is absolutely continuous with respect to* m*, i.e. there exists a continuous density* p *such that*

$$P_t(x, dy) = p_t(x, y)m(dy)$$

- *the occupation time formula is*

$$\int_0^t f(X_u)du = \int f(y)L_t^{s(y)}m(dy)$$

where $(L^x,\ x \in \mathbb{R})$ *denotes the family of (Meyer) local times of the local martingale* $s(Y)$

Show that $\frac{\partial}{\partial t}\mathbb{E}_x\left[L_t^{s(a)}\right] = p_t(x, a)$

c) *Show that, for any* $x \in \mathbb{R}$*,* $P_x(\mathcal{L}_a \in dt) = -\frac{1}{2s(a)}p_t(x, a)dt$

Exercise 11 *(Penalization of BM with its one-sided supremum)[RVY05a]*
Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ denote a probability density on $\mathbb{R}_+$, and consider the family $(W_t^f;\ t \geq 0)$ of probability measures on $\Omega_* = \mathcal{C}([0,\infty), \mathbb{R})$ defined as follows:

$$W_t^f = \frac{f(S_t)}{\mathbb{E}_W\left[f(S_t)\right]}.W \tag{1.2}$$

where W denotes Wiener measure, $S_t = \sup_{s \leq t} X_s$ and $(X_s;\ s \geq 0)$ is the coordinate process.
The aim of this exercise is to show that the family $(W_t^f;\ t \geq 0)$ converges weakly, as $t \to \infty$, to W_∞^f, which is defined as

$$W^f_{\infty|\mathcal{F}_s} = M_s^f.W_{|\mathcal{F}_s} \tag{1.3}$$

i.e., W_∞^f, when restricted to $\mathcal{F}_s$, is absolutely continuous with respect to W, with Radon-Nikodym density

$$M_s^f = (1 - F(S_s)) + f(S_s)(S_s - X_s)$$

where $F(x) = \int_0^x dy\ f(y)$.
In fact, prove that for any $\Gamma_s \in \mathcal{F}_s$:

$$W_t^f(\Gamma_s) \underset{t\to\infty}{\longrightarrow} \mathbb{E}_W\left[1_{\Gamma_s} M_s^f\right] \tag{1.4}$$

***Hint**: In order to prove (1.4), show that, for some universal constant C,*

$$C\sqrt{t}\ \mathbb{E}\left[f(S_t)|\mathcal{F}_s\right] \underset{t\to\infty}{\longrightarrow} M_s^f$$

Exercise 12 *Let $\lambda > 0$ and denote $S_t^{(-\lambda)} = \sup_{s \leq t}(B_s - \lambda s)$*
Prove that $S_\infty^{(-\lambda)}$ is distributed exponentially, and identify the parameter of the exponential law.
***Hint**: For some constant C, the process $(\exp\left(C(B_s - \lambda s)\right); s \geq 0)$ is a martingale, which starts at 1, and goes to 0, as $s \to \infty$. Conclude from there.*

Exercise 13 *Let $(\beta_u;\ u \geq 0)$ denote a one-dimensional Brownian motion, starting from 0, and $(\lambda_u;\ u \geq 0)$ its local time at 0.*
Denote by $(\tau_l^{(\beta)},\ l \geq 0)$ the right-continuous inverse of $(\lambda_u;\ u \geq 0)$.
a) Prove that, for any $A > 0$:

$$\sup_{v \leq \tau_A^{(\beta)}} |\beta_v| \overset{(law)}{=} \frac{A}{\mathbf{e}} \tag{1.5}$$

where $\mathbf{e}$ denotes a standard exponential variable.
***Hint**: Write*

$$\left\{ \sup_{v \leq \tau_A^{(\beta)}} |\beta_v| \geq C \right\} = \left\{ A \geq \lambda_{T_C(|\beta|)} \right\}$$

where $T_C(|\beta|) = \inf\{t \geq 0;\ |\beta_t| = C\}$, and use the fact that $\lambda_{T_C(|\beta|)}$ is exponentially distributed (see Example 1.4).

b) Prove that the process $(W_l := \sup_{v \leq \tau_l^{(\beta)}} |\beta_v|;\ l \geq 0)$ *is a Markov process. Compute its semigroup.*

Exercise 14 *(A mixture of balayage and time-change)*
Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *be such that* $\int_0^a \frac{dx}{\varphi(x)} < \infty$, *for every* $a > 0$*; denote by* $(B_t;\ t \geq 0)$ *a one-dimensional Brownian motion starting at 0, and by* $(L_t;\ t \geq 0)$ *its local time at 0.*

a) The aim of this question is to prove the formula:

$$P(\exists t \geq 0,\ |B_t| \geq \varphi(L_t)) = 1 - \exp\left(-\int_0^\infty \frac{dx}{\varphi(x)}\right) \tag{1.6}$$

or, equivalently, that the variable $\inf_{t\geq 0}\left\{\frac{\varphi(L_t)}{|B_t|}\right\}$ *is exponentially distributed with parameter* $\int_0^\infty \frac{dx}{\varphi(x)}$.

a.1) Prove that $(\frac{1}{\varphi(L_t)}B_t;\ t \geq 0)$ *is a local martingale with Dubins-Schwarz representation*

$$(\beta_{\int_0^t \frac{ds}{\varphi^2(L_s)}};\ t \geq 0)$$

where β *denotes a Brownian motion.*

a.2) Let $(L_u^*; u \geq 0)$ *denote the local time at 0 of* β.
Prove the relationship

$$\int_0^{L_t} \frac{dx}{\varphi(x)} = L^*_{\int_0^t \frac{ds}{\varphi^2(L_s)}}$$

Deduce that

$$\int_0^{\tau_l} \frac{ds}{\varphi^2(L_s)} \overset{(law)}{=} \inf\left\{t \geq 0;\ B_t = \int_0^l \frac{dx}{\varphi(x)}\right\}$$

where $\tau_l = \inf\{t \geq 0;\ L_t > l\}$

a.3) Deduce from Exercise 13 the identity

$$P(\exists t \leq \tau_l,\ |B_t| \geq \varphi(L_t)) = 1 - \exp\left(-\int_0^l \frac{dx}{\varphi(x)}\right) \tag{1.7}$$

b) Find an integral criterion bearing on φ *such that:*

$$P(\forall A > 0,\ \exists t \geq A,\ |B_t| \geq \varphi(L_t)) = \begin{cases} 1 \\ 0 \end{cases}$$

c) Develop some analogous study for the quantities

$$P(\exists t \geq 0,\ B_t \leq \psi(S_t))$$

✠ $$P(\exists t \geq 0,\ \varphi_-(I_t) \leq B_t \leq \varphi_+(S_t))$$

where $S_t = \sup_{s \leq t} B_s$ *and* $I_t = \inf_{s \leq t} B_s$.

Exercise 15 *This exercise extends the results of the preceding exercise to recurrent Bessel processes using excursion theory; more precisely, for* $\nu \in (0,1)$, *let* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *be such that* $\int_0^a \frac{dx}{\varphi(x)^{2\nu}} < \infty$, *for every* $a > 0$*; denote by* $(R_t;\ t \geq 0)$ *a Bessel process with index* $-\nu$, *starting at 0, and by* $(L_t; t \geq 0)$ *its local time at 0, chosen so that* $(R_t^{2\nu} - L_t;\ t \geq 0)$ *is a martingale.*

a) Prove that

$$P(\forall t \leq \tau_l,\ R_t \leq \varphi(L_t)) = \exp - \int_0^l d\lambda\, \mathbf{n}\left(\max_u \varepsilon_u \geq \varphi(\lambda)\right)$$

where ε, *under the Itô measure* $\mathbf{n}$ *of excursions of* R *away from 0, denotes the generic excursion and* τ *is the right-inverse of the local time process* L.

b) Show that $\mathbf{n}\left(\max_u \varepsilon_u \geq a\right) = a^{-2\nu}$. *Conclude that*

$$P(\exists t \leq \tau_l,\ R_t \geq \varphi(L_t)) = 1 - \exp\left(- \int_0^l \frac{dx}{\varphi(x)^{2\nu}}\right)$$

c) Recover this result for $l = \infty$, *without excursion theory, but using instead Doob's maximal identity (Lemma 0.1).*
Hint*: One may use the martingale property of* $(h(L_t)R_t^{2\nu} + 1 - H(L_t),\ t \geq 0)$ *for a conveniently chosen measurable, positive function* h *such that* $\int_0^\infty h(u)du = 1$ *and* $H(x) = \int_0^x h(y)dy$.

Remark 1.5 *This result can also be deduced from the result of Exercise 14 about reflecting Brownian motion. Indeed,* $R^{2\nu}$ *is a time changed reflecting Brownian motion.*

Exercise 16 ✠ *(See [OY05], and [Obł05])*
Show that the only local martingales with respect to the Brownian filtration which are of the form $(f(S_t, B_t);\ t \geq 0)$ *may be written as:*

$$(H(S_t) - h(S_t)(S_t - B_t);\ t \geq 0)$$

where $h : [0, \infty) \to \mathbb{R}$ *is locally integrable, and* $H(x) = \int_0^x dy\ h(y)$.
Hint*: Prove the result in case* f *is regular.*

The following exercise extends to any $\mu > 0$ the result seen in Example 1.8 in this chapter for $\mu = 1/2$.

Exercise 17 *(Initial enlargement with the perpetuity: $A_\infty^{(-\mu)}$; see, e.g. [MY01] for details)*
Let $A_\infty^{(-\mu)} = \int_0^\infty ds \exp 2(B_s - \mu s)$ *be the perpetuity with parameter* μ.

a) *Show Lamperti's relationship:*

$$\exp(B_t - \mu t) = R^{(-\mu)}_{A_t^{(-\mu)}} \tag{1.8}$$

where $(R_u^{(-\mu)}; u \geq 0)$ *denotes a Bessel process with index* $(-\mu)$ *(or dimension* $\delta = 2(1-\mu))$ *starting from 1, considered up to* $A_\infty^{(-\mu)}$ *and* $A_t^{(-\mu)} = \int_0^t \exp 2(B_s - \mu s) ds$ *[Recall that* $BES^{(-\mu)}$ *is a diffusion taking values in* $\mathbb{R}^+$ *with infinitesimal generator* $\frac{1}{2}\frac{d^2}{dx^2} + \frac{2(-\mu)+1}{2x}\frac{d}{dx}$*]*

b) *Deduce from (1.8) that:*

$$A_\infty^{(-\mu)} = T_0(R^{(-\mu)}) = \inf\{u,\ R_u^{(-\mu)} = 0\} \tag{1.9}$$

c) *Prove that* $T_0(R^{(-\mu)})$ *is distributed as* $1/2\gamma_\mu$, *where* γ_μ *denotes a gamma variable with parameter* μ.

d) *Prove, with the notation in Chapter 1, that for* $X = A_\infty^{(-\mu)}$, *one has:*

$$\rho(x,t) = 2\mu - \frac{\exp(2(B_t - \mu t))}{x - A_t^{(-\mu)}}$$

hence that

$$B_t^{(-\mu)} = \widetilde{B}_t^{(\mu)} - \int_0^t ds \frac{\exp\left(2B_s^{(-\mu)}\right)}{A_\infty^{(-\mu)} - A_s^{(-\mu)}} \tag{1.10}$$

where $B_t^{(-\mu)} = B_t - \mu t$ *and* $\widetilde{B}_t^{(\mu)} = \widetilde{B}_t + \mu t$ *denotes a Brownian motion with drift* μ *in the filtration* $(\mathcal{F}_t^{\sigma(A_\infty^{(-\mu)})};\ t \geq 0)$.

e) *Solve equation (1.10), by expressing explicitly* $(B_t^{(-\mu)};\ t \geq 0)$ *in terms of* $(\widetilde{B}_t^{(\mu)};\ t \geq 0)$ *and* $A_\infty^{(-\mu)}$. *That is, show that:*

$$B_t^{(-\mu)} = \widetilde{B}_t^{(\mu)} + \log\left(\frac{A_\infty^{(-\mu)}}{A_\infty^{(-\mu)} + \widetilde{A}_t^{(\mu)}}\right) \tag{1.11}$$

f) *Deduce from formula (1.10) the following identity*[7]

$$\frac{1}{A_t^{(-\mu)}} = \frac{1}{\widetilde{A}_t^{(\mu)}} + \frac{1}{A_\infty^{(-\mu)}} \tag{1.12}$$

[7] This identity has first been proved, without enlargement of filtrations, by Dufresne in [Duf90].

Comment 1.4 *Note that formula (1.11) contains a recipe to "construct" a $B^{(-\mu)}$ process from a $\widetilde{B}^{(\mu)}$ one, once an additional gamma(μ) variable γ_μ is given, independent of $\widetilde{B}^{(\mu)}$; precisely*

$$B_t^{(-\mu)} = \widetilde{B}_t^{(\mu)} - \log\left(1 + 2\gamma_\mu \widetilde{A}_t^{(\mu)}\right)$$

Exercise 18 ✠
Obtain an initial enlargement formula with $\int_0^\infty ds\ \varphi(B_s^{(\mu)})$ for more general functions φ than $\exp(-ax)$.

Exercise 19 *Let $(B_t;\ t \geq 0)$ be a standard Brownian motion, $(S_t;\ t \geq 0)$ its one-sided supremum and $(L_t;\ t \geq 0)$ its local time process at level 0.*

a) 1. Compute the formula for the initial enlargement with (B_T, S_T), with T a fixed time.
2. Deduce the decomposition of $(S_t - B_t;\ t \geq 0)$ in $(\mathcal{F}_t^{\sigma(S_T,B_T)};\ t \geq 0)$.
b) 1. Compute the formula for the initial enlargement with (B_T, L_T), with T a fixed time.
2. Deduce the decomposition of $(|B_t|;\ t \geq 0)$ in $(\mathcal{F}_t^{\sigma(L_T,B_T)};\ t \geq 0)$.
c) Compare the two formulae obtained in question a)2. and b)2.

Exercise 20 *This exercise exploits the computations made at the beginning of Section 1.3 for the initial enlargement with $\overline{N}_\infty$. Define $\dot{\lambda}_t(f)$ with respect to N, namely*

$$\lambda_t(f) = \mathbb{E}\left[f(\overline{N}_\infty)\right] + \int_0^t \dot{\lambda}_s(f) dN_s$$

a) Prove the formulae

$$\lambda_t(dx) = \left(1 - N_t/\overline{N}_t\right) \varepsilon_{\overline{N}_t}(dx) + N_t \frac{dx}{x^2} 1_{[\overline{N}_t,\infty)}(x) \qquad (1.13)$$

$$\dot{\lambda}_t(dx) = -(1/\overline{N}_t)\varepsilon_{\overline{N}_t}(dx) + \frac{dx}{x^2} 1_{[\overline{N}_t,\infty)}(x) \qquad (1.14)$$

b) Compare the progressive enlargement formula with $\mathcal{L}^{(N)} := \sup\{t \geq 0,\ \overline{N}_t = N_t\}$ and the initial enlargement formula with $\overline{N}_\infty$.

The following exercise presents an attempt to shrink/slim the Brownian filtration. Note that, on the contrary to the enlargement theory, the shrinking/slimming operation is still a quite undeveloped topic.

Exercise 21 *Let $(B_t;\ t \geq 0)$ be a standard Brownian motion and $(\mathcal{F}_t;\ t \geq 0)$ its natural filtration. Consider $h \in L^2_{loc}(\mathbb{R}_+)$; this exercise aims at shrinking the filtration $(\mathcal{F}_t;\ t \geq 0)$ with the variables $\int_0^t h(u)dB_u$, i.e. at defining and studying a filtration $(\mathcal{F}_t^{h,-};\ t \geq 0)$ such that:*

- *for any $t \geq 0$, $\mathcal{F}_t^{h,-}$ is independent from the variable $\int_0^t h(u)dB_u$*

- *for any* $t \geq 0$, $\mathcal{F}_t = \mathcal{F}_t^{h,-} \vee \sigma\left(\int_0^t h(u)dB_u\right)$

a) *Show that there exists* $(\mathcal{F}_t^{h,-};\ t \geq 0)$, *a subfiltration of* $(\mathcal{F}_t;\ t \geq 0)$, *which satisfies the above conditions.*
Hint: Consider $\mathcal{F}_t^{h,-} = \sigma\left(\int_0^t g(u)dB_u;\ g \in L^2([0,t]),\ \int_0^t du g(u)h(u) = 0\right)$.

b) *Show that* $B_t^{h,-} := B_t - \frac{\int_0^t h(u)du}{\int_0^t h(u)^2 du} \int_0^t h(u)dB_u$ *is a* $(\mathcal{F}_t^{h,-};\ t \geq 0)$-*martingale.*

c) *Study the process* $(B_t^{h_\alpha,-};\ t \geq 0)$ *for* $h_\alpha(u) = u^\alpha$ *with* $\alpha > -1/2$. *In particular, show that*

$$\left(B_t - \frac{3}{t}\int_0^t B_u du;\ t \geq 0\right)$$

defines a Brownian motion. More generally, prove that, for every $\alpha > -1/2$,

$$\left(B_t^{\alpha,-} := B_t - \frac{2\alpha+1}{t^\alpha}\int_0^t u^{\alpha-1} B_u du;\ t \geq 0\right)$$

is a Brownian motion, which generates the filtration $(\mathcal{F}_t^{\alpha,-},\ t \geq 0)$, *i.e. the filtration* $(\mathcal{F}_t^{h_\alpha,-},\ t \geq 0)$ *with* $h_\alpha(x) = x^\alpha$.

A

Appendix: Some Enlargements Formulae

R. Mansuy and M. Yor: *Random Times and Enlargements of Filtrations in a Brownian Setting*,
Lect. Notes Math. **1873**, 31–39 (2006)
www.springerlink.com

Table 1α: Progressive Enlargements

	Λ	Z_t^Λ	A_t^Λ
1	$\gamma_T = \sup\{u \leq T,\ B_u = 0\}$ T fixed time	$\Phi\left(\frac{\lvert B_t\rvert}{\sqrt{T-t}}\right) 1_{t<T}$	$\sqrt{\frac{2}{\pi}} \int_0^{t\wedge T} \frac{dL_u}{\sqrt{T-u}}$
2	$\gamma_T^a = \sup\{u \leq T,\ B_u = a\}$ T fixed time	$\Phi\left(\frac{\lvert B_t - a\rvert}{\sqrt{T-t}}\right) 1_{t<T}$	$\sqrt{\frac{2}{\pi}} \int_0^{t\wedge T} \frac{dL_u^a}{\sqrt{T-u}}$
3	$\gamma_{T_a^*} = \sup\{u \leq T_a^*,\ B_u = 0\}$ $T_a^* = \inf\{t,\ \lvert B_t\rvert = a\},\ a > 0$	$1 - \frac{1}{a}\lvert B_{t\wedge T_a^*}\rvert$	$\frac{1}{a} L_{t\wedge T_a^*}$
4	$\gamma_{T_a} = \sup\{u \leq T_a,\ B_u = 0\}$ $T_a = \inf\{t,\ B_t = a\},\ a > 0$	$1 - \frac{1}{a} B^+_{t\wedge T_a}$	$\frac{1}{2a} L_{t\wedge T_a}$
5	$\gamma_T(R) = \sup\{u \leq T,\ R_u = 0\}$ R BES$(-\mu)$ with $\mu \in (0,1)$	$\frac{1}{\Gamma(\mu)} \int_{\frac{R_t^2}{2(T-t)}}^{\infty} dv v^{\mu-1} e^{-v} 1_{t<T}$	$\int_0^{t\wedge T} \frac{dL_s(R)}{\mu\Gamma(\mu)(2(T-s))^\mu}$
6	$\mathcal{L}_a = \sup\{u,\ R_u = a\}$ R BES(μ), with $\mu > 0,\ R_0 = 0$	$1 \wedge \left(\frac{a}{R_t}\right)^{2\mu}$	$\frac{\mu}{a} L_t^a(R)$
7	$\mathcal{L}^{(N)} = \sup\{t, \overline{N}_t = N_t\}$ $N \geq 0$ martingale, $N_\infty = 0,\ N_0 = 1$	$N_t/\overline{N_t}$	$\log(\overline{N}_t)$

Here, we denote $\Phi(x) = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-u^2/2} du$.

Table 1β: Associated Enlargement Formulae

We now write the associated formulae, where the process $(\widetilde{B}_t;\ t \geq 0)$ (resp. $(\widetilde{M}_t;\ t \geq 0)$) always indicates a Brownian motion (resp. a martingale) with respect to the enlarged filtration.

Convention A.1 *In the following tables, integrals such as* $\int_\varrho^t ds\ \vartheta_s$ *are equal to 0 for* $t \leq \varrho$.

1	$B_t = \widetilde{B}_t - \int_0^{t\wedge\gamma_T} \frac{ds}{\sqrt{T-s}} \mathrm{sgn}(B_s)\sqrt{\frac{2}{\pi}}\, \frac{e^{-\frac{B_s^2}{2(T-s)}}}{\Phi\left(\lvert B_s\rvert/\sqrt{T-s}\right)}$ $+\mathrm{sgn}(B_T)\int_{\gamma_T}^{t\wedge T} \frac{ds}{\sqrt{T-s}}\sqrt{\frac{2}{\pi}}\, \frac{e^{-\frac{B_s^2}{2(T-s)}}}{1-\Phi\left(\lvert B_s\rvert/(\sqrt{T-s})\right)}$
2	$B_t = \widetilde{B}_t - \int_0^{t\wedge\gamma_T^a} \frac{ds}{\sqrt{T-s}} \mathrm{sgn}(B_s - a)\sqrt{\frac{2}{\pi}}\, \frac{e^{-\frac{(B_s-a)^2}{2(T-s)}}}{\Phi\left(\lvert B_s-a\rvert/\sqrt{T-s}\right)}$ $+\mathrm{sgn}(B_T - a)\int_{\gamma_T^a}^{t\wedge T} \frac{ds}{\sqrt{T-s}}\sqrt{\frac{2}{\pi}}\, \frac{e^{-\frac{(B_s-a)^2}{2(T-s)}}}{1-\Phi\left(\lvert B_s-a\rvert/\sqrt{T-s}\right)}$
3	$B_t = \widetilde{B}_t - \int_0^{t\wedge\gamma_{T_a^*}} \frac{\mathrm{sgn}(B_s)}{a-\lvert B_s\rvert} ds + \int_{\gamma_{T_a^*}}^{t\wedge T_a^*} \frac{1}{B_s} ds$
4	$B_t = \widetilde{B}_t - \int_0^{t\wedge\gamma_{T_a}} \frac{1_{B_s>0}}{a-B_s} ds + \int_{\gamma_{T_a}}^{t\wedge T_a} \frac{ds}{B_s}$
5	$M_t = \widetilde{M}_t - \sqrt{2}\int_0^{t\wedge\gamma_T(R)} \frac{d<M,R>_s}{\sqrt{T-s}} \left(\frac{R_s^2}{2(T-s)}\right)^{\mu-1/2} \frac{e^{-\frac{R_s^2}{2(T-s)}}}{\int_{R_s^2/(2(T-s))}^\infty dv v^{\mu-1}e^{-v}}$ $+\sqrt{2}\int_{\gamma_T(R)}^{t\wedge T} \frac{d<M,R>_s}{\sqrt{T-s}} \left(\frac{R_s^2}{2(T-s)}\right)^{\mu-1/2} \frac{e^{-\frac{R_s^2}{2(T-s)}}}{\int_0^{R_s^2/(2(T-s))} dv v^{\mu-1}e^{-v}}$
6	$M_t = \widetilde{M}_t - 2\mu\int_0^{t\wedge\mathcal{L}_a} \frac{d<M,R>_s}{R_s} 1_{R_s>a} - 2\mu\int_{\mathcal{L}_a}^t \frac{d<M,R>_s}{R_s} \frac{a^{2\mu}}{a^{2\mu}-R_s^{2\mu}}$
7	$M_t = \widetilde{M}_t + \int_0^{t\wedge\mathcal{L}^{(N)}} \frac{d<M,N>_s}{N_s} + \int_{\mathcal{L}^{(N)}}^t d<M,N>_s / (N_s - \overline{N}_s)$

Table 2α: Initial Enlargements

	X	$\rho(t,x)$
1	B_T terminal value	$\frac{x-B_t}{T-t}$
2	S_T supremum at fixed time	$-\frac{1}{\sqrt{T-t}}\varphi\left(\frac{S_t-B_t}{\sqrt{T-t}}\right)1_{S_t=x}+\frac{x-B_t}{T-t}1_{S_t<x}$
3	(B_T,S_T) terminal value and supremum	$\left(\frac{2z-y-B_t}{T-t}-\frac{1}{2z-y-B_t}\right)1_{S_t<z}$ $+\frac{1}{T-t}\left(S_t-B_t-(z-y)\coth\left[\frac{(z-y)(S_t-B_t)}{T-t}\right]\right)\ 1_{z=S_t}\ 1_{y\le S_t}$ with $x=(y,z)$
4	L_T local time at fixed time	$-\frac{\mathrm{sgn}(B_t)}{\sqrt{T-t}}\left(\varphi\left(\frac{\lvert B_t\rvert}{\sqrt{T-t}}\right)1_{L_t=x}+\frac{x-L_t+\lvert B_t\rvert}{\sqrt{T-t}}1_{L_t<x}\right)$
5	(B_T,L_T) terminal value and local time	$\mathrm{sgn}(B_t)\left(\frac{1}{l-L_t+\lvert b\rvert+\lvert B_t\rvert}-\frac{l-L_t+\lvert b\rvert+\lvert B_t\rvert}{T-t}\right)1_{l>L_t}$ $+\frac{1}{T-t}\left(b\coth\left[\frac{bB_t}{T-t}\right]-B_t\right)1_{l=L_t}1_{bB_t>0}$ with $x=(b,l)$
6	γ_T last zero before a fixed time	$-\frac{\mathrm{sgn}(B_t)}{\sqrt{T-t}}\varphi\left(\frac{\lvert B_t\rvert}{\sqrt{T-t}}\right)1_{x=\gamma_t}-\frac{B_t}{x-t}1_{T>x>t}$
7	(γ_T,δ_T) zeros around a fixed time	$\left(\frac{1}{B_t}-\frac{B_t}{z-t}\right)1_{\gamma_t<T}1_{z>t\vee T}1_{y=\gamma_t}-\frac{B_t}{y-t}1_{z>T>y>t}$ with $x=(y,z)$
8	T_a first hitting time of a	$-\frac{1}{a-B_t}+\frac{a-B_t}{x-t}$ $t\le T_a$
9	τ_l inverse of local time at l	$-\frac{1}{l-\lvert B_t\rvert+L_t}+\frac{l-\lvert B_t\rvert+L_t}{x-t}$ $t\le\tau_l$
10	$A_\infty^{(-\mu)}$ perpetuity with drift $-\mu$	$2\mu-\frac{e^{2(B_t-\mu t)}}{x-A_t^{(-\mu)}}$
11	$\overline{N}_\infty$ Supremum of a martingale$N\ge 0$ $N_0=1$, vanishing at $+\infty$	$1/(N_t-\overline{N}_t)1_{\overline{N}_t=x}+\frac{1}{N_t}1_{\overline{N}_t<x}$

We use the notation: $\varphi(x)=\frac{e^{-x^2/2}}{\int_0^x e^{-u^2/2}du}$

Table 2β: Associated Enlargement Formulae

We now write the associated formulae, where the process $(\widetilde{M}_t;\ t \geq 0)$ always indicates a martingale in the enlarged filtration.

1	$M_t = \widetilde{M}_t + \int_0^t d < M, B >_s \frac{B_T - B_s}{T-s}$
2	$M_t = \widetilde{M}_t + \int_0^{t \wedge \sigma_T} d < M, B >_s \frac{S_T - B_s}{T-s} - \int_{\sigma_T}^t d < M, B >_s \frac{1}{\sqrt{T-s}} \varphi\left(\frac{S_s - B_s}{\sqrt{T-s}}\right)$
3	$M_t = \widetilde{M}_t + \int_0^{t \wedge \sigma_T} d < M, B >_s \left(\frac{2S_T - B_T - B_s}{T-s} - \frac{1}{2S_T - B_T - B_s}\right)$ $+ \int_{\sigma_T}^t d < M, B >_s \frac{1}{T-s} \left(S_s - B_s - (S_T - B_T) \coth\left[\frac{(S_T - B_T)(S_s - B_s)}{T-s}\right]\right)$
4	$M_t = \widetilde{M}_t + \int_0^{t \wedge \gamma_T} d < M, B >_s \frac{L_T - L_s + \|B_s\|}{\sqrt{T-s}} - \text{sgn}(B_T) \int_{\gamma_T}^{t \wedge T} \frac{d<M,B>_s}{\sqrt{T-s}} \varphi\left(\frac{\|B_s\|}{\sqrt{T-s}}\right)$
5	$M_t = \widetilde{M}_t + \int_0^{t \wedge \gamma_T} d < M, B >_s \text{sgn}(B_s) \left(\frac{1}{L_T - L_s + \|B_T\| + \|B_s\|} - \frac{L_T - L_s + \|B_T\| + \|B_s\|}{T-s}\right)$ $+ \int_{\gamma_T}^{t \wedge T} d < M, B >_s \frac{1}{T-s} \left(B_T \coth\left[\frac{B_T B_s}{T-s}\right] - B_s\right)$
6	$M_t = \widetilde{M}_t - \int_0^{t \wedge \gamma_T} d < M, B >_s \frac{B_s}{\gamma_T - s} - \text{sgn}(B_T) \int_{\gamma_T}^{t \wedge T} \frac{d<M,B>_s}{\sqrt{T-s}} \varphi\left(\frac{\|B_s\|}{\sqrt{T-s}}\right)$
7	$M_t = \widetilde{M}_t - \int_0^{t \wedge \gamma_T} d < M, B >_s \frac{B_s}{\gamma_T - s} + \int_{\gamma_T}^{t \wedge \delta_T} d < M, B >_s \left(\frac{1}{B_s} - \frac{B_s}{\delta_T - s}\right)$
8	$M_t = \widetilde{M}_t + \int_0^{t \wedge T_a} d < M, B >_s \left(\frac{a - B_s}{T_a - s} - \frac{1}{a - B_s}\right)$
9	$M_t = \widetilde{M}_t + \int_0^{t \wedge \tau_l} d < M, B >_s \left(\frac{l - \|B_s\| + L_s}{\tau_l - s} - \frac{1}{l - \|B_s\| + L_s}\right)$
10	$M_t = \widetilde{M}_t + 2\mu t - \int_0^t d < M, B >_s \frac{e^{2(B_s - \mu s)}}{A_\infty^{(-\mu)} - A_s^{(-\mu)}}$
11	$M_t = \widetilde{M}_t + \int_0^{t \wedge \mathcal{L}^{(N)}} \frac{d<M,N>_s}{N_s} + \int_{\mathcal{L}^{(N)}}^t d < M, N >_s /(N_s - \overline{N}_s)$

Some Comments About Path Decompositions Related to These Formulae

We show how all (or almost all!) of the formulae given in Table 2β may be interpreted in terms of semimartingale decompositions of Markovian bridges. This rests on the following lemma, close to Exercise 5, p.8.

Lemma A.1 *If $P^{(t)}_{x\to y}$ denotes the law of a good diffusion (on $\mathbb{R}$), starting from x, conditioned to be at y at time t, then, for any $s < t$,*

$$P^{(t)}_{x\to y|\mathcal{F}_s} = \frac{p_{t-s}(X_s, y)}{p_t(x,y)} . P_{x|\mathcal{F}_s}$$

where $p_t(.,.)$ denotes the density of the transition probability associated to the considered diffusion.
As a consequence, if under P_x:

$$X_s = x + \beta_s + \int_0^s b(X_u)du$$

then, under $P^{(t)}_{x\to y}$:

$$X_s = x + \widetilde{\beta}_s + \int_0^s b(X_u)du + \int_0^s du \left(\frac{\partial}{\partial z} (\log(p_{t-u}(z,y))) \right)_{|z=X_u}$$

We apply this lemma to three different cases:

Example A.1 *If P_x is Wiener measure with starting point x, then under $P^{(t)}_{x\to y}$:*

$$X_s = x + \widetilde{\beta}_s + \int_0^s du \frac{y - X_u}{t-u} \tag{A.1}$$

Example A.2 *If P_x is the law of a 3-dimensional Bessel process, starting from x (hence $b(z) = 1/z$), we recall that*

$$p_t(x,y) = \frac{2}{\sqrt{2\pi t}} \frac{y}{x} \exp\left(-\frac{x^2+y^2}{2t}\right) \sinh\left(\frac{xy}{t}\right)$$

Hence

$$\frac{\partial}{\partial z} (\log(p_t(z,y))) = -\frac{1}{z} - \frac{z}{t} + \frac{y}{t} \coth\left(\frac{zy}{t}\right)$$

so that, under $P^{(t)}_{x\to y}$

$$X_s = x + \widetilde{\beta}_s + \int_0^s du \left[\frac{y}{t-u} \coth\left(\frac{yX_u}{t-u}\right) - \frac{X_u}{t-u} \right] \tag{A.2}$$

Example A.3 *In particular, letting $y \to 0$, one obtains from* (A.2)*, that, under $P^{(t)}_{x \to 0}$*

$$X_s = x + \widetilde{\beta}_s + \int_0^s du \left[\frac{1}{X_u} - \frac{X_u}{t-u} \right] \tag{A.3}$$

With all these decompositions in mind, we are ready to discuss the enlargement decompositions found in the Tables.

* * * * *

Comments on Table 2β

Line 1

Conditionally on $B_T = y$, $(B_u;\ u \le T)$ is a Brownian bridge of length T, starting from 0 and ending at y.

Line 2

- Let $\sigma_T := \inf\{t > 0;\ B_t = S_T\}$. Conditionally on $S_T = \theta$, $(B_t;\ t \le \sigma_T)$ is a Brownian bridge of length T, starting from 0 and ending at θ, considered up to $\sigma_T := \inf\{t > 0;\ B_t = \theta\}$.
- $(S_{\sigma_T+u} - B_{\sigma_T+u};\ u \le T - \sigma_T)$ is a Brownian meander of length $T - \sigma_T$.

The meander process is studied with more details in Subsection 3.1.3.

Line 3

σ_T still denotes the first hitting time of S_T. Conditionally on $S_T = \theta$ and $B_T = x$,

- $(2S_T - B_T - B_t;\ t \le \sigma_T)$ is a Brownian bridge of length T, starting from 0 and ending at θ, stopped at its first hitting time of $\theta - x$.
- $(S_T - B_{\sigma_T+u};\ u \le T - \sigma_T)$ is a 3-dimensional Bessel bridge of length $T - \sigma_T$ starting from 0 and ending at $\theta - x$.

Lines 4-5

See lines 2-3 thanks to Lévy's equivalence.

Line 6

Conditionally on $\gamma_T = g$,

- $(B_u;\ u \le g)$ is a Brownian bridge of length g starting from 0, ending at 0.
- $(|B_{g+u}|;\ u \le T - g)$ is a Brownian meander of length $T - g$.

See Subsection 3.1.2 and 3.1.3 for details.

Line 7

Conditionally on $\gamma_T = g$ and $\delta_T = d$,

- $(B_u;\ u \le g)$ is a Brownian bridge of length g starting from 0, ending at 0.
- The process $(|B_{g+u}|;\ u \le d - g)$ is a 3-dimensional Bessel bridge of length $d - g$ starting from 0 and ending at 0.

See Subsection 3.1.2 for details.

Line 8

Conditionally on $T_a = t$, $(a - B_u;\ u \leq t)$ is a 3-dimensional Bessel bridge of length t starting at a and ending at 0. This may also be seen since the time-reversed process is a 3-dimensional Bessel bridge starting from 0 and going to a.

Line 9

See line 8 thanks to Lévy's equivalence.

Line 10

Conditionally on $A_\infty^{(-\mu)} = a$, the process $R^{(-\mu)}$ defined from Lamperti's relationship [Lam72] is a $(2(\mu+1))$-dimensional Bessel bridge of length t starting from 1 and ending at 0. Indeed, this formula implies easily $e^{B_t - \mu t} =$

$$1 + \int_0^t e^{B_s - \mu s} d\widetilde{B}_s + (\mu + \frac{1}{2}) \int_0^t e^{B_s - \mu s} ds - \int_0^t e^{B_s - \mu s} \frac{e^{2(B_s - \mu s)}}{A_\infty^{(-\mu)} - A_s^{(-\mu)}} ds$$

Then, time-changing with $u = A_s^{(-\mu)}$ and using Lamperti's relationship, we find

$$R_h^{(-\mu)} = 1 + \widetilde{\beta}_h + \left(\mu + \frac{1}{2}\right) \int_0^h \frac{du}{R_u^{(-\mu)}} - \int_0^h \frac{R_u^{(-\mu)}}{A_\infty^{(-\mu)} - u} du, \qquad h < A_\infty^{(-\mu)}$$

Hence the announced result.

Line 11

Note that this line is precisely the same as line 7 in Table 1β. See Exercise 20 for an explanation.

As mentioned at the beginning of these comments, most of these path decompositions can be read in terms of standard bridges thanks to suitably re-scaled processes. For example, Line 6 can also be stated as follows:
conditionally on $\gamma_T = g$, $\left(\frac{B_{vg}}{\sqrt{g}};\ v \leq 1\right)$ is a standard Brownian bridge, and $\left(\frac{|B_{g+v(T-g)}|}{\sqrt{T-g}};\ v \leq 1\right)$ is a standard Brownian meander.

* * * * *

Comments on Table 1β

Line 1

Conditionally on $\gamma_T = g$,

- The law of the process $(B_t;\ t \leq g)$ is given, for $t \leq g$, by

$$\mathbb{P}_{|\mathcal{F}_t} = \sqrt{\frac{2}{\pi}}.\int_{|B_t|/\sqrt{T-t}}^{\infty} e^{-\frac{u^2}{2}} du\ e^{\sqrt{\frac{2}{\pi}}L_t}.\mathbb{W}_{|\mathcal{F}_t}$$

  where $\mathbb{W}$ denotes the Wiener measure.
- $(|B_{g+u}|;\ u \leq T-g)$ is a Brownian meander of length $T-g$.

Line 2

Similar to Line 1 with $(B_t - a;\ t \geq 0)$ instead of B.

Line 3

Conditionally on $\gamma_{T_a^*} = g$,$(|B_{g+u}|;\ u \leq T_a^* - g)$ is a 3-dimensional Bessel process considered up to its first hitting time of the level a.

Line 4

This result is essentially a consequence of the preceding one since the positive part of B is a time-changed, reflected Brownian motion; more precisely, there exists a reflecting Brownian motion $(\rho_u;\ u \geq 0)$ such that

$$B_t^+ = \rho_{\int_0^t 1_{B_s \geq 0} ds}$$

2

Stopping and Non-stopping Times

This chapter is devoted to the role of stopping times; broadly speaking, the discussion we engage is:

- how fundamental are stopping times in martingale theory?
- which results can be extended outside of the stopping time framework?

This chapter consists of three sections. First we introduce and discuss some random times called pseudo-stopping times. These times generalize stopping times in the sense that they satisfy Doob's optional theorem. Second, we investigate how the BDG inequalities are affected by general random times. Last we attempt to replace in the BDG inequalities the ordinary time process by the local time process taken at a general random time.
From now on, ρ shall denote a general random time, that is a variable

$$\rho : (\Omega, \mathcal{F}) \to (\overline{\mathbb{R}}_+, \mathcal{B}(\overline{\mathbb{R}}_+)).$$

2.1 Stopping Times and Doob's Optional Theorem

2.1.1 The Knight-Maisonneuve Characterization of Stopping Times

Theorem 2.1 *[KM94] ρ is a stopping time if and only if for any bounded martingale M,*

$$\mathbb{E}\left[M_\infty | \mathcal{F}_\rho\right] = M_\rho \tag{2.1}$$

Proof
For simplicity, we assume (**C**) (i.e. all martingales are continuous).

- If ρ is a stopping time, (2.1) is a particular case of Doob's optional stopping theorem.

R. Mansuy and M. Yor: *Random Times and Enlargements of Filtrations in a Brownian Setting*, Lect. Notes Math. **1873**, 41–51 (2006)
www.springerlink.com

- Conversely, we want to show that, under hypothesis (2.1), for any $t \geq 0$, $1_{\rho \leq t}$ is an $\mathcal{F}_t$-measurable variable. Introduce $(A_t^\rho;\ t \geq 0)$ the predictable compensator of $(1_{\rho \leq t};\ t \geq 0)$.

$$\begin{aligned}\mathbb{E}\left[M_\infty 1_{\rho \leq t}\right] = \mathbb{E}\left[M_\rho 1_{\rho \leq t}\right] &= \mathbb{E}\left[\int_0^\infty M_u 1_{u \leq t} dA_u^\rho\right] \\ &= \mathbb{E}\left[\int_0^t M_u dA_u^\rho\right] = \mathbb{E}\left[M_\infty A_t^\rho\right] \quad \text{(integration by parts } I_o)\end{aligned}$$

Thus $1_{\rho \leq t} = A_t^\rho$ and then $1_{\rho \leq t}$ is $\mathcal{F}_t$-measurable.

■

Remark 2.1 *If (**C**) is not satisfied, we use the optional compensator, instead of the predictable one.*

2.1.2 D. Williams' Example of a Pseudo-stopping Time

D. Williams [Wil02] provided an example of a non-stopping time ρ such that for every bounded martingale $(M_t;\ t \geq 0)$,

$$\mathbb{E}\left[M_\infty\right] = \mathbb{E}\left[M_\rho\right]$$

Such a time will be called here a pseudo-stopping time. Before characterizing these times, we detail D. Williams' original example:

$$\rho = \sup\{t \leq \gamma_{T_1},\ B_t = S_t\}$$

where T_1 is the first hitting time of 1 by B and γ_s still denotes the last zero of B before s.

The pseudo-stopping time property of ρ is strongly connected to the well-known Williams' path decomposition of the Brownian trajectory $(B_t;\ t \leq T_1)$, which we partly recall:

- S_ρ is uniformly distributed on [0,1]
- Conditionally on $S_\rho = m$, $(B_u;\ u \leq \rho)$ is a Brownian motion considered up to its first hitting time of m.

Heuristically, if $(M_t;\ t \geq 0)$ is a Brownian martingale,

$$\mathbb{E}\left[M_\rho(B) | S_\rho = m\right] = \mathbb{E}\left[M_{T_m(B)}(B)\right] = \mathbb{E}\left[M_0(B)\right]$$

Such an argument can be made rigorous by proving the result at the random walk level and then passing to the limit.

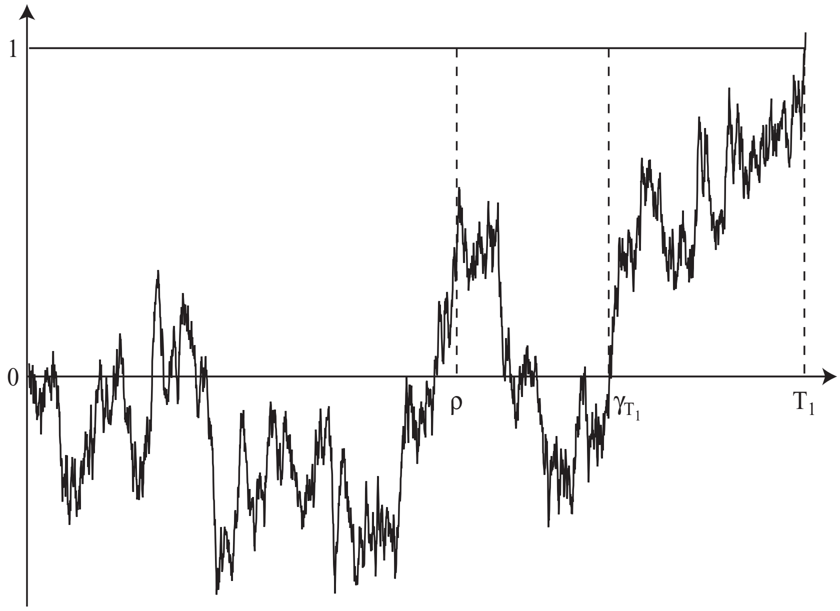

Fig. 2.1. D. Williams' example of a pseudo stopping time ρ

2.1.3 A Characterization of Pseudo-stopping Times

To any random time ρ, we associate the Azéma supermartingale

$$Z_t^\rho = P(\rho > t|\mathcal{F}_t) = \mu_t^\rho - A_t^\rho$$

with $\mu_t^\rho = \mathbb{E}\left[A_\infty^\rho|\mathcal{F}_t\right]$.

Theorem 2.2 *[NY05a] ρ is a pseudo-stopping time if and only if $\mu^\rho = 1$ which is also equivalent to Z^ρ being a decreasing predictable process.*

Proof
As in the proof of Theorem 2.1, we assume hypothesis (**C**) for simplicity. From the definition of the predictable compensator, for any bounded martingale $(M_t;\ t \geq 0)$,

$$\mathbb{E}\left[M_\rho\right] = \mathbb{E}\left[\int_0^\infty M_s dA_s^\rho\right] = \mathbb{E}\left[M_\infty A_\infty^\rho\right]$$

Therefore ρ is a pseudo-stopping time if and only if

$$\mathbb{E}\left[M_\infty A_\infty^\rho\right] = \mathbb{E}\left[M_\infty\right]$$

i.e. if and only if $A_\infty^\rho = 1$.

■

Remark 2.2 *ρ is a pseudo-stopping time if, and only if, for every bounded martingale M, $(M_{t\wedge\rho};\ t \geq 0)$ is a $(\mathcal{F}^{\rho}_t;\ t \geq 0)$-martingale.*

It may be interesting to note that D. Williams' pseudo-stopping time can be recovered as a particular case of the following procedure:
Let Λ be the end of a predictable set, $(\Delta_t;\ t \geq 0)$ a decreasing, continuous $(\mathcal{F}_t;\ t \geq 0)$-adapted process starting at 1 and ending at 0 and define the random time $\rho = \sup\{t < \Lambda,\ Z^{\Lambda}_t = \Delta_t\}$.
Indeed, if we set $\Delta_t = 1 - S_{t\wedge T_1}$ and $\Lambda = \sup\{t \leq T_1,\ B_t = 0\}$, we have already found that $Z^{\Lambda}_t = 1 - B^{+}_{t\wedge T_1}$, and we recover D. Williams' example.
We can also provide, following [NY05a], some other examples of $(\Delta_t;\ t \geq 0)$ such that $Z^{\rho}_t = \Delta_t$ (and therefore some new examples of pseudo-stopping times). More precisely, with $\Delta_t = \inf_{u \leq t\wedge\Lambda} Z^{\Lambda}_u$, and under the additional assumption (**CA**), we have the following generalization of D. Williams' example:

Proposition 2.3

i) $I_{\Lambda} := \inf_{u\leq\Lambda} Z^{\Lambda}_u$ *is uniformly distributed on [0,1].*
ii) for any $t \geq 0$, $Z^{\rho}_t = \inf_{u\leq t} Z^{\Lambda}_u$.

Proof

i) for any $0 \leq b \leq 1$, $P(I_{\Lambda} \leq b) = P(T_b \leq \Lambda)$ where T_b denotes the first hitting time of b by Z^{Λ}.
Then:
$$b = \mathbb{E}\left[Z^{\Lambda}_{T_b}\right] = P(T_b \leq \Lambda) = P(I_{\Lambda} \leq b),$$
which yields the result.

ii)
$$\begin{aligned} P(\rho > t|\mathcal{F}_t) &= P(T^t < \Lambda|\mathcal{F}_t) \qquad \text{with } T^t = \inf\{s \geq t;\ Z^{\Lambda}_s \leq \inf_{u\leq t} Z^{\Lambda}_u\} \\ &= \mathbb{E}\left[Z^{\Lambda}_{T^t}|\mathcal{F}_t\right] = I_t \end{aligned}$$

■

2.2 How Badly are the BDG Inequalities Affected by a General Random Time?

We first recall the classical Burkholder-Davis-Gundy (BDG) inequalities in the Brownian framework, for $p = 1$:

Proposition 2.4 *There exist two universal constants c and C such that for any stopping time T,*

$$c\mathbb{E}\left[\sqrt{T}\right] \leq \mathbb{E}\left[B^*_T\right] \leq C\mathbb{E}\left[\sqrt{T}\right] \tag{2.2}$$

where $B^*_t = \sup_{s\leq t} |B_s|$

Our aim in this section is to study what remains of these inequalities when T is replaced by any random time Λ.

2.2.1 A Global Approach (Common to all Λ's)

First we note that there does not exist a universal constant C such that for all random times Λ:

$$\mathbb{E}\left[|B_\Lambda|\right] \leq C\mathbb{E}\left[\sqrt{\Lambda}\right]$$

Indeed, assuming that C exists, if we replace Λ by 1_A, we obtain that $|B_1|$ is bounded, which is absurd.
Nonetheless, some ersatz of the BDG inequalities may be obtained for all random times Λ, in some different frameworks, namely:

a) $(L^1 - L^p)$ inequality
b) Orlicz space inequality (See [Nev72] for some results about Orlicz spaces).

a) The $(L^1 - L^p)$ inequality

For any random time Λ, one has:

$$\begin{aligned}\mathbb{E}\left[|B_\Lambda|\right] &= \mathbb{E}\left[|B_\Lambda| - \mu\Lambda^{p/2}\right] + \mu\mathbb{E}\left[\Lambda^{p/2}\right] \\ &\leq \mathbb{E}\left[\sup_t\{|B_t| - \mu t^{p/2}\}\right] + \mu\mathbb{E}\left[\Lambda^{p/2}\right]\end{aligned}$$

But

$$\begin{aligned}\sup_t\{|B_t| - \mu t^{p/2}\} &= \sup_v\{|B_{\lambda^2 v}| - \mu\lambda^p v^{p/2}\} \\ &\overset{(law)}{=} \sup_v\{\lambda|B_v| - \mu\lambda^p v^{p/2}\} \\ &= \mu^{-\frac{1}{p-1}}\sup_v\{|B_v| - v^{p/2}\} \qquad \text{by taking } \lambda = \mu^{\frac{1}{1-p}}\end{aligned}$$

The quantity $\sigma_p = \mathbb{E}\left[\sup_v\{|B_v| - v^{p/2}\}\right]$ is finite and will be studied in Exercise 22.
Therefore,

$$\mathbb{E}\left[|B_\Lambda|\right] \leq \mu^{-\frac{1}{p-1}}\sigma_p + \mu\mathbb{E}\left[\Lambda^{p/2}\right]$$

Minimizing in μ, we obtain

$$\mathbb{E}\left[|B_\Lambda|\right] \leq C_p||\sqrt{\Lambda}||_p$$

with $C_p = \sigma_p^{(p-1)/p}(p-1)^{1/p}\left(1 + (p-1)^{-1}\right)$

b) An inequality involving Orlicz spaces (see [BJY84, BJY86])

We can replace the L^p norm for $p > 1$, by some precise Orlicz space norm, namely: for any Orlicz function φ that satisfies $||\mathcal{N}||_\psi < \infty$ with ψ the conjugate of φ and $\mathcal{N}$ a standard normal variable, there exists a universal constant C_φ such that

$$\mathbb{E}\left[|B_\Lambda|\right] \leq C_\varphi ||\sqrt{\Lambda}||_\varphi \tag{2.3}$$

We shall now show how an enlargement of filtration allows to prove this result.

2.2.2 An "Individual" Approach (Depending on Λ)

The idea which "governs" this subsection is to deal with a given random time Λ (which is not a stopping time) by making it a stopping time in a larger filtration.
Up to time Λ, the enlargement formula is (see [Yor85] and references therein):

$$M_t = \widetilde{M_t} + \int_0^{t\wedge\Lambda} \frac{d < M, Z >_s}{Z_{s-}}$$

A rough attempt: Fefferman's inequality

At first, we would like to control the term $\int_0^\Lambda \frac{d<M,Z>_s}{Z_{s-}}$ in terms of $\sqrt{< M >_\Lambda}$. Thus, applying Fefferman's inequality, one obtains:

$$\begin{aligned}\mathbb{E}\left[|M_\Lambda|\right] &\leq C\mathbb{E}\left[\sqrt{< M >_\Lambda}\right] + \mathbb{E}\left[\int_0^\Lambda \left|\frac{d < M, \widetilde{Z} >_s}{Z_{s-}}\right|\right] \\ &\leq C\mathbb{E}\left[\sqrt{< M >_\Lambda}\right] + C\mathbb{E}\left[\sqrt{< M >_\Lambda}\right] \left\|\int_0^{\cdot\wedge\Lambda} \frac{d\widetilde{Z}_u}{Z_{u-}}\right\|_{BMO}\end{aligned}$$

where BMO refers to the BMO space with respect to $(\mathcal{F}_t^\Lambda;\ t \geq 0)$ and $\widetilde{Z}$ denotes the local martingale part of Z.
In fact, it seems difficult to find examples for which $\int_0^{\cdot\wedge\Lambda} \frac{d\widetilde{Z}_u}{Z_{u-}}$ is in BMO, so that the preceding inequality is useless.

A refinement of Fefferman's inequality *[Cho84]*

Next, the following refinement of Fefferman's inequality will turn out to be the right tool. Assume that M and N are two continuous local martingales with respect to a filtration $(\mathcal{G}_t;\ t \geq 0)$; then

$$\mathbb{E}\left[\int_0^\infty |d < M, N >_s|\right] \leq \mathbb{E}\left[\sqrt{< M >_\infty\ \rho_2(N)}\right] \tag{2.4}$$

with $\rho_2(N) = esssup_{T\in\mathcal{T}}\mathbb{E}\left[< N >_\infty - < N >_T |\mathcal{G}_T\right]$ and $\mathcal{T}$ the set of $(\mathcal{G}_t;\ t \geq 0)$ stopping times.
Note that the random variable $\rho_2(N)$ is bounded if and only if N belongs to BMO.
Applying identity (2.4) with $N_. = \int_0^{.\wedge L} \frac{d\widetilde{Z}_u}{Z_{u-}}$, we estimate $\rho_2(N)$ and find

$$\rho_2(N) \leq C\,(1 - \log I_\Lambda)$$

One can then show that $\frac{1}{I_\Lambda}$ is stochastically dominated by $\frac{1}{U}$, where U is a uniform variable; hence, formula (2.3) follows from (2.4), with the help of the general Hölder inequality involving pairs (φ, ψ) of Young functions.

Comment 2.1 *This result can be slightly strengthened by using more carefully some BMO inequalities; more precisely, let $f : (0, 1] \to \mathbb{R}_+$ be a function such that $\int_0^1 f(x)dx < \infty$ and define*

$$F(z) = \int_z^1 \frac{dx}{x} f(x) + \frac{1}{z}\int_0^z dx f(x) - \int_0^1 dx f(x)$$

There exists a universal constant C (independent of f) such that for every $(\mathcal{F}_t;\ t \geq 0)$-martingale $(M_t;\ t \geq 0)$, we have

$$\mathbb{E}\left[|M_\Lambda|\right] \leq C\mathbb{E}\left[< M >_\Lambda^{1/2} + \left(\int_0^\Lambda \frac{d < M >_s}{f(Z_s^\Lambda)}\right)^{1/2} F(I_\Lambda)^{1/2}\right]$$

We recover the preceding result by taking for f a constant fonction.

2.3 Local Time Estimates

We shall now attempt to obtain some variants of (2.2), or rather (2.3), when we replace on the right hand side of (2.2)-(2.3) $\sqrt{t}$ by $\sqrt{L_t}$.
The following proposition is reminiscent of Knight's study (see [Kni73]) related with some laws of the iterated logarithm (see [Shi96] or [Kho96]).

Proposition 2.5 *Let B be a standard Brownian motion, L its local time at level 0. Define τ the right continuous inverse of L, namely for any $l \geq 0$,*

$$\tau_l = \inf\{t \geq 0/\ L_t \geq l\}$$

Then for any $l > 0$ and any positive function g, we have

$$\mathbb{P}(\exists t;\ |B_t| \geq g(L_t)) = 1 - e^{-\int_0^\infty \frac{dx}{g(x)}}$$

As a consequence,

$$\mathbb{P}(\exists t \leq \tau_l,\ |B_t| \geq g(L_t)) = 1 - e^{-\int_0^l \frac{dx}{g(x)}} \tag{2.5}$$

Proof

Here is a proof which relies upon the balayage formula of Proposition 2.5 (for proofs by time change or excursion theory, see Exercises 14 and 15).

- First we assume that $\int_0^\infty \frac{dx}{g(x)} < \infty$.
  Consider H the solution of the first order ODE
  $$y(x) - g(x)y'(x) = 1$$
  with the boundary condition $\lim_{x\to\infty} y(x) = 0$; namely
  $$H(x) = 1 - \exp\left(-\int_x^\infty \frac{du}{g(u)}\right)$$
  The balayage formula (see Lemma 0.3) implies that $(M_t := H(L_t) - |B_t|h(L_t);\ t \geq 0)$, where $h = H'$, is a local martingale.
  Moreover, if for some $u \leq \tau_l$, we have $|B_u| \geq g(L_u)$, then
  $$M_{t_0} \geq H(L_{t_0}) - g(L_{t_0})h(L_{t_0}) = 1$$
  Thus
  $$\begin{aligned}\mathbb{P}(\exists t,\ |B_t| \geq g(L_t)) &= \mathbb{P}(\sup_t\{M_t\} \geq 1)\\ &= \mathbb{P}\left(\sup_t \left\{\frac{M_t}{M_0}\right\} \geq \frac{1}{M_0}\right)\\ &= M_0 \qquad \text{(Doob's identity, Lemma 0.1)}\end{aligned}$$
  Replacing g by the function $g^{(l)}$ defined as
  $$g^{(l)}(x) = \begin{cases} g(x) & \text{if } x < l \\ \infty & \text{otherwise}\end{cases}$$
  we obtain the first part of Proposition 2.5.
- By passing to the limit, the result remains true even if $\int_0^\infty \frac{dx}{g(x)} = \infty$.

■

Application 2.1 *[DÉY91] Consider for $q > p > 0$, the random variable*

$$\Sigma_{p,q} = \sup_{t\geq 0}\{|B_t|^p - (L_t)^q\}$$

Then

$$\Sigma_{p,q} \overset{(law)}{=} (\mathbf{e}_{p,q})^{\frac{pq}{p-q}} \overset{(law)}{=} \sup_{t\geq 0}\left(\frac{|B_t|}{1+(L_t)^q}\right)^{\frac{q}{q-p}}$$

with $\mathbf{e}_{p,q}$ an exponential random variable with parameter

$$c_{p,q} = \frac{1}{q}\int_0^\infty \frac{dz}{z^{1-\frac{1}{q}}(1+z)^{\frac{1}{p}}} = \frac{1}{q}B\left(\frac{1}{p} - \frac{1}{q}, \frac{1}{q}\right)$$

2.4 Further References

Although researches about moment inequalities between two random processes are not as active as they used to be in the 70's and 80's, this topic still plays an important rôle in a number of applications. We suggest the following further reading: [Bur02] [CSK99] [dlPE97] [JY93] [Kaz94] [Yan02].

2.5 Exercises

Exercise 22 *Let $r > 1$ and ρ its conjugate, i.e.*

$$\frac{1}{r} + \frac{1}{\rho} = 1$$

a) Then, prove that

$$\sup_t \{|B_t| - t^{r/2}\} \overset{(law)}{=} \sup_t \left(\frac{|B_t|}{1 + t^{r/2}} \right)^\rho \tag{2.6}$$

***Hint**: Consider*

$$P(\sup_t \{|B_t| - t^{r/2}\} \le a) = P(\forall t \ge 0,\ |B_t| \le a + t^{r/2})$$
$$= P(\forall t \ge 0,\ |B_{\lambda^2 t}| \le a + \lambda^r t^{r/2}),\ \textit{for any } \lambda > 0$$

Then, choose λ conveniently.

b) Prove that

$$\sup_t \{|B_t| - t\} \overset{(law)}{=} \sup_{u \le 1} b_u^2$$

where $(b_u;\ u \le 1)$ denotes a standard Brownian bridge.
***Hint**: Take $r = \rho = 2$ in a) and make the change of variable $t = u/(1-u)$.*

***Remark 2.3** The variable $K := \sup_{u \le 1} b_u^2$ is the limit variable involved in the Kolmogorov-Smirnov statistical test, well-known in studies of Empirical Processes. Its distribution function (see e.g. [CY03] p110) is*

$$P(\sup_{u \le 1} |b_u| \le x) = 1 - 2 \sum_{n=1}^{+\infty} (-1)^{n-1} \exp(-2n^2 x^2)$$

Moreover, the Gauss transform of K, which is defined as the law of $|\mathcal{N}|\sqrt{K}$ where $\mathcal{N}$ denotes a standard Gaussian variable independent from K, is given by $P(|\mathcal{N}| \sup_{u \le 1} |b_u| \le a) = \tanh(a)$.
It is also known (e.g. [BPY01]) that the variable K is distributed as the first hitting time of level $\frac{\pi}{2}$ by a 3-dimensional Bessel process[1].

[1] It is even more surprising that the sum of two independent copies of the first hitting time of level $\frac{\pi}{2}$ by a 3-dimensional Bessel process is distributed as $\sup_{u \le 1} e_u^2$ with e a standard Brownian excursion. This remark is originally due to Chung [Chu76]

c) ✠ *What is the law of* $\sup_t\{|B_t| - t^{r/2}\}$ *for* $r \neq 2$?

Exercise 23 *This is a variant of Exercise 22, where we replace* $\sqrt{t}$ *by* $(L_t;\ t \geq 0)$ *the local time of* $(B_t;\ t \geq 0)$.
The aim of this exercise is to compute (or to estimate) the laws of:

$$\Sigma^\varphi = \sup_{t\geq 0}(|B_t| - \varphi(L_t))$$

for suitable φ*'s.*

a) *Prove the formula*

$$P(\Sigma^\varphi \geq a) = 1 - \exp\left(-\int_0^\infty \frac{dx}{a+\varphi(x)}\right) \tag{2.7}$$

b) *Discuss for which* $\alpha > 0$, $\beta > 1$, *one has:*

$$\mathbb{E}\left[\sup_t\{|B_t|^\alpha - L_t^{\alpha\beta}\}\right] < \infty \tag{2.8}$$

Hint*: Use Application 2.1.*

c) *Discuss for which* $\alpha > 0$, $\beta > 1$, *the inequality*

$$\mathbb{E}\left[|B_\Lambda|^\alpha\right] \leq C\left(\mathbb{E}\left[(L_\Lambda)^{\alpha\beta}\right]\right)^{1/\beta} \tag{2.9}$$

holds, where C *denotes a universal constant, and* $\Lambda \geq 0$ *any random time. Compare the answers for b) and c).*

d) *Compute the best constants in (2.9).*

Exercise 24 *(About the BDG inequalities...)*
Consider a Brownian motion $(B_t;\ t \geq 0)$ *with respect to a filtration* $(\mathcal{F}_t;\ t \geq 0)$ *(for simplicity, you may take for* $(\mathcal{F}_t;\ t \geq 0)$ *the natural filtration of* B*).*
We say that two continuous, increasing,adapted processes $(A_t;\ t \geq 0)$ *and* $(C_t;\ t \geq 0)$ *are moment-equivalent if, for every* $p > 0$, *there exist* $\alpha_p > 0$ *and* $\beta_p > 0$ *such that*

$$\alpha_p \mathbb{E}\left[A_T^p\right] \leq \mathbb{E}\left[C_T^p\right] \leq \beta_p \mathbb{E}\left[A_T^p\right]$$

for every $(\mathcal{F}_t;\ t \geq 0)$ *stopping time* T.
We recall, using our terminology, that the classical BDG inequalities assert that $(A_t^{(1)} = sup_{s\leq t}|B_s|,\ t \geq 0)$ *and* $(C_t^{(1)} = \sqrt{t},\ t \geq 0)$ *are moment-equivalent.*

a) *Are* $(A_t^{(2)} = sup_{s\leq t}B_s;\ t \geq 0)$ *and* $(C_t^{(2)} = sup_{s\leq t}|B_s|,\ t \geq 0)$ *moment-equivalent?*

b) *Are* $(A_t^{(3)} = L_t;\ t \geq 0)$, *the local time of* B *at 0, and* $(C_t^{(3)} = sup_{s\leq t}|B_s|,\ t \geq 0)$ *moment-equivalent?*

Exercise 25
Let $(\rho_t^{(n)};\ t \geq 0)$ *be the unique strong solution of the following stochastic differential equation*

$$X_t = \frac{2}{\sqrt{n}} \int_0^t \sqrt{X_s} d\beta_s + t \tag{2.10}$$

with $(\beta_t;\ t \geq 0)$ *a standard Brownian motion. Note that* $(n\rho_t^{(n)};\ t \geq 0)$ *is a squared Bessel process of dimension* n.
Let $L_1^{(n)}$ *denote the last hitting time of 1 by* $\rho^{(n)}$, *i.e.*

$$L_1^{(n)} = \sup\{t \geq 0;\ \rho_t^{(n)} = 1\}$$

a) Show that for any $p > 0$, *there exist an universal constant* $C_p > 0$ *such that, for any* $n > 0$:

$$\mathbb{E}\left[\sup_{t \leq L_1^{(n)}} |\rho_t^{(n)} - t|^p\right] \leq C_p n^{-p/2} \tag{2.11}$$

b) Show the following refinement of (2.11): *for any* $p > 0$, *there exists a universal constant* $\widetilde{C}_p > 0$ *such that, for any* $n > 0$,

$$\mathbb{E}\left[\sup_{t \leq L_1^{(n)}} \left|\sqrt{n}(\rho_t^{(n)} - t) - 2\int_0^t \sqrt{s} d\beta_s\right|^p\right] \leq \widetilde{C}_p n^{-p/4}$$

Comment 2.2 *This exercise is closely related to the so-called Poincaré lemma; see [DF87] and [Str93] for some historical comments about it.*

Exercise 26 ✠
Find the best constants $C_p^{(1)}$ *and* $C_p^{(2)}$ *(with* $p > 1$*) such that*

a) $\mathbb{E}[|B_T|] \leq C_p^{(1)}(\mathbb{E}[T^{p/2}])^{1/p}$
b) $\mathbb{E}[|B_T|] \leq C_p^{(2)}(\mathbb{E}[L_T^p])^{1/p}$

for every stopping time T.

3

On the Martingales which Vanish on the Set of Brownian Zeroes

$(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0), P)$ still denotes a filtered probability space, L the end of a predictable set and

$$\mathcal{F}_{L-} := \sigma\{z_L, z \in b(\mathcal{P})\}$$

To any integrable random variable X, we associate the $(\mathcal{F}_t)$-martingale

$$(X_t := \mathbb{E}\left[X|\mathcal{F}_t\right];\ t \geq 0).$$

X_L is the value at time L of the process $(X_t;\ t \geq 0)$.
In this chapter, for simplicity, rather than developing a full discussion about general ends of predictable sets, we focus on the particular case:

$$L = \gamma := \sup\{s \leq 1,\ B_s = 0\}$$

considered with respect to $(\mathcal{F}_t;\ t \geq 0)$ the natural filtration of $(B_t;\ t \geq 0)$ a Brownian motion. We propose to compare the quantities $\mathbb{E}\left[X|\mathcal{F}_\gamma\right]$ and X_γ for any $X \in L^1(\mathcal{F}_1)$ (note that in this case, $\mathcal{F}_\gamma = \mathcal{F}_{\gamma-}$).
This chapter begins with a brief reminder of the γ-progressive enlargement, leading to a well-known path decomposition[1] of Brownian motion before and after γ, and related topics. In Section 3.2, we give some examples of martingales which vanish on the zero set of a given Brownian motion. Section 3.4 is devoted to the characterization of all such martingales; in particular, we show that $X_\gamma = 0$ if and only if $\mathbb{E}\left[X|\mathcal{F}_\gamma\right] = 0$. Finally, in Section 3.6, we investigate how, in general, the quantities X_γ and $\mathbb{E}\left[X|\mathcal{F}_\gamma\right]$ differ.

3.1 Some Quantities Associated with γ

Most of the results in this section may be read from Tables 1α and 2α; however, the following discussion is essentially self-contained.

[1] This is different from, but related to, Williams' path decomposition [Wil74] before and after $\gamma_{T_1} = \sup\{t < T_1;\ B_t = 0\}$

R. Mansuy and M. Yor: *Random Times and Enlargements of Filtrations in a Brownian Setting*, Lect. Notes Math. **1873**, 53–69 (2006)
www.springerlink.com

3.1.1 Azéma Supermartingale and the Predictable Compensator Associated with γ

These quantities will play an essential role in the remainder of this chapter.

Proposition 3.1 *For any* $t \leq 1$,

$$Z_t^\gamma = \sqrt{\frac{2}{\pi}} \int_{\frac{|B_t|}{\sqrt{1-t}}}^{\infty} e^{-\frac{x^2}{2}} dx, \quad \textit{and} \quad A_t^\gamma = \sqrt{\frac{2}{\pi}} \int_0^t \frac{dL_u}{\sqrt{1-u}}$$

Proof
We easily obtain:

$$Z_t^\gamma = P(\gamma > t|\mathcal{F}_t) = 1 - P(\gamma < t|\mathcal{F}_t) = 1 - P(\delta_t > 1|\mathcal{F}_t)$$

with

$$\delta_t = \inf\{u \geq t;\ B_u = 0\} = t + \inf\{v \geq 0;\ B_{t+v} - B_t = -B_t\} = t + \hat{T}_{-B_t},$$

where $\hat{T}_a = \inf\{v,\ \hat{B}_v = a\}$ and $\hat{B}_v = B_{t+v} - B_t$.
Therefore $Z_t^\gamma = 1 - \hat{P}(\hat{T}_{-B_t} > 1 - t)$.
The formula for Z_t^γ now follows from the fact that $\hat{T}_a \stackrel{(law)}{=} \frac{a^2}{\mathcal{N}^2}$.
It then remains to apply Itô-Tanaka formula to obtain the expression for A_t^γ.

■

Corollary 3.1.1 *(Lévy's Arcsine law for* γ*)*

$$P(\gamma \in du) = \frac{du}{\pi\sqrt{u(1-u)}} 1_{[0,1]}(u)$$

Proof
Let φ be a generic positive Borel function; then:

$$\begin{aligned} \mathbb{E}\left[\varphi(\gamma)\right] &= \mathbb{E}\left[\int_0^1 \varphi(u) dA_u^\gamma\right] = \sqrt{\frac{2}{\pi}} \mathbb{E}\left[\int_0^1 \varphi(u) \frac{dL_u}{\sqrt{1-u}}\right] \\ &= \sqrt{\frac{2}{\pi}} \int_0^1 \frac{\varphi(u)}{\sqrt{1-u}} d\mathbb{E}\left[L_u\right] \end{aligned}$$

Now, we deduce from Tanaka's formula that: $\mathbb{E}\left[L_u\right] = \mathbb{E}\left[|B_u|\right] = \sqrt{\frac{2u}{\pi}}$.
Thus

$$\mathbb{E}\left[\varphi(\gamma)\right] = \int_0^1 \frac{\varphi(u)}{\pi\sqrt{u(1-u)}} du$$

■

3.1.2 Path Decomposition Relative to γ

Notation 3.1 *For any pair of random variables* $0 \le a \le b$ *and any given process* X*, we define the Brownian-scaled process* X *over* $[a,b]$ *by*

$$X_u^{[a,b]} = \frac{1}{\sqrt{b-a}} X_{a+(b-a)u}, \; u \le 1$$

We now recall some properties of the Brownian-scaled Brownian motion B over $[0,\gamma]$, $[\gamma,1]$ and $[\gamma,\delta]$. Namely,

- $B^{[0,\gamma]}$ is a standard Brownian bridge independent of γ.
- $B^{[\gamma,1]}$ is closely related to Chung's definition of the Brownian meander[2], namely
$$\left(m_u = \frac{1}{\sqrt{1-\gamma}} |B_{\gamma+u(1-\gamma)}|; \; u \le 1 \right)$$
Hence, $B_u^{[\gamma,1]} = \mathrm{sgn}(B_1) m_u$; moreover m, $\mathrm{sgn}(B_1)$ and $\mathcal{F}_\gamma$ are independent.
- The absolute value of $B^{[\gamma,\delta]}$ is a standard 3-dimensional Bessel bridge independent of $\mathcal{F}_\gamma \vee \sigma(B_s; s \ge \delta)$.

See Exercise 27 (see also [Imh85], [RY05], [Yor95] or [BP94]) for a sketch of the proof of these results.

3.1.3 Brownian Meander

Let $\mathbb{M}$ denote the law of the Brownian meander m (just defined).

Proposition 3.2 *(Imhof's relation, [Imh84], see also [MY05a])*

$$\mathbb{M} = \sqrt{\frac{2}{\pi}} \frac{1}{R_1} . \mathbb{P}_0^{(3)} \tag{3.1}$$

with $\mathbb{P}_0^{(3)}$ *the law of the standard 3-dimensional Bessel process.*

Comment 3.1 *We leave to the reader the extension of* (3.1) *to the laws* $\mathbb{M}^{(t)}$ *and* $\mathbb{P}_{0|\mathcal{R}_t}^{(3)}$*, where* $\mathbb{M}^{(t)}$ *denotes the law of* $m^{(t)}$*.*

Proof
Here is a sketch of the proof which combines enlargement of filtration and Girsanov's theorem[3].

---

[2] It is also natural to consider the meander $(m_v^{(t)}; \; v \le t)$ of length t, which may be defined by:
$$m_v^{(t)} = \sqrt{t} m_{v/t}; \qquad v \le t$$

[3] For a discussion in the same vein, see also [MW91].

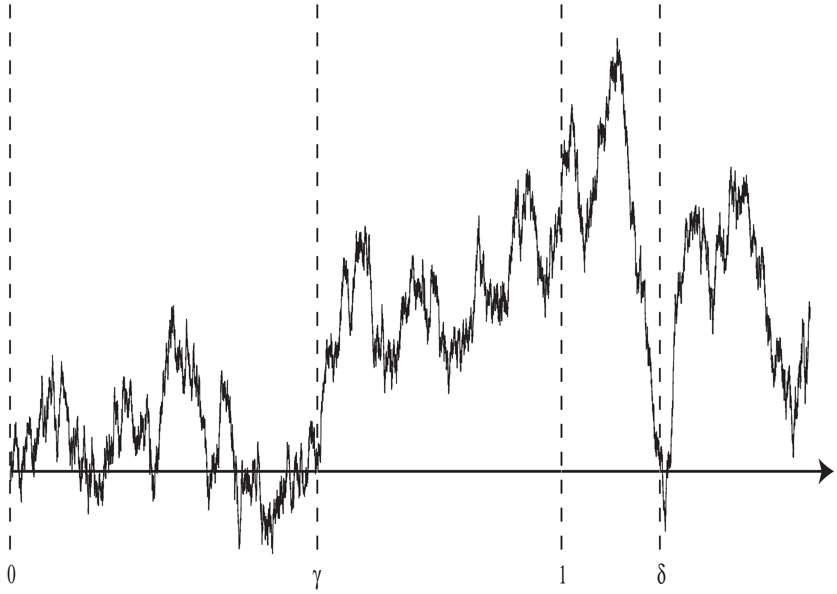

Fig. 3.1. Path decomposition relative to γ and δ

- On one hand, the enlargement formula after γ is

$$B_{\gamma+u} = B_{\gamma+u} - B_\gamma = \widetilde{\beta}_u + \int_0^u \frac{d\langle B, 1-Z^\gamma\rangle_{\gamma+v}}{1-Z^\gamma_{\gamma+v}}$$
$$= \widetilde{\beta}_u + \int_0^u \frac{\operatorname{sgn}(B_{\gamma+v}) e^{-B^2_{\gamma+v}/2(1-\gamma-v)}}{\int_0^{|B_{\gamma+v}|/\sqrt{1-\gamma-v}} e^{-x^2/2}dx} . \frac{dv}{\sqrt{1-\gamma-v}}$$

  where $(\widetilde{\beta}_u;\ u \geq 0)$ is a $(\mathcal{F}_{\gamma+u},\ u \geq 0)$ Brownian motion.
  We then make the change of variables $u = t(1-\gamma)$, $v = h(1-\gamma)$.
- On the other hand, we can use Girsanov's theorem in order to understand the probability on the right hand side of (3.1), which we call temporarily $\widetilde{\mathbb{M}}$. We have, for $u < 1$:

$$\widetilde{\mathbb{M}}_{|\mathcal{R}_u} = \sqrt{\frac{2}{\pi}} \mathbb{E}_0^{(3)} \left[\frac{1}{R_1} | \mathcal{R}_u \right] . \mathbb{P}_0^{(3)}{}_{|\mathcal{R}_u} = \sqrt{\frac{2}{\pi}} Q_{1-u} \left(\frac{1}{r} \right) (R_u) . \mathbb{P}_0^{(3)}{}_{|\mathcal{R}_u},$$

  where Q_t denotes the BES(3) semigroup, which satisfies:

$$Q_t(f)(x) = \frac{1}{x} \mathbb{E}_x \left[f(B_t) B_t 1_{T_0 > t} \right]$$

  i.e. the 3-dimensional Bessel process is Doob's h-transform of Brownian motion killed at 0, with $h(x) = x$.
  Hence

$$Q_t\left(\frac{1}{r}\right)(x) = \frac{1}{x}P_x(T_0 > t) = \frac{1}{x}P\left(\frac{x^2}{\mathcal{N}^2} > t\right)$$
$$= \frac{1}{x}\sqrt{\frac{2}{\pi}}\int_0^{x/\sqrt{t}} dy e^{-\frac{y^2}{2}}$$

Therefore,

$$\frac{d}{dx}\left(\log Q_t\left(\frac{1}{r}\right)(x)\right) = -\frac{1}{x} + \frac{1}{\sqrt{t}}\left(\frac{e^{-\frac{x^2}{2t}}}{\int_0^x e^{-\frac{y^2}{2t}}\,dy}\right)$$

It remains to compare the two expressions thus obtained for $(R_u;\ u \leq 1)$ under $\mathbb{M}$ and $\widetilde{\mathbb{M}}$ to conclude.

■

Remark 3.1 *From Imhof's result, one can easily deduce the fact that m_1 is Rayleigh distributed* [4], *i.e.*

$$P(m_1 \in d\rho) = \rho e^{-\frac{\rho^2}{2}}\,d\rho \tag{3.2}$$

A deeper fact which also yields (3.2) *is:*

$$(m_u;\ u \leq 1) \stackrel{(law)}{=} \left(\sqrt{b_u^2 + \rho_u^2};\ u \leq 1\right)$$

where $(b_u;\ u \leq 1)$ is a standard Brownian bridge, independent of $(\rho_u;\ u \geq 0)$, a two-dimensional Bessel process. A number of results about the Brownian meander are presented in [BY88].

3.2 Some Examples of Martingales which Vanish on $\mathcal{Z} = \{t;\ B_t = 0\}$

This section aims at describing[5] $\mathcal{M}_0$ the set of local martingales $(X_t;\ t \geq 0)$ which vanish on the zero set of a given Brownian motion $(B_t;\ t \geq 0)$. A number of examples are easily discovered.

Example 3.1 *If $z \in b(\mathcal{P})$, then $(z_{\gamma_t} B_t;\ t \geq 0) \in \mathcal{M}_0$*

Example 3.2 *Let us look for elements of $\mathcal{M}_0$ of the form*[6] $(B_t H_t;\ t \geq 0)$ *where H is a semimartingale with canonical decomposition $N + V$. Then, $(B_t H_t;\ t \geq 0)$ is a local martingale if and only if, using Itô's formula:*

[4] A Rayleigh distributed random variable can be obtained from two independent standard normal variables $\mathcal{N}$ and $\mathcal{N}'$, as $\sqrt{\mathcal{N}^2 + \mathcal{N}'^2} \stackrel{(law)}{=} \sqrt{2\mathbf{e}}$.

[5] Later, we shall also consider $\mathcal{M}_0^t$ the set of local martingales $(X_u;\ u \leq t)$ which vanish on $\mathcal{Z}_t = \{u \leq t;\ B_u = 0\}$.

[6] Note that, because of the predictable representation property for Brownian motion (see Chapter 4), $(H_t;\ t \geq 0)$ cannot be a $(\mathcal{F}_t;\ t \geq 0)$ martingale, unless it is constant.

$$\int_0^t B_s dV_s + \langle B, N \rangle_t = 0$$

Assuming that N is given, we find that $V_t := \int_0^t 1_{B_s \neq 0} dV_s + \int_0^t 1_{B_s = 0} dV_s$ must satisfy

$$1_{B_s \neq 0} dV_s = -\frac{1}{B_s} d\langle N, B \rangle_s$$

We know that $N_t = \int_0^t n_s dB_s$, with $\int_0^t n_s^2 ds < \infty$.
Hence $\frac{1}{B_s} d < N, B >_s = \frac{n_s}{B_s} ds$, and we need to assume

$$\int_0^t \left| \frac{n_s}{B_s} \right| ds < \infty \tag{3.3}$$

We remark that this condition is satisfied for $n_s = |B_s|^\alpha$, for any $\alpha > 0$, but not for $\alpha = 0$.
Note that if we denote: $V_t^0 = \int_0^t 1_{B_s=0} dV_s = \int_0^{\gamma_t} 1_{B_s=0} dV_s$, then the fact that $(V_t^0 B_t;\ t \geq 0)$ belongs to $\mathcal{M}_0$ is a particular instance of Example 3.1.
Thus to summarize Example 3.2, we have found that, with a careful restriction (3.3) placed upon the predictable integrands $(n_s;\ s \geq 0)$, the processes $\left(\left[\int_0^t n_s \left(dB_s - \frac{ds}{B_s} \right) \right] B_t;\ t \geq 0 \right)$ belong to $\mathcal{M}_0$.

Example 3.3 *To continue with the discussion started in Example 3.2, it may even be possible to define as* ***principal values*** *$\int_0^t n_s \frac{ds}{B_s}$, for some predictable n's such that $\int_0^t n_s^2 ds < \infty$ but which do not necessarily satisfy $\int_0^t \left| \frac{n_s}{B_s} \right| ds < \infty$: more precisely, thanks to the Hölder property of Brownian local times,*

$$\int_0^t \frac{ds}{B_s} = \lim_{\varepsilon \to 0} \int_0^t \frac{ds}{B_s} 1_{|B_s| \geq \varepsilon} = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \frac{dx}{x} (L_t^x - L_t^{-x}) \qquad \textit{exists}$$

Then, carefully passing to the limit as $\varepsilon \to 0$, we find that $\left(B_t (B_t - \int_0^t \frac{ds}{B_s});\ t \geq 0 \right)$ is a martingale, hence, it belongs to $\mathcal{M}_0$.
We leave to the reader the task of defining, more generally, $\int_0^t \frac{n_s}{B_s} ds$, for suitable integrands[7] n, and also $\int_0^t \frac{ds}{B_s^\alpha}$, with $\alpha < 3/2$, where we denote x^α for $|x|^\alpha sgn(x)$.

In fact, the different local martingales of $\mathcal{M}_0$ which have been exhibited in Examples 3.1-3.2-3.3 are particular cases of the most general element of $\mathcal{M}_0$ whose decomposition is presented in the following theorem (see [AY92], for details and proof).

[7] In fact, it suffices that $(n_t;\ t \geq 0)$ is a continuous $(\mathcal{F}_t;\ t \geq 0)$-semimartingale, which thanks to Kolmogorov's continuity criterion implies that $\left(\int_0^t n_s dL_s^x;\ x \in \mathbb{R} \right)$ admits a Hölder continuous version.

Theorem 3.3 *$M \in \mathcal{M}_0$ if, and only if, M may be written as:*

$$M_t = z_{\gamma_t} B_t \ \exp\left(\int_{\gamma_t}^t u_s \left(dB_s - \frac{ds}{B_s}\right) - \frac{1}{2}\int_{\gamma_t}^t u_s^2 ds\right) \tag{3.4}$$

where $(z_s;\ s \geq 0)$ and $(u_s;\ s \geq 0)$ are two $(\mathcal{F}_s;\ s \geq 0)$-predictable processes, with suitable integrability properties.
In that case, M may also be represented in the additive form:

$$M_t = z_{\gamma_t} B_t + B_t \int_{\gamma_t}^t \eta_s \left(dB_s - \frac{ds}{B_s}\right) \tag{3.5}$$

where η is a predictable process.
In both these representations, the associated process $(z_{\gamma_t};\ t \leq 1)$ may be obtained as

$$z_{\gamma_t} = \lim_{u \downarrow\downarrow \gamma_t} \frac{M_u}{B_u}$$

Comment 3.2 *Comparing formulae* (3.4) *and* (3.5) *with Examples 3.2 and 3.3, we see that the difficulties encountered in these examples with the singularity $1/B_s$ over $\mathbb{R}_+$ are now taken care of in the pre-γ_t parts of formulae* (3.4) *and* (3.5), *while the post-γ_t part exhibits a semimartingale*

$$\left(B_{\gamma_t+u} - \int_0^u \frac{ds}{B_{\gamma_t+s}};\ u \leq t - \gamma_t\right)$$

which, thanks to Imhof's relation (3.1), *is "close" to Brownian motion.*

3.3 Some Brownian Martingales with a Given Local Time, or Supremum Process

We begin with the definition of $\mathcal{M}_0^{strict}$, the set of Brownian martingales whose zero set coincides exactly with that of Brownian motion.
We note that, from Theorem 3.3, a martingale $(M_t;\ t \geq 0)$ belongs to $\mathcal{M}_0^{strict}$ if, and only if M may be written

$$z_{\gamma_t} B_t \exp\left(\int_{\gamma_t}^t u_s \left(dB_s - \frac{ds}{B_s}\right) - \frac{1}{2}\int_{\gamma_t}^t u_s^2 ds\right)$$

with $P(\exists u \geq 0,\ z_{\gamma_u} = 0;\ u \neq \gamma_u) = 0$.

Proposition 3.4 *If M belongs to $\mathcal{M}_0^{strict}$, then its local time at level 0 is given by*

$$L_t^0(M) = \int_0^t |z_s| dL_s^0(B)$$

Conversely, if the local time at level 0 of a Brownian martingale $(M_t;\ t \geq 0)$ is equivalent to the local time of the underlying Brownian motion $(B_t;\ t \geq 0)$, then M belongs to $\mathcal{M}_0^{strict}$.

Remark 3.2 *Note that if $M \in \mathcal{M}_0^{strict}$ and if z denotes the associated predictable process, then $\left(\frac{1}{z_{\gamma_t}} M_t;\ t \geq 0\right)$ has the same local time process at level 0 as Brownian motion.*

Example 3.4 *Let $\alpha \in \mathbb{R}^*$; $M_t := \sinh(\alpha B_t) \exp\left(-\frac{\alpha^2 t}{2}\right)$ defines a $\mathcal{M}_0^{strict}$-martingale and the associated predictable process is given by*

$$z_{\gamma_t} = \alpha \exp\left(-\frac{\alpha^2}{2}\gamma_t\right)$$

Therefore $\left(\frac{1}{\alpha}\sinh(\alpha B_t)\exp\left(-\frac{\alpha^2}{2}(t-\gamma_t)\right);\ t \geq 0\right)$ has the same local time at level 0 as Brownian motion.
More generally, if $h(x,t)$ is a space time harmonic function such that $h(x,t) = 0$ if and only if $x = 0$, then $(h(B_t,t);\ t \geq 0)$ is a $\mathcal{M}_0^{strict}$-martingale and $\left(\frac{h(B_t,t)}{h'_x(0,\gamma_t)};\ t \geq 0\right)$ has the same local time at 0 as Brownian motion.

Combining Theorem 3.3 with Lévy's equivalence, we obtain the following

Proposition 3.5 *Let $(v_t;\ t \geq 0)$ a predictable process, with suitable integrability property. We define the local martingale $(N_t^v;\ t \geq 0)$ as*

$$N_t^v := S_t - (S_t - B_t)\exp\left(\int_{\gamma'_t}^t v_s\left(dB_s + \frac{ds}{S_s - B_s}\right) - \frac{1}{2}\int_{\gamma'_t}^t v_s^2 ds\right)$$

where $\gamma'_t = \sup\{u \leq t;\ B_u = S_u\}$, and $S_t = \sup_{u \leq t} B_u$.
Clearly, for any $t \geq 0$,

$$\sup_{s \leq t} N_s^v = S_t$$

3.4 A Remarkable Coincidence between $\mathbb{E}[X|\mathcal{F}_\gamma]$ and X_γ

Theorem 3.6 *For any $X \in L^1(\mathcal{F}_1)$, the three following properties are equivalent*

1. *$(X_t;\ t \leq 1)$ vanishes on $\mathcal{Z}_1 = \{t \leq 1; B_t = 0\}$*
2. *$\mathbb{E}[X|\mathcal{F}_\gamma] = 0$*
3. *$X_\gamma = 0$*

Proof

- $(1 \Rightarrow 2)$ Consider a generic $z \in b(\mathcal{P})$; the balayage formula yields to:

$$z_{\gamma_t} X_t = \int_0^t z_{\gamma_s} dX_s$$

Since $(X_t;\ t \leq 1)$ is a martingale, it follows easily that $(z_{\gamma_t} X_t;\ t \leq 1)$ is also a martingale, hence: $\mathbb{E}[z_\gamma X] = 0$ and therefore that $\mathbb{E}[X|\mathcal{F}_\gamma] = 0$.

- $(2 \Rightarrow 3)$ We first use the following lemma

  Lemma 3.7 *For any $X \in L^1(\mathcal{F}_1)$ and any $z \in b(\mathcal{P})$,*

  $$\mathbb{E}[z_\gamma X_\gamma] = \mathbb{E}\left[\mathbb{E}[X|\mathcal{F}_\gamma] \int_0^1 dA_u^\gamma z_u\right]$$

  From Lemma 3.7, one deduces that $\mathbb{E}[X|\mathcal{F}_\gamma] = 0$ implies: $\mathbb{E}[z_\gamma X_\gamma] = 0$; therefore $X_\gamma = 0$.

  Proof of the lemma

  $$\begin{aligned}\mathbb{E}[z_\gamma X_\gamma] &= \mathbb{E}\left[\int_0^1 z_u X_u dA_u^\gamma\right] \\ &= \mathbb{E}\left[X \int_0^1 z_u dA_u^\gamma\right] \qquad \text{by integration by parts } (I_o) \\ &= \mathbb{E}\left[\mathbb{E}[X|\mathcal{F}_\gamma] \int_0^1 dA_u^\gamma z_u\right],\end{aligned}$$

  since $\int_0^1 dA_u^\gamma z_u = \int_0^\gamma dA_u^\gamma z_u$ is $\mathcal{F}_\gamma$ measurable .

  ♦

- $(3 \Rightarrow 1)$ Since $X_\gamma = 0$, $\mathbb{E}[|X_\gamma|] = 0$
  Then, from the balayage formula, we deduce that $\mathbb{E}\left[\int_0^1 dA_u^\gamma |X_u|\right] = 0$
  Therefore (due to Proposition 3.1), $X_u = 0 \; dL_u \; dP$ a.s.
  This yields to $X \in \mathcal{M}_0^1$, since the support of dL_u is precisely $\mathcal{Z}_1$.

■

Example 3.5 *If we consider the random variable $X^{(f)} = f(B_1)$, we obtain $X_t^{(f)} = \mathbb{E}(f(B_1)|\mathcal{F}_t) = P_{1-t}f(B_t)$.*
Therefore $X_\gamma^{(f)} = P_{1-\gamma}f(0) = \frac{1}{\sqrt{2\pi(1-\gamma)}} \int_{\mathbb{R}} dx e^{-\frac{x^2}{2(1-\gamma)}} f(x)$
Moreover, if m still denotes the Brownian meander (recall formulae (3.1) *and* (3.2)*), we have*

$$\begin{aligned}\mathbb{E}[f(B_1)|\mathcal{F}_\gamma] &= \mathbb{E}\left[f(\sqrt{1-\gamma}\varepsilon m_1)|\mathcal{F}_\gamma\right] \\ &= \frac{1}{2(1-\gamma)} \int_{\mathbb{R}} dy \; |y| e^{-\frac{y^2}{2(1-\gamma)}} f(y)\end{aligned}$$

Accordingly, if f is odd, then $X_\gamma^{(f)} = 0$ and $(X_t^{(f)};\; t \le 1) \in \mathcal{M}_0^1$.

Remark 3.3 *In general, starting from a variable $X \in L^1(\mathcal{F}_1)$, the variable $Y := X - \mathbb{E}[X|\mathcal{F}_\gamma]$ is associated to a martingale in $\mathcal{M}_0^1$. Why?*

As a consequence of Theorem 3.6, we may now state a partial converse to the balayage formula; the proof of this proposition is left to the reader.

Proposition 3.8 *For any Brownian martingale $(M_t;\ t \le 1)$, the following properties are equivalent:*

1. $(M_t; t \le 1)$ *vanishes on* $\mathcal{Z}_1$
2. $(M_t; t \le 1)$ *satisfies the balayage formula, i.e. for any bounded predictable process* $(z_s;\ s \le 1)$*:*

$$z_{\gamma_t} M_t = \int_0^t z_{\gamma_u} dM_u;\ t \le 1$$

3. *For any bounded predictable process* $(z_s;\ s \le 1)$, $(z_{\gamma_t} M_t;\ t \le 1)$ *is a martingale.*

Remark 3.4 *Even if* $X \in L^1(\mathcal{F}_\gamma)$, *it is not true in general that* $X_\gamma = X$. *For instance, if we consider the random variable* $X = L_1$, *we can compute explicitly* $(L_1)_\gamma$ *and show that*

$$(L_1)_\gamma \ne L_1$$

Indeed,

$$\begin{aligned}
\mathbb{E}\left[L_1 | \mathcal{F}_t\right] &= -\int_0^t sgn(B_s) dB_s + \mathbb{E}\left[|B_1||\mathcal{F}_t\right] \\
&= -\int_0^t sgn(B_s) dB_s + \hat{\mathbb{E}}\left[|B_t + \sqrt{1-t}\hat{\mathcal{N}}|\right]
\end{aligned}$$

Hence,

$$\begin{aligned}
(L_1)_\gamma &= -\int_0^\gamma sgn(B_s) dB_s + \sqrt{1-\gamma}\sqrt{\frac{2}{\pi}} \\
&= L_1 + \sqrt{1-\gamma}\sqrt{\frac{2}{\pi}} \qquad (3.6)
\end{aligned}$$

The last identity follows from Tanaka's formula taken at time γ.

3.5 Resolution of Some Conditional Equations

First we may state the following corollary of Theorem 3.6.

Theorem 3.9 *For any variable* $X \in L^1(\mathcal{F}_1)$, *define* $X^\bullet = \int_0^1 dA_u^\gamma\, x_u$, *where* $(x_u;\ u \le 1)$ *is the predictable process such that* $x_\gamma = \mathbb{E}\left[X | \mathcal{F}_\gamma\right]$.
Then, the following properties are equivalent:

1. $(X_t; t \le 1)$ *vanishes on* $\mathcal{Z}_1$
2. $X_\gamma = 0$
3. $X^\bullet = 0$
4. $X^\bullet = (X^\bullet)_\gamma$

In order to understand better the equivalent properties of this theorem, we shall attempt to solve as precisely as possible the three following conditional equations[8]:

$$\begin{array}{ll}(C_-) & X_\gamma = \mathbb{E}[X|\mathcal{F}_{\gamma-}] \\ (C_+) & X_\gamma = \mathbb{E}[X|\mathcal{F}_{\gamma+}] \\ (C_{-/+}) & \mathbb{E}[X|\mathcal{F}_{\gamma-}] = \mathbb{E}[X|\mathcal{F}_{\gamma+}]\end{array}$$

where $X \in L^1(\mathcal{F}_1)$ is the unknown.
To do this, we first transform this problem into a similar one involving the filtration[9] $(\mathcal{F}^\gamma_t;\ t \leq 1)$. It is not difficult to show the following equalities $\mathcal{F}_\gamma = \mathcal{F}^\gamma_{\gamma-}$, $\mathcal{F}_{\gamma+} = \mathcal{F}^\gamma_{\gamma+} = \mathcal{F}^\gamma_\gamma$ and also, for a generic integrable variable $X \in L^1(\mathcal{F}_1)$, $X_\gamma = \widetilde{X}_{\gamma-}$ with $\widetilde{X}_t = \mathbb{E}[X|\mathcal{F}^\gamma_t]$.

With these remarks, the three conditional equations may be rewritten as:

$$\begin{array}{ll}(C_-) & \widetilde{X}_{\gamma-} = \mathbb{E}[X|\mathcal{F}^\gamma_{\gamma-}] \\ (C_+) & \widetilde{X}_{\gamma-} = \mathbb{E}[X|\mathcal{F}^\gamma_{\gamma+}] \\ (C_{-/+}) & \mathbb{E}[X|\mathcal{F}^\gamma_{\gamma-}] = \mathbb{E}[X|\mathcal{F}^\gamma_{\gamma+}]\end{array}$$

Our main ingredient to solve these conditional equations will be the following representation for $(\mathcal{F}^\gamma_t;\ t \geq 0)$-martingales.

Theorem 3.10 *Every square integrable $(\mathcal{F}^\gamma_t;\ t \geq 0)$-martingale $(\widetilde{M}_t;\ t \geq 0)$ with $M_0 = 0$, may be written in a unique way as the sum of four square integrable, orthogonal $(\mathcal{F}^\gamma_t;\ t \geq 0)$-martingales*

$$\forall t \geq 0, \qquad \widetilde{M}_t = \widetilde{M}^{(1)}_t + \widetilde{M}^{(2)}_t + \widetilde{M}^{(3)}_t + \widetilde{M}^{(4)}_t$$

such that these martingales are of the form:

$$\widetilde{M}^{(1)}_t = \int_0^{t\wedge\gamma} J^{(1)}_s d\widetilde{B}_s \qquad \widetilde{M}^{(2)}_t = \int_\gamma^{t\vee\gamma} J^{(2)}_s d\widetilde{B}_s$$

$$\widetilde{M}^{(3)}_t = J^{(3)}_\gamma 1_{\gamma\leq t} - \int_0^{t\wedge\gamma} J^{(3)}_s dA^\gamma_s \qquad \widetilde{M}^{(4)}_t = sgn(B_1)\Phi 1_{\gamma\leq t}$$

where $\Phi \in L^2(\mathcal{F}_\gamma)$, $\tilde{B}$ is the martingale port of B in its $(\mathcal{F}^r_t;\, t \geq 0)$ semi-martingale decomposition and $J^{(i)}$, $i = 1, 2, 3$, are three $(\mathcal{F}_t;\ t \geq 0)$-predictable processes which satisfy the following integrability conditions:

$$\mathbb{E}\left[\int_0^\infty (J^{(1)}_s)^2 Z^\gamma_s ds\right] < \infty; \quad \mathbb{E}\left[\int_0^\infty (J^{(2)}_s)^2 (1 - Z^\gamma_s) ds\right] < \infty;$$

$$\mathbb{E}\left[\int_0^\infty (J^{(3)}_s)^2 dA^\gamma_s\right] < \infty$$

---

[8] For the definition of the σ-fields $\mathcal{F}_{\gamma-}$ and $\mathcal{F}_{\gamma+}$, see Section 0.4.

[9] Recall that $\mathcal{F}^\gamma$ is the progressive enlargement of $\mathcal{F}$ with the random time γ.

This theorem is easily deduced from Barlow's representation results, in [Bar78]. The following theoretical resolution of the conditional equations is now straightforward.

Theorem 3.11 *Let* $X \in L^2(\mathcal{F}_1)$ *and the associated decomposition* $X = \widetilde{X}_\infty^{(1)} + \widetilde{X}_\infty^{(2)} + \widetilde{X}_\infty^{(3)} + \widetilde{X}_\infty^{(4)}$

1. *X solves (C_-) if, and only if* $\widetilde{X}_\infty^{(3)} = 0$
2. *X solves (C_+) if, and only if* $\widetilde{X}_\infty^{(3)} = \widetilde{X}_\infty^{(4)} = 0$
3. *X solves $(C_{-/+})$ if, and only if* $\widetilde{X}_\infty^{(4)} = 0$

Consequently, X solves (C_+) (hence, it solves both (C_-) and $(C_{-/+})$) if, and only if it may be represented as a stochastic integral with respect to $d\widetilde{B}$, with a $(\mathcal{F}_t^\gamma;\ t \geq 0)$-predictable integrand.

Comment 3.3 *In order to study the solutions of these equations, some light should be shed on the natural filtration of $\widetilde{B}$. This filtration is also the natural filtration of the "Brownian snake"*[10] $(\Sigma_t;\ t \leq 1)$*, i.e. the solution of*

$$\Sigma_t = \widetilde{B}_t + \int_0^t \frac{ds}{\sqrt{1-s}} u\left(\frac{\Sigma_s}{\sqrt{1-s}}\right), \qquad \text{where } u(x) = sgn(x)\frac{\Phi'(x)}{\Phi(x)}$$

This process is closely linked with the canonical decomposition of B in $(\mathcal{F}_t^\gamma; t \geq 0)$ (see Table 1β, line 1). Moreover, this process enjoys the following regeneration property: $\Sigma^{[\gamma,1]}$ is independent of $\mathcal{F}_{\gamma+}$ and has the same distribution as $(\Sigma_u;\ u \leq 1)$.

3.6 Understanding how $\mathbb{E}[X|\mathcal{F}_\gamma]$ and X_γ Differ

Above we were mainly concerned with the difference between the quantities X_γ and $\mathbb{E}[X|\mathcal{F}_\gamma]$, for X a generic $\mathcal{F}_1$-measurable, bounded variable.
Here, we provide a "global" explanation of this discrepancy by showing that the distributions[11]:

$$\Gamma \in \mathcal{F}_1 \mapsto P_{\gamma,u}(\Gamma) := \mathbb{E}[(1_\Gamma)_\gamma | \gamma = u]$$
$$\Gamma \in \mathcal{F}_1 \mapsto P'_{\gamma,u}(\Gamma) := \mathbb{P}(\Gamma|\mathcal{F}_\gamma, \gamma = u)$$

are different; in fact, we give a simple identification of each of them.
To present this result, we need a few notations. The different probabilities involved are defined on the canonical space of continuous functions, considered

[10] This process has nothing to do with the celebrated super-process studied by Le Gall. See [RV95] for a complete study of the process $(\Sigma_t;\ t \leq 1)$.

[11] Let us be more precise as to the meaning of the second quantity: we know there exists z, a predictable process such that $P(\Gamma|\mathcal{F}_\gamma) = z_\gamma$, and we compute z_u.

only on a finite time interval; if R and S are two such probabilities, $R \circ S$ is the probability of the process obtained by concatenating the first process (with law R) with the second (with law S). With this notation, we may state:

Proposition 3.12 *The following holds*

(*i*) $P_{\gamma,u} = Q^u \circ P^{1-u}$

(*ii*) $P'_{\gamma,u} = Q^u \circ \widetilde{M}^{1-u}$

where Q^u denotes the law of the Brownian bridge with length u, P^{1-u} the law of Brownian motion considered on the time interval $[0, 1-u]$, $\widetilde{M}^{1-u}$ the law of the symmetric meander of duration $1-u$.

Proof
(*i*) For any $X \geq 0$, $\mathcal{F}_1$-measurable, and for every $f : [0,1] \to \mathbb{R}^+$, Borel, one has:

$$\begin{aligned}
\mathbb{E}[X_\gamma f(\gamma)] &= \mathbb{E}\left[\int_0^1 dA_u^\gamma X_u f(u)\right] \\
&= \mathbb{E}\left[\sqrt{\frac{2}{\pi}}\int_0^1 \frac{dL_u}{\sqrt{1-u}} X_u f(u)\right] \\
&= \mathbb{E}\left[\sqrt{\frac{2}{\pi}}\int_0^1 \frac{dL_u}{\sqrt{1-u}} \mathbb{E}[X_u|B_u = 0] f(u)\right] \\
&= \int_0^1 \frac{du}{\pi\sqrt{u(1-u)}} \mathbb{E}[X_u|B_u = 0] f(u)
\end{aligned}$$

It follows that

$$\mathbb{E}[X_\gamma|\gamma = u] = \mathbb{E}[X_u|B_u = 0] \tag{3.7}$$

We recall that $X_u = \mathbb{E}[X|\mathcal{F}_u]$. It is now easy to show that the right hand side of (3.7) coincides with the expectation of X with respect to $Q^u \circ P^{1-u}$ (use the Markov property at time u).
(*ii*) This follows from the conditional independence of $(B_s; s \leq \gamma)$ and $(B_{\gamma+s}, s \leq 1-\gamma)$ given $\gamma = u$ and from the identification of their laws.

■

The striking identity for $(f(\gamma))_\gamma$ which is discussed in Exercise 29, namely

$$(f(\gamma))_\gamma = \frac{1}{\pi}\int_0^1 \frac{dw}{\sqrt{w(1-w)}} f(\gamma + w(1-\gamma)) \tag{3.8}$$

now follows simply from (3.7); indeed,

$$(f(\gamma))_{\gamma=u} := \mathbb{E}_{\gamma,u}[f(\gamma)]$$

Under $Q^u \circ P^{1-u}$,

$$\gamma \overset{(law)}{=} u + \gamma_{1-u} \overset{(law)}{=} u + (1-u)\gamma$$
$$\text{(a)} \qquad\qquad \text{(b)}$$

where, in (a) and (b), γ_{1-u} and γ, are considered under the canonical Wiener measure.
It is of some interest to look at the statement of Proposition 3.6 in the light of Proposition 3.12 (or of its proof), i.e.

$$(X_u) \in \mathcal{M}_0^1 \text{ if and only if } \mathbb{E}\left[|X_u| \, |B_u = 0\right] = 0 \quad du \ a.e.$$

We note that the validity of Proposition 3.12 may be extended to a large class of Markov processes, say diffusion processes, with, for given a, t, x, $\gamma := \sup\{s \leq t, \ X_s = a\}$, $X_0 = x$.
Then the increasing process A will satisfy:

$$dA_u << dL_u$$

where (L_u) is the local time of X at a, and we obtain:

$$P_{x;\gamma,u} = Q^u_{x\to a} \circ P^{t-u}_a$$

with obvious notation.
In particular, this applies to all Bessel processes, with dimension $2(1-\mu) < 2$, and $\gamma = \sup\{u \leq 1, \ R_u = 0\}$. The formula (3.8) now becomes

$$(f(\gamma))_\gamma = c_\mu \int_0^1 \frac{dt}{t^{1-\mu}(1-t)^\mu} f(\gamma + t(1-\gamma))$$

with $c_\mu = \frac{1}{B(\mu, 1-\mu)}$.

3.7 Exercises

Recall the notation of Brownian scaling: $X_u^{[a,b]} = \frac{1}{\sqrt{b-a}} X_{a+(b-a)u}$, $u \leq 1$.
Also, $(B_u; u \geq 0)$ is a standard Brownian motion, and

$$\gamma = \sup\{t \leq 1, \ B_t = 0\}; \quad \delta = \inf\{t \geq 1, \ B_t = 0\}$$

Exercise 27 *In this chapter, the following results have been presented:*

(r-1) $(B_u^{[0,\gamma]}; \ u \leq 1)$ *is a standard Brownian bridge, independent of* γ.
(r-2) $(|B_u^{[\gamma,1]}|; \ u \leq 1)$*, the Brownian meander (by definition!), the* σ*-field* $\mathcal{F}_\gamma$ *and* $\varepsilon = sgn(B_1)$*, are independent.*

(r-3) $(|B_u^{[\gamma,\delta]}|;\ u \le 1)$ *is a BES(3) bridge independent of* $\sigma(B_u;\ u \le \gamma) \vee \sigma(B_u;\ u \ge \delta)$.

a) *Prove (r-1) using time inversion only;*
 Hint*: Write* $\hat{B}_t = tB_{1/t}$*; first show that* $\gamma = 1/\hat{\delta}$*, and proceed...*
b) *Which of the results about the meander can you recover using the time inversion approach? e.g. the fact that* $m_1 = \frac{1}{\sqrt{1-\gamma}}|B_1|$ *is Rayleigh distributed?*
c) ✠ *Can you prove (r-3) by time-inversion?*

Exercise 28 *Prove, with arguments similar to those of Exercise 27, question b), that* γ *is Arcsine distributed, i.e.*

$$\gamma \overset{(law)}{=} \frac{B_1^2}{B_1^2 + \hat{B}_1^2} \overset{(law)}{=} \cos^2(\theta)$$

with B *and* $\hat{B}$ *two independent Brownian motions and* θ *a uniform variable.*

Exercise 29

a) *Recover (r-1) in Exercise 27 by obtaining the initial enlargement formulae for* $(\mathcal{F}_t^{\sigma(\gamma)};\ t \ge 0)$*. That is, compute, for any* $f : [0,1] \to \mathbb{R}^+$*, Borel:*

$$\lambda_t(f) = \mathbb{E}\left[f(\gamma)|\mathcal{F}_t\right], \qquad t < 1$$

Compare with the progressive enlargement formula for $(\mathcal{F}_t^{\gamma};\ t \ge 0)$*.*
Prove that

$$(f(\gamma))_\gamma = \frac{1}{\pi}\int_0^1 \frac{dw}{\sqrt{w(1-w)}} f(\gamma + w(1-\gamma))$$

b) *Obtain the initial enlargement formulae with the variables* (γ_T, δ_T) *and recover (r-3) in Exercise 27; more precisely, first show that there exists a* $(\mathcal{F}_t^{\sigma(\gamma_T,\delta_T)};\ t \ge 0)$ *Brownian motion* $(\widetilde{B}_t;\ t \ge 0)$ *such that for any* $t \ge 0$*:*

$$B_t = \widetilde{B}_t - \int_0^{t\wedge\gamma_T} ds \frac{B_s}{\gamma_T - s} + \int_{\gamma_T}^{t\wedge\delta_T} ds \left(\frac{1}{B_s} - \frac{B_s}{\delta_T - s}\right) \tag{3.9}$$

Exercise 30 *(Denisov's result [Den83]) Prove Denisov's result asserting that if* σ *denotes the time at which a Brownian motion* $(B_t; t \le 1)$ *reaches its maximum* $S_1 = \sup_{s\le 1} B_s$*, then the two processes prior to* σ *and posterior to* σ*, correctly viewed, i.e.:* (σ, B_σ) *becoming the origin in time-space, and Brownian scaled, are two independent meanders (Give the precise statement).*

Exercise 31 *This exercise provides an extension of formula (3.6):*

$$(L_1)_\gamma = L_1 + \sqrt{\frac{2}{\pi}(1-\gamma)} \tag{3.10}$$

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$, Borel. Prove the formula

$$(f(L_1))_\gamma = \int_0^\infty dx\ f(L_1 + x) e^{-x^2/2(1-\gamma)} \sqrt{\frac{2}{\pi(1-\gamma)}} \tag{3.11}$$

***Hint**: With our usual notation*

$$\mathbb{E}\left[f(L_1)|\mathcal{F}_t\right] = \hat{\mathbb{E}}_{B_t(\omega)}\left[f(L_t(\omega) + \hat{L}_{1-t})\right]$$

In particular, from (3.11), we see that, in general

$$(f(L_1))_\gamma \neq \mathbb{E}\left[f(L_1)|\mathcal{F}_\gamma\right] = f(L_1)$$

Exercise 32 *(Cauchy's principal value of Brownian local time and loss of information). In Example 3.3, the existence of $\int_0^t \frac{ds}{B_s} = \lim_{\varepsilon\to 0}\int_0^t \frac{ds}{B_s} 1_{|B_s|\geq\varepsilon}$ was established. Let us denote $\hat{B}_t = B_t - \int_0^t \frac{ds}{B_s}$ and $(\hat{\mathcal{B}}_t;\ t \geq 0)$ its natural filtration.*
Consider $f : \mathbb{R}^+ \to \mathbb{R}$, a simple function, i.e. a function which takes a finite number of values on a finite number of intervals, and $(z_u;\ u \geq 0)$ a bounded predictable process.

a) Prove that

$$z_{\gamma_t} B_t \exp\left(i\int_0^t f(s)d\hat{B}_s + \tfrac{1}{2}\int_0^t f(s)^2 ds\right) \tag{3.12}$$

is a $(\mathcal{F}_t;\ t \geq 0)$-martingale.
Note that there is no problem in defining the stochastic integral $\int_0^t f(s)d\hat{B}_s$, since f is a simple function.

b) Consequently, show that if $B_0 = 0$, then:

$$\mathbb{E}\left[B_t|\mathcal{F}_{\gamma_t} \vee \hat{\mathcal{B}}_t\right] = 0 \tag{3.13}$$

As a consequence,

$$\hat{\mathcal{B}}_t \vee \mathcal{F}_{\gamma_t} \subsetneq \mathcal{F}_t \tag{3.14}$$

A fortiori, this implies $\hat{\mathcal{B}}_t \subsetneq \mathcal{F}_t$.

c) As a complement to (3.14), show that $\mathcal{F}_t = \hat{\mathcal{B}}_t \vee \mathcal{F}_{\gamma_t} \vee \sigma(sgn(B_t))$, and also:$\mathcal{F}_t = \mathcal{S}_t \vee \hat{\mathcal{B}}_t$, where $\mathcal{S}_t = \sigma\{sgn(B_s),\ s \leq t\}$.

d) If $(R_u;\ u \geq 0)$ is a 3-dimensional Bessel process, with driving Brownian motion $(\beta_u;\ u \geq 0)$, i.e.

$$R_u = \beta_u + \int_0^u \frac{ds}{R_s} \tag{3.15}$$

Since R is a strong solution of (3.15), we may represent: $R_1 = \Phi(\beta_u;\ u \leq 1)$, for some functional Φ on $\mathcal{C}([0,1],\mathbb{R})$. Prove the following formula: for any $f;\mathbb{R} \to \mathbb{R}_+$, Borel,

$$\mathbb{E}[f(B_t)|\mathcal{F}_{\gamma_t} \vee \hat{\mathcal{B}}_t] = f(\sqrt{t-\gamma_t}\lambda_t^+)\frac{\lambda_t^-}{\lambda_t^+ + \lambda_t^-} + f(-\sqrt{t-\gamma_t}\lambda_t^-)\frac{\lambda_t^+}{\lambda_t^+ + \lambda_t^-}$$

where $\lambda_t^+ = \Phi(\hat{B}_u^{[\gamma_t,t]};\ u \leq 1)$*;* $\lambda_t^- = \Phi(-\hat{B}_u^{[\gamma_t,t]};\ u \leq 1)$

e) *We now assume* $B_0 = a \neq 0$*, and let* P_a *denote the law of* $(B_t;\ t \leq 1)$*. Then, introduce the signed measure*

$$Q_{a|\mathcal{F}_1} = \frac{B_1}{a} P_{a|\mathcal{F}_1}$$

Prove that, under the signed measure Q_a*,* $(\hat{B}_u;\ u \leq 1)$ *is a Brownian motion.*
Explain the difference with the situation in questions a) and b).

Comment 3.4

- *Despite the results obtained in question d), the following question is open:* ✠ *Is* $(\hat{B}_t;\ t \geq 0)$ *a semimartingale with respect to* $(\hat{\mathcal{B}}_t;\ t \geq 0)$*, its own filtration? If so, can one express its canonical decomposition?*
- *A number of results concerning this principal value are discussed in [BY87].*
- *More general principal values related to Brownian motion and Bessel processes are discussed in [Yor97a], second part.*
- *Stochastic calculus with respect to signed measures has been developed first by Ruiz de Chavez [RdC84], then in [BS03a].*

4

Predictable and Chaotic Representation Properties for Some Remarkable Martingales Including the Azéma and the Dunkl Martingales

In this chapter, we still consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0), P)$ and $(M_t;\ t \geq 0)$ a locally square integrable $((\mathcal{F}_t;\ t \geq 0), P)$ martingale, i.e. there exists a sequence of stopping times $(T_n,\ n \in \mathbb{N})$ with

$$T_n \nearrow \infty \qquad \text{and} \qquad \mathbb{E}\left[\sup_{t \leq T_n} |M_t|^2\right] < \infty$$

It will be convenient to assume that $\mathcal{F} = \mathcal{F}_\infty := \lim_{t \uparrow \infty} \mathcal{F}_t$.
We shall use the abbreviation CRP (resp. PRP) to denote the Chaotic (resp. Predictable) Representation Property for such a given martingale $(M_t;\ t \geq 0)$. After a short Section 4.1 consisting of definitions, we explore successively the PRP and the CRP in Section 4.2. In Section 4.3, we focus on the particular example of the first Azéma martingale. Section 4.4 consists of some exercises, including further examples of martingales which enjoy the CRP.

4.1 Definition and First Example

We first recall the Kunita-Watanabe orthogonal decomposition of a square integrable martingale with respect to another such martingale.

Theorem 4.1 *For any locally square integrable martingale N, there is a unique orthogonal decomposition:*

$$N_t = \int_0^t n_s dM_s + R_t$$

where R is a local martingale and RM is a local martingale (i.e. R is orthogonal to M) and the predictable process $(n_s;\ s \geq 0)$ is obtained via

$$< N, M >_t = \int_0^t n_s \, d < M, M >_s \qquad i.e.\ n_s = \frac{d < N, M >_s}{d < M, M >_s}$$

R. Mansuy and M. Yor: *Random Times and Enlargements of Filtrations in a Brownian Setting*, Lect. Notes Math. **1873**, 71–86 (2006)
www.springerlink.com

Definition 4.1 *(PRP with respect to M)*[1]
M enjoys the PRP relatively to $(\mathcal{F}_t;\ t \geq 0)$, if for every locally square integrable $(\mathcal{F}_t;\ t \geq 0)$-martingale N, there exist $c \in \mathbb{R}$ and n a predictable process such that $\int_0^t n_s^2 d < M >_s < \infty$ for every $t \geq 0$ and

$$N_t = c + \int_0^t n_s dM_s$$

We now define the notion of CRP with respect to M; in order that multiple dM-integrals be well defined, we make the additional assumption that $< M >_t = \alpha t$, for some α; the restriction to work with such a martingale M comes from the fact that, although it would be possible to define multiple stochastic integrals with respect to martingales such that $d < M >_t \leq Cdt$, variables thus defined and belonging to different chaoses need not to be orthogonal. See [DMM92] Remarque 8 a) p.203.

Definition 4.2 *For any $k \geq 1$, the k-th chaos of M is defined as*

$$C_k = \left\{ \int_0^\infty dM_{t_1} \int_0^{t_1^-} \cdots \int_0^{t_{k-1}^-} dM_{t_k} f(t_1, \ldots, t_k) \right\}$$

with $\Delta_k = \{(t_1, \ldots, t_k),\ t_1 \geq t_2 > \ldots > t_k > 0\}$ and f any deterministic, square integrable function on Δ_k.
M enjoys the CRP, if

$$L^2(\sigma(M_u;\ u \geq 0)) = \bigoplus_{k=0}^\infty C_k$$

where $C_0 = \mathbb{R}$.

Remark 4.1 *In the set-up of Definition 4.1, we shall simply write that M enjoys the PRP when $(\mathcal{F}_t;\ t \geq 0)$ is the natural filtration of M. Clearly, the distinguo is no longer meaningful in the case of the CRP (see Definition 4.2).*

Remark 4.2 *For both definitions, one can switch from a formulation with random variables in $L^2(\mathcal{F}_\infty)$, resp. $L^2(\sigma(M_u;\ u \geq 0))$ to a formulation with square integrable martingales simply by considering the martingale associated with a terminal value in $L^2(\mathcal{F}_\infty)$, resp. $L^2(\sigma(M_u; u \geq 0))$.*

Remark 4.3 *Note that, if M enjoys the CRP, then M enjoys the PRP.*

Theorem 4.2 *Brownian motion enjoys the CRP, hence the PRP.*

[1] In all generality, local square integrability arguments are not necessary- the whole discussion might be developed with local martingales only. See, e.g., Jacod-Yor [JY77].

Proof
Let $(B_t;\ t \geq 0)$ be a Brownian motion, and denote $\mathcal{B}_\infty = \sigma\{B_u;\ u \geq 0\}$. The set of all random variables $X \in L^2(\mathcal{B}_\infty)$ which admit a chaotic representation

$$X = c + \sum_{k=1}^{\infty} \int_0^\infty dB_{t_1} \int_0^{t_1} dB_{t_2} ... \int_0^{t_{k-1}} dB_{t_k}\, f_k(t_1, ..., t_k)$$

is closed in $L^2(\mathcal{B}_\infty)$ since

$$\mathbb{E}\left[X^2\right] = c^2 + \sum_{k=1}^{\infty} \int_0^\infty dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{k-1}} dt_k\, f_k(t_1, ..., t_k)^2$$

Therefore, it suffices to show the chaotic decomposition for the random variables

$$\mathcal{E}^{(f)} = \exp\left(\int_0^\infty f(u)dB_u - \frac{1}{2}\int_0^\infty f(u)^2 du\right)$$

with $f \in L^2(\mathbb{R}_+, du)$, since the set $\left\{\mathcal{E}^{(f)};\ f \in L^2(\mathbb{R}_+, du)\right\}$ is total in $L^2(\mathcal{B}_\infty)$.

An incomplete argument
From Itô's formula:

$$\mathcal{E}_t^{(f)} = \exp\left(\int_0^t f(u)dB_u - \frac{1}{2}\int_0^t f(u)^2 du\right) = 1 + \int_0^t \mathcal{E}_s^{(f)} f(s) dB_s;$$

then iterate the use of Itô's formula to develop $\mathcal{E}_s^{(f)}$...
Question: What is missing in this argument?

A rigorous proof
The decomposition of the function $(x,t) \mapsto \exp(\lambda x - \frac{\lambda^2}{2} t)$ in terms of the Hermite polynomials $((H_k(x,t);\ x,t \in \mathbb{R}),\ k \in \mathbb{N})$ yields to

$$\mathcal{E}_t^{(\lambda f)} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} H_k\left(\int_0^t f(s)dB_s; \int_0^t f(s)^2 ds\right) \tag{4.1}$$

and $H_k\left(\int_0^t f(s)dB_s; \int_0^t f(s)^2 ds\right)$ is a variable in the k-th chaos with the integrand

$$f_k(t_1, \ldots, t_k) = f(t_1) \ldots f(t_k) 1_{t > t_1 > \ldots > t_k > 0}$$

■

Remark 4.4 *Schoutens ([Sch00] Section 5.4) remarks that for the compensated Poisson process, one can obtain an analogue to formula* (4.1) *by replacing Hermite polynomials with Charlier polynomials, a remark which goes back at least to Ogura[Ogu72].*

4.2 PRP and Extremal Martingale Distributions

We are interested in understanding[2] when or why M may have the PRP with respect to $(\mathcal{F}_t;\ t \geq 0)$. Let us begin with the following remark, due to Dellacherie [Del74], [Del75], concerning the Wiener measure.

Proposition 4.3 *The Wiener measure W is extremal in the set of martingale laws.*
More precisely, if we denote by Ω_* the set of càdlàg functions over $\mathbb{R}_+$, $X_t(\omega) = \omega(t)$ the coordinate process, by $\mathcal{X}_t = \sigma(X_s;\ s \leq t)$ the associated filtration and $\mathcal{M}$ the set of probability measures on $(\Omega_*, \mathcal{X}_\infty)$ which make X a local martingale. Then W is extremal in $\mathcal{M}$.

Proof
Indeed, if $W = \alpha P_1 + (1-\alpha)P_2$ with $0 < \alpha < 1$ and P_1, $P_2 \in \mathcal{M}$, then $P_1 << W$.
Moreover, the quadratic variation of X under W is t, so t is also the quadratic variation of X under P_1.
Hence, under P_1, $(X_t;\ t \geq 0)$ is a continuous (local) martingale with quadratic variation t. Accordingly (Lévy's theorem), $(X_t;\ t \geq 0)$ is a P_1-Brownian motion; in other terms:

$$P_1 = W$$

■

Theorem 4.4 *Let P belong to $\mathcal{M}$. P is an extremal point in $\mathcal{M}$ if and only if, under P, X enjoys the PRP.*

This proof will use in a fundamental manner the $\mathbf{H^1}$**-BMO** duality, which is recalled in Section 0.6.

Proof
We want to prove

$$\mathrm{Ext}(\mathcal{M}) = \Pi$$

where Π denotes the set of laws P under which $(X_t;\ t \geq 0)$ admits the PRP.

$(\supseteq)$

Let $P \in \Pi$ and assume $P = \alpha P_1 + (1-\alpha)P_2$ with $0 < \alpha < 1$ and P_1, $P_2 \in \mathcal{M}$.
One has $P_1 << P$ that is $P_{1|\mathcal{X}_t} = D_t . P_{|\mathcal{X}_t}$, with $(D_t;\ t \geq 0)$ a $(P, (\mathcal{X}_t;\ t \geq 0))$ uniformly integrable martingale (in fact, $(D_t;\ t \geq 0)$ is bounded by $1/\alpha$).
$(X_t;\ t \geq 0)$ is a P_1 martingale if, and only if, $(X_t D_t;\ t \geq 0)$ is a P martingale, i.e. $(D_t;\ t \geq 0)$ is orthogonal to $(X_t;\ t \geq 0)$.

[2] See also [Dav05], for a survey type discussion

Since $P \in \mathfrak{T}$, D is a stochastic integral with respect to X; hence, $(D_t;\ t \geq 0)$ is constant. Therefore: $\forall t \geq 0$, $D_t = 1$ and $P = P_1$; hence P is extremal.

$(\subseteq)$
Conversely, assume that $P \in \mathrm{Ext}(\mathcal{M})$.
We will now use the duality result between $\mathbf{H^1}$ and **BMO** (cf. Section 0.6) and the Hahn-Banach theorem.
Let $\mathbf{K^1} = \{c + \int_0^{\cdot} n_s dX_s \in \mathbf{H^1}\}$; if $\mathbf{K^1} \subsetneq \mathbf{H^1}$, then there would exist (Hahn-Banach theorem with the duality $\mathbf{H^1}$-**BMO**) a non-zero **BMO** martingale R which is orthogonal to $\mathbf{K^1}$, and may be chosen bounded, since R is already locally bounded and we can stop it.
Consider the decomposition

$$P = \frac{1}{2}\left(P_+ + P_-\right), \text{ where } P_+ = \left(1 + \frac{R_\infty}{2k}\right) P \text{ and } P_- = \left(1 - \frac{R_\infty}{2k}\right) P$$

with k such that $|R_\infty| \leq k$.
But this contradicts the extremality of P. Thus, $\mathbf{K^1} = \mathbf{H^1}$, and finally, every local martingale is a stochastic integral with respect to P. Thus, $P \in \mathfrak{T}$.

■

Remark 4.5

- *The relationship between extremality and the PRP has been discovered for Brownian motion and the compensated Poisson process by Dellacherie [Del74] [Del75], then developed in the generality of Theorem 4.4 in Jacod-Yor [JY77].*
- *Following Dellacherie's arguments [Del68] about the laws of martingales indexed by $\mathbb{N}$, it is shown in Jacod-Yor [JY77] that every element of $\mathcal{M}$ is an integral of extremal points of $\mathcal{M}$.*

4.3 CRP: An Attempt Towards a General Discussion

Assumption 4.1 *We assume that $(M_t;\ t \geq 0)$ admits the following decomposition in its continuous and purely discontinuous parts:*

$$M_t = M_t^{(c)} + M_t^{(d)}$$

with $< M^{(c)} >_t = \alpha t$ and $< M^{(d)} >_t = \beta t$.

By now, we change slightly our definition of the CRP, the original one (as given in Definition 4.2) being too rigid:

Definition 4.3 $(M_t;\ t \geq 0)$ *enjoys the CRP if*

$$L^2(\mathcal{M}_\infty) = \bigoplus_{n=0}^{\infty} \widetilde{C}_n$$

with

$$\widetilde{C}_n = \left\{ \int_0^\infty dM_{t_1}^{\varepsilon_1} \int_0^{t_1^-} \cdots \int_0^{t_{n-1}^-} dM_{t_n}^{\varepsilon_n} f(t_1, \cdots, t_n) \right\}$$

with $\Delta_n = \{(t_1, \cdots, t_n),\ t_1 > t_2 > \ldots > t_n > 0\}$ *and where* $(M_t^\varepsilon;\ t \geq 0)$ *is either* $M^{(c)}$ *or* $M^{(d)}$ *and* f *any deterministic, square integrable function defined on* Δ_n.

Remark 4.6 *In turn, this definition of the CRP may prove to be too restrictive, and in a more general scheme, it may be of some interest to replace* $M^{(c)}$ *and* $M^{(d)}$ *in Assumption 4.1 and Definition 4.3 by a set of mutually orthogonal martingales* $(M_k;\ k \geq 0)$ *such that* $< M_k >_t = \alpha_k t$. *See Émery [Éme91] [Éme94], Nualart-Schoutens [NS00] or Jamshidian [Jam05]. [Éme96] and [Éme05] present the "big" picture...*

4.3.1 An Attempt to Understand the CRP in Terms of a Generalized Moments Problem

The preceding characterization of the martingales which enjoy the PRP may also be understood in terms of a particular case of some generalized moments problem; see, e.g. [Yor78b].
We would like to develop a similar discussion for the martingales which enjoy the CRP, or at least point out the remaining difficulties.
First, we recall the generalized moments problem solution (independently proved by Naimark [Nai47] and Douglas [Dou66]; see [CY03] Exercise 1.9 for details and a complete review).

Proposition 4.5 *Let* $\Phi = (\varphi_i;\ i \in I)$ *a family of real valued variables on a measurable space* $(\Omega, \mathcal{F})$. *We define* $\mathcal{M}_\Phi$ *as the family of probability measures on* $(\Omega, \mathcal{F})$ *such that:*

- $\forall i \in I, \qquad \varphi_i \in L^1(P)$
- $\forall i \in I, \qquad \mathbb{E}_P(\varphi_i) = 0$

Then P *is extremal in* $\mathcal{M}_\Phi$ *if and only if span*$(1; \varphi_i,\ i \in I)$ *is dense in* $L^1(P)$.

We now consider two generalized moments problems related with the PRP, resp. the CRP.
First problem
Consider a process $(M_t;\ t \geq 0)$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t;\ t \geq 0))$ and introduce $\Phi^{(\mathcal{P})} = \{1_{\Gamma_s}(M_t - M_s);\ s \leq t,\ \Gamma_s \in \mathcal{F}_s\}$; then we obtain:

- a probability measure P is in $\mathcal{M}_{\Phi^{(\mathcal{P})}}$ if, and only if, under P, $(M_t;\ t \geq 0)$ is a $(\mathcal{F}_t;\ t \geq 0)$-martingale.
- $\text{span}(1, \Phi^{(\mathcal{P})})$ is dense in $L^1(P)$ if, and only if, M enjoys the PRP under P. In fact, this is somewhat subtle as one needs to go from $L^1(P)$ to $H^1(P)$ by localization... For details, see [Yor78b].

Hence, we recover the characterization of the PRP. This parallels somehow the argument given in the proof of Theorem 4.4.
Second problem
A similar approach to the CRP can be tried. Let us introduce the following family

$$\Phi^{(\mathcal{C})} = \left\{ \prod_{i=1}^{k} (M_{t_{i+1}} - M_{t_i});\ k \in \mathbb{N}^*,\ t_1 < .... < t_{k+1} \right\}$$

Consider a probability measure P such that, under P, $(M_t;\ t \geq 0)$ is a square integrable martingale and there exists $c \in \mathbb{R}_+$ such that $< M >_t = ct$.

- If, under P, M enjoys the CRP, it is easy to show (L^2 convergence implies L^1 convergence...) that $P \in \text{Ext}(\mathcal{M}_{\Phi^{(\mathcal{C})}})$.
- Conversely, if $P \in \text{Ext}(\mathcal{M}_{\Phi^{(\mathcal{C})}})$, then $\text{span}(1, \Phi^{(\mathcal{C})})$ is dense in $L^1(P, \mathcal{F})$, that is any integrable variable can be represented as a limit of multiple stochastic integrals with respect to $(M_t;\ t \geq 0)$ with deterministic integrands. To deduce the CRP from the assumption $P \in \text{Ext}(\mathcal{M}_{\Phi^{(\mathcal{C})}})$, "morally", it should then suffice to localize and therefore to prove that $\text{span}(1, \Phi^{(\mathcal{C})})$ is dense in $L^2(P, \mathcal{F})$. Nevertheless, we have not been able to push this localization argument further...

4.3.2 Some Sufficient Conditions for the CRP

We are now ready to examine rigorously some **sufficient** conditions for the CRP.
First, assume that $(M_t;\ t \geq 0)$ enjoys the CRP (in the sense of Definition 4.3). As a consequence, for any square integrable martingale $(N_t;\ t \geq 0)$, there exist $a \in \mathbb{R}$, and two predictable processes $(n_t^{(c)};\ t \geq 0)$ and $(n_t^{(d)};\ t \geq 0)$ such that

$$N_t = a + \int_0^t n_s^{(c)} dM_s^{(c)} + \int_0^t n_s^{(d)} dM_s^{(d)}$$

In particular

$$\sum_{s \leq t} (\Delta M_s)^2 - \beta t = \int_0^t \varphi_s dM_s^{(d)}$$

for some predictable process $(\varphi_s;\ s \geq 0)$.

Then $(\Delta M_t)^2 = \varphi_t \Delta M_t$, which is equivalent to $\Delta M_t = \varphi_t 1_{\{\Delta M_t \neq 0\}}$. Therefore the amplitude of the jumps is predictable (but not the whole jump process, and in particular the times of jumps!).
Conversely, let us assume that $\Delta M_t = \varphi_t 1_{\{\Delta M_t \neq 0\}}$ with φ predictable, $\varphi_t \neq 0$, everywhere, together with Assumption 4.1. Thanks to these assumptions, we may write a "predictable" Itô's formula:

Proposition 4.6 *Under the preceding hypothesis:*

$$< M^{(c)} >_t = \alpha t; \quad < M^{(d)} >_t = \beta t; \quad \Delta M_t = \varphi_t 1_{\Delta M_t \neq 0} \ ,$$

there is the following "predictable" Itô's formula for any function f of class $\mathcal{C}^2$ with f''_{xx} bounded[3]*:*

$$\begin{aligned} f(M_t, t) - f(M_0, 0) = \\ &= \int_0^t f'_x(M_s, s) dM_s^{(c)} + \int_0^t \frac{f(M_{s-} + \varphi_s, s) - f(M_{s-}, s)}{\varphi_s} dM_s^{(d)} \qquad (4.2) \\ &+ \int_0^t ds \left\{ \frac{\alpha}{2} f''_{xx}(M_s, s) + f'_s(M_s, s) \right. \\ &\quad \left. + \beta \frac{f(M_s + \varphi_s, s) - f(M_s, s) - f'_x(M_s, s)\varphi_s}{\varphi_s^2} \right\} \end{aligned}$$

Proof
Indeed, the "ordinary" Itô's formula yields to

$$\begin{aligned} f(M_t, t) &= f(M_0, 0) + \\ &+ \int_0^t f'_x(M_{s-}, s) dM_s + \frac{\alpha}{2} \int_0^t f''_{xx}(M_s, s) ds + \int_0^t f'_s(M_s, s) ds \\ &+ \sum_{s \leq t} \left(f(M_s, s) - f(M_{s-}, s) - f'_x(M_{s-}, s)\varphi_s \right) 1_{\{\Delta M_s \neq 0\}} \end{aligned}$$

Thus, together with the property $M_s = M_{s-} + \varphi_s 1_{\Delta M_s \neq 0}$, we use the martingale property of $(\sum_{s \leq t} (\Delta M_s)^2 - \beta t;\ t \geq 0)$ in the expression $\sum_{s \leq t}$ after replacing $1_{\Delta M_s \neq 0}$ by $\frac{(\Delta M_s)^2}{\varphi_s^2}$.

■

We continue to look for **sufficient** conditions on M which ensure the CRP. Assume that the family of random variables of the form $\prod_{i=1}^n (M_{t_i})^{m_i}$ is total in $L^2(\mathcal{M}_\infty)$ where $n \in \mathbb{N}$, $t_1 < t_2 < ... < t_n$, and $m_i \in \mathbb{N}$, $i \leq n$. We decompose the variable $\prod_{i=1}^n (M_{t_i})^{m_i} = \prod_{i=1}^{n-1} (M_{t_i})^{m_i} \times (M_{t_n})^{m_n}$ and apply the predictable Itô formula between t_{n-1} and t_n for $M_t^{m_n}$.

[3] In [Éme89], such an Itô's formula is obtained in the more general framework of Schwartz formal semi-martingales.

Let us first consider the particular case $m_n = 2$.

$M_{t_n}^2 - M_{t_{n-1}}^2 =$

$$= 2\int_{t_{n-1}}^{t_n} M_{s-} dM_s^{(c)} + \int_{t_{n-1}}^{t_n} (2M_{s-} + \varphi_s) dM_s^{(d)} + (\alpha+\beta)(t_n - t_{n-1})$$

Hence in order to be able to use again our assumption, we need that $\varphi_u = aM_{u-} + b$.

This assumption being made, we now obtain, for a general integer exponent m_n, the formula:

$M_{t_n}^{m_n} - M_{t_{n-1}}^{m_n} =$

$$= m_n \int_{t_{n-1}}^{t_n} M_{s-}^{m_n-1} dM_s^{(c)} + \int_{t_{n-1}}^{t_n} \sum_{k=0}^{m_n-1} \binom{m_n}{k} \varphi_s^{m_n-k-1} M_{s-}^k dM_s^{(d)} +$$

$$\int_{t_{n-1}}^{t_n} ds \left[\frac{\alpha\ m_n(m_n-1)}{2} M_s^{m_n-2} + \beta \left(\sum_{k=0}^{m_n-2} \binom{m_n}{k} \varphi_s^{m_n-k-2} M_{s-}^k \right) \right]$$

This formula shows that, by induction, all variables of the form $\prod_{i=1}^n (M_{t_i})^{m_i}$ may be written as a sum of iterated deterministic multiple integrals.

Finally, we have obtained the following theorem:

Theorem 4.7 *If* $(M_t;\ t \geq 0)$ *satisfies*

- *The family of random variables of the form* $\prod_{i=1}^n (M_{t_i})^{m_i}$ *is total in* $L^2(\mathcal{M}_\infty)$ *(in particular, this is satisfied when for fixed t,* M_t *is uniformly bounded, or more generally, admits some exponential moments).*
- $\forall t \geq 0, \qquad \Delta M_t = (aM_{t-} + b)1_{\{\Delta M_t \neq 0\}}.$
- $\forall t \geq 0, \qquad < M^{(c)} >_t = \alpha t.$
- $\forall t \geq 0, \qquad < M^{(d)} >_t = \beta t.$

Then $(M_t;\ t \geq 0)$ *enjoys the CRP (in the sense of Definition 4.3)*

4.3.3 The Case of the Azéma Martingale

Now, we shall study an example of process that fits with the framework of Theorem 4.7.

Theorem 4.8 *Let* $(B_t;\ t \geq 0)$ *be a Brownian motion,* $\gamma_t = \sup\{s \leq t;\ B_s = 0\}$.

Define $\mu_t = sgn(B_t)\sqrt{t - \gamma_t}$. *Then*

- μ *is a* $(\mathcal{F}_{(\gamma_t)+},\ t \geq 0)$*-martingale.*
- μ *is purely discontinuous.*
- $< \mu >_t = t/2$.

- *μ enjoys the CRP with respect to its own filtration which is ($\mathcal{S}_t = \sigma\{sgn(B_u),\ u \leq t\};\ t \geq 0$).*

($\mu_t;\ t \leq 1$) is called Azéma's martingale[4].

Proof
We shall project ($\mathcal{F}_t;\ t \geq 0$)-Brownian martingales on ($\mathcal{F}_{(\gamma_t)+};\ t \geq 0$). Thanks to the independence for each time $t > 0$ of the meander ($\widetilde{m}_u^{(t)} = \frac{1}{\sqrt{t-\gamma_t}}|B_{\gamma_t+u(t-\gamma_t)}|;\ u \leq 1$) and $\mathcal{F}_{(\gamma_t)+}$ (see Chapter 3, Subsection 3.1.2), we obtain:

$$\begin{aligned}\mathbb{E}\left[f(B_t)|\mathcal{F}_{(\gamma_t)+}\right] &= \mathbb{E}\left[f(\sqrt{t-\gamma_t}\varepsilon_t\widetilde{m}_1^{(t)})|\mathcal{F}_{(\gamma_t)+}\right] \\ &= \int_0^\infty \rho e^{-\rho^2/2} f(\mu_t\rho)d\rho \end{aligned} \tag{4.3}$$

With $f(x) = x$, this identity becomes $C\mu_t = \mathbb{E}\left[B_t|\mathcal{F}_{\gamma_t+}\right]$; with $f(x) = x^2$, we find that $<\mu>_t = \beta t$, and since $\mathbb{E}[t-\gamma_t] = \mathbb{E}[\gamma_t] = t/2$, we obtain $\beta = 1/2$. Remark, moreover, that ($\mu_t;\ t \geq 0$) jumps only to 0; therefore

$$\Delta\mu_t = -\mu_{t-}1_{\Delta\mu_t\neq 0}$$

The totality assumption is satisfied, since for a given t, μ_t is bounded; thus Theorem 4.7 yields to the announced result.
To prove that μ is purely discontinuous, we reproduce the Janson-Protter argument found in Theorem 67, p.184 of [Pro04]:
There is the general formula

$$<\mu^c>_t = \int_{-\infty}^{\infty} L_t^a(\mu)da \tag{4.4}$$

where ($L_t^a(\mu),\ a \in \mathbb{R}$) denotes Meyer's continuous part of the local times ($\mathcal{L}_t^a(\mu),\ \mu \in \mathbb{R}$) constructed by Meyer in his Course (See e.g. [AY78] p.20). Now it is argued in Protter [Pro04] that, for every $a \neq 0$, $L_t^a(\mu) = 0$, hence from (4.4): $\mu^c = 0$.
In turn, that $L_t^a(\mu) = 0$ follows from the fact that

$$L_t^a(\mu) = \int_0^t 1_{\mu_s=\mu_{s-}=a}dL_s^a(\mu)$$

and, for any $a \neq 0$, there are a.s. only countable times s such that

$$\mu_s = \mu_{s-} = a\ .$$

■

[4] It was discovered by Azéma in his general study of closed random sets [Azé85a]; its properties have been discussed in a number of papers (see, e.g. [AY89], [Éme89], [Éme90]).

Remark 4.7 *In fact, much more generally, all* $\left(\mathcal{F}_{(\gamma_t)+};\ t \geq 0\right)$*-martingales are purely discontinuous.*
Indeed, from the totality argument developed in the proof of Theorem 4.2, it suffices to show that, for $f \in \mathcal{C}_b^1(\mathbb{R}_+)$*,*

$$\Theta_t^f := \mathbb{E}\left[\mathcal{E}_t^f | \mathcal{F}_{(\gamma_t)+}\right]$$

is purely discontinuous. Indeed, we get:

$$\Theta_t^f = \exp\left(-\int_0^{\gamma_t} f'(u) B_u du - \frac{1}{2}\int_0^t f^2(u) du\right) \varphi(\mu_t, t, \gamma_t)$$

where: $\varphi(\mu_t, t, \gamma_t) = \mathbb{E}\left[\exp\left(\int_{\gamma_t}^t f(u) dB_u\right) | \mathcal{F}_{(\gamma_t)+}\right]$*. Details are left to the reader.*

Proposition 4.9 $(\mu_t;\ t \geq 0)$ *is a Markov process with respect to* $(\mathcal{F}_{\gamma_t +},\ t \geq 0)$*; its semigroup* $(Q_t;\ t \geq 0)$ *is intertwined with the Brownian semigroup* P_t*:*

$$Q_t \Lambda = \Lambda P_t \text{ with } \Lambda f(\mu) = \int_0^\infty d\rho\, \rho\, e^{-\rho^2/2} f(\mu\rho), \qquad (\mu \in \mathbb{R}) \tag{4.5}$$

Proof
We shall obtain simultaneously the Markov property of $(\mu_t;\ t \geq 0)$ with respect to the filtration $(\mathcal{F}_{\gamma_t +};\ t \geq 0)$, and the intertwining relation (4.5).
For a polynomial function f, we consider $\mathbb{E}\left[f(B_t)|\mathcal{F}_{\gamma_s +}\right]$ for $s < t$.

- On one hand, we get

$$\begin{aligned}\mathbb{E}\left[f(B_t)|\mathcal{F}_{\gamma_s +}\right] &= \mathbb{E}\left[P_{t-s} f(B_s)|\mathcal{F}_{\gamma_s +}\right] \\ &= \Lambda P_{t-s} f(\mu_s), \qquad \text{from (4.3)}\end{aligned}$$

- On the other hand

$$\mathbb{E}\left[f(B_t)|\mathcal{F}_{\gamma_s +}\right] = \mathbb{E}\left[\Lambda f(\mu_t)|\mathcal{F}_{\gamma_s +}\right], \qquad \text{also from (4.3)}$$

Hence

$$\mathbb{E}\left[\Lambda f(\mu_t)|\mathcal{F}_{\gamma_s +}\right] = \Lambda P_{t-s} f(\mu_s)$$

In particular, for $f(x) = x^n$, we get

$$m_n\, \mathbb{E}\left[\mu_t^n|\mathcal{F}_{\gamma_s +}\right] = \Pi_{t-s}^{(n)}(\mu_s)$$

with $m_n := \mathbb{E}\left[m_1^n\right]$ and $\Pi_t^{(n)}$ a polynomial function.
With the help of the boundedness of μ_t, an application of Weierstrass' theorem yields that there exists a semigroup $(Q_t;\ t \geq 0)$ such that, for any continuous function g:

$$\mathbb{E}\left[g(\mu_t)|\mathcal{F}_{\gamma_s+}\right] = Q_{t-s}g(\mu_s)$$

Moreover, we get

$$Q_{t-s}\Lambda f(\mu_s) = \Lambda P_{t-s}f(\mu_s)$$

from which we deduce $Q_u\Lambda = \Lambda P_u$, for any $u \geq 0$.

■

Remark 4.8 *Various generalizations of Azéma's martingale (sometimes called Azéma's first martingale) can be found in the literature (see the articles by Émery mentioned in the references and particularly [Éme05]; [AÉ96] discusses multidimensional Azéma martingales). Here is a short list of examples:*

- *We have seen in the proof of Theorem 4.8 that Azéma's martingale can be obtained from the projection of $(B_t;\ t \geq 0)$ on the "slow filtration" $(\mathcal{F}_{(\gamma_t)+};\ t \geq 0)$, namely $\sqrt{\frac{\pi}{2}}\mu_t = \mathbb{E}\left[B_t|\mathcal{F}_{(\gamma_t)+}\right]$.*
  It may be of interest to introduce the projections of other martingales. For example, Azéma's second martingale $(\nu_t;\ t \geq 0)$ is defined as

$$\sqrt{\frac{\pi}{2}}\ \nu_t := \mathbb{E}\left[|B_t| - L_t^0|\mathcal{F}_{(\gamma_t)+}\right] = \sqrt{\frac{\pi}{2}}\sqrt{t-\gamma_t} - L_t^0$$

  This process enjoys the PRP, its natural filtration is generated by the process $(\gamma_t;\ t \geq 0)$ and is immersed (see Section 5.4 for a discussion about immersion of filtrations) in the filtration of $(\mu_t;\ t \geq 0)$.
  ✠ *To our knowledge, the question of knowing whether $(\nu_t;\ t \geq 0)$ enjoys the CRP, which was posed in [Yor97b], is still open.*
- *Let $(X_t;\ t \geq 0)$ be a regular diffusion which is symmetric at 0 and is in natural scale. Let $(\mathcal{F}_t;\ t \geq 0)$ denote its natural filtration and $\gamma_t(X) = \sup\{u \leq t;\ X_u = 0\}$. Then the projection of $(X_t;\ t \geq 0)$ on the filtration $(\mathcal{F}_{\gamma_t(X)};\ t \geq 0)$ is given by*

$$\mathbb{E}\left[X_t|\mathcal{F}_{\gamma_t(X)}\right] = \frac{sgn(X_t)}{\overline{N}(t-\gamma_t(X))}$$

  where $\overline{N}$ is the tail-distribution of the lifetime of the generic excursion, under Itô's measure associated with the excursions of $(X_t;\ t \geq 0)$ (See [Azé85b], [Rai96] and [Rai97]).
  In particular, if $(\widetilde{R}_t;\ t \geq 0)$ is a symmetrized Bessel process with dimension $d = 2(1-\alpha)$, $0 < \alpha < 1$, then, $(sgn(\widetilde{R}_t)(t-\gamma_t(\widetilde{R}))^\alpha;\ t \geq 0)$ is a martingale with respect to the slow filtration associated with $\widetilde{R}$.

Remark 4.9 *Although in this Chapter, we tried to develop general criteria to ensure that a probability $P(\in \mathcal{M})$ enjoys the CRP, or the PRP, some rather special arguments were developed in Theorem 4.2 and Proposition 4.3 to prove*

that Wiener measure satisfies CRP and PRP. In particular, this led P.A. Meyer to define the notion, and study existence and uniqueness for structure equations (thus, looking for rather special martingales with PRP, and in fact CRP):
Consider a martingale $(M_t;\ t \geq 0)$ *such that* $(M_t^2 - t;\ t \geq 0)$ *is also a martingale. Then, if* $(M_t,\ t \geq 0)$ *(or its law) enjoys the PRP, there exists a predictable functional* $H(M_u,\ u \leq s)$ *such that:*

$$[M, M]_t = \int_0^t H(M_u,\ u \leq s) dM_s + t \tag{4.6}$$

This is the so-called structure equation indexed by H*:* H *being given, one may consider* (4.6) *as an equation for the law of* M.

- *If* $H = 0$*, then* M *has no jumps, and from* (4.6)*and Lévy's theorem, it is a Brownian motion.*
- *In general,* H *being given, if* (4.6) *enjoys uniqueness in law, then this law satisfies the PRP ([DMM92], top of p.265)*
- *It is natural to study the case where* $H(M_u,\ u \leq s) = h(M_{s-})$*. In this case, for any continuous* h*, there is at least one solution to* (4.6) *(see [Mey89]).*
- *Finally, for any* $\beta \in \mathbb{R}$*, the structure equation for* $h(x) = \beta x$ *enjoys existence and uniqueness.*

Among some remarkable examples, see [AR94].

4.4 Exercises

Exercise 33 *(PRP under probability measures equivalent to Wiener measure)*
Let Q *be a probability on the canonical space of continuous functions, which is locally equivalent to the Wiener measure* W*, i.e.:*

$$Q_{|\mathcal{F}_t} = D_t . W_{|\mathcal{F}_t}$$

with $D_t > 0$.
Denote $X_t^D = X_t - \int_0^t \frac{d\langle D, X\rangle_s}{D_s};\ t \geq 0$*, the Girsanov transform of* X *from* W *to* Q.

a) What is the law of X^D *under* Q*?*
b) Prove that $(X_t^D;\ t \geq 0)$ *enjoys the PRP for* $(\mathcal{F}_t;\ t \geq 0)$ *under* Q*, i.e. every* $(Q, (\mathcal{F}_t;\ t \geq 0))$ *martingale* $(M_t;\ t \geq 0)$ *may be written as:*

$$M_t = c + \int_0^t m_s dX_s^D$$

for $(m_s;\ s \geq 0)$ *a* $(\mathcal{F}_s;\ s \geq 0)$ *predictable process, and some* $c \in \mathbb{R}$.

Comment 4.1 *Despite the fact that X^D enjoys the PRP under Q, there are many examples of such Q's with $\mathcal{X}_t^D \subsetneq \mathcal{F}_t$. See Tsirel'son's first example in Chapter 6.*

Exercise 34 *To a $\mathbb{C}$-valued Brownian motion $Z_t = X_t + iY_t$, $t \geq 0$, with $Z_0 = z_0 \neq 0$, i.e. X and Y are two real-valued, independent Brownian motions, we associate the local martingales:*

$$\mathcal{A}_t = \int_0^t (X_s dY_s - Y_s dX_s) \tag{4.7}$$

$$\theta_t = \int_0^t \frac{X_s dY_s - Y_s dX_s}{|Z_s|^2} \tag{4.8}$$

($(\mathcal{A}_t;\ t \geq 0)$ is Lévy's stochastic area process and $(\theta_t;\ t \geq 0)$ the continuous determination of the argument of Z)

a) Prove that the filtrations of these two local martingales are equal.
Show that this common filtration is also the natural filtration of the Brownian motion $(Z_t;\ t \geq 0)$.
b) Prove that the laws of $(\mathcal{A}_t;\ t \geq 0)$ and of $(\theta_t;\ t \geq 0)$ are not extremal in $\mathcal{M}^c$, the set of all continuous local martingale laws.
c) Give some modification of these statements when $z_0 = 0$. [SY80]

Exercise 35 *(Some examples of intertwinings of Markov semigroups) [CPY98] Recall that if $(P_t(x, dx');\ t \geq 0)$ and $(Q_t(y, dy');\ t \geq 0)$ are two Markov semigroups on $(E, \mathcal{E})$, resp. $(F, \mathcal{F})$, then we say that $(Q_t;\ t \geq 0)$ and $(P_t;\ t \geq 0)$ are intertwined via Λ, a Markov kernel: $\Lambda := (\Lambda(y, dx))$, if:*

$$Q_t \Lambda = \Lambda P_t$$

In this exercise, we give a number of examples of such intertwinings.

a) $P_t = (P_t^{(n)})$ is the semigroup of n-dimensional standard Brownian motion $(B_t^{(n)};\ t \geq 0)$; $Q_t = (Q_t^{(n)})$ is the semigroup of n-dimensional Bessel process $(|B_t^{(n)}|;\ t \geq 0)$.
b) $P_t = (Q_t^{(m)})$ and $Q_t = (Q_t^{(n)})$ with $m > n$ (not necessarily integers).

***Hint**: If $(X_t;\ t \geq 0)$ and $(Y_t;\ t \geq 0)$ are two Markov processes with respective semigroups $(P_t;\ t \geq 0)$ and $(Q_t;\ t \geq 0)$, such that moreover*

i) $\mathcal{Y}_t \subset \mathcal{X}_t$
ii) $\mathbb{E}_0\left[f(X_t)|\mathcal{Y}_t\right] = \Lambda f(Y_t)$

for some Markov kernel Λ, prove that:

$$Q_t \Lambda = \Lambda P_t$$

under some additional mild hypothesis...
Check that this holds for the example a) and use a modified argument for b).

Exercise 36 *(Émery martingales; see [Éme89])*
For any $\beta \in [-2, 0)$, define $(X_t;\ t \geq 0)$ the strong Markov process with infinitesimal generator

$$\mathcal{L}^{(\beta)} f(x) = \frac{f(x(1+\beta)) - f(x) - \beta x f'(x)}{(\beta x)^2} = \int_0^1 dv (1-v) f''(x(1+\beta v)) \qquad (4.9)$$

when f is of class $\mathcal{C}^2$ with compact support.
Note that we recover Azéma's martingale (up to a constant multiplicative factor) when $\beta = -1$.

a) Show that $(X_t;\ t \geq 0)$ is a martingale.
b) Show that $(X_t;\ t \geq 0)$ enjoys the CRP.
c) If, moreover $\beta > -1$, show that there exists a Markov kernel M such that the following intertwining relationship holds

$$Q_t M = M P_t$$

where $(Q_t;\ t \geq 0)$ denotes the Markovian semigroup of $(X_t;\ t \geq 0)$, $(P_t;\ t \geq 0)$ the Brownian semigroup.

Comment 4.2 *For any $\beta \in \mathbb{R}$, there exists a unique martingale $(X_t;\ t \geq 0)$ which satisfies the structure equation:*

$$d[X, X]_t = \beta X_{t-} dX_t + dt$$

✠ *It is not known whether this martingale enjoys the CRP for β outside the interval $[-2, 0]$.*

Exercise 37 *(An introduction to the Dunkl martingales, inspired from [GY05c]; see also [GY05b] and [GY05a])*
The Dunkl processes $(X_t^{(k)};\ t \geq 0)$, $k \in \mathbb{R}^+$, are a family of (discontinuous) Markov processes, with infinitesimal generators

$$\mathcal{L}_k :\ f \in \mathcal{C}^2(\mathbb{R}^*) \mapsto \frac{1}{2} f''(x) + k \left(\frac{f'(x)}{x} + \frac{f(-x) - f(x)}{2x^2} \right)$$

a) Check that $(X_t^{(k)};\ t \geq 0)$ and $((X_t^{(k)})^2 - (1+2k)t;\ t \geq 0)$ are martingales.
b) Check that $(|X_t^{(k)}|;\ t \geq 0)$ is a Bessel process with dimension $\delta = (1+2k)$, and that $\Delta X_t^{(k)} = -2 X_{t-}^{(k)} 1_{\Delta X_t^{(k)} \neq 0}$.
c) Prove that $(X_t^{(k)};\ t \geq 0)$ may be decomposed as

$$X_t^{(k)} = \beta_t + \sqrt{2k}\gamma_t, \qquad t \geq 0$$

where γ is a purely discontinuous martingale, with $\langle \gamma \rangle_t = t$ and $(\beta_t;\ t \geq 0)$ is a standard Brownian motion.

d) *Prove, for $k \geq \frac{1}{2}$, the skew product decomposition for $X^{(k)}$, starting from $x > 0$:*

$$X_t^{(k)} = Y_{A_t} = |X_t^{(k)}|(-1)^{\mathcal{N}_{A_t}^{(k/2)}}$$

where: $A_t = \int_0^t \frac{ds}{(X_s^{(k)})^2}$, $Y_u = \exp\left(\beta_u + (k - \frac{1}{2})u\right)(-1)^{\mathcal{N}_u^{(k/2)}}$ and $\mathcal{N}^{(k/2)}$ is a Poisson process with parameter $k/2$, independent from the Brownian motion β.

e) *Prove that $X^{(k)}$ enjoys the chaotic representation property in the sense of Definition 4.3.*

Remark 4.10 *Some multidimensional generalizations of Dunkl processes have been introduced and studied in [GY05c], [Éme05] and [Chy05].*

5

Unveiling the Brownian Path (or history) as the Level Rises

Let $(B_t;\ t \geq 0)$ denote a Brownian motion. We have already discussed some deep properties of its time filtration $(\mathcal{F}_t = \sigma(B_s;\ s \leq t),\ t \geq 0)$. We shall now investigate some filtrations indexed by the space variable (or level). Here are the definitions of two such level-indexed filtrations found in the literature:

Definition 5.1 *The family of σ-fields $\mathcal{E}_{\mathbb{A}}^a = \sigma\{(B_s \wedge a),\ s \geq 0\}$ is a filtration indexed by $a \in \mathbb{R}$.*

This definition is in fact Azéma's counterpart (hence the subscript $\mathbb{A}$), studied in depth in [Hu96], to the following (more involved) definition due to D. Williams hence the subscript $\mathbb{W}$).
For any $a \in \mathbb{R}$, the process $(B_t \wedge a; t \geq 0)$ is a supermartingale which decomposes as :

$$B_t \wedge a = a \wedge 0 + \int_0^i 1_{B_u < a} dB_u - \frac{1}{2} L_t^a \tag{5.1}$$

There exists a Brownian motion $(\beta_s^{(a)};\ s \geq 0)$ such that :

$$\int_0^t 1_{B_u < a} dB_u = \beta^{(a)}_{\int_0^t 1_{B_u < a} du} \tag{5.2}$$

We are now in a position to introduce D. Williams' filtration.

Definition 5.2 *The family $\mathcal{E}_{\mathbb{W}}^a = \sigma\{\beta_u^{(a)};\ u \geq 0\}$ of σ-fields indexed by $a \in \mathbb{R}$ is increasing in a, i.e.: it is a filtration.*

It easily follows from (5.1) and (5.2) that :

$$\mathcal{E}_{\mathbb{W}}^a \vee \sigma \left\{ A_u^{a,-} = \int_0^u ds 1_{B_s < a};\ u \geq 0 \right\} = \mathcal{E}_{\mathbb{A}}^a$$

In particular, $\mathcal{E}_{\mathbb{W}}^a \subset \mathcal{E}_{\mathbb{A}}^a$. Both filtrations are of interest as shown in the following results.

R. Mansuy and M. Yor: *Random Times and Enlargements of Filtrations in a Brownian Setting*, Lect. Notes Math. **1873**, 87–102 (2006)
www.springerlink.com

Theorem 5.1 *[RW91]*
All $(\mathcal{E}^a_{\mathbb{W}},\ a \in \mathbb{R})$*-martingales are continuous and the multiplicity of* $(\mathcal{E}^a_{\mathbb{W}},\ a \in \mathbb{R})$ *is infinite*[1].

Theorem 5.2 *[Hu95][Hu96]*
All $(\mathcal{E}^a_{\mathbb{A}},\ a \in \mathbb{R})$*-martingales are purely discontinuous.*

We begin this chapter with some elementary computations which are useful for our purpose. The two following sections provide some insight into the proofs of Theorem 5.1 and Theorem 5.2 respectively. Section 5.4 consists in a series of examples of immersed and non-immersed filtrations (the notion of immersion was introduced in *Notation and convention*, p.1).

5.1 Above and Under a Given Level

Before we engage in the study of these filtrations, let us fix a level, $a = 0$, say, and look above and under this level.

5.1.1 First Computations in Williams' Framework

With obvious notation, Tanaka's formula yields to

$$B^+_t = -\beta^{(+)}_{A^+_t} + \frac{1}{2}L_t \tag{5.3}$$

$$B^-_t = -\beta^{(-)}_{A^-_t} + \frac{1}{2}L_t \tag{5.4}$$

where $\beta^{(+)}$ and $\beta^{(-)}$ are two independent Brownian motions such that $\mathcal{B}_\infty = \mathcal{B}^{(-)}_\infty \vee \mathcal{B}^{(+)}_\infty$ (this is a consequence of Knight's theorem about continuous orthogonal martingales, applied to: $M^+_t = \int_0^t 1_{B_s>0} dB_s$, and $M^-_t = \int_0^t 1_{B_s<0} dB_s$). We have denoted $A^+_t = \int_0^t ds 1_{B_s>0}$ and $A^-_t = \int_0^t ds 1_{B_s<0}$.

Notation 5.1 *Throughout this paragraph, we shall use the convention that* $\mathcal{Y}^{(+)}$ *denotes the object* $\mathcal{Y}$ *related to* $\beta^{(+)}$*, and similarly for* $\beta^{(-)}$*.*
We may also use the notation $\mathcal{Y}^+$*, for another object, which should be understood without difficulty, via its relation with* B^+*.*

Lévy's Arcsine law for A^+_1 may now be easily recovered:

Theorem 5.3 *([Lév39], [Wil69], [PY92]...)* A^+_1 *follows the Arcsine law*[2]*, as a consequence of the double identity in low:*
for any fixed $l > 0$*, and* $t > 0$*,*

[1] For the definition of multiplicity of a filtration, which is due to Davis-Varaiya [DV74], we refer the reader to Proposition 6.1 in the next Chapter 6. It is proven in Proposition 4.2 of [Wal78b] that the multiplicity of $(\mathcal{E}^a_{\mathbb{W}},\ a \in \mathbb{R})$ is infinite.

[2] The Arcsine law is the law of the variable $\frac{\mathcal{N}'^2}{\mathcal{N}^2+\mathcal{N}'^2}$ where $\mathcal{N}$ and $\mathcal{N}'$ two independent standard normal variables; the density of this law is $\frac{1}{\pi\sqrt{x(1-x)}} 1_{0<x<1}$, see also Exercise 28 and Corollary 3.1.1.

$$A_1^+ \underset{(a)}{\overset{(law)}{=}} \frac{t}{\alpha_t^+} \underset{(b)}{\overset{(law)}{=}} \frac{A_{\tau_l}^+}{\tau_l}$$

with

$$\tau_l = \inf\{u \geq 0,\ L_u > l\} \quad \text{and} \quad \alpha_t^+ = \inf\{u \geq 0, A_u^+ > t\}$$

Remark 5.1 *A multidimensional extension relative to the Brownian spider is obtained in [BPY89b]. See also [Wat95] for a general discussion concerning diffusions.*

Proof

- The identity
$$A_1^+ \overset{(law)}{=} \frac{A_{\tau_l}^+}{\tau_l}$$
easily implies the Arcsine law for A_1^+; indeed, from Tanaka's formula (5.3) and Skorohod's lemma (Lemma 0.6), $A_{\tau_l}^+$ is equal to $T_{l/2}^{(+)}$. With the obvious independence properties, we have:
$$\begin{aligned} A_1^+ &= \frac{T_{l/2}^{(+)}}{T_{l/2}^{(+)} + T_{l/2}^{(-)}} \\ &\overset{(law)}{=} \frac{T_1^{(+)}}{T_1^{(+)} + T_1^{(-)}} \qquad \text{using the scaling property} \\ &\overset{(law)}{=} \frac{\mathcal{N}'^2}{\mathcal{N}^2 + \mathcal{N}'^2} \end{aligned}$$
with $\mathcal{N}$ and $\mathcal{N}'$ two independent normal variables, hence A_1^+ is Arcsine distributed.
- As an intermediate result to obtain (b), we first prove (a).
$$\{A_1^+ > s\} = \{1 > \alpha_s^+\} \overset{(law)}{=} \{1 > s\alpha_1^+\}$$
Therefore, $A_1^+ \overset{(law)}{=} \frac{1}{\alpha_1^+}$ and we deduce (a) by scaling. Now, we prove (b): since $t = A_t^+ + A_t^-$, we get:
$$\alpha_s^+ = s + A_{\alpha_s^+}^- = s + A_{\tau(L_{\alpha_s^+})}^-$$
But from Tanaka's formula (5.4), we obtain
$$\begin{aligned} A_{\tau(2\frac{1}{2}L_{\alpha_s^+})}^- &= T_{\frac{1}{2}L_{\alpha_s^+}}^{(-)} \\ &\overset{(law)}{=} \left(\frac{1}{2}L_{\alpha_s^+}\right)^2 T_1^{(-)} \\ &\overset{(law)}{=} s\frac{T_1^{(-)}}{T_1^{(+)}}, \qquad \text{by scaling} \end{aligned}$$

Hence: $\alpha_s^+ \stackrel{(law)}{=} s\frac{T_1^{(+)}+T_1^{(-)}}{T_1^{(+)}}$.

Finally, $A_1^+ \stackrel{(law)}{=} \frac{1}{\alpha_1^+} \stackrel{(law)}{=} \frac{T_1^{(+)}}{T_1^{(+)}+T_1^{(-)}}$.

■

The arguments used in the proof of Theorem 5.3 allow to understand better the difference between $\mathcal{E}_{\mathbb{A}}^a$ and $\mathcal{E}_{\mathbb{W}}^a$ (see [KS91]).

5.1.2 First Computations in Azéma-Hu's Framework

Now, the attention is being focused on $(\mathcal{E}_{\mathbb{A}}^a,\ a \in \mathbb{R})$.
First we consider the case $a = 0$ and introduce the σ-fields

$$\mathcal{F}_t^+ = \sigma\{B_s^+,\ s \le t\} \qquad \text{and} \qquad \mathcal{F}_t^- = \sigma\{B_s^-,\ s \le t\}$$

These two σ-fields are not independent since, e.g. A_t^+ is measurable with respect to both $\mathcal{F}_t^+$ and $\mathcal{F}_t^-$.
Consider $\mathcal{M}_t = \sigma\{\mu_s;\ s \le t\} = \sigma\{\text{sgn}(B_s),\ s \le t\}$, where $(\mu_t;\ t \ge 0)$ denotes the Azéma martingale (see Subsection 4.3.3).

Proposition 5.4

1. *For every t, conditionally on $\mathcal{M}_t$, the σ-fields $\mathcal{M}_\infty$ and $\mathcal{F}_{\gamma_t} \vee \sigma(sgn(B_t))$ are independent.*
2. *Conditionally on $\mathcal{M}_\infty$, the σ-fields $\mathcal{F}_\infty^+$ and $\mathcal{F}_\infty^-$ are independent.*
3. *For every t, conditionally on $\mathcal{M}_t$, the σ-fields $\mathcal{F}_t^+$ and $\mathcal{F}_t^-$ are independent.*

Proof

1. If $\Phi \in \mathcal{F}_{\gamma_t} \vee \sigma(\text{sgn}(B_t))$, then

$$\begin{aligned} \mathbb{E}[\Phi|\mathcal{M}_\infty] &= \mathbb{E}[\Phi|\mathcal{M}_t, \delta_t] \qquad \text{(strong Markov property at } \delta_t) \\ &= \mathbb{E}[\Phi|\mathcal{M}_t] \end{aligned}$$

since $\sigma(\delta_t)$ and $\mathcal{F}_{\gamma_t} \vee \sigma(\text{sgn}(B_t))$ are independent conditionally on $\mathcal{M}_t$ (from Subsection 3.1.2).

2. Let us denote $S_t^{(+)} = \sup_{s \le t} \beta_s^{(+)}$ and $S_t^{(-)} = \sup_{s \le t} \beta_s^{(-)}$. Then, it is not difficult to show that

$$\mathcal{M}_\infty = \sigma(S^{(+)}, S^{(-)}), \qquad \mathcal{F}_\infty^+ = \sigma(\beta^{(+)}, S^{(-)}), \qquad \mathcal{F}_\infty^- = \sigma(\beta^{(-)}, S^{(+)})$$

Then, for any $\Phi \in L^2(\sigma(\beta^{(+)}))$ and $\Psi \in L^2(\mathcal{M}_\infty)$, we obtain

$$\begin{aligned} \mathbb{E}\left[\Phi\Psi|\mathcal{F}_\infty^-\right] &= \Psi\mathbb{E}\left[\Phi|\beta^{(-)}, S^{(+)}\right] = \Psi\mathbb{E}\left[\Phi|S^{(+)}\right] \\ &= \Psi\mathbb{E}\left[\Phi|S^{(+)}, S^{(-)}\right] = \mathbb{E}\left[\Phi\Psi|S^{(+)}, S^{(-)}\right] = \mathbb{E}[\Phi\Psi|\mathcal{M}_\infty] \end{aligned}$$

Hence the result.

3. Let $(\Phi_t;\ t \geq 0)$ (resp. $(\Psi_t;\ t \geq 0)$) a bounded $(\mathcal{F}_t^+;\ t \geq 0)$-adapted (resp. $(\mathcal{F}_t^-;\ t \geq 0)$-adapted) process.

$$\begin{aligned}
\mathbb{E}\left[\Phi_t \Psi_t 1_{B_t>0}\right] &= \mathbb{E}\left[\Phi_t \Psi_{\gamma_t} 1_{B_t>0}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\Phi_t | \mathcal{M}_\infty\right] \mathbb{E}\left[\Psi_{\gamma_t} 1_{B_t>0} | \mathcal{M}_\infty\right]\right], \qquad \text{from 1.} \\
&= \mathbb{E}\left[\mathbb{E}\left[\Phi_t | \mathcal{M}_\infty\right] \mathbb{E}\left[\Psi_{\gamma_t} | \mathcal{M}_\infty\right] 1_{B_t>0}\right], \qquad \text{from 2.} \\
&= \mathbb{E}\left[\mathbb{E}\left[\Phi_t | \mathcal{M}_t\right] \mathbb{E}\left[\Psi_{\gamma_t} | \mathcal{M}_t\right] 1_{B_t>0}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\Phi_t | \mathcal{M}_t\right] \mathbb{E}\left[\Psi_t | \mathcal{M}_t\right] 1_{B_t>0}\right] \qquad (5.5)
\end{aligned}$$

By symmetry

$$\mathbb{E}\left[\Phi_t \Psi_t 1_{B_t<0}\right] = \mathbb{E}\left[\mathbb{E}\left[\Phi_t | \mathcal{M}_t\right] \mathbb{E}\left[\Psi_t | \mathcal{M}_t\right] 1_{B_t<0}\right] \qquad (5.6)$$

Therefore, (5.5) and (5.6) imply

$$\mathbb{E}\left[\Phi_t \Psi_t\right] = \mathbb{E}\left[\mathbb{E}\left[\Phi_t | \mathcal{M}_t\right] \mathbb{E}\left[\Psi_t | \mathcal{M}_t\right]\right]$$

■

We would like to describe all the $(\mathcal{F}_t^-;\ t \geq 0)$ (resp. $(\mathcal{F}_t^+;\ t \geq 0)$) martingales. In each of these filtrations, there are both continuous and purely discontinuous martingales.

Notation 5.2 *Consider* $M_t^+ = B_t^+ - \frac{1}{2}L_t$ *and* $M_t^- = B_t^- - \frac{1}{2}L_t$.
Remark that, when time-changed respectively with α^+ *and* α^-*, these two processes are independent and that*

$$B_t = M_t^+ - M_t^-$$

Let us compute the projection of B on the filtration $(\mathcal{F}_t^-;\ t \geq 0)$:

$$\begin{aligned}
\mathbb{E}\left[B_t | \mathcal{F}_t^-\right] &= \mathbb{E}\left[M_t^+ | \mathcal{F}_t^-\right] - M_t^- \\
&= \mathbb{E}\left[B_t^+ | \mathcal{F}_t^-\right] - \frac{1}{2}L_t - M_t^- \\
&= \mathbb{E}\left[B_t^+ | \mathcal{M}_t\right] - \frac{1}{2}L_t - M_t^- \qquad \text{from Proposition 5.4} \\
&= 1_{B_t>0}\sqrt{\frac{\pi}{2}}(t-\gamma_t)^{1/2} - \frac{1}{2}L_t - M_t^-, \qquad \text{from formula (4.3)}
\end{aligned}$$

Define

$$Y_t^- := \mathbb{E}\left[B_t | \mathcal{F}_t^-\right] + M_t^- = 1_{B_t>0}\sqrt{\frac{\pi}{2}}(t-\gamma_t)^{1/2} - \frac{1}{2}L_t$$

and $Y_t^+ = \mathbb{E}\left[B_t | \mathcal{F}_t^+\right] + M_t^+$.
Remark that $Y_t^- = \sqrt{\frac{\pi}{2}} \int_0^t 1_{\mu_{s-}>0} d\mu_s$ with $(\mu_t;\ t \geq 0)$ the Azéma martingale and therefore Y^- is purely discontinuous.

Theorem 5.5 *All* $(\mathcal{F}_t^-;\ t \geq 0)$ *martingales may be represented as*

$$\Phi_t = c + \int_0^t \varphi_s dM_s^- + \int_0^t \psi_s dY_s^-$$

with obvious integrability conditions on the $\mathcal{F}^-$*-predictable processes* φ *and* ψ.

Remark 5.2 *A similar result holds for* $\mathcal{F}^+$.

Proof
To prove this representation theorem, it suffices to show that the projection of

$$N_t = \exp\left(\int_0^t f(u)dB_u - \frac{1}{2}\int_0^t f(u)^2 du\right)$$

with $f \in L^2(\mathbb{R}_+)$, onto the filtration $(\mathcal{F}_t^-;\ t \geq 0)$ is a sum of stochastic integrals with respect to Y^- and M^-.
First we decompose $N_t = N_t^+ N_t^-$ with

$$N_t^{\pm} = \exp\left(\int_0^t f(u)1_{B_u \in \mathbb{R}_\pm} dB_u - \frac{1}{2}\int_0^t f(u)^2 1_{B_u \in \mathbb{R}_\pm} du\right)$$

Then

$$\begin{aligned} \mathbb{E}\left[N_t | \mathcal{F}_t^-\right] &= N_t^- \mathbb{E}\left[N_t^+ | \mathcal{F}_t^-\right] \\ &= N_t^- \mathbb{E}\left[N_t^+ | \mathcal{M}_t\right] \\ &= N_t^- \left(1 + \int_0^t n_s d\mu_s\right) \end{aligned}$$

where $(n_s;\ s \geq 0)$ is a $(\mathcal{M}_s;\ s \geq 0)$ predictable process, since the Azéma martingale μ enjoys the PRP.
We would like to prove that $n_s 1_{\mu_{s-}<0} = 0$; this will be a simple consequence of

$$\mathbb{E}\left[\left(\int_0^t n_s 1_{\mu_{s-}<0} d\mu_s\right)^2\right] = 0$$

Indeed

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^t n_s 1_{\mu_{s-}<0} d\mu_s\right)^2\right] &= \mathbb{E}\left[\left(\int_0^t n_s 1_{\mu_{s-}<0} d\mu_s\right)\left(\int_0^t n_s d\mu_s\right)\right] \\ &= \mathbb{E}\left[\int_0^t n_s 1_{\mu_{s-}<0} d\mu_s\, N_t^+\right] = 0, \end{aligned}$$

since $(\int_0^t n_s 1_{\mu_{s-}<0} d\mu_s;\ t \geq 0)$ is purely discontinuous, whereas $(N_t^+,\ t \geq 0)$ is continuous.

■

5.2 Rogers-Walsh Theorem about Williams' Filtration

This section aims at giving some elements of a proof of Theorem 5.1. This result was first stated by D. Williams; the main artisans of the proof of this theorem are Williams [Wil79], Rogers [Rog87], Walsh [Wal78b] [RW91] and McGill [McG86b],[McG84] or [McG86a].
The proofs of this theorem often involve some complicated mathematical object: the key argument of Williams' proof is the CMO formula (see Exercise 42), McGill's proof relies on the decomposition of a supermartingale...
The whole story ends (temporarily?) with a representation theorem of the $(\mathcal{E}^a_{\mathbb{W}},\ a \in \mathbb{R})$-martingales as space-time integrals (in a particular sense) with respect to the local time process[3].
In this section, we develop some of the arguments of [Rog87]. In fact, we construct a family of continuous $(\mathcal{E}^a_{\mathbb{W}},\ a \in \mathbb{R})$-martingales, the result will then follow thanks to a density argument.
Let $(L^x_t;\ x \in \mathbb{R},\ t \geq 0)$ be the (jointly continuous) local time process of the Brownian motion $(B_t;\ t \geq 0)$ and let τ^x denote the right-continuous inverse of L^x, i.e. for any $t \geq 0$, $\tau^x_t = \inf\{u \geq 0,\ L^x_u \geq t\}$.
In the following, we consider $(t_n)_{n\in\mathbb{N}}$ an increasing sequence of real numbers, with the additional assumption that $t_0 = 0$, and $t_n \underset{n\to\infty}{\longrightarrow} \infty$.

5.2.1 Some Space Martingales which are Constant up to a Fixed Level

For any fixed $a \in \mathbb{R}$, let us denote $Z^0_x = L^x_{\tau^a_0} 1_{x\geq a}$, and, for any $k \geq 1$,

$$Z^k_x = (L^x_{\tau^a_{t_k}} - L^x_{\tau^a_{t_{k-1}}})1_{x\geq a} + (t_k - t_{k-1})1_{x<a}$$

These processes will be the "building blocks" of our desired continuous (constant up to the level a) martingales as can be seen in Proposition 5.7 below. The following lemma, which is clearly related to the Ray-Knight theorem for Brownian local times, will play a key role in the proof of Proposition 5.7.

Lemma 5.6 $(Z^0_x - 2(x \wedge 0) + 2(a \wedge 0), Z^k_x,\ k \in \mathbb{N})$ *is a family of continuous, orthogonal* $(\mathcal{E}^x_{\mathbb{W}},\ x \in \mathbb{R})$*-martingales and* $< Z^k >_x = 4\int_a^x Z^k_v dv$.
Moreover, $\mathbb{E}\left[e^{\alpha Z^k_x}\right] < \infty$ *as soon as* $\alpha < \frac{1}{4(x-a)}$.

Proposition 5.7 *For any finite set* $(n_k)_{k\in\mathbb{N}}$ *of positive integers, the family of random variables*

$$(x, y) \mapsto \mathbb{E}\left[\prod_{k\in\mathbb{N}} (Z^k_x)^{n_k} | \mathcal{E}^y_{\mathbb{W}}\right]$$

admits a bi-continuous version.

[3] This result clearly implies Theorem 5.1

Moreover, for any $y \le a$, *the quantity* $\mathbb{E}\left[\prod_{k\in\mathbb{N}}(Z_x^k)^{n_k}|\mathcal{E}_{\mathbb{W}}^y\right]$ *does not depend on* y.

Proof

This proposition shall be proved by induction on the degree $N := \sum_k n_k$.

First we use a particular case of Itô's formula [with respect to the space variable]:

$\prod_{k\in\mathbb{N}}(Z_x^k)^{n_k} - \prod_{k\in\mathbb{N}}(Z_a^k)^{n_k} =$

$$= \sum_k n_k \int_a^x \prod_{l\in\mathbb{N}}(Z_v^l)^{n_l}\frac{dZ_v^k}{Z_v^k} + 2\sum_k n_k(n_k-1)\int_a^x \prod_{l\in\mathbb{N}}(Z_v^l)^{n_l}\frac{dv}{Z_v^k}$$

Using BDG inequalities, we obtain the martingale property of the local martingale part.

Taking expectations, we obtain

$$\mathbb{E}\left[\prod_{k\in\mathbb{N}}(Z_x^k)^{n_k}|\mathcal{E}_{\mathbb{W}}^y\right] = \prod_{k\in\mathbb{N}}(Z_a^k)^{n_k} + \sum_k n_k \int_a^{x\wedge y}\prod_{l\in\mathbb{N}}(Z_v^l)^{n_l}\frac{dZ_v^k}{Z_v^k}$$
$$+2\sum_k n_k(n_k-1)\int_a^x \mathbb{E}\left[\frac{\prod_{l\in\mathbb{N}}(Z_v^l)^{n_l}}{Z_v^k}|\mathcal{E}_{\mathbb{W}}^y\right]dv,$$

Then, thanks to the induction argument, we deduce the desired continuity property over $[a,\infty)\times\mathbb{R}$ and therefore on $\mathbb{R}^2$ since these martingales are constant up to a.

The second point may also be proved by induction in a similar way.

■

5.2.2 A Dense Family of Continuous Martingales

This result for a given fixed level a should now be extended to the whole of $\mathbb{R}$. We shall now consider a fixed partition $(a_j)_{j\in\mathbb{Z}}$ and then define $Z_{k,j} = L^{a_{j+1}}_{\tau_{t_k}^{a_j}} - L^{a_{j+1}}_{\tau_{t_{k-1}}^{a_j}}$ and $Z_{0,j} = L^{a_{j+1}}_{\tau_0^{a_j}}$.

Theorem 5.8 *For any sequence* $(n_{j,k})_{j,k\in\mathbb{N}}$ *of positive integers, all but finitely many of which are zero, the martingale* $(\mathbb{E}\left[\prod_{j\in\mathbb{Z}}\prod_{k\in\mathbb{N}}(Z_{j,k})^{n_{j,k}}|\mathcal{E}_{\mathbb{W}}^y\right],\ y\in\mathbb{R})$ *admits a continuous version.*

Proof
This result is a particular consequence of Proposition 5.7. Indeed, let us denote $Y_j = \prod_{k\in\mathbb{N}} (Z_{j,k})^{n_{j,k}}$.
For any $m \in \mathbb{Z}$, if $y \in [a_m, a_{m+1}]$, then we have

$$\begin{aligned}\mathbb{E}\left[\prod_{j\in\mathbb{Z}} Y_j|\mathcal{E}_{\mathbb{W}}^y\right] &= \left(\prod_{j<m} Y_j\right)\mathbb{E}\left[\prod_{j\geq m} Y_j|\mathcal{E}_{\mathbb{W}}^y\right]\\ &= \left(\prod_{j<m} Y_j\right)\left(\prod_{j>m+1}\mathbb{E}\left[Y_j|\mathcal{E}_{\mathbb{W}}^{a_j}\right]\right)\mathbb{E}\left[Y_m|\mathcal{E}_{\mathbb{W}}^y\right]\end{aligned}$$

where, in the last equality, we use that $\mathbb{E}\left[Y_j|\mathcal{E}_{\mathbb{W}}^{a_j}\right]$ is a deterministic constant (see Proposition 5.7).
Now, the continuity on $[a_m, a_{m+1}]$ (hence on $\mathbb{R}$) follows simply.

■

Let us introduce $\mathcal{H} = \left\{Y \in L^2;\ (\mathbb{E}[Y|\mathcal{E}_{\mathbb{W}}^x];\ x \in \mathbb{R}) \text{ is continuous}\right\}$ and $\mathcal{A}_n = \sigma\left(L_{\tau_t^{j2^{-n}}}^{(j+1)2^{-n}};\ t \geq 0\right)$. Rogers [Rog87] concludes the proof of Theorem 5.1 by showing that $b(\mathcal{F}_\infty) \subset \mathcal{H}$, where $b(\mathcal{F}_\infty)$ denotes the space of all bounded $\mathcal{F}_\infty$-measurable variables. The key steps of this last point are the following:

* All $\mathcal{A}_n$-measurable, square integrable variables are in $\mathcal{H}$. This key-statement relies on the following properties: $\mathcal{A}_n$ is a family of positive variables, with exponential moments and the algebra generated by these variables in contained in $\mathcal{H}$.
* $(\mathcal{A}_n;\ n \in \mathbb{N})$ is an increasing sequence of σ-fields and $\bigvee_{n\in\mathbb{N}} \mathcal{A}_n \subset \mathcal{H}$
* $\mathcal{F}_\infty = \bigvee_{n\in\mathbb{N}} \mathcal{A}_n$

5.3 A Discussion Relative to $(\mathcal{E}_{\mathbb{A}}^a,\ a \in \mathbb{R})$

The results found in this section are due to [Hu96].

5.3.1 Hu's Result about $(\mathcal{E}_{\mathbb{A}}^a,\ a \in \mathbb{R})$-Martingales

In this subsection, we study a dense family of purely discontinuous martingales with respect to $(\mathcal{E}_{\mathbb{A}}^a,\ a \in \mathbb{R})$. More precisely, we are interested in

$$\left(\mathbb{E}\left[\prod_{i=1}^n f_i(B_{t_i})|\mathcal{E}_{\mathbb{A}}^a\right],\ a \geq 0\right)$$

for generic bounded functions f_i, and we shall prove that these martingales are purely discontinuous.

The following proposition focuses on the particular case $n = 1$.

Proposition 5.9 *For a generic bounded Borel function $f : \mathbb{R} \to \mathbb{R}$,*

$$\mathbb{E}\left[f(B_t)|\mathcal{E}^a_{\mathbb{A}}\right] = f(B_t)1_{B_t<a} + 1_{B_t>a}\sqrt{\frac{2}{\pi}}\int_a^{\infty} dx\, f(x)\frac{(x-a)^2}{(\eta^a_t)^3}e^{-\frac{(x-a)^2}{2(\eta^a_t)^2}} \quad (5.7)$$

with $\eta^a_t = 1_{B_t>a}\left(\frac{(t-\gamma^a_t)(\delta^a_t-t)}{\delta^a_t-\gamma^a_t}\right)^{1/2}$ and γ^a_t (resp. δ^a_t) the last (resp. first) time before (resp. after) t when B reaches the level a.

It may be worth noticing that this result is closely related to the normalization seen in the preceding Chapter 4: $r^{(t)}_u = \frac{1}{\sqrt{\delta_t-\gamma_t}}|B_{\gamma_t+u(\delta_t-\gamma_t)}|$ which defines the standardized excursion. Moreover $r^{(t)}$ is independent of

$$\mathcal{F}_{\gamma_t} \vee \sigma(\mathrm{sgn}(B_t)) \vee \sigma(\delta_t,\ B_{\delta_t+u},\ u \geq 0)$$

Proof
It suffices to prove formula (5.7) for $a = 0$.
Moreover, since t is a fixed time, we shall write r instead of $r^{(t)}$.
Now, for every bounded Borel f, one has:

$$\begin{aligned}\mathbb{E}\left[f(B_t)|\mathcal{E}^0_{\mathbb{A}}\right] &= f(B_t)1_{B_t<0} + \mathbb{E}\left[f(B_t)1_{B_t>0}|\mathcal{E}^0_{\mathbb{A}}\right] \\ &= f(B_t)1_{B_t<0} + \mathbb{E}\left[f(\sqrt{\delta_t-\gamma_t}\,r_{\frac{t-\gamma_t}{\delta_t-\gamma_t}})1_{B_t>0}|\mathcal{E}^0_{\mathbb{A}}\right] \\ &= f(B_t)1_{B_t<0} + \hat{\mathbb{E}}\left[f(\sqrt{\delta_t-\gamma_t}\,\hat{r}_{\frac{t-\gamma_t}{\delta_t-\gamma_t}})\right]1_{B_t>0},\end{aligned}$$

since r is independent of $\mathcal{E}^0_{\mathbb{A}}$.
Time-changing and using the distribution of $\hat{r}_u = (1-u)R_{u/(1-u)}$, where R is a 3-dimensional Bessel process, yields to the stated result.

■

Theorem 5.10 *For generic bounded functions $(f_i,\ i = 1, ..., n)$, the $(\mathcal{E}^a_{\mathbb{A}},\ a \in \mathbb{R})$-martingale $(\Phi_n(a) := \mathbb{E}\left[\prod_{i=1}^n f_i(B_{t_i})|\mathcal{E}^a_{\mathbb{A}}\right],\ a \in \mathbb{R})$ has bounded variation, and therefore is purely discontinuous.*

This result turns out to be more or less a consequence of the following explicit computations of finite-dimensional marginals of 3-dimensional Bessel bridges.

Lemma 5.11 *Let $(r_t;\ t \leq 1)$ be a standard 3-dimensional Bessel bridge. Then, for $0 = t_0 < t_1 < ... < t_n$, and any bounded Borel function φ_j, one has:*

$$\mathbb{E}\left[\prod_{j=1}^n \varphi_j(r_{t_j})\right] = \int_{(\mathbb{R}^3)^n} dx_1 \cdots dx_n \prod_{j=1}^n \left\{\varphi_j\left((1-t_j)||\sum_{i=1}^j a_i x_i||\right)\frac{e^{-\frac{||x_j||^2}{2}}}{(2\pi)^{3/2}}\right\}$$

with $||.||$ the Euclidean norm in $\mathbb{R}^3$ and $a_i = \left(\frac{t_i-t_{i-1}}{(1-t_i)(1-t_{i-1})}\right)^{1/2}$

Proof
This follows, as before, from the representation $r_t = (1-t)||\mathbb{B}_{\frac{t}{1-t}}||$ with $(\mathbb{B}_t;\ t \geq 0)$ a 3-dimensional Brownian motion.

■

Proof of Theorem 5.10
We shall show that the variation $V(\Phi_n, b)$ of the process $(\Phi_n(a);\ a \leq b)$ satisfies

$$V(\Phi_n, b) \leq c(n)(b+1)$$

for a constant $c(n)$ depending only on n ($f_1, ..., f_n$ and $t_1, ..., t_n$ being fixed). Using the equality

$$1_{B_{t_n}>a, t_{j-1}<\gamma_{t_n}^a<t_j} \Phi_n(a) = 1_{B_{t_n}>a, t_{j-1}<\gamma_{t_n}^a<t_j} \Phi_{j-1}(a) \mathbb{E}\left[\prod_{i=j}^{n} f_i(B_{t_i}) | \mathcal{E}_{\mathbb{A}}^a\right]$$

we deduce the following recurrence property

$$\begin{aligned}\Phi_n(a) &= f_n(B_{t_n}) 1_{B_{t_n}\leq a} \Phi_{n-1}(a) + 1_{B_{t_n}>a} \Phi_n(a) \\ &= f_n(B_{t_n}) 1_{B_{t_n}\leq a} \Phi_{n-1}(a) + 1_{B_{t_n}>a} \sum_{j=1}^{n} \Phi_{j-1}(a) H_j(a)\end{aligned}$$

with $H_j(a) = 1_{B_{t_n}>a, t_{j-1}<\gamma_{t_n}^a<t_j} \mathbb{E}\left[\prod_{k=j}^{n} f_k(B_{t_k}) | \mathcal{E}_{\mathbb{A}}^a\right]$.
Now, it suffices, thanks to the recurrence argument to show that each of the processes H_j has bounded variation on the finite interval $[0, b]$, which is a consequence of Lemma 5.11.

■

5.3.2 Some Markov Processes with Respect to $(\mathcal{E}_{\mathbb{A}}^a,\ a \in \mathbb{R})$

It may be worth underlying that Proposition 5.9 also provides us with a family of Markov processes with respect to $(\mathcal{E}_{\mathbb{A}}^a,\ a \in \mathbb{R})$, as we now show.

Theorem 5.12 *For any fixed $t > 0$, $(\eta_t^a;\ a \in \mathbb{R})$ is an homogeneous $(\mathcal{E}_{\mathbb{A}}^a,\ a \in \mathbb{R})$ Markov process whose semigroup is independent of t and is given by*

$$P_b(x, dy) = \sqrt{\frac{2}{\pi}} \frac{1}{x} \int_0^1 dz \left(e^{-z^2/(2x^2)} - e^{-1/(2x^2)}\right) \varepsilon_0(dy) + p\left(\frac{x}{b}, \frac{y}{b}\right) \frac{dy}{x} 1_{\{0<y\leq x\}}$$

where $p(x,y) = \sqrt{\frac{2}{\pi}} \frac{1}{x} \left[\frac{(2x^2-y^2)y}{(x^2-y^2)^{3/2}} e^{-1/(2(x^2-y^2))} + \frac{1}{x^2} \int_{y/\sqrt{x^2-y^2}}^{\infty} dz \exp\left(-\frac{1+z^2}{2x^2}\right)\right]$
The infinitesimal generator $\mathcal{A}$ associated to this Markov process contains every $\mathcal{C}_c^1$ function $f : \mathbb{R}_+ \to \mathbb{R}$ in its domain and satisfies:

- $\mathcal{A}f(0) = 0$
- $\mathcal{A}f(z) = \sqrt{\frac{2}{\pi}} \int_0^1 dx \frac{(2-x^2)x}{(1-x^2)^{3/2}} \frac{f(xz)-f(z)}{z} = -\sqrt{\frac{2}{\pi}} \int_0^1 dx f'(xz) \frac{x^2}{\sqrt{(1-x^2)}}$

Proof

- Since $(\eta_t^a,\ a \in \mathbb{R})$ is uniformly bounded, to prove the Markov property of $(\eta_t^a;\ a \in \mathbb{R})$, it suffices to show that for any $k \in \mathbb{N}^*$, there exists a Borel function $\phi_k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\text{for } 0 \le a \le b, \qquad \mathbb{E}\left[(\eta_t^b)^k | \mathcal{E}_{\mathbb{A}}^a\right] = \phi_k(b-a, \eta_t^a) \tag{5.8}$$

First, Proposition 5.9 implies that, for each $k \in \mathbb{N}^*$, the process:

$$(\eta_t^a)^k + \frac{kc_{k+1}}{c_{k+2}} \int_0^{a \wedge B_t^+} db (\eta_t^b)^{k-1}$$

is a $(\mathcal{E}_{\mathbb{A}}^a;\ a \in \mathbb{R})$-martingale, with $c_n = \sqrt{\frac{2}{\pi}} 2^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right)$.
Then, the result follows for a simple recurrence argument. Indeed, if we assume that (5.8) holds for $k = n-1$, we obtain

$$\begin{aligned}\mathbb{E}\left[(\eta_t^b)^n | \mathcal{E}_{\mathbb{A}}^a\right] &= (\eta_t^a)^n - \frac{nc_{n+1}}{c_{n+2}} \int_0^b dx \mathbb{E}\left[(\eta_t^x)^{n-1} | \mathcal{E}_{\mathbb{A}}^a\right] \\ &= (\eta_t^a)^n - \frac{nc_{n+1}}{c_{n+2}} \int_0^{b-a} dy \phi_{n-1}(y, \eta_t^a)\end{aligned}$$

hence, ϕ_n exists, and is given by $\phi_n(b,z) = z^n - \frac{nc_{n+1}}{c_{n+2}} \int_0^b dy \phi_{n-1}(y,z)$.
- In order to find the corresponding semigroup, we introduce the function $\Phi_x(y) = \frac{x^2}{y^3} e^{-\frac{x^2}{2y^2}}$.
It follows from Proposition 5.9 that, for $b \le a \le x$,:

$$\mathbb{E}\left[\Phi_{x-a}(\eta_t^a) | \mathcal{E}_{\mathbb{A}}^b\right] = \Phi_{x-b}(\eta_t^b)$$

Then

$$\mathbb{E}\left[\Phi_y(\eta_t^a) | \mathcal{E}_{\mathbb{A}}^b\right] = \Phi_{y+a-b}(\eta_t^b)$$

that is

$$\int_0^\infty \frac{y^2}{x^3} \exp\left(-\frac{y^2}{2x^2}\right) P_b(z, dx) = \frac{(y+b)^2}{z^3} \exp\left(-\frac{(y+b)^2}{2z^2}\right)$$

The explicit formula for $P_b(x, dy)$ is now deduced by inverting this Laplace transform.
- The expression for $\mathcal{A}$ follows easily from the explicit expression of P_b.

■

5.4 Some Subfiltrations of the Brownian Filtration

In the following section, the symbol $\hookrightarrow$ between two filtrations indicates **immersion**, i.e. the small filtration is included in the big one and all martingales with respect to the small filtration remain martingales with respect to the big one (this notion is also known in the literature as the (**H**) hypothesis; see Brémaud-Yor [BY78] for the beginning of the story).

Exercise 38

a) Show that a filtration $\mathcal{A}$ is immersed in the filtration $\mathcal{B}$ if and only if $\mathcal{A}_\infty$ is conditionally independent of $\mathcal{B}_t$ knowing $\mathcal{A}_t$.
b) Now, we only assume that $\mathcal{A}_t \subset \mathcal{B}_t$, for any $t \geq 0$.
Show that, if an $(\mathcal{A}_t;\ t \geq 0)$-martingale $(a_t;\ t \geq 0)$ is also a $(\mathcal{B}_t;\ t \geq 0)$-martingale, and $(a_t;\ t \geq 0)$ enjoys the PRP with respect to $(\mathcal{A}_t;\ t \geq 0)$, then $(\mathcal{A}_t;\ t \geq 0)$ is immersed in $(\mathcal{B}_t;\ t \geq 0)$.

Example 5.1 *Let $\mathcal{M}$ denote the natural filtration of the Azéma martingale. Then*

$$\mathcal{M}_t \hookrightarrow \mathcal{G}_t = \mathcal{F}_{\gamma_t} \vee \sigma(sgn(B_t))$$

Indeed, μ_t is obtained as the projection of B_t on $\mathcal{G}_t$, thus is a $(\mathcal{G}_t;\ t \geq 0)$ martingale and $(\mu_t;\ t \geq 0)$ enjoys the CRP, hence the PRP with respect to $(\mathcal{M}_t;\ t \geq 0)$. As a consequence, $(\mathcal{M}_t;\ t \geq 0)$ is immersed in $(\mathcal{G}_t;\ t \geq 0)$.

The following exercise provides a list of classical examples of immersed or non-immersed pairs of filtrations:

Exercise 39 *Show that:*

a) $\sigma(|B_s|,\ s \leq t) = \sigma(\int_0^s sgn(B_u)dB_u;\ s \leq t) \hookrightarrow \mathcal{F}_t$
b) More generally, if $\mathcal{F}^{(n)}$ denotes the natural filtration of a n-dimensional Brownian motion and $\mathcal{R}$ the natural filtration of its norm (i.e. of the associated n-dimensional Bessel process), then

$$\mathcal{R}_t \hookrightarrow \mathcal{F}_t^{(n)}$$

c) $\mathcal{F}_t^+ \not\hookrightarrow \mathcal{F}_t$

Example 5.2 *Consider the filtration of Brownian bridges*

$$\forall t > 0, \qquad \daleth_t = \sigma(B_s - \frac{s}{t}B_t,\ s \leq t) \qquad \textit{(hence, the name of the filtration)}$$
$$= \sigma(\frac{B_s}{s} - \frac{B_t}{t},\ s \leq t)$$

We have already remarked that $\daleth_t \subsetneq \mathcal{F}_t$ since, for any $t \geq 0$, B_t is independent of $\daleth_t$.
Nevertheless $(\daleth_t;\ t \geq 0)$ is the natural filtration of another Brownian motion

$B^{\#}$ *(which already appeared in Exercise 21, and was denotes there as* $B^{0,-}$*) defined by*

$$B_t^{\#} = B_t - \int_0^t \frac{ds}{s} B_s = \int_0^t ds \left(\frac{B_t}{t} - \frac{B_s}{s} \right)$$

but $B^{\#}$ *is not a martingale for* $(\mathcal{F}_t;\ t \geq 0)$.
Therefore,

$$\Pi_t \not\hookrightarrow \mathcal{F}_t$$

With these various examples at hand, we now define the notion of canonical and non-canonical decomposition of a continuous semimartingale.

Definition 5.3 *The canonical decomposition of a continuous process* $(X_t;\ t \geq 0)$ *is its Doob-Meyer decomposition in its natural filtration* $(\mathcal{X}_t;\ t \geq 0)$*, assuming that* $(X_t;\ t \geq 0)$ *is a semimartingale with respect to* $(\mathcal{X}_t;\ t \geq 0)$.
We shall call non-canonical decomposition of $(X_t;\ t \geq 0)$ *any Doob-Meyer decomposition of* $(X_t;\ t \geq 0)$*, in any superfiltration of* $(\mathcal{X}_t;\ t \geq 0)$ *with respect to which* $(X_t;\ t \geq 0)$ *is a semimartingale.*

The following statement clarifies the relationships between two Doob-Meyer decompositions of a semimartingale $(X_t;\ t \geq 0)$, with respect to two filtrations.

Proposition 5.13 *Let* $X_t = M_t^{\mathcal{G}} + A_t^{\mathcal{G}} = M_t^{\mathcal{F}} + A_t^{\mathcal{F}}$ *be two (non-canonical) decompositions of* $(X_t;\ t \geq 0)$ *relative to* $(\mathcal{G}_t;\ t \geq 0)$ *and* $(\mathcal{F}_t;\ t \geq 0)$*, respectively, with* $\mathcal{G}_t \subseteq \mathcal{F}_t$.
Assume that $\mathbb{E}\left[\int_0^t |dA_s^{\mathcal{F}}|\right] < \infty$ *for any* $t > 0$*, and that* $(M_t^{\mathcal{F}},\ t \geq 0)$ *is a* $(\mathcal{F}_t;\ t \geq 0)$*-martingale.*
Then,

$$A^{\mathcal{G}} = (A^{\mathcal{F}})^{(p)},$$

where $(\Sigma_t^{(p)};\ t \geq 0)$ *denotes the predictable compensator (see Definition 0.1) of* $(\Sigma_t;\ t \geq 0)$*, with respect to* $(\mathcal{G}_t;\ t \geq 0)$.
Consequently,

$$M_t^{\mathcal{G}} = M_t^{\mathcal{F}} + (A_t^{\mathcal{F}} - (A^{\mathcal{F}})_t^{(p)})$$

In particular, if $A^{\mathcal{F}} \neq A^{\mathcal{G}}$*, then* $(\mathcal{G}_t;\ t \geq 0)$ *is not immersed in* $(\mathcal{F}_t;\ t \geq 0)$.

Comment 5.1 *In the vicinity of this chapter, we should mention a recent article by Picard [Pic05a] which brings a new light on stochastic calculus. Indeed, Picard provides a unified construction of stochastic integrals with respect to predictable or anticipative processes by studying simultaneously all excursions at all levels. This point of view also revisits some aspects of Malliavin calculus connected with the Clark-Ocone formula.*

5.5 Exercises

Exercise 40 *(Two examples of non-canonical decompositions of semimartingales)*

a) Let $(B_t^{(\mu)} := B_t + \mu t;\ t \geq 0)$ *be a Brownian motion with drift* μ*.*

1. *Show that* $(|B_t^{(\mu)}|,\ t \geq 0)$ *decomposes in the natural filtration of* B *as:*

$$|B_t^{(\mu)}| = \int_0^t sgn(B_s^{(\mu)}) dB_s + \mu \int_0^t sgn(B_s^{(\mu)}) ds + L_t^0(B^{(\mu)}) \quad (5.9)$$

2. *Show that there exists a Brownian motion* $(\beta_t;\ t \geq 0)$ *adapted to the natural filtration of* $|B^{(\mu)}|$*, such that*

$$|B_t^{(\mu)}| = \beta_t + \mu \int_0^t \tanh(\mu |B_s^{(\mu)}|) ds + L_t^0(B^{(\mu)}) \quad (5.10)$$

Prove that, in fact, $\sigma(|B_s^{(\mu)}|,\ s \leq t) = \sigma(\beta_s;\ s \leq t)$*.*

3. *Deduce that the filtration of* $|B^{(\mu)}|$ *is not immersed in* $\mathcal{F}_t$*, the natural filtration of* B *(hence of* $B^{(\mu)}$*).*

b) $B^{(\mu)}$ *still denotes a Brownian motion with drift* μ*. Let* $S_t^{(\mu)} := \sup_{s \leq t} B_s^{(\mu)}$ *be its one-sided supremum and define* $R^{(\mu)} = 2S^{(\mu)} - B^{(\mu)}$*.*
Recall that $R^{(0)}$ *is a 3-dimensional Bessel process (Pitman's theorem). In fact, an underlying essential result, which may be a useful tool for the following question, is that, for any fixed* $t > 0$*, the conditional law of* B_t *knowing* $\sigma(R_s^{(0)};\ s \leq t)$ *and* $R_t^{(0)} = r$ *is the uniform distribution on the interval* $[-r, r]$*.*

1. *Show that there exists a Brownian motion* $(\gamma_t;\ t \geq 0)$*, adapted to the natural filtration of* $R^{(\mu)}$*, such that*

$$R_t^{(\mu)} = \gamma_t + \mu \int_0^t \coth(\mu R_s^{(\mu)}) ds \quad (5.11)$$

2. *Deduce that the natural filtration of* $R^{(\mu)}$ *is not immersed in* $\mathcal{F}$*.*
3. *Give the conditional law of* $B_t^{(\mu)}$ *knowing* $\sigma(R_s^{(\mu)};\ s \leq t)$ *and* $R_t^{(\mu)} = r$*.*

Comment 5.2 *In fact, many extensions of Pitman's theorem, among which those in [ST90], may be recovered thanks to enlargements of filtrations; See [RVY05b] for a number of examples.*

Exercise 41 *(Arcsine law and switching identity)*

a) *Using the path decomposition for Brownian motion* $(B_u;\ u \leq 1)$ *at time* $\gamma = \gamma_1$, *deduce from the fact that* $A_1^+ = \int_0^1 du 1_{B_u>0}$ *is Arcsine distributed, that* $a_1^+ := \int_0^1 du 1_{b_u>0}$, *where* b *is a standard Brownian bridge, is uniformly distributed on* $(0,1)$.
Hint*: one can compute the Gauss transform of* a_1^+, *namely the law of* $\sqrt{a_1^+}\mathcal{N}$,*where* $\mathcal{N}$ *is a standard normal variable.*

b) *Give another proof by using the* ***switching identity****:*

$$\forall F \geq 0, \qquad \mathbb{E}\left[F(B_u;\ u \leq \tau_l)|\tau_l = t\right] = \mathbb{E}\left[F(B_u;\ u \leq t)|B_t = 0, L_t = l\right]$$

the fact $A_{\tau_l}^+$ *and* $A_{\tau_l}^-$ *are iid as well as the law of the total local time of a standard Brownian bridge*[4], *namely*

$$P(L_t \in dl|B_t = 0) = \frac{l}{t}e^{-\frac{l^2}{2t}}dl \qquad (5.12)$$

c) *Prove the switching identity.*
Hint*: Consider*

$$\mathbb{E}\left[\int_0^\infty dL_t F(B_u;\ u \leq t)\varphi(L_t)\psi(t)\right]$$

for a generic functional F *and Borel functions* φ *and* ψ*:* $\mathbb{R}^+ \to \mathbb{R}^+$.

Exercise 42 *Consider a standard Brownian motion* $(B_t;\ t \geq 0)$ *and* $x \in \mathbb{R}$. *Introduce the times spent below and above the level* x

$$A_t^{x,-} = \int_0^t du 1_{B_u<x} \qquad A_t^{x,+} = \int_0^t du 1_{B_u \geq x}$$

Denote by $(L_t^x;\ t \geq 0)$ *the local time process of* $(B_t;\ t \geq 0)$ *at level* x *and* $(\tau_u^x;\ u \geq 0)$ *its right-continuous inverse.*

a) *Show that* $(A_{\tau_{2u}^x}^{x,+};\ u \geq 0)$ *is a stable (1/2) process independent of* $\mathcal{E}_{\mathbb{W}}^x$.
b) *If* ξ *denotes an independent exponential time with parameter* $\lambda > 0$, *show that, for any bounded Borel function* f,

$$\mathbb{E}\left[f(B_\xi)1_{B_\xi<x}|\mathcal{E}_{\mathbb{W}}^x\right] = \lambda \int_0^\infty dt 1_{B_t<x} f(B_t) e^{-\lambda A_t^{x,-} - \sqrt{\frac{\lambda}{2}} L_t^x} \qquad (5.13)$$

Exercise 43 ✠ *Can you find a Markov process* $\{\Pi^a, a \in \mathbb{R}\}$ *which generates* $\{\mathcal{E}_{\mathbb{A}}^a\}$*?*

[4] This law may be obtained either as a consequence of the **switching identity** which will be shown in the following question or as a simple application of the Beta-Gamma algebra together with the representation of Brownian bridge as a Brownian motion rescaled up to time γ.

6

Weak and Strong Brownian Filtrations

This last chapter is concerned with the following questions: can one discuss the nature of "a generic filtration" $(\mathcal{F}_t;\ t \geq 0)$ on a probability space $(\Omega, \mathcal{F}, P)$? Can one find some fundamental invariants which ensure that $(\mathcal{F}_t;\ t \geq 0)$ is generated by certain fundamental processes, such as Brownian motions and Poisson processes?
A first step towards such a characterization is the following result of Davis and Varaiya.

Proposition 6.1 *[DV74]*
There exists an at most countable set of square integrable non-zero $(\mathcal{F}_t;\ t \geq 0)$-martingales $(M^i)_{i\in I}$ such that

1. *The martingales $(M^i)_{i\in I}$ are orthogonal.*
2. *For any $i \in I$, $d < M^i > \succ d < M^{i+1} >$, where $\succ$ indicates absolute continuity between the two random measures involved.*
3. *Any square integrable $(\mathcal{F}_t;\ t \geq 0)$-martingale $(M_t;\ t \geq 0)$, with $M_0 = 0$, can be represented as a sum (or series) of stochastic integrals with respect to $(M^i)_{i\in I}$ with predictable integrands, i.e. $M_\cdot = \sum_{i\in I} \int_0^\cdot H_i(s) dM^i_s$.*

Moreover, if we consider two sets $(M^i)_{i\in I}$ and $(N^j)_{j\in J}$ which satisfy conditions 1.-3., then $card(I) = card(J)$; Davis and Varaiya [DV74] called this common value the multiplicity of the filtration.

Our working hypothesis in this chapter is

Assumption 6.1 *$card(I) = 1$ and M^1 is a one-dimensional Brownian motion.*

Then two situations are possible:

- either $(\mathcal{F}_t;\ t \geq 0)$ is the natural filtration of M^1, then the filtration is called a strong Brownian filtration (in short, SBF)

R. Mansuy and M. Yor: *Random Times and Enlargements of Filtrations in a Brownian Setting*, Lect. Notes Math. **1873**, 103–116 (2006)
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- or, $(\mathcal{F}_t;\ t \geq 0)$ is larger than the natural filtration of any martingale M^1 which may be featured in Assumption 6.1; the filtration is then said to be a weak Brownian filtration (in short, WBF).

Section 6.1 gives some more precisions about the notions of strong and weak Brownian filtrations. In Section 6.2, we present some striking examples of weak Brownian filtrations. Finally, Section 6.3 provides some elements of the theory of invariants of filtrations, which shed some general light on the previous examples.

6.1 Definitions

We now give some more details and provide some examples about the two notions we just presented.

Definition 6.1 *A Strong Brownian Filtration (SBF) is a filtration which is generated by a standard Brownian motion*

Be aware that the filtration (strongly) generated by a Brownian motion β is also generated by any of the following Brownian motions

$$(\beta_t^{(s)} = \int_0^t s(u)\, d\beta_u;\ t \geq 0),\ \text{where } s : \mathbb{R}_+ \to \{\pm 1\} \text{ is deterministic}$$

More generally, we might consider ε, any $(\mathcal{F}_t;\ t \geq 0)$ predictable process taking values in $\{\pm 1\}$ and associate

$$(\beta_t^{(\varepsilon)} = \textstyle\int_0^t \varepsilon(u)\, d\beta_u;\ t \geq 0) \tag{6.1}$$

Then the natural filtration of $\beta^{(\varepsilon)}$ satisfies (recall the immersion symbol $\hookrightarrow$ introduced in Section 5.4)

$$\mathcal{F}(\beta^{(\varepsilon)}) \hookrightarrow \mathcal{F}(\beta)$$

But the inclusion may be strict. For example, with $\varepsilon_s = \text{sgn}(\beta_s)$, the natural filtration of $\beta^{(\varepsilon)}$ is the filtration of $|\beta|$ according to Tanaka formula. Hence, for a given t, $\text{sgn}(\beta_t)$ is independent of $\mathcal{F}_t(\beta^{(\varepsilon)})$ and in fact of $\mathcal{F}_\infty(\beta^{(\varepsilon)})$.
Nonetheless, in the generality of the set-up of (6.1), every $(\mathcal{F}_t;\ t \geq 0)$ martingale may be written

$$M_t = c + \int_0^t m_s^{(\varepsilon)} d\beta_s^{(\varepsilon)} = c + \int_0^t m_s^{(\varepsilon)} \varepsilon(s) d\beta_s$$

for some $(\mathcal{F}_s;\ s \geq 0)$-predictable process $(m_s^{(\varepsilon)};\ s \geq 0)$.
Indeed, since, by Itô's representation: $M_t = c + \int_0^t m_s d\beta_s$, we obtain $m_s^{(\varepsilon)} = m_s \varepsilon(s)$. This simple remark shall help us to formulate the next Definition 6.2.

Exercise 44 ✠ *Remark that Lévy's transform* $\beta \mapsto \int_0^{\cdot} sgn(\beta_s) d\beta_s$ *preserves Wiener measure*[1]*; is this transformation ergodic? more generally, for which* $\mathcal{F}(\beta)$*-predictable processes* ε *taking values in* $\{\pm 1\}$*, is the transformation* $\beta \mapsto \beta^{(\varepsilon)}$ *ergodic?*
This seems to be an extremely difficult question, the solution to which has escaped so far both Brownian motion and ergodic theory experts.

Comment 6.1 *The previous open question may be compared with the following easier study. The transform* $T : \beta \mapsto \beta - \int_0^{\cdot} \frac{ds}{s} \beta_s$ *preserves Wiener measure and is ergodic.*
Indeed, $(\beta_1, T(\beta)_1, \ldots, T^n(\beta)_1, \ldots)$ *is a sequence of orthogonal Gaussian variables with*

$$T^n(\beta)_1 = \int_0^1 d\beta_s L_n(\log \frac{1}{s})$$

where L_n *is the* n*-th Laguerre polynomial (the Laguerre polynomials are a total family of orthonormal polynomials with respect to the measure* $e^{-x}dx$*). For details about* T*, see Jeulin-Yor [JY90] or [Yor92a] chapter 1.*

Definition 6.2 $(\mathcal{F}_t;\ t \geq 0)$ *is a Weak Brownian Filtration (WBF) if there exists a* $(\mathcal{F}_t;\ t \geq 0)$ *Brownian motion* $(\beta_t;\ t \geq 0)$ *such that every* $(\mathcal{F}_t;\ t \geq 0)$ *local martingale* $(M_t;\ t \geq 0)$ *may be represented as*

$$M_t = c + \int_0^t m_s d\beta_s; \qquad t \geq 0$$

for some constant c *and some* $(\mathcal{F}_t;\ t \geq 0)$ *predictable process* $(m_t;\ t \geq 0)$*.*

Remark 6.1

- *From Lévy's example, we know that such a Brownian motion* $(\beta_t;\ t \geq 0)$ *may not strongly generate* $(\mathcal{F}_t;\ t \geq 0)$*. We shall say that* $(\beta_t;\ t \geq 0)$ *weakly generates* $(\mathcal{F}_t;\ t \geq 0)$*.*
- *Clearly, from Itô's representation of Brownian martingales, we deduce:*

$$SBF \Rightarrow WBF$$

6.2 Examples of Weak Brownian Filtrations

Here are three families of examples of Weak Brownian Filtrations which are not necessarily Strong Brownian Filtrations. To create such families, we shall perturb Brownian motion either by changing probability, or changing time, or disturbing its excursions...

[1] Malric ([Mal95] and [Mal03]) has obtained a number of properties of the Lévy transform. See also [DÉY93].

6.2.1 Change of Probability

We shall work on the canonical space $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ endowed with the canonical filtration $(\mathcal{F}_t;\ t \geq 0)$ generated by the coordinate process $X_t(\omega) = \omega(t)$, $t \geq 0$, and consider a strictly positive martingale $(D_t = 1 + \int_0^t \delta_s dX_s;\ t \geq 0)$ under the Wiener measure.
Then we consider the measure W^D constructed from the Wiener measure W with

$$W^D_{|\mathcal{F}_t} = D_t . W_{|\mathcal{F}_t}$$

Theorem 6.2 *Under* W^D, $(X_t^D := X_t - \int_0^t \frac{\delta_s ds}{D_s},\ t \geq 0)$ *is a* $(\mathcal{F}_t;\ t \geq 0)$*-Brownian motion which weakly generates* $(\mathcal{F}_t;\ t \geq 0)$.
"In general", it does not strongly generate $(\mathcal{F}_t;\ t \geq 0)$.

Proof

- The fact that $(X_t^D;\ t \geq 0)$ is a $((\mathcal{F}_t;\ t \geq 0), W^D)$-Brownian motion is a well-known consequence of Girsanov's theorem.
- Let $(Y_t;\ t \geq 0)$ be a W^D-martingale starting at 0.
  Then $\widetilde{Y} = (Y_t D_t;\ t \geq 0)$ is a W-martingale so it can be expressed as an integral with respect to X (for the PRP of Brownian motion, see Chapter 5). That is
  $$\widetilde{Y}_t = \int_0^t \widetilde{y}_s dX_s$$
  with $\widetilde{y}$ a predictable process. Hence:
  $$\begin{aligned} Y_t &= \frac{1}{D_t} \int_0^t \widetilde{y}_s dX_s \\ &= \int_0^t \widetilde{y}_s \frac{dX_s}{D_s} + \int_0^t \left(\int_0^s \widetilde{y}_u dX_u \right) d\left(\frac{1}{D_s} \right) + \int_0^t \widetilde{y}_s d < X, \frac{1}{D} >_s \qquad (I_o) \\ &= \int_0^t \frac{\widetilde{y}_s}{D_s} dX_s^D - \int_0^t \left(\int_0^s \widetilde{y}_u dX_u \right) \frac{\delta_s}{D_s^2} dX_s^D \end{aligned}$$
  since $d\left(\frac{1}{D_s}\right) = -\frac{\delta_s}{D_s^2} dX_s^D$ by Itô's formula.

■

It may happen that X^D does not strongly generate $(\mathcal{F}_t;\ t \geq 0)$ under W^D. In 1975, B. Tsirel'son gave such an example of D. This study originated from the following result of Zvonkin:

Proposition 6.3 *[Zvo74] Let* b *denote a bounded Borel function. Then the solution* Y *of the stochastic differential equation:*

$$Y_t = B_t + \int_0^t ds\, b(Y_s) \tag{6.2}$$

has the same filtration as B.

"Tsirel'son's (first) example" [Tsi75] consists in replacing in (6.2) b by $T(s,.)$ a functional depending of the whole past of Y, up to time s:

$$T(s,Y_.) = \sum_{k\in -\mathbb{N}} \left\{ \frac{Y_{t_k} - Y_{t_{k-1}}}{t_k - t_{k-1}} \right\} 1_{]t_k,t_{k+1}]}(s)$$

where $\{x\}$ denotes the fractional part of x and the sequence $(t_k,\ k \in -\mathbb{N})$ decreases towards 0.
Then, for any realization[2] of this "Tsirel'son stochastic differential equation", the natural filtration of Y strictly contains that of B, as shown by the following theorem:

Theorem 6.4 *([SY80] or [RY05] Chapter IX)*
For any $k \in -\mathbb{N}$, $\theta_k = \left\{\frac{Y_{t_k}-Y_{t_{k-1}}}{t_k-t_{k-1}}\right\}$ is uniformly distributed in $[0,1]$ and independent of B.

Proof
We proceed in two steps, by showing first the uniformity of θ_k, then the independence property.

i) **Uniformity of θ_k**
For any $p \in \mathbb{Z} - \{0\}$,

$$\begin{aligned}\mathbb{E}\left[e^{2i\pi p\theta_k}\right] &= \mathbb{E}\left[\exp\left(2i\pi p \frac{Y_{t_k} - Y_{t_{k-1}}}{t_k - t_{k-1}}\right)\right] \\ &= \mathbb{E}\left[\exp\left(2i\pi p \frac{B_{t_k} - B_{t_{k-1}}}{t_k - t_{k-1}}\right)\right] \mathbb{E}\left[e^{2i\pi p\theta_{k-1}}\right] \\ &= e^{-\frac{2\pi^2 p^2}{t_k - t_{k-1}}} \mathbb{E}\left[e^{2i\pi p\theta_{k-1}}\right]\end{aligned}$$

Therefore, by induction and passage to the limit, since $\sum_j \frac{1}{t_j - t_{j-1}} = +\infty$, we obtain that $\mathbb{E}\left[e^{2i\pi p\theta_k}\right] = 0$, i.e. θ_k is uniform.

ii) **Independence property**
We consider a bounded Borel function f and show, using similar arguments to the previous ones, that, for any $p \in \mathbb{Z} - \{0\}$,

$$\mathbb{E}\left[e^{2i\pi p\theta_k + i\int_0^{t_k} f(u)dB_u}\right] = 0$$

which easily implies the independence property.

■

[2] As is well known, such a realization may be obtained thanks to Girsanov's theorem, and the law of $(Y_t;\ t \geq 0)$ is uniquely determined.

Remark 6.2 *Clearly, the above argument rests on the study of the sequence ratios of increments of* Y *between* t_{k-1} *and* t_k, $k \in -\mathbb{N}$. *This led [Yor92b] to develop a "general" study of Tsirel'son's equation in discrete time.*

Nevertheless, it has been shown in [ABÉH95] (see also [Éme02a]) that the natural filtration $(\mathcal{F}_t;\ t \geq 0)$ of $(\mathcal{Y}_t;\ t \geq 0)$ in Toirel'son's first example is a Strong Brownian Filtration.
Thus, it cannot be deduced from this example, that equivalent changes of probability may induce the loss of the SBF property; however, twenty years after the publication of this first example, there appeared "Tsirel'son's second example":

Theorem 6.5 *[DFST96]*
For any $\eta \in (0,1)$, *there exists a martingale* $D^{(\eta)}$ *valued in* $(1-\eta, 1+\eta)$ *such that under* $W^{D^{(\eta)}}$, *the canonical filtration* $(\mathcal{F}_t;\ t \geq 0)$ *is a Weak but not a Strong Brownian Filtration.*

Since the proof requires an important machinery, we refer to [DFST96] for the details. It is worth noticing -as it was understood later in [BSÉ99]- that the hard-core of the proof is the property of standardness, or of cosiness of a SBF[3]. Indeed, [DFST96] manage to produce an example of $D^{(\eta)}$, such that $(\mathcal{F}_t;\ t \geq 0)$ is no longer standard under $W^{D^{(\eta)}}$.

Remark 6.3 ✠ *It would be nice to be able to construct explicitly such a* $D^{(\eta)}$.

6.2.2 Change of Time

Once this surprising result about the loss of SBF under change of measure was obtained, it became interesting to know whether a similar phenomenon could occur with time-changing. More precisely, the question became: does there exist some strictly increasing (absolutely continuous with respect to Lebesgue measure[4]) time change which changes a Strong Brownian Filtration $(\mathcal{F}_t;\ t \geq 0)$ into a filtration which is not strongly Brownian? The following theorem from [ÉS99a] answers this question positively.

Theorem 6.6 *For any* $\eta \in (0,1)$, *there exists* $(T_u^{(\eta)};\ u \geq 0)$ *a strictly increasing* $(\mathcal{F}_t;\ t \geq 0)$ *time change with*

$$1 - \eta \leq \frac{dT_u^{(\eta)}}{du} \leq 1 + \eta$$

such that the associated time-changed filtration $(\mathcal{F}_{T_u^{(\eta)}},\ u \geq 0)$ *is a Weak Brownian Filtration but not a Strong Brownian Filtration.*

---

[3] The notions of standardness, and of cosiness are introduced in Section 6.3.

[4] Time changes which are not absolutely continuous are of no interest since they transform the original Wiener process into a martingale whose increasing process is not absolutely continuous with respect to Lebesgue's measure, hence the time changed filtration is not even a WBF.

As for Theorem 6.5, we are not able to give a succinct proof of this result and we refer to [ÉS99a]. This article details an example of a non-cosy time-changed filtration. As is shown in Section 6.3, this implies that this time-changed filtration is not a SBF.

6.2.3 Walsh's Brownian Motion and Spider Martingales

Walsh [Wal78a] introduced a curious random object $(W_t;\ t \geq 0)$ taking its values in the plane, say; its definition can be stated, loosely speaking, in the following way:
consider N rays (i.e. N half lines with the same origin 0) $I_1,\ldots, I_N$

- On a ray I_i, away from 0, the process $(W_t;\ t \geq 0)$ behaves as Brownian motion;
- When coming to 0, it chooses I_j as its "next" ray with probability p_j.

Such a process (indexed by the probability laws $p = (p_1, \ldots, p_N)$ on the set $\{1, 2, \ldots, N\}$) is called Walsh's Brownian motion and can be defined rigorously using either a martingale problem (as done in the sequel), or constructing explicitly its markovian semigroup or again using excursion theory (see [BPY89a] for a detailed presentation).

Definition 6.3 *Let $\theta_1,\ldots,\theta_N$ be N different angles and consider the rays defined, for $i \leq N$ by*

$$I_i = \{(r, \theta_i),\ r > 0\}$$

Introduce the functions

$$\begin{cases} h_i(z) = (1_{arg(z)=\theta_i} - p_i)1_{z\neq 0} \\ \\ g_i(z) = |z|h_i(z) \end{cases}$$

There is only one probability measure under which $Z_0 = z_0$ a.s. and for every $i \leq N$, the processes $(g_i(Z_t);\ t \geq 0)$ and $(g_i(Z_t)^2 - \int_0^t h_i(Z_s)^2 ds;\ t \geq 0)$ are martingales. Under this probability measure, $(Z_t;\ t \geq 0)$ is the Walsh Brownian motion with parameter $p = (p_1, \ldots, p_N)$, starting from z_0.

Example 6.1 *With $N = 2$, $p_1 = \alpha$ (and $p_2 = 1 - \alpha$), one recovers the Skew Brownian motion with parameter α (see [BPY89a] for many references about the skew Brownian motions).*

From now on, we shall assume for simplicity that

Assumption 6.2 *$p = (p_1, \ldots, p_N)$ is the uniform distribution (i.e. $p_i = 1/N$ for any $i \leq N$) and $N \geq 3$.*

It may be fruitful to think in terms of more general random objects called spider martingales

Definition 6.4 *A spider martingale* $(M_t := (M_t^{(1)}; \dots, M_t^{(N)});\ t \geq 0)$ *is a* $\mathbb{R}_+^N$ *valued process such that*

- *At most one component at any given time is strictly positive.*
- *For any* i, j, *the process* $(M_t^i - M_t^j;\ t \geq 0)$ *is a martingale.*
- M^i *is a submartingale (not a martingale) whose increasing process* A^i *satisfies*

$$1_{M_s \neq 0} dA_s^i = 0$$

Remark 6.4 *This definition implies that all the increasing processes* A^i *are equal. Call A this common increasing process; then, for each* i, $(M_t^i - A_t; t \geq 0)$ *is a martingale.*

For example, Walsh's Brownian motion is a spider martingale which generates a Weak Brownian Filtration. Indeed, $(|W_t|,\ t \geq 0)$ is a reflecting one-dimensional Brownian motion[5], and its (Brownian) martingale part weakly generates the filtration of $(W_t;\ t \geq 0)$. However

Theorem 6.7 *[Tsi97]*
For $N \geq 3$, *the filtration of Walsh's Brownian motion is not a Strong Brownian Filtration.*

Proof
We provide sketches of two different proofs which rely respectively on:

1. a splitting multiplicity argument
2. a joining (or coupling) argument

1. **A splitting multiplicity argument**

   Theorem 6.8 *(Barlow's conjecture holds)[BÉK+ 98]*
   If $(\mathcal{F}_t;\ t \geq 0)$ *is a Strong Brownian Filtration and* L *is the end of a predictable set, then:*

$$\mathcal{F}_{L+} = \mathcal{F}_L \vee \sigma(A) \tag{6.3}$$

   with at most one[6] *non trivial set* $A \in \mathcal{F}_{L+}$.

   Comment 6.2 *Barlow's conjecture was precisely the statement of Theorem 6.8. In Azéma-Yor [AY92], the result could only be obtained for L a last zero of Brownian motion in its natural filtration.*

   Admitting this result, it is clear that the filtration of Walsh's Brownian motion cannot be a Strong Brownian Filtration since, with $L = \sup\{t < 1,\ Z_t = 0\}$:

$$\mathcal{F}_{L+} = \mathcal{F}_L \vee \sigma(Z_1 \in I_i,\ i = 1, .., N)$$

[5] We denote by $|W_t|$ the distance of W_t to the origin.
[6] To clarify:there may be several A's which satisfy (6.3); indeed, if A does, so does ${}^cA$; but any such A cannot be decomposed in $A_1 + A_2$ with A_1, A_2 non trivial.

2. **Tsirel'son's joining argument**
The following theorem reinforces the statement of Theorem 6.7.

Theorem 6.9 *[Tsi97] and [Tsi98] There is no non-zero spider martingale living in a Strong Brownian filtration.*

We follow the arguments of [ÉY98].
Consider $B^{(0)}$ and $B^{(1)}$ two independent real-valued Brownian motions and $\mathcal{B}^{(0)}$ and $\mathcal{B}^{(1)}$ their associated natural filtrations. We shall **join** these two frameworks in order to prove the result for the filtration $\mathcal{B}^{(0)}$.
Assume that a spider martingale Y is defined as a functional of $B^{(0)}$, namely, that there exists Φ such that

$$\forall t \geq 0, \qquad Y_t = \Phi_t(B^{(0)})$$

We define some other spider martingales for $0 \leq r \leq 1$ by

$$\forall t \geq 0, \qquad Y_t^{(r)} = \Phi_t(B^{(r)})$$

with $B^{(r)} = \sqrt{1-r^2}\; B^{(0)} + rB^{(1)}$.
Slutsky's lemma (see e.g. [Del98]) implies that

$$\mathbb{E}\left[\delta(Y_t, Y_t^{(r)})\right] \underset{r\to 0}{\longrightarrow} 0 \tag{6.4}$$

where $\delta(x,y)$ denotes the distance on the spider's graph between x and y.
We shall show that this is impossible if $N \geq 3$.
Indeed, if Y and Y' are two spider martingales, then it is not difficult to show that the process:

$$M_t := \delta(Y_t, Y_t') - \left(\frac{1}{2}\Lambda_t + (N-2)\left[\int_0^t 1_{Y_s' \neq 0} dA_s^Y + \int_0^t 1_{Y_s \neq 0} dA_s^{Y'}\right]\right)$$

defines a martingale, where A^Y and $A^{Y'}$ are the predictable compensators associated respectively to Y and Y' (see Remark 6.4), and Λ is the local time at 0 of $(\delta(Y_t, Y_t'),\ t \geq 0)$; we then deduce by balayage that

$$N\delta(Y_t, Y_t') - \frac{N}{2}\Lambda_t - (N-2)\left\{1_{Y'_{\gamma_t} \neq 0}|Y_t| + 1_{Y_{\gamma_t'} \neq 0}|Y_t'|\right\}$$

is a martingale (We use that: $(|Y_t| - NA_t^Y;\ t \geq 0)$ is a martingale).
Therefore

$$N\mathbb{E}\left[\delta(Y_t, Y_t')\right] \geq (N-2)\mathbb{E}\left[1_{\gamma_t \neq \gamma_t'}\left(|Y_t| \wedge |Y_t'|\right)\right]$$

But the event $\{\gamma_t \neq \gamma_t'\}$ is of probability 1, since the laws are diffuse [EY98]. As a consequence, the preceding inequality may be written, with $Y' = Y^{(r)}$:

$$N\mathbb{E}\left[\delta(Y_t, Y_t^{(r)})\right] \geq (N-2)\mathbb{E}\left[|Y_t| \wedge |Y_t^{(r)}|\right]$$

The left side vanishes as r tends to zero (see (6.4)) whereas the right side admits the limit $(N-2)\mathbb{E}[|Y_t|]$. This is absurd.

■

Remark 6.5 *In fact, we can state, following Tsirel'son and co-authors, the following reinforcement of Theorem 6.9:*
There is no non-zero spider martingale (with at least 3 rays) living in the filtration of a strong solution of a stochastic differential equation driven by a (d-dimensional) Brownian motion.
This result does not remain true if we consider weak solutions of SDEs. Indeed, Watanabe [Wat99] exhibited some diffusion in $\mathbb{R}^2$ whose natural filtration contains a spider martingale. Again, as expected, this example is somewhat complicated, as the general theory of Dirichlet forms is required to express some simple quantities such as the infinitesimal generator of such diffusions.

Remark 6.6

- *Recently, Picard [Pic05b] developed a stochastic calculus for tree-valued martingales. This quite general framework provides with some stochastic calculus for spider martingales. It may be worth noting that Picard uses a coupling method which is deeply linked with Tsirel'son's argument and its variants which we just detailed.*
- *We should also mention another work about spider martingales by Najnudel [Naj05]. In this paper, the law of Walsh's Brownian motion is penalized with functions of the local time process at the origin, and some asymptotical results for such penalized laws are obtained.*

6.3 Invariants of a Filtration

The preceding studies have led several authors to introduce (or to rediscover) some deep properties enjoyed by the (strong) Brownian filtration. In this section, we present the two main notions of a **cosy** filtration and of a **standard** filtration.
The notion of cosiness was invented as a necessary condition for a filtration to be strongly Brownian. There are several variants for this notion (sometimes called I-cosiness, D-cosiness or T-cosiness) which are related to various notions of separability for filtrations (see point 1. in Definitions 6.6 and 6.7). Our aim in this section being to give a glimpse of this general theory, the interested reader should consult the original articles by Émery (or his review [Éme02a]) on this topic for more details.

Definition 6.5 *A joint-copy of the filtered*[7] *probability space $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$ is a triplet $((\overline{\Omega}, \overline{\mathcal{A}}, \overline{\mathbb{P}}), \mathcal{F}', \mathcal{F}'')$ where $(\overline{\Omega}, \overline{\mathcal{A}}, \overline{\mathbb{P}})$ is a probability space endowed with two filtrations $\mathcal{F}'$ and $\mathcal{F}''$ such that:*

[7] The set of indices for the filtration may be $\mathbb{R}_+$, $-\mathbb{N}$,. . . .

1. *The three filtered probability spaces* $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$, $(\overline{\Omega}, \overline{\mathcal{A}}, \overline{\mathbb{P}}, \mathcal{F}')$, $(\overline{\Omega}, \overline{\mathcal{A}}, \overline{\mathbb{P}}, \mathcal{F}'')$ *are isomorphic.*
2. $\mathcal{F}'$ *and* $\mathcal{F}''$ *are jointly immersed, i.e. both the* $\mathcal{F}'$*-martingales and the* $\mathcal{F}''$*-martingales are* $\mathcal{F}' \vee \mathcal{F}''$*-martingales.*

We are now ready to introduce the notion of a **standard** filtration.

Definition 6.6 *(Standardness or I-cosiness)*
A filtration $(\mathcal{F}_{-n};\ n \in \mathbb{N})$ *on our reference probability space* $(\Omega, \mathcal{A}, \mathbb{P})$ *is standard if for any* $\varepsilon > 0$ *and any* $U \in L^0(\mathcal{F}_0)$*, there exists a joint-copy*[8] $((\overline{\Omega}, \overline{\mathcal{A}}, \overline{\mathbb{P}}), \mathcal{F}' := \varphi'(\mathcal{F}), \mathcal{F}'' := \varphi''(\mathcal{F}))$ *such that:*

1. *there exists* $n \in \mathbb{N}$ *such that* $\mathcal{F}'_{-n}$ *and* $\mathcal{F}''_{-n}$ *are independent.*
2. *the random variables* $\varphi'(U)$ *and* $\varphi''(U)$ *are* ε*-close in probability, namely*

$$\overline{\mathbb{P}}(|\varphi'(U) - \varphi''(U)| > \varepsilon) \leq \varepsilon$$

Remark 6.7 *The notion of standardness was introduced by A. Vershik in the early 70's in an ergodic theory framework (see [Éme02a] for a review of Vershik's thesis). This definition differs from Vershik's original definition, but corresponds in fact to the property called I-cosiness*[9]*, to which this definition is equivalent, as proven in [BSÉ99].*

Remark 6.8 *One may naturally wonder how to use this property stated for filtrations indexed by* $-\mathbb{N}$ *to study filtrations indexed by* $\mathbb{R}^+$*.*
In fact, to obtain the non SBF property in Subsection 6.2.2, one may consider $\widetilde{\mathcal{F}}_{-n} := \mathcal{F}_{T_{t_n}}$ *for a conveniently chosen decreasing sequence* $(t_n;\ n \in \mathbb{N})$ *and find a time change* T *such that* $(\widetilde{\mathcal{F}}_{-n};\ n \in \mathbb{N})$ *is not standard.*

Cosiness now appears as a variant of Definition 6.6.

Definition 6.7 *(D-cosiness)*
A filtration $(\mathcal{F}_t;\ t \geq 0)$ *on our reference probability space* $(\Omega, \mathcal{A}, \mathbb{P})$ *is D-cosy*[10] *if for any* $\varepsilon > 0$ *and any* $U \in L^0(\mathcal{F}_\infty)$*, there exists a joint-copy* $((\overline{\Omega}, \overline{\mathcal{A}}, \overline{\mathbb{P}}), \mathcal{F}' := \varphi'(\mathcal{F}), \mathcal{F}'' := \varphi''(\mathcal{F}))$ *such that:*

1. *for all pairs* $(X', X'') \in L^0(\mathcal{F}'_\infty) \times L^0(\mathcal{F}''_\infty)$ *of random variables with diffuse laws*

$$\mathbb{P}(X' = X'') = 0$$

2. *the random variables* $\varphi'(U)$ *and* $\varphi''(U)$ *are* ε*-close in probability, namely*

$$\overline{\mathbb{P}}(|\varphi'(U) - \varphi''(U)| > \varepsilon) \leq \varepsilon$$

---
[8] φ' and φ'' are isomorphisms between the relevant filtered probability spaces.
[9] I-cosy: I stands for independence in point 1.
[10] D-cosy: D stands for diffuse laws in point 1.

These two properties (standard and cosy) turn out to be necessary conditions for a filtration to be a SBF. We shall now focus on the proof of the following proposition to shed light on how crucial the notion of cosiness is in the joining argument sketched in the previous section.

Proposition 6.10 *A Strong Brownian Filtration is D-cosy.*

Remark 6.9 *In fact, any filtration generated by a Gaussian process is D-cosy (see [BSÉ99]).*

Proof of Proposition 6.10
Let $(B_t;\ t \geq 0)$ denote a Brownian motion which strongly generates a given filtration $(\mathcal{F}_t;\ t \geq 0)$.
Let $\varepsilon > 0$ and $U \in L^0(\mathcal{F}_\infty)$. Consider the filtered probability space

$$(\overline{\Omega}, \overline{\mathcal{A}}, \overline{\mathbb{P}}, \overline{\mathcal{F}}) := (\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F}) \otimes (\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$$

On this probability space, we introduce the following Brownian motions $B^0(\omega_1, \omega_2) = B(\omega_1)$, $B^{\pi/2}(\omega_1, \omega_2) = B(\omega_2)$ and, for any $\theta \in [0, \pi/2]$, $B^\theta = B^0 \cos(\theta) + B^{\pi/2} \sin(\theta)$.
Since $U = u(B_t;\ t \geq 0)$, we define $U^\theta = u(B_t^\theta;\ t \geq 0)$ and notice from Slutsky's lemma that U^θ converges in probability toward U^0 as θ tends to 0. Therefore, there exists θ_0 such that

$$\overline{\mathbb{P}}(|U^{\theta_0} - U^0)| > \varepsilon) \leq \varepsilon$$

The filtrations $\mathcal{F}' := \mathcal{F}^0$ and $\mathcal{F}'' := \mathcal{F}^{\theta_0}$, which are respectively generated by B^0 and B^{θ_0}, satisfy the properties in Definition 6.7.

■

6.4 Further References

A rich variety of papers on this topic has appeared in recent years, among which:
[BS00] [BS02a] [BS03a] [BS03b] [BS03c] [BS03d] [DM99] [Éme02b] [ÉS99b] [ÉS01] [GR91] [Kni95]

6.5 Exercises

Exercise 45 *Give different examples of ends of predictable sets in the Brownian filtration such that either a)or b)holds, with:*

a) $\mathcal{F}_{L^+} = \mathcal{F}_L$
b) $\mathcal{F}_{L^+} = \mathcal{F}_L \vee \sigma(A)$ *for some non-trivial set* A.

Exercise 46 *A weak solution of the stochastic differential equation*

$$X_t = x + \int_0^t 1_{X_s>0} dB_s + \theta \int_0^t 1_{X_s=0} ds \tag{6.5}$$

where $\theta > 0$, $x \geq 0$ *and* $(B_t;\ t \geq 0)$ *is a standard Brownian motion, is called a sticky Brownian motion with parameter* θ.

a) Let $(X_t;\ t \geq 0)$ *be a sticky Brownian motion with parameter* θ *starting from 0. Show that, for any* $\lambda > 0$,

$$\left(e^{-\sqrt{2\lambda}\ X_t - \lambda t} + (\sqrt{2\lambda}\ \theta + \lambda) \int_0^t e^{-\lambda s} 1_{X_s=0} ds;\ t \geq 0 \right)$$

is a martingale.
Deduce that $\mathbb{E}\left[\int_0^\infty e^{-\lambda s} 1_{X_s=0} ds\right] = \frac{1}{\sqrt{2\lambda}\ \theta + \lambda}$.

b) Consider X *and* $\widetilde{X}$ *two different solutions of* (6.5) *starting from 0 such that the local time process at level 0 of* $X \vee \widetilde{X}$ *satisfies*

$$L_t^0(X \vee \widetilde{X}) = 4\theta \int_0^t 1_{(X_s=0)\bigcap(\widetilde{X}_s=0)} ds \tag{6.6}$$

Show that, for any $\lambda > 0$,

$$\left(e^{-\sqrt{2\lambda}\ X_t \vee \widetilde{X}_t - \lambda t} + (2\sqrt{2\lambda}\ \theta + \lambda) \int_0^t e^{-\lambda s} 1_{(X_s=0)\bigcap(\widetilde{X}_s=0)} ds;\ t \geq 0 \right)$$

is a super-martingale. Deduce that

$$\mathbb{E}\left[\int_0^t e^{-\lambda s} 1_{(X_s=0)\bigcap(\widetilde{X}_s=0)} ds\right] \leq \frac{1}{2\sqrt{2\lambda}\ \theta + \lambda}.$$

c) Show that there is no sequence of pairs of sticky Brownian motions

$$((X^{(n)}, \widetilde{X}^{(n)});\ n \in \mathbb{N}^*)$$

satisfying (6.6) *and the following additional assumption:*
for any $n \geq 1$, $1_{X^{(n)}=0}$ *and* $1_{X^{(n)}=0}$ *are* $1/n$*-close in probability.*
Hint*: Show from the preceding questions that*

$$\mathbb{E}\left[\int_0^t e^{-\lambda s} 1_{(X_s^{(n)}=0)\bigcap(\widetilde{X}_s^{(n)}=0)} ds\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[\int_0^t e^{-\lambda s} 1_{X_s=0} ds\right]$$

Comment 6.3 *Warren [War99] shows that if the natural filtration of sticky Brownian motion were cosy, there would exist a sequence such as the one defined in question c). Admitting this, the above exercise shows that the natural filtration of a sticky Brownian motion is not cosy.*
This result has also been recovered in [Wat99], by time-changing the natural filtration and using some already exhibited non-cosy filtrations.

Exercise 47 ✠ *Is the filtration of a symmetric α-stable process cosy? (More informally, could you start developing Tsirel'son arguments in this framework?) See [Éme02a].*

Exercise 48 *Given a spider martingale, construct other such spider-martingales, by balayage, stochastic integrals, time changes,...*
***Hints**: Under which condition on functions*

$$f_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}, \qquad i \leq n$$

if (Z_t) is a n-legged spider martingale, then $(f_i(Z_t^{(i)}, t);\ i \leq n, t \geq 0)$ is also one?
As an application, show that:
For any $z_1, \ldots, z_n \in \mathbb{R}^+$, if we consider the stopping time $T_{\{z_1,\ldots,z_n\}} = \inf\{t \geq 0;\ \exists i,\ Z_t^i = z_i\}$, then

$$\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}\, T_{\{z_1,\ldots,z_n\}}\right) 1_{Z_{T_{\{z_1,\ldots,z_n\}}} = z_i}\right] = \frac{\frac{1}{\sinh(\lambda z_i)}}{\sum_{j=1}^n \coth(\lambda z_j)} \qquad (6.7)$$

$$\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}\, T_{\{z_1,\ldots,z_n\}}\right)\right] = \frac{\sum_{j=1}^n \frac{1}{\sinh(\lambda z_j)}}{\sum_{j=1}^n \coth(\lambda z_j)} \qquad (6.8)$$

Note that, if $z_1 = z_2 = \ldots = z_n = z$, then $T_{\{z,\ldots,z\}}$ and $Z_{T_{\{z,\ldots,z\}}}$ are independent.

7

Sketches of Solutions for the Exercises

This chapter is devoted to giving solutions, hints and references for the exercises of the seven preceding chapters, from Chapter 0 to Chapter 6. For ease of the reader, we have indicated the page where the text of a given exercise may be found.

S_0 Preliminaries

• **Solution to Exercise 1** *(p.8)*
Using Itô's formula, we obtain

$$Z_t = Z_0 + \int_0^t c'(S_u) dB_u$$

The conclusion follows easily.

Comment 7.1 *This generalization of the balayage formula was introduced by [DÉY91] to study sets in which multidimensional martingales may have surprising behaviors.*

• **Solution to Exercise 2** *(p.8)*
For any $f \in \mathcal{C}^1$, $(F(W_t) - f(W_t)Y_t;\ t \geq 0)$ *is a local martingale if and only if*

$$(f(W_t) - f'(W_t)Y_t)dW_t - f(W_t)dV_t = 0$$

Taking $f = 1$, *we obtain* $W = V$. *Then, we are left with* $Y_t dV_t = 0$.
Hence, a necessary and sufficient condition is $W = V$ *and* dV_t *is carried by the zero set of* Y.

Remark 7.1 *Moreover, if* $Y_t \geq 0$, *then* $V_t = 2L_t^0(Y)$.

• **Solution to Exercise 3** *(p.8)*
For a "reasonable" function f:

R. Mansuy and M. Yor: *Random Times and Enlargements of Filtrations in a Brownian Setting*, Lect. Notes Math. **1873**, 117–139 (2006)
www.springerlink.com

$$f(N_t) = f(0) + \sum_{s \le t} (f(N_s) - f(N_{s-}))$$
$$= f(0) + \sum_{s \le t} (f(N_{s-} + 1) - f(N_{s-}))\, \Delta N_s$$

which is compensated by $\int_0^t (f(N_{s-}+1) - f(N_{s-}))\, \lambda ds$.

Thus, for $f_1(x) = x^2$, $f_2(x) = e^x$, *the compensators are* $\int_0^t (2N_s + 1)\lambda ds$ *and* $\int_0^t e^{N_s}(e-1)\lambda ds$.

• **Solution to Exercise 4** *(p.8)*
Let $s < t$, *then*

$$\mathbb{E}\left[{}^{(o)}A_t | \mathcal{F}_s\right] = \mathbb{E}\left[\mathbb{E}\left[A_t | \mathcal{F}_t\right] | \mathcal{F}_s\right]$$
$$= \mathbb{E}\left[A_t | \mathcal{F}_s\right] \ge \mathbb{E}\left[A_s | \mathcal{F}_s\right] = {}^{(o)}A_s$$

Hence ${}^{(o)}A$ *is a submartingale.*
Moreover,

$$\mathbb{E}\left[{}^{(o)}A_t - {}^{(o)}A_s | \mathcal{F}_s\right] = \mathbb{E}\left[A_t - A_s | \mathcal{F}_s\right] = \mathbb{E}\left[A_t^{(p)} - A_s^{(p)} | \mathcal{F}_s\right]$$

Hence ${}^{(o)}A - A^{(p)}$ *is a martingale.*

• **Solution to Exercise 5** *(p.8)*

a) We compute in two different manners $\mathbb{E}_x\left[F(X_u;\ u \le s) f(X_t)\right]$ *for* $F(X_u;$ $u \le s)$ *a generic, positive* $\mathcal{F}_s$*-measurable function, and* $f : \mathbb{R} \mapsto \mathbb{R}^+$ *a Borel function;*
- *on one hand, we use the definition of* $P^t_{x \to y}$
- *on the other hand, we replace* $f(X_t)$ *by*

$$P_{t-s} f(X_s) = \int dy \phi_{t-s}(y) f(X_s + y)$$

(For a rigorous definition of the bridges of Markov processes, see [FPY93]).

b) Let ν *denote the Lévy measure of* X.
Under P_x, $M^f_u := \sum_{s \le u} f(\Delta X_s) - u < \nu, f >$ *is a martingale (for suitable* f*'s). We then use Girsanov's theorem to see how the martingale* $(M^f_u; u \ge 0)$ *is transformed when passing from* P_x *to* $P^t_{x \to y}$.

$$M^f_u = \widetilde{M}^f_u + \int_0^u \frac{d < M^f, \phi_{t-.}(y - X_.) >_s}{\phi_{t-s}(y - X_s)}$$

where $(\widetilde{M}^f_u;\ u \ge 0)$ *is a* $P^t_{x \to y}$*- local martingale.*
It remains to compute $< M^f, \phi_{t-.}(y - X_.) >_u$ *which is the compensator under* P_x *of*

$$\sum_{s\leq u} f(\Delta X_s)\left(\phi_{t-s}(y-X_s)-\phi_{t-s}(y-X_{s-})\right)=$$

$$=\sum_{s\leq u} f(\Delta X_s)\left(\phi_{t-s}(y-X_{s-}-\Delta X_s)-\phi_{t-s}(y-X_{s-})\right)$$

This compensator is precisely given by

$$\int_0^u ds\int \nu(dz)f(z)\left(\phi_{t-s}(y-X_s-z)-\phi_{t-s}(y-X_s)\right)$$

Finally the compensator we are looking for is:

$$u<\nu,f>+\int_0^u ds\int \nu(dz)f(z)\left(\frac{\phi_{t-s}(y-X_s-z)}{\phi_{t-s}(y-X_s)}-1\right)$$

Comment 7.2 *The reader may find it interesting to extend such computations starting from a general Markov process, not necessarily a Lévy process. See Kunita [Kun69] and [Kun76], and Sato [Sat99] for such general discussions.*

• **Solution to Exercise 6** *(p.9)*

a) It is a simple consequence of the integration by parts formula (I_o) *for the conveniently stopped local martingale* $(M_t;\ t\geq 0)$.

b) Idem with the integration by parts formula (I_p).

c) From integration by parts formula (0.1), *we deduce*

$$[M,A]_t=\left(M_tA_t-\int_0^t M_{s-}dA_s\right)-\int_0^t A_s\,dM_s+\int_0^t \Delta A_s dM_s$$

We deduce that $[M,A]$ *is a local martingale. Since* $[M,A]$ *is purely discontinuous and has the same jumps as* $\int_0^{\cdot}\Delta A_s dM_s$, *the result follows.*

• **Solution to Exercise 7** *(p.9)*

a) The martingale associated with the variable B_1 *is* $(B_{t\wedge 1};\ t\geq 0)$*; hence, its increasing process is bounded, which implies the result. The result for* $(|B_t|;\ t\leq 1)$ *follows from Tanaka's formula.*

b) This result follows from the John-Nirenberg inequality, see [DM80] for a proof.

c) First note that, for any $t\leq 1$, $M_t:=\mathbb{E}\left[B_1^2|\mathcal{F}_t\right]=B_t^2+1-t$. *Consequently,*

$$\begin{aligned}\mathbb{E}\left[(M_1-M_t)^2|\mathcal{F}_t\right]&=\mathbb{E}\left[(B_1^2-B_t^2-1+t)^2|\mathcal{F}_t\right]=\mathbb{E}\left[(2\int_t^1 B_u dB_u)^2|\mathcal{F}_t\right]\\&=\mathbb{E}\left[4\int_t^1 B_u^2du|\mathcal{F}_t\right]=4\int_t^1\mathbb{E}\left[B_u^2|\mathcal{F}_t\right]du\\&\geq 4B_t^2(1-t)\end{aligned}$$

Therefore $(M_t;\ t\leq 1)$ *is not in BMO.*

S_1 Chapter 1

• **Solution to Exercise 8** *(p.22)*

a) *Apply Lemma 0.3.*

b) *Let* $(T_n, n \in \mathbb{N})$ *denote an increasing sequence of stopping times which reduces the local martingale* $(\int_0^t k_{\gamma_u} dN_u;\ t \geq 0)$. *We take* k *predictable, bounded. Then:*

$$\mathbb{E}\left[k_{\gamma_{T_n}}(\overline{N}_{T_n} - N_{T_n})\right] = \mathbb{E}\left[\int_0^{T_n} k_{\gamma_u} d\overline{N}_u\right] = \mathbb{E}\left[\int_0^{T_n} k_u d\overline{N}_u\right]$$

In the preceding formula, we now replace $(k_u;\ u \geq 0)$ *by* $(k_u/(\alpha + \varepsilon\overline{N}_u),\ u \geq 0)$ *for two positive constants* α *and* ε. *Then, by dominated convergence, we get:*

$$\mathbb{E}\left[k_\Lambda \frac{\overline{N}_\infty}{\alpha + \varepsilon\overline{N}_\infty}\right] = \mathbb{E}\left[\int_0^\infty k_u \frac{d\overline{N}_u}{\alpha + \varepsilon\overline{N}_u}\right] \tag{7.1}$$

The result in b) is now obtained by letting $\varepsilon \downarrow 0$, *and applying Beppo-Levi.*

c) *Similarly, in* (7.1), *we let* α *decrease to 0, which yields to the result.*

The distribution of A_∞^Λ *follows from Lemma 0.1, since* $\overline{N}_\infty \overset{(law)}{=} 1/U$ *with* U *a uniform variable.*

d) *With the same care as before, we arrive to the formula*

$$\mathbb{E}\left[k_\Lambda(\overline{N}_\infty - N_\infty)\right] = \mathbb{E}\left[\int_0^\infty k_u d\overline{N}_u\right]$$

hence

$$\mathbb{E}\left[k_\Lambda(\overline{N}_\infty - n_\Lambda)\right] = \mathbb{E}\left[\int_0^\infty k_u d\overline{N}_u\right]$$

and finally, $A_t^\Lambda = \int_0^t \frac{d\overline{N}_u}{\overline{N}_u - n_u}$.

• **Solution to Exercise 9** *(p.23)*

a) *Using that* $Z_\Lambda = 1$ *and that* $(A_{\Lambda+t}^\Lambda;\ t \geq 0)$ *is constant, we find*

$$\begin{aligned} 1 - Z_{\Lambda+t} = \mu_\Lambda - \mu_{\Lambda+t} &= \hat{\mu}_t - \int_\Lambda^{\Lambda+t} \frac{d < \mu, 1 - Z >_s}{1 - Z_{s-}} \\ &= \hat{\mu}_t + \int_0^t \frac{d < \hat{\mu} >_s}{1 - Z_{\Lambda+s}} \end{aligned} \tag{7.2}$$

b) *From* (7.2), *we deduce that there exists a 3-dimensional Bessel process* $R^{(3)}$ *such that* $1 - Z_{\Lambda+t} = R^{(3)}_{<\hat{\mu}>_t}$.

• **Solution to Exercise 10** *(p.23)*

a) *Let* $(L^x,\ x \in \mathbb{R})$ *be the family of (Meyer) local times of the local martingale* $s(Y)$.
Using the Markov property of Y *and the local martingale property of* $s(Y)$, *we obtain that*

$$Z_t^{\mathcal{L}_a} = 1 \wedge \left(\frac{s(Y_t)}{s(a)} \right)$$

Applying Itô-Tanaka's formula to the RHS of this equality, we find

$$A_t^{\mathcal{L}_a} = -\frac{1}{2s(a)} L_t^{s(a)}$$

b) *Taking the expectation in the occupation time formula, we obtain* $\int_0^t p_u(y,a)\, du = \mathbb{E}_y \left[L_t^{s(a)} \right]$ *and the result follows by differentiation.*

c) *Applying the definition of the predictable compensator (Definition 0.1) with the predictable process* $k_u = 1_{0 \le u \le t}$, *we deduce*

$$P_y(0 \le \mathcal{L}_a \le t) = -\frac{1}{2s(a)} \mathbb{E}_y \left[L_t^{s(a)} \right]$$

and then we use b).

• **Solution to Exercise 11** *(p.24)*

The proof of (1.4) *follows easily from the Hint, which itself is a consequence of the Markov property;* $C = \sqrt{\pi/2}$.

• **Solution to Exercise 12** *(p.24)*

$C = 2\lambda$; *Lemma 0.1 yields* $S_\infty^{(-\lambda)} \overset{(law)}{=} -\frac{1}{2\lambda} \log U \overset{(law)}{=} \mathbf{e}_{2\lambda} \overset{(law)}{=} \frac{1}{2\lambda} \mathbf{e}$ *where* $\mathbf{e}_\alpha$, *resp.* $\mathbf{e}$, *denotes an exponential variable with parameter* α, *resp. 1 and* U *is a uniform variable on* $[0,1]$.

• **Solution to Exercise 13** *(p.24)*

a) *It easily follows from the Hint and Example 1.4:* $L_{T_a^*}$ *is* $\exp(a)$.

b) *It suffices to apply the strong Markov property at time* $\tau_l^{(\beta)}$. *One then obtains (thanks to a)):*

$$P_t(x, dy) = e^{-\frac{t}{x}} \varepsilon_x(dy) + e^{-\frac{t}{y}} \frac{t}{y^2} 1_{\{y \ge x\}} dy$$

For more details about the process $(W_l;\ l \ge 0)$, *which has been introduced by S. Watanabe, see [RY05] Exercice 4.11 Chapter XII and references therein.*

• **Solution to Exercise 14** *(p.25)*

a) a.1) *From Itô's formula (or the balayage formula),* $(M_t^\varphi := \frac{B_t}{\varphi(L_t)},\ t \ge 0)$ *is a local martingale with quadratic variation* $(\int_0^t \frac{ds}{\varphi^2(L_s)},\ t \ge 0)$.

a.2) Using Tanaka's formula for B, we deduce that

$$\frac{1}{\varphi(L_t)}|B_t| = \int_0^{L_t} \frac{dx}{\varphi(x)} + \int_0^t \frac{sgn(B_s)dB_s}{\varphi(L_s)}$$

Hence, the local time of M^φ *is* $(\int_0^{L_t} \frac{dx}{\varphi(x)},\ t \geq 0)$, *and it is also the local time* L^* *of* β, *time-changed with* $\int_0^{\cdot} \frac{ds}{\varphi^2(L_s)}$.

a.3)

$$P(\exists t \leq \tau_l,\ |B_t| \geq \varphi(L_t)) = P(\sup_{v \leq \tau_A^{(\beta)}} |\beta_v| \geq 1)$$

with $A = \int_0^l \frac{dx}{\varphi(x)}$. *The result follows from Exercise 13 a).*

b) The event $\Gamma = \{\forall A > 0,\ \exists t \geq A,\ |B_t| \geq \varphi(L_t)\}$ *is in the tail σ-field of the process* $(|B|_\cdot, L_\cdot)$, *so its probability is 0 or 1.*

- *If* $\int_0^\infty \frac{dx}{\varphi(x)} < \infty$, *then* $P(\exists t \geq 0,\ |B_t| \geq \varphi(L_t)) < 1$ *and, consequently,* $P(\Gamma) = 0$.
- *If* $\int_0^\infty \frac{dx}{\varphi(x)} = \infty$, *then*

$$1 = P(\exists t \geq \tau_l,\ |B_t| \geq \varphi(L_t))(= P(\exists t \geq 0,\ |\hat{B}_t| \geq \varphi(l + \hat{L}_t)))$$

Finally, $P(\Gamma) = 1$ *since* Γ *is the decreasing limit of* $\{\exists t \geq \tau_l,\ |B_t| \geq \varphi(L_t)\}$ *when* l *tends to* ∞.

c) To compute $P(\exists t \geq 0,\ B_t \leq \psi(S_t))$, *we use both Lévy's theorem (Proposition 0.5) and question a) with* $\varphi(x) = x - \psi(x)$

Comment 7.3 *In [Kni78], Knight studied some more general quantities such as*

$$\mathbb{E}\left[e^{-\int_0^{\tau_l} ds f(B_s, L_s)}\right]$$

and the preceding exercise focuses on a particular case of this study (Corollary 1.3 p.435). Nevertheless, the methods developed here are more directly taken from [JY81].

• **Solution to Exercise 15** *(p.26)*

a) We consider Itô's excursion process $e_l(t) := B_{\tau_{l-}+t} 1_{t \leq \tau_l - \tau_{l-}}$, $l \geq 0$. *From excursion theory, we know that* $N_l^\varphi := \sum_{\lambda < l} 1_{\{\max_u e_l(u) \geq \varphi(\lambda)\}}$ *is a Poisson variable with parameter* $\int_0^l d\lambda\ \mathbf{n}(\max_u \varepsilon_u \geq \varphi(\lambda))$.

b) Apply a) with constant φ *and use the martingale property of* $(R_t^{2\nu} - L_t;\ t \geq 0)$ *to compute the expectation of* L *at the first hitting time of* a *by* R.

c) Applying Doob's maximal identity to the martingale $(h(L_t)R_t^{2\nu} + 1 - H(L_t),\ t \geq 0)$, *we obtain, for any* $x \in [0, 1]$,

$$1 - x = P(\exists t \geq 0,\ h(L_t)R_t^{2\nu} + 1 - H(L_t) \geq x)$$
$$= P\left(\exists t \geq 0,\ R_t \geq \left(\frac{H(L_t) + x - 1}{h(L_t)}\right)_+^{1/2\nu}\right)$$

Then if we consider $\varphi(l) = \left(\frac{H(l)+x-1}{h(l)}\right)_+^{1/2\nu}$*, the desired result is obtained by finding* h*, the positive solution of the ODE* $\varphi(l)^{2\nu} y'(l) = y(l) + x - 1$.

- **Solution to Exercise 16** *(p.26)*✠

If f *is twice differentiable, Itô's formula yields to linear ODE for* f *and the result follows from the resolution of these equations.*
In the general case, the question turns to be more tricky; nevertheless Jan Obłoj [OY05] [Obł05] recently managed to solve this problem and to prove the announced result. The main arguments used in Obłoj's proof are Motoo's theorem and the fact that the scale functions of Brownian motion are the affine functions.

- **Solution to Exercise 17** *(p.27)*

a) Let ρ *be a process such that* $\exp\{B_t - \mu t\} = \rho_{A_t^{(-\mu)}}$*. After applying Itô's formula to* $\exp\{2(B_t - \mu t)\}$ *and time-changing with the inverse of* $A_t^{(-\mu)}$*,* ρ *is seen to satisfy the Bessel SDE with index* $-\mu$*.*
b) Easy since $\lim_{t\to\infty} \exp\{B_t - \mu t\} = 0$.
c) Using time-reversal at T_0 *for the Bessel process* $R^{(-\mu)}$*,* T_0 *may be seen as the last passage time at level 1 for a Bessel process with index* μ*; then, the result is a particular case of Exercise 10*
d) The computations of λ_t *and* $\dot{\lambda}_t$ *are simple consequences of the decomposition* $A_\infty^{(-\mu)} = A_t^{(-\mu)} + \hat{A}_\infty^{(-\mu)} \exp\{B_t - \mu t\}$ *where we use the distribution found in question c).*
e) From (1.10)*, we deduce*

$$B_t^{(-\mu)} = \widetilde{B}_t^{(\mu)} - \int_0^t \frac{dA_s^{(-\mu)}}{A_\infty^{(-\mu)} - A_s^{(-\mu)}} = \widetilde{B}_t^{(\mu)} + \log\left(\frac{A_\infty^{(-\mu)} - A_s^{(-\mu)}}{A_\infty^{(-\mu)}}\right)$$

Replacing $\widetilde{B}$ *by this expression in the computation of* $\widetilde{A}_t^{(\mu)}$*, we obtain*

$$\widetilde{A}_t^{(\mu)} = \int_0^t \left(\frac{A_\infty^{(-\mu)}}{A_\infty^{(-\mu)} - A_s^{(-\mu)}}\right)^2 dA_s^{(-\mu)}$$

- **Solution to Exercise 18** *(p.28)*✠

- **Solution to Exercise 19** *(p.28)*

a) 1. See Table 2 p.34 line 3.

2. We deduce from the preceding question that

$$S_t - B_t =$$

$$= \widetilde{B}_t - \int_0^t ds \left(\frac{2S_T - B_T - B_s}{T-s} - \frac{1}{2S_T - B_T - B_s} \right) 1_{S_s < S_T} + S_t$$

$$- \int_0^t ds \frac{1}{T-s} \left(S_s - B_s - (S_T - B_T) \coth \left[\frac{(S_T - B_T)(S_s - B_s)}{T-s} \right] \right) 1_{S_T = S_s}$$

b) 1. See Table 2 p.34 line 5.
2. Combined with Tanaka's formula, the preceding question yields to

$$|B_t| = \widetilde{B}_t + \int_0^t ds \left(\frac{1}{L_T - L_s + |B_T| + |B_s|} - \frac{L_T - L_s + |B_T| + |B_s|}{T-s} \right) 1_{s < \gamma_T}$$

$$+ \int_0^t ds \frac{1}{T-s} \left(|B_T| \coth \left[\frac{|B_T||B_s|}{T-s} \right] - |B_s| \right) 1_{T \ge s \ge \gamma_T} + L_t$$

c) These two formulae are identical thanks to Lévy's equivalence (Lemma 0.5).

• **Solution to Exercise 20** *(p.28)*

a) See Remark 1.4.
*b) The two formulae are identical; in [Jeu80] p58-59 or [JY85], Jeulin deals with a more general framework involving any honest time L which satisfies assumption (**A**). Then the coincidence between the formulae obtained by enlarging progressively with L (i.e. w.r.t. $\mathcal{F}^L$) and by enlarging initially with the terminal value of the predictable compensator A^L_∞ follows from the fact that the (**H**) hypothesis is satisfied between the filtration $\mathcal{F}^L$ and $\mathcal{F}^L \vee \sigma(A^L_\infty)$, i.e. the first one is immersed in the second one.*

Remark 7.2 *In fact, the coincidence observed in this exercise turns out to be the general case under (**CA**) hypothesis. Indeed, the key argument is the multiplicative representation of Z^L, the Azéma supermartingale of an honest time L, obtained in Proposition 1.3. With $Z_t = N_t/\overline{N}_t$, enlarging progressively with L (resp. initially with A^L_∞) is equivalent to enlarging progressively with the random time $\sup\{t \ge 0,\ N_t = \overline{N}_t\}$ (resp. initially with $\overline{N}_\infty$).*

• **Solution to Exercise 21** *(p.28)*

a) Consider the closed subspace $\mathbb{G}_t^{h,-}$ of the Gaussian space $\mathbb{G}_t$ generated by $(B_u,\ u \le t)$:

$$\mathbb{G}_t^{h,-} = span \left\{ \int_0^t f(u) dB_u \text{ with } f \in L^2([0,t]) \text{ and } \int_0^t f(u)h(u)du = 0 \right\}$$

Then $\mathbb{G}_t$ *is the orthogonal sum of* $\mathbb{G}_t^{h,-}$ *and the one-dimensional space:* $\left\{\lambda \int_0^t h(u)dB_u;\ \lambda \in \mathbb{R}\right\}$. *Then, it suffices to consider* $\mathcal{F}_t^{h,-}$ *the* σ*-field generated by* $\mathbb{G}_t^{h,-}$.

The fact that $\mathcal{F}_t^{h,-} \subset \mathcal{F}_{t+t'}^{h,-}$ *follows from the remark that* $\mathbb{G}_t^{h,-} \subset \mathbb{G}_{t+t'}^{h,-}$.

b) It suffices to remark that $B^{h,-}$ *is the projection of* B *on the filtration* $\mathcal{F}^{h,-}$.

c) i) From question b), using integration by parts, we get that

$$B_t - \left(\frac{t^{\alpha+1}}{\alpha+1}\frac{2\alpha+1}{t^{2\alpha+1}}\right)\int_0^t u^\alpha dB_u = B_t - \left(\frac{2\alpha+1}{\alpha+1}\right) \times \frac{1}{t^\alpha}\left[t^\alpha B_t - \int_0^t \alpha B_s s^{\alpha-1} ds\right] = -\left(\frac{\alpha}{\alpha+1}\right) B_t + \left(\frac{\alpha}{\alpha+1}\right)(2\alpha+1)\frac{1}{t^\alpha}\int_0^t B_s s^{\alpha-1} ds$$

is a $(\mathcal{F}_t^{\alpha,-},\ t \geq 0)$*-martingale whose quadratic variation is* $\left(\frac{\alpha}{\alpha+1}\right)^2 t$.

- *For* $\alpha \neq 0$, *we deduce that:*

$$\left(B_t^{\alpha,-} := B_t - \frac{2\alpha+1}{t^\alpha}\int_0^t B_s s^{\alpha-1} ds,\ t \geq 0\right)$$

is a $(\mathcal{F}_t^{\alpha,-},\ t \geq 0)$*-Brownian motion.*

- *For* $\alpha = 0$, *by passage to the limit, we obtain that:*

$$\left(B_t - \int_0^t ds \frac{B_s}{s},\ t \geq 0\right)$$

is a $(\mathcal{F}_t^{0,-},\ t \geq 0)$*-Brownian motion.*

ii) In order to show $\left(B_t^{\alpha,-},\ t \geq 0\right)$ *generates* $(\mathcal{F}_t^{\alpha,-},\ t \geq 0)$, *it suffices to prove that the orthogonal in* $\mathbb{G}_t$, *the Gaussian space generated by* $(B_u,\ u \leq t)$, *of* $\mathbb{G}_t^{\alpha,-}$, *is precisely the one-dimensional space* $\left\{\lambda \int_0^t u^\alpha dB_u,\ \lambda \in \mathbb{R}\right\}$.

Indeed, in general, $\int_0^t k(u)dB_u$ *is orthogonal to* $\mathbb{G}_t^{h,-}$ *if and only if:*

$$\forall s \leq t, \qquad \mathbb{E}\left[\left(\int_0^t k(u)dB_u\right)\left\{B_s - \sigma_h(s)\int_0^s h(u)dB_u\right\}\right] = 0$$

where $\sigma_h(s) = \frac{\int_0^s h(u)du}{\int_0^s h(u)^2 du}$.

Hence:

$$\int_0^s k(u)du = \sigma_h(s)\int_0^s h(u)k(u)du \tag{7.3}$$

In the case $h(s) = s^\alpha$*, we get* $\sigma_h(s) = \frac{2\alpha+1}{\alpha+1}\frac{1}{s^\alpha}$*, hence* (7.3) *is equivalent to:*

$$s^\alpha \int_0^s k(u)du = \frac{2\alpha+1}{\alpha+1}\int_0^s u^\alpha k(u)du$$

i.e., for $\alpha \neq 0$*:*

$$-\frac{1}{\alpha+1}u^\alpha k(u) + u^{\alpha-1}\int_0^u k(t)dt = 0, \qquad du \ a.s.$$

and $k(u) = \frac{\alpha+1}{u}\int_0^u k(t)dt$ *which yields:* $\int_0^u k(t)dt = Cu^{\alpha+1}$*, hence the result.*

A similar computation holds for $\alpha = 0$*, starting with*

$$B_t^{0,-} = B_t - \int_0^t ds\frac{B_s}{s}$$

Comment 7.4 *We note that, at least concerning the functions* $h_\alpha(s) = s^\alpha$*, one may recover the original Brownian filtration by enlarging adequately* $(\mathcal{F}_t^{h_\alpha,-},\ t \geq 0)$*.*

Indeed, note that, for $0 < s < t$*:*

$$\frac{B_t}{t} - \frac{B_s}{s} = \int_s^t \frac{dB_u}{u} - \int_s^t \frac{B_u}{u^2}du$$

which yields: $\frac{B_s}{s} = -\int_s^\infty \frac{dB_u^{0,-}}{u}$*;hence:* $\mathcal{F}_t = \mathcal{F}_t^{0,-} \vee \sigma\left\{\int_t^\infty \frac{dB_u^{0,-}}{u}\right\}$.

A similar computation may be done with h_α*,* $\alpha > -1/2$*.*

S_2 Chapter 2

- **Solution to Exercise 22** *(p.49)*

a) $\lambda = a^{\frac{1}{r}}$.
b) $(b_u := uB_{u/(1-u)},\ u \leq 1)$ *is a standard Brownian bridge.*
c) ✠

It may be worth noticing that

$$P(\sup_t\{|B_t| - t^{r/2}\} \geq x) = P(\exists t \geq 0,\ B_t \geq x + t^{r/2})$$
$$= P(\inf\{t \geq 0,\ B_t \geq x + t^{r/2}\} < \infty)$$

and that the density of the variable $\inf\{t \geq 0,\ B_t \geq x + t^{r/2}\}$ *is shown, in [dlPHDV04] (following [RSS84]), to satisfy a Volterra integral equation of second type.*

Note that this question has already been solved for $r = 4$ *by Groeneboom [Gro89] which relates first passage times for* $B_t - ct^2$ *to Bessel processes, and then find their densities explicitly in terms of Airy functions.*

Remark 7.3 *It follows from a well-known theorem due to Fernique about the supremum of any centered Gaussian process that both quantities in equation* (2.6) *have moments of all orders.*

- **Solution to Exercise 23** *(p.50)*

a) *See Exercise 14.*
b) *From Example 2.1, we obtain*

$$\mathbb{E}\left[\sup_t\{|B_t|^\alpha - L_t^{\alpha\beta}\}\right] = \mathbb{E}\left[(\mathbf{e}_{\alpha,\alpha\beta})^{\frac{\alpha\beta}{1-\beta}}\right]$$

with $\mathbf{e}_{a,b}$ *an exponential random variable with parameter* $c_{a,b}$ *defined in Example 2.1. Therefore, this quantity is finite if and only if* $\frac{\alpha\beta}{\beta-1} < 1$

c) *For any* $c > 0$*, we write:*

$$\begin{aligned}\mathbb{E}\left[|B_\Lambda|^\alpha\right] &= \mathbb{E}\left[|B_\Lambda|^\alpha - cL_\Lambda^{\alpha\beta}\right] + c\mathbb{E}\left[L_\Lambda^{\alpha\beta}\right] \\ &\le \mathbb{E}\left[\sup_{t\in\mathbb{R}^+}\{|B_t|^\alpha - cL_t^{\alpha\beta}\}\right] + c\mathbb{E}\left[L_\Lambda^{\alpha\beta}\right] \\ &\le c^{-\frac{1}{\beta-1}}\mathbb{E}\left[\sup_{t\in\mathbb{R}^+}\{|B_t|^\alpha - L_t^{\alpha\beta}\}\right] + c\mathbb{E}\left[L_\Lambda^{\alpha\beta}\right] \qquad \textit{(scaling)}\end{aligned}$$

Minimizing with respect to c, we obtain $\mathbb{E}\left[|B_\Lambda|^\alpha\right] \le K.\mathbb{E}\left[L_\Lambda^{\alpha\beta}\right]^{\frac{1}{\beta}}$, *with*

$$K = \beta\left(\frac{\mathbb{E}\left[\sup_{t\in\mathbb{R}^+}\{|B_t|^\alpha - L_t^{\alpha\beta}\}\right]}{\beta-1}\right)^{\frac{\beta-1}{\beta}}$$

d) *We prove that the constant K found in c) is the best constant, simply by considering the random time*

$$\inf\{u \ge 0,\ |B_u|^\alpha - L_u^{\alpha\beta}\} = \sup_t\{|B_t|^\alpha - L_t^{\alpha\beta}\}\ .$$

For more details, see Song-Yor [SY87].

- **Solution to Exercise 24** *(p.50)*

a) $A^{(2)}$ *and* $C^{(2)}$ *are not moment-equivalent, as may be seen by considering* $T = T_a := \inf\{t;\ B_t = a\}$, *and the moment-equivalence between* $A^{(1)}$ *and* $C^{(1)}$.
b) *Take* $T = \tau_l$. $C_{\tau_l}^{(3)}$ *is distributed as the reciprocal of an exponential variable (see Example 13); hence,* $A^{(3)}$ *and* $C^{(3)}$ *are not moment-equivalent.*

Remark 7.4 *Although* $A^{(i)}$ *and* $C^{(i)}$ *are not moment-equivalent for* $i = 2, 3$, *if we consider only small exponents p, that is* $p < 1$, *then*

$$\mathbb{E}\left[(C_T^{(2)})^p\right] \le \gamma_p^{(2)} \mathbb{E}\left[(A_T^{(2)})^p\right] \qquad \mathbb{E}\left[(C_T^{(3)})^p\right] \le \gamma_p^{(3)} \mathbb{E}\left[(A_T^{(3)})^p\right]$$

with universal constants $\gamma_p^{(2)}$ *and* $\gamma_p^{(3)}$. *These inequalities are particular consequences of the domination relation introduced by Lenglart [Len77] to whom we refer the reader.*

- **Solution to Exercise 25** *(p.51)*

a) From (2.10), *we deduce that*

$$\rho_t^{(n)} - t = \frac{2}{\sqrt{n}} \int_0^t \sqrt{\rho_s^{(n)}}\, d\beta_s$$

and we conclude thanks to some ersatz of BDG inequalities applied to the martingale $\left(\frac{2}{\sqrt{n}} \int_0^t \sqrt{\rho_s^{(n)}}\, d\beta_s;\ t \ge 0\right)$

b) Similar arguments yields to the result.

- **Solution to Exercise 26** *(p.51)*✠

It may be helpful to write these inequalities in an equivalent form, namely

a) $\mathbb{E}\left[|M_T|\right] \le C_p^{(1)} \mathbb{E}\left[< M >_T^{p/2}\right]^{1/p}$ *for any martingale* $(M_t;\ t \ge 0)$

b) $\mathbb{E}\left[\varphi(L_T)|B_T|\right] \le C_p^{(2)} \mathbb{E}\left[\Phi(L_T)^p\right]^{1/p}$ *for any function* φ *and* $\Phi(x) = \int_0^x \varphi(y)dy$

S_3 Chapter 3

- **Solution to Exercise 27** *(p.66)*

a) $B_u^{[0,\gamma]} = u\beta_{\frac{1}{u}-1}$ *where* $\beta_v = \frac{1}{\sqrt{\hat{\delta}}}(\hat{B}_{\hat{\delta}+\hat{\delta}v} - \hat{B}_{\hat{\delta}})$.

b) $m_1 = \sqrt{\hat{B}_1^2 + \frac{\hat{B}_1^2}{\hat{\delta}-1}}$; *moreover,* $\frac{\hat{\delta}-1}{\hat{B}_1^2}$ *is independent of* $\hat{B}_1^2$ *and is distributed as the first hitting time of 1 by a Brownian motion* $\widetilde{B}$; *finally:*

$$\frac{1}{\widetilde{T}_1} \overset{(law)}{=} \widetilde{B}_1^2$$

c) ✠

- **Solution to Exercise 28** *(p.67)*

We note the equalities between events:

$$\begin{aligned}(\gamma \le u) &= (1 \le \delta_u) = (1 \le u + \hat{T}_{B_u}) \\ &= (1 \le u + B_u^2 \hat{T}_1) = \left(1 \le u\left(1 + \frac{B_1^2}{\hat{B}_1^2}\right)\right)\end{aligned}$$

with $\delta_u = \inf\{v \ge u,\ B_v = 0\}$ *and* $\hat{T}_{B_u}$ *(resp.* $\hat{T}_1$*) the first hitting time of* B_u *(resp. 1) by a Brownian motion independent of* $(B_s;\ s \le u)$.

• **Solution to Exercise 29** *(p.67)*

a) Using the expression of Z^γ computed at the beginning of Chapter 3, we find

$$\lambda_t(f) = f(\gamma_t)\left(1 - \psi\left(\frac{|B_t|}{\sqrt{1-t}}\right)\right) + \mathbb{E}\left[f(\gamma)1_{\gamma>t}|\mathcal{F}_t\right]$$

To compute the conditional law of γ knowing $\mathcal{F}_t$, we note that

$$P(\gamma \in du|\mathcal{F}_t) = \sqrt{\frac{2}{\pi}}\,\frac{\mathbb{E}\left[dL_u|\mathcal{F}_t\right]}{\sqrt{1-u}} = \sqrt{\frac{2}{\pi}}\,\frac{\mathbb{E}_{B_t}\left[dL_{u-t}\right]}{\sqrt{1-u}}, \qquad t < u$$

Then, we use $\mathbb{E}_x\left[L_h^y\right] = \int_0^h ds p_s(x,y)$ to show that

$$\lambda_t(f) = f(\gamma_t)\left(1 - \psi\left(\frac{|B_t|}{\sqrt{1-t}}\right)\right) + \int_t^1 du \frac{f(u)e^{-\frac{B_t^2}{2(u-t)}}}{\pi\sqrt{(1-u)(u-t)}}$$

The two enlargement formulae are identical (see the solution to Exercise 20).

b) In order to compute $\mathbb{E}\left[f(\gamma_T,\delta_T)|\mathcal{F}_t\right]$ for any Borel function f, we compute successively:

$$\mathbb{E}\left[f(\gamma_T,\delta_T)1_{t>\delta_T}|\mathcal{F}_t\right] = f(\gamma_T,\delta_T)1_{t>\delta_T}$$

$$\mathbb{E}\left[f(\gamma_T,\delta_T)1_{\delta_T>t>T}|\mathcal{F}_t\right] = \frac{1}{\sqrt{2\pi}}\int_t^\infty f(\gamma_t,u)e^{-\frac{B_t^2}{2(u-t)}}\frac{|B_t|}{(u-t)^{3/2}}du 1_{\gamma_t<T<t}$$

$$\mathbb{E}\left[f(\gamma_T,\delta_T)1_{T>t>\gamma_T}|\mathcal{F}_t\right] = \frac{1}{\sqrt{2\pi}}\int_T^\infty f(\gamma_t,u)e^{-\frac{B_t^2}{2(u-t)}}\frac{|B_t|}{(u-t)^{3/2}}du 1_{t<T}$$

$$\mathbb{E}\left[f(\gamma_T,\delta_T)1_{\gamma_T>t}|\mathcal{F}_t\right] = \frac{1}{2\pi}\int_t^T du \int_t^\infty dv f(u,v)\frac{e^{-\frac{B_t^2}{2(u-t)}}}{\sqrt{(v-u)^3(u-t)}}$$

where we use the conditional distribution of γ_T knowing $\mathcal{F}_t$ already computed in question a) above and where the last equality is obtained by conditioning successively on $\mathcal{F}_T$ and $\mathcal{F}_{\gamma_T}$.
Then, the result follows easily.

• **Solution to Exercise 30** *(p.67)*

Thanks to Lévy's equivalence (Proposition 0.5), we may reduce our discussion to path decomposition after and before γ. More precisely, we deduce from Lévy's equivalence that $(m_u^{(1)} := \frac{B_\sigma - B_{\sigma+u(1-\sigma)}}{\sqrt{1-\sigma}},\ u \leq 1)$ and $(m_u^{(2)} := \frac{B_\sigma - B_{\sigma-u\sigma}}{\sqrt{\sigma}},\ u \leq 1)$ are independent and that $m^{(1)}$ is a meander.
$m^{(2)}$ is also a meander since Wiener measure is invariant by time-reversal at fixed time 1.

• **Solution to Exercise 31** *(p.67)*

The result follows easily from the Hint and the explicit law of $\hat{L}_{1-t}$ (obtained e.g. via Lévy's equivalence, Proposition 0.5).

• **Solution to Exercise 32** *(p.68)*

a) and b): The process in (3.12) *is the limit in* L^2*, uniformly in* t *(*$\leq T$*) of*

$$z_{\gamma_t} B_t \exp\left(i \int_0^t f(s) 1_{|B_s| \geq \frac{1}{n}} d\hat{B}_s + \frac{1}{2} \int_0^t f(s)^2 ds\right)$$

and we use Itô's formula to obtain in the limit:

$$z_{\gamma_t} B_t \widehat{\mathcal{E}}_t = \int_0^t z_{\gamma_s} \widehat{\mathcal{E}}_s (1 + i f(s) B_s) dB_s$$

with $\widehat{\mathcal{E}}_t = \exp\left(i \int_0^t f(s) d\hat{B}_s + \frac{1}{2} \int_0^t f(s)^2 ds\right)$.

Therefore, $(z_{\gamma_t} B_t \widehat{\mathcal{E}}_t;\ t \geq 0)$ *is a martingale, which yields the desired result.*

c) • *We write:*

$$|B_{\gamma_t+u}| = sgn(B_t)\left(\hat{B}_{\gamma_t+u} - \hat{B}_{\gamma_t}\right) + \int_0^u \frac{dh}{|B_{\gamma_t+h}|}$$

The result now follows from Imhof's relation (3.1) *and the fact that the three dimensional Bessel process generates the same filtration as its driving Brownian motion.*

• *We have shown previously that* B_t *is measurable with respect to*

$$\sigma(\gamma_t) \vee \sigma(sgn(B_t)) \vee \hat{\mathcal{B}}_t$$

But, $\gamma_t = \inf\{s < t,\ sgn(B_u) \text{ is constant on } [s,t]\}$*, hence the result.*

d) We write:

$$f(B_t) = f(|B_t|) 1_{s_t=+1} + f(-|B_t|) 1_{s_t=-1},$$

where $s_t = sgn(B_t)$*, and the desired formula will follow from the computation of* $\sigma_t^+ := P(s_t = +1 | \widehat{\mathcal{B}}_t \vee \mathcal{F}_{\gamma_t})$.

For this purpose, we use: $\mathbb{E}[B_t | \widehat{\mathcal{B}}_t \vee \mathcal{F}_{\gamma_t}] = 0$

This implies: $\lambda_t^+ \sigma_t^+ - \lambda_t^- (1 - \sigma_t^+) = 0$*, hence:*

$$\sigma_t^+ = \frac{\lambda_t^-}{\lambda_t^+ + \lambda_t^-}$$

e) Under the signed measure Q_a, $\widehat{\mathcal{E}}_t := \exp\left(i \int_0^t f(s) d\hat{B}_s + \frac{1}{2} \int_0^t f(s)^2 ds\right)$ *defines a martingale since* $(\frac{B_t}{a} \widehat{\mathcal{E}}_t;\ t \geq 0)$ *is a* P_a*-martingale. Moreover, since* $\mathbb{E}_{P_a}\left[\frac{B_t}{a} \widehat{\mathcal{E}}_t\right] = 1$*, we deduce that* $\mathbb{E}_{Q_a}\left[\widehat{\mathcal{E}}_t\right] = 1$*, so that:*

$$\mathbb{E}_{Q_a}\left[\exp\left(i \int_0^t f(s) d\hat{B}_s\right)\right] = \exp\left(-\frac{1}{2} \int_0^t f(s)^2 ds\right)$$

The main differences with the preceding questions can be explained using T_0*, the first hitting time by* B *of the level 0. More precisely, in the following decomposition*

$$\frac{B_1}{a} = \frac{B_{1\wedge T_0}}{a} + \frac{B_1}{a} 1_{T_0<1}$$

the first part on the right corresponds to the density between the Wiener measure P_a and the law of the three dimensional Bessel process starting from a.

S_4 Chapter 4

• **Solution to Exercise 33** *(p.83)*

a) $(X_t^D;\ t \geq 0)$ *is a* $(Q, (\mathcal{F}_t;\ t \geq 0))$*-Brownian motion.*

b) *It is easy to show that every* $(Q, (\mathcal{F}_t;\ t \geq 0))$ *local martingale* $(M_t^D;\ t \geq 0)$ *may be obtained as the Girsanov transform of* $(M_t; t \geq 0)$*, a* $(W, (\mathcal{F}_t; t \geq 0))$ *local martingale, namely,*

$$M_t^D = M_t - \int_0^t \frac{d < D, M >_s}{D_s}$$

From the representation of $(M_t;\ t \geq 0)$ *as a stochastic integral with respect to* $(X_t;\ t \geq 0)$*, we deduce the representation of* $(M_t^D;\ t \geq 0)$ *as a stochastic integral with respect to* $(X_t^D;\ t \geq 0)$*.*

• **Solution to Exercise 34** *(p.84)*

a) *The first point follows from*

$$< \mathcal{A} >_t = \int_0^t |Z_s|^2 ds \qquad < \theta >_s = \int_0^t \frac{ds}{|Z_s|^2}$$
$$\mathcal{A}_t = \int_0^t |Z_s|^2 d\theta_s \qquad \theta_t = \int_0^t \frac{d\mathcal{A}_s}{|Z_s|^2}$$

The second point follows from the polar representation of $(Z_t;\ t \geq 0)$*, namely* $Z_t = |Z_t| \exp(i\theta_t)$*, also called skew product decomposition of* $(Z_t;\ t \geq 0)$*, see, e.g. [IM74] and [PR88].*

b) *We note that it is not possible to represent either* X *or* Y *as a stochastic integral with respect to* $\mathcal{A}$*(or* θ*); why?*
Then, use Theorem 4.4.

c) *When* $z_0 = 0$*,* $(\theta_t;\ t \geq 0)$ *cannot be defined via formula* (4.8) *(see, e.g. [IM74] Section 7.16 p276).*
On the other hand, the definition of $(\mathcal{A}_t;\ t \geq 0)$ *makes sense as before; furthermore,* $(|Z_t|,\ t \geq 0)$ *is adapted to the filtration of* $(\mathcal{A}_t;\ t \geq 0)$ *since*

$$|Z_t|^2 = \frac{d}{dt}(< \mathcal{A} >_t)$$

The natural filtration of $|Z|$ *is that of* $\beta_t = \int_0^t \frac{X_s dX_s + Y_s dY_s}{|Z_s|}$ *and* $\gamma_t = \int_0^t \frac{d\mathcal{A}_s}{|Z_s|}$ *is another Brownian motion independent from* β*. The filtration of*

(β,γ) is precisely that of $\mathcal{A}$.
Finally, for any fixed time t, $Z_t/|Z_t|$ is uniformly distributed on the unit circle, and independent from the process $(\mathcal{A}_s;\ s \geq 0)$. Indeed, for any positive, Borel function f and any positive, bounded functional Φ:

$$\mathbb{E}\left[\Phi(\mathcal{A})f\left(\frac{Z_t}{|Z_t|}\right)\right]=\mathbb{E}\left[\Phi(\mathcal{A})f\left(\frac{e^{i\theta}Z_t}{|Z_t|}\right)\right]=\mathbb{E}\left[\Phi(\mathcal{A})\frac{1}{2\pi}\int_0^{2\pi}d\theta f\left(\frac{e^{i\theta}Z_t}{|Z_t|}\right)\right]$$
$$=\mathbb{E}\left[\Phi(\mathcal{A})\right]\frac{1}{2\pi}\int_0^{2\pi}d\mu f(e^{i\mu})$$

• **Solution to Exercise 35** *(p.84)*
We first detail the arguments suggested in the Hint:
for $f:(E,\mathcal{E})\to(\mathbb{R}^+,\mathcal{B}(\mathbb{R}^+))$, compute, for any $t,s>0$, $\mathbb{E}\left[f(X_{t+s})|\mathcal{Y}_s\right]$ in two different manners, which shall yield

$$\Lambda(P_tf)(Y_s)=Q_t(\Lambda f)(Y_s) \qquad \text{a.s.}$$

We now consider points a) and b):

a) $X_t=B_t^{(n)}$, $Y_t=|B_t^{(n)}|$.
Then, under P_0, $X_t=Y_t\Theta_t$, with $\Theta_t=X_t/Y_t$ uniformly distributed on the unit sphere, and independent of $(Y_u;\ u\geq 0)$. Hence $\Lambda(x,dy)$ is the uniform law on the sphere of radius y.

b) It will be easier to work with the squared Bessel processes $(Z_t^{(m)};\ t\geq 0)$ and $(Z_t^{(n)};\ t\geq 0)$. From the additivity property of squared Bessel processes, we may construct the pair $(Z^{(m)},Z^{(n)})$ as:

$$Z_t^{(m)}=Z_t^{(m-n)}+Z_t^{(n)}, \qquad t\geq 0$$

where, on the right hand side, the two processes $(Z_t^{(m-n)};\ t\geq 0)$ and $(Z_t^{(n)};\ t\geq 0)$ are independent.
We now take

$$\mathcal{X}_t=\sigma\{Z_u^{(m-n)},\ Z_u^{(n)},\ u\leq t\}$$
$$\mathcal{Y}_t=\sigma\{Z_u^{(m)},\ u\leq t\}$$

Then, we use the following properties:

- *$(Z_t^{(n)};\ t\geq 0)$ is a Markov process with respect to $(\mathcal{X}_t;\ t\geq 0)$.*
- *For any $t>0$, the variable $Z_t^{(n)}/Z_t^{(m)}$ is independent from $\mathcal{Y}_t$ and $\beta(\frac{n}{2},\frac{m-n}{2})$ distributed.*

The second property may be proven by using time inversion for the two-dimensional process $(Z^{(m)},Z^{(n)})$, an elementary beta-gamma result which yields $Z_t^{(n)}/Z_t^{(m)}$ is $\beta(\frac{n}{2},\frac{m-n}{2})$ distributed, and finally the Markov property of $(Z_t^{(m)};\ t\geq 0)$ with respect to $(\mathcal{X}_t;\ t\geq 0)$.
This argument is taken from [CPY98].

• **Solution to Exercise 36** *(p.85)*

a) $\mathcal{L}^{(\beta)} f = 0$ *with* $f(x) = x$.

b) *Noting that* $< X >_t = t$ *and that* $\Delta X_t = \beta X_{t-} 1_{\Delta X_t \neq 0}$ *and using Theorem 4.7, it suffices to prove that* X_t *is bounded; from the equations* $d(X_t^2) = 2X_{t-} dX_t + d[X, X]_t$ *and* $d[X, X]_t = \beta X_{t-} dX_t + dt$, *we deduce that*

$$(\beta + 2) d[X, X]_t - \beta d(X_t^2) = 2dt$$

therefore $X_t^2 \leq X_0^2 \leq \frac{2t}{-\beta}$.

c) *The fact that there exists a Markov kernel* M *such that* $\mathcal{L}^{(\beta)} M = M \left(\frac{1}{2} D^2\right)$ *follows from the representation of* $\mathcal{L}^{(\beta)}$ *as* $\mathcal{L}^{(\beta)} f(x) = \frac{1}{2} \mathbb{E}\left[f''(xV)\right]$ *where the law of the random variable* V *is given by*

$$\mathbb{P}(V \in dv) = \frac{2}{\beta^2} (v - 1 - \beta)^+ 1_{v \leq 1} dv$$

and $Mf(x) = \mathbb{E}[f(xV)]$. *The announced result follows easily.*

Remark 7.5 *In the particular case* $\beta = -2$, $(X_t;\ t \geq 0)$ *is called the parabolic martingale[Val95]; its paths belong to the parabola* $x^2 = t$. *More precisely,* $P(X_t = \sqrt{t}) = P(X_t = -\sqrt{t}) = 1/2$ *and its jumps, which occur when* X *changes signs, happen at times distributed according to a Poisson point process with intensity* $dt/(4t)$. $(X_t;\ t \geq 0)$ *may be realized as* $X_t = B_{T^*_{\sqrt{t}}}$ *where* B *is a Brownian motion and* $T^*_a = \inf\{u \geq 0;\ |B_u| = a\}$.

• **Solution to Exercise 37** *(p.85)*

a) $\mathcal{L}_k f_1 = 0$; $\mathcal{L}_k f_2(x) = (1 + 2k)$, *with* $f_1(x) = x$, $f_2(x) = x^2$.

b) $\mathcal{L}_k$ *restricted to even functions is the infinitesimal generator of the Bessel process.*
Since $|X^{(k)}|$ *is a continuous process, the jump process satisfies* $\Delta X_t^{(k)} = -2X_{t-}^{(k)} 1_{\Delta X_t^{(k)} \neq 0}$.

c) *Since* $(X^{(k)})^2$ *is a squared Bessel process with dimension* $(2k+1)$, *there exists a Brownian motion* B *such that*

$$(X_t^{(k)})^2 = x^2 + 2 \int_0^t \sqrt{(X_s^{(k)})^2} dB_s + (2k+1)t$$

Identifying this formula with Itô's formula, we obtain:

$$\begin{cases} \int_0^t X_{s-}^{(k)} d(X^{(k)})_s^c = \int_0^t |X_s^{(k)}| dB_s \\ 2 \int_0^t X_{s-}^{(k)} d(X^{(k)})_s^d + \sum\limits_{s \leq t} \left(\Delta X_s^{(k)}\right)^2 = 2kt \end{cases}$$

Therefore $1_{X_s^{(k)} \neq 0} d < (X^{(k)})^c >_s = ds$ *and* $1_{X_s^{(k)} = 0} d < (X^{(k)})^c >_s = 0$, *but the zero set of the squared Bessel process* $(X^{(k)})^2$ *(hence of* $X^{(k)}$*)*

has 0-Lebesgue measure. Hence $(X^{(k)})^c$ *is a Brownian motion (Lévy's characterization theorem). The result then follows.*

d) *It suffices to show that the processes* $(Y_u;\ u \geq 0)$ *and* $(X^{(k)}_{\tau_u},\ u \geq 0)$*, where* $\tau_u = \inf\{t \geq 0,\ A_t > u\}$*, are both Markovian with the same infinitesimal generator, namely*

$$\widetilde{\mathcal{L}}f(x) = \frac{x^2}{2} f''(x) + k \left(x f'(x) + \frac{f(-x) - f(x)}{2} \right)$$

e) *The hypotheses made in Theorem 4.7 apply:*
- *the totality property follows from the fact that each variable* $(X^{(k)}_t)^2$ *admits some exponential moments*
- $\Delta X^{(k)}_t = -2X^{(k)}_{t-} 1_{\Delta X^{(k)}_t \neq 0}$
- *we have just seen previously that*

$$< (X^{(k)})^{(c)} >_t = t, \qquad < (X^{(k)})^{(d)} >_t = 2kt$$

S_5 Chapter 5

• **Solution to Exercise 38** *(p.99)*

a) $(\mathcal{A}_t;\ t \geq 0)$ *is immersed in* $(\mathcal{B}_t;\ t \geq 0)$ *if and only if every uniformly integrable* $(\mathcal{A}_t;\ t \geq 0)$*-martingale* $(M^{\mathcal{A}}_t;\ t \geq 0)$ *is a* $(\mathcal{B}_t;\ t \geq 0)$*-martingale; hence, it may be written as* $M^{\mathcal{A}}_t = \mathbb{E}\left[M^{\mathcal{A}}_\infty | \mathcal{B}_t\right]$*, that is*

$$\mathbb{E}\left[M^{\mathcal{A}}_\infty | \mathcal{A}_t\right] = \mathbb{E}\left[M^{\mathcal{A}}_\infty | \mathcal{B}_t\right]$$

Since $M^{\mathcal{A}}_\infty$ *may be taken to be any variable in* $L^1(\mathcal{A}_\infty)$*, we get the desired result.*

b) *Using both the PRP and the* $(\mathcal{B}_t;\ t \geq 0)$*-martingale properties of* $(a_t;\ t \geq 0)$*, the result follows from the fact that* $(\mathcal{A}_t;\ t \geq 0)$*-predictable processes remain* $(\mathcal{B}_t;\ t \geq 0)$*-predictable processes.*

• **Solution to Exercise 39** *(p.99)*

a) *We give two different arguments:*
- *Since the filtration of* $|B|$ *is that of* $\beta := \int_0^\cdot sgn(B_s) dB_s$*, which is a* $(\mathcal{F}_t;\ t \geq 0)$*-Brownian motion, we can use the predictable representation property of* β*, and use Exercise 38.*
- $|B|$ *is a* $(\mathcal{F}_t;\ t \geq 0)$*-Markov process (Dynkin's criteria, see e.g. [RP81]). Then, the martingale additive functionals of* $|B|$ *are also* $(\mathcal{F}_t;\ t \geq 0)$*-martingales; since they generate (in the sense of Kunita-Watanabe) all* $(\sigma\{|B_u|,\ u \leq t\},\ t \geq 0)$*-martingales, the latter are also* $(\mathcal{F}_t;\ t \geq 0)$*-martingales.*

b) *For general integers n, both arguments may be adequately extended.*

c) This is immediate since there exist discontinuous $(\mathcal{F}_t^+,\ t \geq 0)$*-martingales as proven in Theorem 5.5.*

- **Solution to Exercise 40** *(p.101)*

a) 1. *Use Tanaka's formula.*
2. *For any bounded functional* F*, the Cameron-Martin formula yields*

$$\begin{aligned}\mathbb{E}\left[F(|B_s^{(\mu)}|,\ s \leq t)\right] &= \mathbb{E}\left[F(|B_s|,\ s \leq t)e^{\mu B_t - \frac{\mu^2}{2}t}\right] \\ &= \mathbb{E}\left[F(|B_s|,\ s \leq t)e^{-\mu B_t - \frac{\mu^2}{2}t}\right] \\ &= \mathbb{E}\left[F(|B_s|,\ s \leq t)\cosh(\mu B_t)e^{-\frac{\mu^2}{2}t}\right]\end{aligned}$$

Formula (5.9) *now follows from Girsanov's theorem.*
The equality of the two filtrations follows by considering $(B_t^{(\mu)})^2$ *which appears as a strong solution of a SDE driven by* β*.*
3. *Comparing the two decomposition formulae* (5.9) *and* (5.10)*, we see that* β *is not a* $(\mathcal{F}_t;\ t \geq 0)$*-martingale.*

Comment 7.5 *With the notation of Proposition 5.13, we have*

$$A_t^{\mathcal{F}} = \mu \int_0^t sgn(B_s^{(\mu)})ds + L_t^0(B^{(\mu)}), \quad (A^{\mathcal{F}})_t^{(p)} = \mu \int_0^t \tanh(\mu |B_s^{(\mu)}|)ds + L_t^0(B^{(\mu)})$$

b) 1. *With the conditional law recalled in the exercise, we deduce*

$$\begin{aligned}\mathbb{E}\left[F(R_s^{(\mu)};\ s \leq t)\right] &= \mathbb{E}\left[F(R_s^{(0)};\ s \leq t)e^{\mu B_t - \frac{\mu^2}{2}t}\right] \\ &= \mathbb{E}\left[F(R_s^{(0)};\ s \leq t)\frac{1}{2R_t^{(0)}}\int_{-R_t^{(0)}}^{R_t^{(0)}} dx\ e^{\mu x}e^{-\frac{\mu^2}{2}t}\right] \\ &= \mathbb{E}\left[F(R_s^{(0)};\ s \leq t)\frac{\sinh(\mu R_t^{(0)})}{\mu R_t^{(0)}}e^{-\frac{\mu^2}{2}t}\right]\end{aligned}$$

Equation (5.11) *is obtained, once more, from Girsanov's theorem.*
2. *This is a consequence of the non-canonical decomposition*

$$R_t^{(\mu)} = -B_t + (2S_t^{(\mu)} - \mu t).$$

3. *Using the Cameron-Martin formula together with the result recalled for processes with 0-drift, we obtain that the law of* $B_t^{(\mu)}$ *conditionally on* $\sigma(R_s^{(\mu)};\ s \leq t)$ *and* $R_t^{(\mu)} = r$ *is*

$$\frac{\mu e^{\mu x}}{2\sinh(\mu r)}1_{x \in (-r,r)}dx$$

• **Solution to Exercise 41** *(p.102)*

a) Recall that $A_1^+ \overset{(law)}{=} \gamma a_1^+ + \varepsilon(1-\gamma)$ *(see Subsection 3.1.2) and multiply both sides of this identity with* $2\mathbf{e}$*, where* $\mathbf{e}$ *denotes a standard exponential variable. From the beta-gamma algebra, we obtain*

$$\mathcal{N}^2 \overset{(law)}{=} \mathcal{N}^2 a_1^+ + \varepsilon \mathcal{N}'^2 \tag{7.4}$$

where on the RHS, the four variables are independent, and $\mathcal{N}$ *and* $\mathcal{N}'$ *are standard normal variables. We then easily deduce from* (7.4) *that*

$$\mathbb{E}\left[e^{-\lambda a_1^+ \mathcal{N}^2}\right] = \frac{\sqrt{2\lambda+1}-1}{\lambda}$$

with $\mathcal{N}$ *a standard Gaussian variable independent from* a_1^+*. Therefore,* $a_1^+\mathcal{N}^2 \overset{(law)}{=} U\mathcal{N}^2$ *with* U *a uniform variable. The result follows from the injectivity of the Gauss transform.*

b) For any bounded function f *on* $[0,1]$,

$$\begin{aligned}
\mathbb{E}(f(a_1^+)) &= \int_0^\infty dl\ l e^{-\frac{l^2}{2}} \mathbb{E}\left[f(A_1^+)|L_1 = l, B_1 = 0\right] \\
&= \int_0^\infty dl\ l e^{-\frac{l^2}{2}} \mathbb{E}\left[f(A_1^+)|\tau_l = 1\right] && \textit{(switching identity)} \\
&= \int_0^\infty dl\ l e^{-\frac{l^2}{2}} \mathbb{E}\left[f(A_{\tau_l}^+)|\tau_l = 1\right] \\
&= \int_0^\infty dm\ e^{-m} \mathbb{E}\left[f(2mA_{\tau_1}^+)|2m\tau_1 = 1\right] && \textit{(scaling)} \\
&= \int_0^\infty \frac{dm}{\sqrt{m}} e^{-m} \mathbb{E}\left[\sqrt{m} f(\frac{A_{\tau_1}^+}{\tau_1})|\frac{1}{2\tau_1} = m\right] \\
&= \int_0^\infty \frac{dm}{\sqrt{m}} e^{-m} \mathbb{E}\left[\frac{1}{\sqrt{2\tau_1}} f(\frac{A_{\tau_1}^+}{\tau_1})|\frac{1}{2\tau_1} = m\right] \\
&= \sqrt{\frac{\pi}{2}} \mathbb{E}\left[\frac{1}{\sqrt{\tau_1}} f(\frac{A_{\tau_1}^+}{\tau_1})\right]
\end{aligned}$$

Now, the result follows from simple computations with the variables $\mathcal{N}^2 \overset{(law)}{=} \frac{1}{2A_{\tau_1}^+}$ *and* $\mathcal{N}'^2 \overset{(law)}{=} \frac{1}{2A_{\tau_1}^-}$*, where* $\mathcal{N}$ *and* $\mathcal{N}'$ *may be chosen to be independent standard Gaussian variables.*

Remark 7.6 *In [Yor95], using the absolute continuity between the standard Brownian bridge and the pseudo bridge*[1] $(\frac{1}{\sqrt{\tau_1}} B_{u\tau_1},\ u \le 1)$*, one also arrives at the identity*

[1] See [BLGY87] for more details about this process; in fact, this absolute continuity result may also be obtained as a consequence of the switching identity, see e.g. [PY92].

$$\mathbb{E}(f(a_1^+)) = \sqrt{\frac{\pi}{2}}\mathbb{E}\left[\frac{1}{\sqrt{\tau_1}} f\left(\frac{A^+_{\tau_1}}{\tau_1}\right)\right]$$

c) On one hand,
$\mathbb{E}\left[\int_0^\infty dL_t F(B_u;\ u \le t)\varphi(L_t)\psi(t)\right] =$

$$= \mathbb{E}\left[\int_0^\infty dL_t \mathbb{E}\left[F(B_u;\ u \le t)\varphi(L_t)|B_t = 0\right]\psi(t)\right]$$
$$= \int_0^\infty \frac{dt}{\sqrt{2\pi t}}\mathbb{E}\left[F(B_u;\ u \le t)\varphi(L_t)|B_t = 0\right]\psi(t)$$
$$= \int_0^\infty \frac{dt\psi(t)}{\sqrt{2\pi t}}\int_0^\infty P(L_t \in dl|B_t = 0)\varphi(l)\mathbb{E}\left[F(B_u;\ u \le t)|B_t = 0, L_t = l\right]$$

On the other hand,

$$\mathbb{E}\left[\int_0^\infty dL_t F(B_u;\ u \le t)\varphi(L_t)\psi(t)\right] = \mathbb{E}\left[\int_0^\infty dl F(B_u;\ u \le \tau_l)\varphi(l)\psi(\tau_l)\right]$$
$$= \int_0^\infty dl\varphi(l)\int_0^\infty P(\tau_l \in dt)\psi(t)\mathbb{E}\left[F(B_u;\ u \le \tau_l)|\tau_l = t\right]$$

We can identify these two quantities for any φ, ψ. Thus
$\frac{dt}{\sqrt{2\pi t}}P(L_t \in dl|B_t = 0)\mathbb{E}\left[F(B_u;\ u \le t)|B_t = 0, L_t = l\right] =$

$$dlP(\tau_l \in dt)\mathbb{E}\left[F(B_u;\ u \le \tau_l)|\tau_l = t\right]$$

Taking $F = 1$, we get $\frac{dt}{\sqrt{2\pi t}}P(L_t \in dl|B_t = 0) = dlP(\tau_l \in dt)$ (note that this entails (5.12) *since: $P(\tau_l \in dt) = \frac{le^{-l^2/2t}dt}{\sqrt{2\pi t^3}}$) and, therefore, the switching identity:*

$$\mathbb{E}\left[F(B_u;\ u \le t)|B_t = 0, L_t = l\right] = \mathbb{E}\left[F(B_u;\ u \le \tau_l)|\tau_l = t\right]$$

• **Solution to Exercise 42** *(p.102)*

a) This result is a simple consequence of Tanaka's formula for $(B_t - x)^+$.
b) Let $\alpha^{x,-}_{\cdot}$ denote the right-continuous inverse of $A^{x,-}$.

$$\begin{aligned}
\mathbb{E}\left[f(B_\xi)1_{B_\xi<x}|\mathcal{E}^x_{\mathbb{W}}\right] &= \lambda\mathbb{E}\left[\int_0^\infty f(B_t)1_{B_t<x}e^{-\lambda t}dt|\mathcal{E}^x_{\mathbb{W}}\right]\\
&= \lambda\int_0^\infty du f(B_{\alpha_u^{x,-}})\mathbb{E}\left[e^{-\lambda\alpha_u^{x,-}}|\mathcal{E}^x_{\mathbb{W}}\right]\\
&= \lambda\int_0^\infty du f(B_{\alpha_u^{x,-}})e^{-\lambda u}\mathbb{E}\left[e^{-\lambda A^{x,+}(\alpha_u^{x,-})}|\mathcal{E}^x_{\mathbb{W}}\right]\\
&= \lambda\int_0^\infty du f(B_{\alpha_u^{x,-}})e^{-\lambda u}e^{-\sqrt{2\lambda}\frac{1}{2}L^x_{\alpha_u^{x-}}} \qquad \textit{(using a))}\\
&= \lambda\int_0^\infty dt 1_{B_t<x}f(B_t)e^{-\lambda A_t^{x,-}-\sqrt{\frac{\lambda}{2}}L_t^x}
\end{aligned}$$

Remark 7.7 *In fact, this exercise gives a glimpse of D. Williams' attempt to prove the continuity of martingales in the filtration* $\mathcal{E}^{\cdot}_{\mathbb{W}}$ *of Definition 5.2. More precisely, D. Williams considers the martingales* $\left(\mathbb{E}\left[\prod_{i=1}^{N} f_i(B_{\xi_i})|\mathcal{E}^{x}_{\mathbb{W}}\right];\ x\in\mathbb{R}\right)$ *and uses arguments similar to those in this exercise (or their excursion theory counterpart) to obtain the a.s. continuity.*

• **Solution to Exercise 43** *(p.102)*✠

S_6 Chapter 6

• **Solution to Exercise 44** *(p.105)*✠

• **Solution to Exercise 45** *(p.114)*
The main difficulty is to justify the exchange of the symbols $\vee$ *and* $\bigcap$ *in*

$$\mathcal{F}_{L+}\left(=\bigcap_{\varepsilon>0}\mathcal{F}_{L+\varepsilon}\right)=\bigcap_{\varepsilon>0}\left(\mathcal{F}_L\vee\sigma\{B_{L+u},\ u\le\varepsilon\}\right)$$

This exchange (which has been the cause of many errors! See some discussion in Chaumont-Yor [CY03], Exercise 2.5 p.29) is licit in the following cases because of the following independence property.

a) $L=\sup\{t\le\gamma_{T_1},\ B_t=S_t\}$ *or* $L=\gamma_{T_1}$*; then, the independence property between* $\mathcal{F}_L$ *and* $(B_{L+u};\ u\ge 0)$ *holds.*
b) $L=\gamma$*. then,* $\mathcal{F}_\gamma$ *and* $\bigcap_{\eta>0}\sigma\left\{B_u^{[\gamma,1]},\ u\le\eta\right\}$ *are independent and the second* σ*-field reduces to* $\sigma(A)$ *where* $A=\{B_1>0\}$*.*

• **Solution to Exercise 46** *(p.115)*
First note that, from the comparison theorem, the sticky Brownian motion starting from $x\ge 0$ *turns out to be a* $\mathbb{R}_+$*-valued diffusion.*

a) A simple application of Itô's formula yields the desired martingale property. Then, we deduce, for any $t\ge 0$*,*

$$\mathbb{E}\left[e^{-\sqrt{2\lambda}\ X_t-\lambda t}+(\sqrt{2\lambda}\ \theta+\lambda)\int_0^t e^{-\lambda s}1_{X_s=0}ds\right]=1$$

Since X *is a positive process, the right-hand side tends to* $(\sqrt{2\lambda}\ \theta+\lambda)\mathbb{E}\left[\int_0^\infty e^{-\lambda s}1_{X_s=0}ds\right]$ *as* t *tends to* $+\infty$*.*
b) Once again, the result is deduced from Itô's formula and from the positivity of $X\vee\widetilde{X}$*.*
c) It suffices to use the L^2*-convergence of* $1_{X_t^{(n)}=0}-1_{\widetilde{X}_t^{(n)}=0}$ *towards 0.*

- **Solution to Exercise 47** *(p.116)*✠

- **Solution to Exercise 48** *(p.116)*

Here is an interesting example:
Consider $(f_i)_{i=1,..,n}$ *a set of* n *space-time harmonic functions for Brownian motion, i.e.*

$$\frac{\partial}{\partial t} f_i + \frac{1}{2}\frac{\partial^2}{\partial x^2} f_i = 0$$

such that

$$\begin{cases} f_i(x,t) \geq 0 & \text{if } x \geq 0 \\ f_i(x,t) = 0 & \text{if and only if } x = 0 \\ \frac{\partial}{\partial x} f_i(0,t) & \text{is independent of } i \end{cases}$$

Then, if $(Z_t^i;\ t \geq 0)_{i=1,..,n}$ *denotes Walsh's Brownian motion, then* $(f_i(Z_t^i, t),\ t \geq 0)_{i=1,..,n}$ *is a spider-martingale.*
The example $f_i(x,t) = \sinh(\lambda x) e^{-\frac{\lambda^2 t}{2}}$ *(with the same* λ*, independent of* i*) is of particular interest in the following*

Application:

To prove formulae (6.7) *and* (6.8)*, we consider the previous spider martingale. There exists a constant* C*, independent of* i*, such that*

$$C = \sinh(\lambda z_i)\mathbb{E}\left[e^{-\frac{\lambda^2}{2}T_{\{z_1,\ldots,z_n\}}} 1_{Z_{T_{\{z_1,\ldots,z_n\}}}=z_i}\right]$$

Moreover, using the martingale $(\cosh(\lambda|Z_t|)e^{-\frac{\lambda^2 t}{2}};\ t \geq 0)$*, we obtain from Doob's stopping theorem*

$$1 = \sum_{i=1}^{n} \cosh(\lambda z_i)\mathbb{E}\left[e^{-\frac{\lambda^2}{2}T_{\{z_1,\ldots,z_n\}}} 1_{Z_{T_{\{z_1,\ldots,z_n\}}}=z_i}\right]$$

Thus,
$$1 = \sum_{i=1}^{n} \cosh(\lambda z_i)\frac{C}{\sinh(\lambda z_i)}$$

The result (6.7) *follows.*

References

For each reference, we have indicated the relevant chapter and page(s) where it is discussed.

[ABÉH95] S. Attal, K. Burdzy, M. Émery, and Y. Hu. Sur quelques filtrations et transformations browniennes. In *Séminaire de Probabilités, XXIX*, volume 1613 of *Lecture Notes in Math.*, pages 56–69. Springer, Berlin, 1995. →**Chapter 6**. Quoted in page(s) 108

[AÉ96] S. Attal and M. Émery. Martingales d'Azéma bidimensionnelles. *Astérisque*, (236):9–21, 1996. Hommage à P. A. Meyer et J. Neveu, →**Chapter 4**. Quoted in page(s) 82

[AJK+96] J. Azéma, T. Jeulin, F. Knight, G. Mokobodzki, and M. Yor. Sur les processus croissants de type injectif. In *Séminaire de Probabilités, XXX*, volume 1626 of *Lecture Notes in Math.*, pages 312–343. Springer, Berlin, 1996. →**Chapter 0**. Quoted in page(s) 6

[AJKY93] J. Azéma, T. Jeulin, F. Knight, and M. Yor. Le théorème d'arrêt en une fin d'ensemble prévisible. In *Séminaire de Probabilités, XXVII*, volume 1557 of *Lecture Notes in Math.*, pages 133–158. Springer, Berlin, 1993. →**Chapter 1**. Quoted in page(s) 22

[AJKY98] J. Azéma, T. Jeulin, F. Knight, and M. Yor. Quelques calculs de compensateurs impliquant l'injectivité de certains processus croissants. In *Séminaire de Probabilités, XXXII*, volume 1686 of *Lecture Notes in Math.*, pages 316–327. Springer, Berlin, 1998. →**Chapter 0**. Quoted in page(s) 6

[AR94] J. Azéma and C. Rainer. Sur l'équation de structure $d[X,X]_t = dt - X_{t-}^{+} dX_t$. In *Séminaire de Probabilités, XXVIII*, volume 1583 of *Lecture Notes in Math.*, pages 236–255. Springer, Berlin, 1994. →**Chapter 4**. Quoted in page(s) 83

[AY78] J. Azéma and M. Yor, editors. *Temps locaux*, volume 52 of *Astérisque*. Société Mathématique de France, Paris, 1978. Exposés du Séminaire J. Azéma-M. Yor, Held at the Université Pierre et Marie Curie, Paris, 1976–1977, With an English summary, →**Chapter 4**. Quoted in page(s) 80

[AY89] J. Azéma and M. Yor. Étude d'une martingale remarquable. In *Séminaire de Probabilités, XXIII*, volume 1372 of *Lecture Notes in Math.*, pages 88–130. Springer, Berlin, 1989. →**Chapter 4**. Quoted in page(s) 80

[AY92] J. Azéma and M. Yor. Sur les zéros des martingales continues. In *Séminaire de Probabilités, XXVI*, volume 1526 of *Lecture Notes in Math.*, pages 248–306. Springer, Berlin, 1992. →**Chapter 4**. Quoted in page(s) 58, 110

[Azé85a] J. Azéma. Sur les fermés aléatoires. In *Séminaire de probabilités, XIX, 1983/84*, volume 1123 of *Lecture Notes in Math.*, pages 397–495. Springer, Berlin, 1985. →**Chapter 4**. Quoted in page(s) 80

[Azé85b] J. Azéma. Sur les fermés aléatoires. In *Séminaire de Probabilités XIX, 1983/84*, volume 1123 of *Lecture Notes in Math.*, pages 397–495. Springer, Berlin, 1985. →**Chapter 4**. Quoted in page(s) 82

[Bar78] M. Barlow. Study of a filtration expanded to include an honest time. *Z. Wahrsch. Verw. Gebiete*, 44(4):307–323, 1978. →**Chapter 1**. Quoted in page(s) 64

[Bau02] F. Baudoin. Conditioned stochastic differential equations: theory, examples and application to finance. *Stochastic Process. Appl.*, 100:109–145, 2002. →**Chapter 1**. Quoted in page(s) 22

[Bau03] F. Baudoin. Modeling anticipations on financial markets. In *Paris-Princeton Lectures on Mathematical Finance, 2002*, volume 1814 of *Lecture Notes in Math.*, pages 43–94. Springer, Berlin, 2003. →**Chapter 1**. Quoted in page(s) 22

[Bau04] F. Baudoin. Conditioning and initial enlargement of filtration on a Riemannian manifold. *Ann. Probab.*, 32(3A):2286–2303, 2004. →**Chapter 1**. Quoted in page(s) 22

[BÉK+98] M. T. Barlow, M. Émery, F. B. Knight, S. Song, and M. Yor. Autour d'un théorème de Tsirelson sur des filtrations browniennes et non browniennes. In *Séminaire de Probabilités, XXXII*, volume 1686 of *Lecture Notes in Math.*, pages 264–305. Springer, Berlin, 1998. →**Chapter 6**. Quoted in page(s) 110

[Ber91] J. Bertoin. Sur la décomposition de la trajectoire d'un processus de Lévy spectralement positif en son infimum. *Ann. Inst. H. Poincaré Probab. Statist.*, 27(4):537–547, 1991. →**Chapter 1**. Quoted in page(s) 18

[Ber92] J. Bertoin. An extension of Pitman's theorem for spectrally positive Lévy processes. *Ann. Probab.*, 20(3):1464–1483, 1992. →**Chapter 1**. Quoted in page(s) 18

[Bia94] P. Biane. Quelques propriétés du mouvement brownien dans un cone. *Stochastic Process. Appl.*, 53(2):233–240, 1994. →**Chapter 1**. Quoted in page(s) 18

[BJY84] M. Barlow, S. Jacka, and M. Yor. Inégalités pour un couple de processus arrêtés à un temps quelconque. *C. R. Acad. Sci. Paris Sér. I Math.*, 299(8):351–354, 1984. →**Chapter 2**. Quoted in page(s) 46

[BJY86] M. Barlow, S. Jacka, and M. Yor. Inequalities for a pair of processes stopped at a random time. *Proc. London Math. Soc. (3)*, 52(1):142–172, 1986. →**Chapter 2**. Quoted in page(s) 46

[BLGY87] P. Biane, J.-F. Le Gall, and M. Yor. Un processus qui ressemble au pont brownien. In *Séminaire de Probabilités, XXI*, volume 1247 of *Lecture*

Notes in Math., pages 270–275. Springer, Berlin, 1987. →**Chapter 3**. Quoted in page(s) 136

[BM82] M. Berger and V. Mizel. An extension of the stochastic integral. *Ann. Probab.*, 10(2):435–450, 1982. →**Chapter 1**. Quoted in page(s) 22

[BP94] J. Bertoin and J. Pitman. Path transformations connecting Brownian bridge, excursion and meander. *Bull. Sci. Math.*, 118(2):147–166, 1994. →**Chapter 3**. Quoted in page(s) 55

[BPY89a] M. Barlow, J. Pitman, and M. Yor. On Walsh's Brownian motions. In *Séminaire de Probabilités, XXIII*, volume 1372 of *Lecture Notes in Math.*, pages 275–293. Springer, Berlin, 1989. →**Chapter 6**. Quoted in page(s) 109

[BPY89b] M. Barlow, J. Pitman, and M. Yor. Une extension multidimensionnelle de la loi de l'arc sinus. In *Séminaire de Probabilités, XXIII*, volume 1372 of *Lecture Notes in Math.*, pages 294–314. Springer, Berlin, 1989. →**Chapter 5**. Quoted in page(s) 89

[BPY01] P. Biane, J. Pitman, and M. Yor. Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions. *Bull. Amer. Math. Soc. (N.S.)*, 38(4):435–465 (electronic), 2001. →**Chapter 2**. Quoted in page(s) 49

[BS00] S. Beghdadi-Sakrani. *Martingales continues,filtrations faiblement Browniennes et mesures signées.* PhD thesis, Université Paris VI, 2000. →**Chapter 6**. Quoted in page(s) 114

[BS02a] S. Beghdadi-Sakrani. The uniqueness class of continuous local martingales. *Bernoulli*, 8(2):207–217, 2002. →**Chapter 6**. Quoted in page(s) 114

[BS02b] A. Borodin and P. Salminen. *Handbook of Brownian motion—facts and formulae.* Probability and its Applications. Birkhäuser Verlag, Basel, second edition, 2002. Quoted in page(s) 23

[BS03a] S. Beghdadi-Sakrani. Calcul stochastique pour des mesures signées. In *Séminaire de Probabilités, XXXVI*, volume 1801 of *Lecture Notes in Math.*, pages 366–382. Springer, Berlin, 2003. →**Chapter 3,6**. Quoted in page(s) 69, 114

[BS03b] S. Beghdadi-Sakrani. On pathwise uniqueness of stochastic differential equations with drift. *J. Theoret. Probab.*, 16(4):789–812 (2004), 2003. →**Chapter 6**. Quoted in page(s) 114

[BS03c] S. Beghdadi-Sakrani. Some remarkable pure martingales. *Ann. Inst. H. Poincaré Probab. Statist.*, 39(2):287–299, 2003. →**Chapter 6**. Quoted in page(s) 114

[BS03d] S. Beghdadi-Sakrani. Une martingale non pure, dont la filtration est brownienne. In *Séminaire de Probabilités, XXXVI*, volume 1801 of *Lecture Notes in Math.*, pages 348–359. Springer, Berlin, 2003. →**Chapter 6**. Quoted in page(s) 114

[BS04] A. N. Borodin and P. Salminen. On some exponential integral functionals of BM(μ) and BES(3). *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 311(Veroyatn. i Stat. 7):51–78, 298–299, 2004. →**Chapter 1**. Quoted in page(s) 21

[BSÉ99] S. Beghdadi-Sakrani and M. Émery. On certain probabilities equivalent to coin-tossing, d'après Schachermayer. In *Séminaire de Probabilités, XXXIII*, volume 1709 of *Lecture Notes in Math.*, pages 240–256. Springer, Berlin, 1999. →**Chapter 6**. Quoted in page(s) 108, 113, 114

[Bur02] D. Burkholder. The best constant in the Davis inequality for the expectation of the martingale square function. *Trans. Amer. Math. Soc.*, 354(1):91–105 (electronic), 2002. →**Chapter 2**. Quoted in page(s) 49

[BY78] P. Brémaud and M. Yor. Changes of filtrations and of probability measures. *Z. Wahrsch. Verw. Gebiete*, 45(4):269–295, 1978. →**Chapter 5**. Quoted in page(s) 99

[BY87] P. Biane and M. Yor. Valeurs principales associées aux temps locaux browniens. *Bull. Sci. Math. (2)*, 111(1):23–101, 1987. →**Chapter 3**. Quoted in page(s) 69

[BY88] P. Biane and M. Yor. Quelques précisions sur le méandre brownien. *Bull. Sci. Math. (2)*, 112(1):101–109, 1988. →**Chapter 3**. Quoted in page(s) 57

[Cho84] C. S. Chou. Sur certaines généralisations de l'inégalité de Fefferman. In *Seminar on probability, XVIII*, volume 1059 of *Lecture Notes in Math.*, pages 219–222. Springer, Berlin, 1984. →**Chapter 2**. Quoted in page(s) 46

[Chu72] K. L. Chung. Some universal field equations. In *Séminaire de Probabilités, VI (Univ. Strasbourg, année universitaire 1970-1971)*, pages 90–97. Lecture Notes in Math., Vol. 258. Springer, Berlin, 1972. →**Chapter 0**. Quoted in page(s) 7

[Chu76] K. L. Chung. Excursions in Brownian motion. *Ark. Mat.*, 14(2):155–177, 1976. →**Chapter 2**. Quoted in page(s) 49

[Chy05] O. Chybiryakov. Skew-product representations of multidimensional Dunkl Markov processes. 2005. →**Chapter 4**. Quoted in page(s) 86

[CPY98] P. Carmona, F. Petit, and M. Yor. Beta-gamma random variables and intertwining relations between certain Markov processes. *Rev. Mat. Iberoamericana*, 14(2):311–367, 1998. →**Chapter 4**. Quoted in page(s) 84, 132

[CSK99] T. Choulli, C. Stricker, and L. Krawczyk. On Fefferman and Burkholder-Davis-Gundy inequalities for $\mathcal{E}$-martingales. *Probab. Theory Related Fields*, 113(4):571–597, 1999. →**Chapter 2**. Quoted in page(s) 49

[CY03] L. Chaumont and M. Yor. *Exercises in probability: a guided tour from measure theory to random processes, via conditioning*, volume 13 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 2003. Quoted in page(s) 49, 76, 138

[Dav05] M. Davis. *Martingale representation and all that*, volume Advances in Control, Communication Networks, and Transportation Systems: In Honor of Pravin Varaiya of *Systems and Control: Foundations and Applications Series*. Birkhauser, 2005. →**Chapter 4**. Quoted in page(s) 74

[Del68] C. Dellacherie. Une représentation intégrale des surmartingales à temps discret. *Publ. Inst. Statist. Univ. Paris*, 17(2):1–17, 1968. →**Chapter 4**. Quoted in page(s) 75

[Del72] C. Dellacherie. *Capacités et processus stochastiques*. Springer-Verlag, Berlin, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 67. Quoted in page(s) VII, 6

[Del74] C. Dellacherie. Intégrales stochastiques par rapport aux processus de Wiener ou de Poisson. In *Séminaire de Probabilités, VIII (Univ. Strasbourg, année universitaire 1972-1973)*, pages 25–26. Lecture Notes in Math., Vol. 381. Springer, Berlin, 1974. →**Chapter 4**. Quoted in page(s) 74, 75

[Del75] C. Dellacherie. Correction à: "Intégrales stochastiques par rapport aux processus de Wiener ou de Poisson" (*séminaire de probabilités, viii* (Univ. Strasbourg, année universitaire 1972-1973), pp. 25–26, Lecture Notes in Math., Vol. 381, Springer, Berlin, 1974). In *Séminaire de Probabilités, IX (Seconde Partie, Univ. Strasbourg, Strasbourg, années universitaires 1973/1974 et 1974/1975)*, pages p. 494. Lecture Notes in Math., Vol. 465. Springer, Berlin, 1975. →**Chapter 4**. Quoted in page(s) 74, 75

[Del98] F. Delbaen. A remark on Slutsky's theorem. In *Séminaire de Probabilités, XXXII*, volume 1686 of *Lecture Notes in Math.*, pages 313–315. Springer, Berlin, 1998. →**Chapter 6**. Quoted in page(s) 111

[Den83] I. V. Denisov. Random walk and the Wiener process considered from a maximum point. *Teor. Veroyatnost. i Primenen.*, 28(4):785–788, 1983. →**Chapter 3**. Quoted in page(s) 67

[DÉY91] L.E. Dubins, M. Émery, and M. Yor. A continuous martingale in the plane that may spiral away to infinity. *Séminaire de Probabilités XXV*, pages 284–290, 1991. →**Chapter 1, 2**. Quoted in page(s) 48, 117

[DÉY93] L. E. Dubins, M. Émery, and M. Yor. On the Lévy transformation of Brownian motions and continuous martingales. In *Séminaire de Probabilités, XXVII*, volume 1557 of *Lecture Notes in Math.*, pages 122–132. Springer, Berlin, 1993. →**Chapter 6**. Quoted in page(s) 105

[DF87] P. Diaconis and D. Freedman. A dozen de Finetti-style results in search of a theory. *Ann. Inst. H. Poincaré Probab. Statist.*, 23(2, suppl.):397–423, 1987. →**Chapter 2**. Quoted in page(s) 51

[DFST96] L. Dubins, J. Feldman, M. Smorodinsky, and B. Tsirel'son. Decreasing sequences of σ-fields and a measure change for Brownian motion. *Ann. Probab.*, 24(2):882–904, 1996. →**Chapter 6**. Quoted in page(s) 108

[dlPE97] V. de la Peña and N. Eisenbaum. Exponential Burkholder Davis Gundy inequalities. *Bull. London Math. Soc.*, 29(2):239–242, 1997. →**Chapter 2**. Quoted in page(s) 49

[dlPHDV04] V. de la Peña and G. Hernandez-Del-Valle. On first-passage time densities of brownian motion over a non decreasing, continuous barrier. Technical report, Columbia University, 2004. →**Chapter 2**. Quoted in page(s) 126

[DM78] C. Dellacherie and P.-A. Meyer. À propos du travail de Yor sur le grossissement des tribus. In *Séminaire de Probabilités, XII (Univ. Strasbourg, Strasbourg, 1976/1977)*, volume 649 of *Lecture Notes in Math.*, pages 70–77. Springer, Berlin, 1978. →**Chapter 1**. Quoted in page(s) 14

[DM80] C. Dellacherie and P.-A. Meyer. *Probabilités et potentiel. Chapitres V à VIII*, volume 1385 of *Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics]*. Hermann, Paris, revised edition, 1980. Théorie des martingales. [Martingale theory]. Quoted in page(s) 119

[DM99] B. De Meyer. Une simplification de l'argument de Tsirelson sur le caractère non-brownien des processus de Walsh. In *Séminaire de Probabilités, XXXIII*, volume 1709 of *Lecture Notes in Math.*, pages 217–220. Springer, Berlin, 1999. →**Chapter 6**. Quoted in page(s) 114

[DMM92] C. Dellacherie, B. Maisonneuve, and P.A. Meyer. *Probabilités et potentiel, Chapitres XVII-XXIV: Processus de Markov (fin), Compléments de calcul stochastique*. Hermann, 1992. Quoted in page(s) VII, 72, 83

[Dou66] R. G. Douglas. On extremal measures and subspace density. II. *Proc. Amer. Math. Soc.*, 17:1363–1365, 1966. →**Chapter 4**. Quoted in page(s) 76

[Duf90] D. Dufresne. The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand. Actuar. J.*, (1-2):39–79, 1990. →**Chapter 1**. Quoted in page(s) 27

[DV74] M. H. A. Davis and P. Varaiya. The multiplicity of an increasing family of σ-fields. *Ann. Probability*, 2:958–963, 1974. →**Chapter 6**. Quoted in page(s) 88, 103

[Éme89] M. Émery. On the Azéma martingales. In *Séminaire de Probabilités, XXIII*, volume 1372 of *Lecture Notes in Math.*, pages 66–87. Springer, Berlin, 1989. →**Chapter 4**. Quoted in page(s) VIII, 78, 80, 85

[Éme90] M. Émery. Sur les martingales d'Azéma (suite). In *Séminaire de Probabilités XXIV, 1988/89*, volume 1426 of *Lecture Notes in Math.*, pages 442–447. Springer, Berlin, 1990. →**Chapter 4**. Quoted in page(s) 80

[Éme91] M. Émery. Quelques cas de représentation chaotique. In *Séminaire de Probabilités XXV*, volume 1485 of *Lecture Notes in Math.*, pages 10–23. Springer, Berlin, 1991. →**Chapter 4**. Quoted in page(s) 76

[Éme94] M. Émery. Correction to: "Some cases of chaos representation" [in Séminaire de Probabilités XXV, 10–23, Lecture Notes in Math., 1485, Springer, Berlin, 1991]. In *Séminaire de Probabilités XXVIII*, volume 1583 of *Lecture Notes in Math.*, page 334. Springer, Berlin, 1994. →**Chapter 4**. Quoted in page(s) 76

[Éme96] M. Émery. On the chaotic representation property for martingales. In *Probability theory and mathematical statistics (St. Petersburg, 1993)*, pages 155–166. Gordon and Breach, Amsterdam, 1996. →**Chapter 4**. Quoted in page(s) 76

[Éme02a] M. Émery. Espaces probabilisés filtrés: de la théorie de Vershik au mouvement brownien, via des idées de Tsirelson. *Astérisque*, (282):Exp. No. 882, vii, 63–83, 2002. Séminaire Bourbaki, Vol. 2000/2001 →**Chapter 6**. Quoted in page(s) 108, 112, 113, 116

[Éme02b] M. Émery. Old and new tools in the theory of filtrations. In *Dynamics and randomness (Santiago, 2000)*, volume 7 of *Nonlinear Phenom. Complex Systems*, pages 125–146. Kluwer Acad. Publ., Dordrecht, 2002. →**Chapter 6**. Quoted in page(s) 114

[Éme05] M. Émery. Chaotic representation property of certain Azéma martingales. *Submitted to Illinois Math. Journal*, 2005. →**Chapter 4**. Quoted in page(s) 76, 82, 86

[ÉS99a] M. Émery and W. Schachermayer. Brownian filtrations are not stable under equivalent time-changes. In *Séminaire de Probabilités, XXXIII*, volume 1709 of *Lecture Notes in Math.*, pages 267–276. Springer, Berlin, 1999. →**Chapter 6**. Quoted in page(s) 108, 109

[ÉS99b] M. Émery and W. Schachermayer. A remark on Tsirelson's stochastic differential equation. In *Séminaire de Probabilités, XXXIII*, volume 1709 of *Lecture Notes in Math.*, pages 291–303. Springer, Berlin, 1999. →**Chapter 6**. Quoted in page(s) 114

[ÉS01] M. Émery and W. Schachermayer. On Vershik's standardness criterion and Tsirelson's notion of cosiness. In *Séminaire de Probabilités, XXXV*, volume 1755 of *Lecture Notes in Math.*, pages 265–305. Springer, Berlin, 2001. →**Chapter 6**. Quoted in page(s) 114

[ÉY98] M. Émery and M. Yor. Sur un théorème de Tsirelson relatif à des mouvements browniens corrélés et à la nullité de certains temps locaux. In *Séminaire de Probabilités, XXXII*, volume 1686 of *Lecture Notes in Math.*, pages 306–312. Springer, Berlin, 1998. →**Chapter 6**. Quoted in page(s) 111

[FI93] H. Föllmer and P. Imkeller. Anticipation cancelled by a Girsanov transformation: a paradox on Wiener space. *Ann. Inst. H. Poincaré Probab. Statist.*, 29(4):569–586, 1993. →**Chapter 1**. Quoted in page(s) 22

[FPY93] P. Fitzsimmons, J. Pitman, and M. Yor. Markovian bridges: construction, Palm interpretation, and splicing. In *Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992)*, volume 33 of *Progr. Probab.*, pages 101–134. Birkhäuser Boston, Boston, MA, 1993. →**Chapter 0**. Quoted in page(s) 118

[GR91] A. Goswami and B. V. Rao. Conditional expectation of odd chaos given even. *Stochastics Stochastics Rep.*, 35(4):213–214, 1991. →**Chapter 6**. Quoted in page(s) 114

[Gro89] P. Groeneboom. Brownian motion with a parabolic drift and Airy functions. *Probab. Theory Related Fields*, 81(1):79–109, 1989. →**Chapter 2**. Quoted in page(s) 126

[GS72] R. K. Getoor and M. J. Sharpe. Conformal martingales. *Invent. Math.*, 16:271–308, 1972. →**Chapter 0**. Quoted in page(s) 8

[GY05a] L. Gallardo and M. Yor. A chaotic representation property of the multidimensional Dunkl processes. *To appear in Ann. Prob.*, 2005. →**Chapter 4**. Quoted in page(s) 85

[GY05b] L. Gallardo and M. Yor. Some new examples of Markov processes which enjoy the time-inversion property. *Probability Theory and Related Fields*, 132(1):150–162, 2005. →**Chapter 4**. Quoted in page(s) 85

[GY05c] L. Gallardo and M. Yor. Some remarkable properties of the Dunkl martingales. *To appear in Séminaire de Probabilités, XXXIX (hommage to P.A. Meyer)*, 2005. →**Chapter 4**. Quoted in page(s) 85, 86

[Hit75] M. Hitsuda. *Formula for Brownian partial derivatives*, volume Second Japan-USSR Symp. of *Lecture Notes in Maths.*, pages 111–114. Springer, 1975. →**Chapter 1**. Quoted in page(s) 11

[Hu95] Y. Hu. Sur la représentation des $(\mathcal{F}_t^- = \sigma\{B_s^-,\ s \le t\})$-martingales. In *Séminaire de Probabilités, XXIX*, volume 1613 of *Lecture Notes in Math.*, pages 290–296. Springer, Berlin, 1995. →**Chapter 5**. Quoted in page(s) 88

[Hu96] Y. Hu. *Sur le mouvement Brownien: calculs de lois, études asymptotiques, filtrations, relations avec certaines équations paraboliques.* PhD thesis, Université Paris VI, 1996. →**Chapter 5**. Quoted in page(s) 87, 88, 95

[IM74] K. Itô and H. McKean. *Diffusion processes and their sample paths*. Springer-Verlag, Berlin, 1974. Second printing, corrected, Die Grundlehren der mathematischen Wissenschaften, Band 125. Quoted in page(s) 131

[Imh84] J.-P. Imhof. Density factorizations for Brownian motion, meander and the three-dimensional Bessel process, and applications. *J. Appl. Probab.*, 21(3):500–510, 1984. →**Chapter 3**. Quoted in page(s) 55

[Imh85] J.-P. Imhof. On Brownian bridge and excursion. *Studia Sci. Math. Hungar.*, 20(1-4):1–10, 1985. →**Chapter 3**. Quoted in page(s) 55

[Imk96] P. Imkeller. Enlargement of the Wiener filtration by an absolutely continuous random variable via Malliavin's calculus. *Probab. Theory Related Fields*, 106(1):105–135, 1996. →**Chapter 1**. Quoted in page(s) 22

[Itô78] K. Itô. Extension of stochastic integrals. In *Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976)*, pages 95–109, New York, 1978. Wiley. →**Chapter 1**. Quoted in page(s) 11

[Jam05] F. Jamshidian. Chaotic expansion of powers and martingale representation. 2005. → **Chapter 4**. Quoted in page(s) 76

[Jeu79] T. Jeulin. Un théorème de J. W. Pitman. In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78)*, volume 721 of *Lecture Notes in Math.*, pages 521–532. Springer, Berlin, 1979. With an appendix by M. Yor, →**Chapter 1**. Quoted in page(s) 17

[Jeu80] T. Jeulin. *Semi-martingales et grossissement d'une filtration*, volume 833 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980. →**Chapter 1**. Quoted in page(s) VII, 14, 17, 124

[JP88] J. Jacod and P. Protter. Time reversal on Lévy processes. *Ann. Probab.*, 16(2):620–641, 1988. →**Chapter 1**. Quoted in page(s) 20

[JY77] J. Jacod and M. Yor. Étude des solutions extrémales et représentation intégrale des solutions pour certains problèmes de martingales. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 38(2):83–125, 1977. →**Chapter 4**. Quoted in page(s) 72, 75

[JY79] T. Jeulin and M. Yor. Inégalité de Hardy, semimartingales, et faux-amis. In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78)*, volume 721 of *Lecture Notes in Math.*, pages 332–359. Springer, Berlin, 1979. →**Chapter 1**. Quoted in page(s) 20

[JY81] T. Jeulin and M. Yor. Sur les distributions de certaines fonctionnelles du mouvement brownien. In *Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980) (French)*, volume 850 of *Lecture Notes in Math.*, pages 210–226. Springer, Berlin, 1981. →**Chapter 1**. Quoted in page(s) 122

[JY85] T. Jeulin and M. Yor, editors. *Grossissements de filtrations: exemples et applications*, volume 1118 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1985. Papers from the seminar on stochastic calculus held at the Université de Paris VI, Paris, 1982/1983. Quoted in page(s) VII, VIII, 124

[JY90] T. Jeulin and M. Yor. Filtration des ponts browniens et équations différentielles stochastiques linéaires. In *Séminaire de Probabilités,* *XXIV, 1988/89*, volume 1426 of *Lecture Notes in Math.*, pages 227–265. Springer, Berlin, 1990. →**Chapter 1**. Quoted in page(s) 105

[JY93] S. Jacka and M. Yor. Inequalities for nonmoderate functions of a pair of stochastic processes. *Proc. London Math. Soc. (3)*, 67(3):649–672, 1993. →**Chapter 2**. Quoted in page(s) 49

[Kaz94] N. Kazamaki. *Continuous Exponential Martingales and BMO*, volume 1579 of *Lecture Notes in Maths*. Springer-Verlag, 1994. →**Chapter 2**. Quoted in page(s) 49

[KH04] A. Kohatsu-Higa. *Enlargement of Filtrations and Models for Insider Trading*, volume Stochastic processes and applications to mathematical finance, pages 151–165. World Scientific, 2004. Proceedings of the Ritsumeikan international symposium, →**Chapter 1**. Quoted in page(s) 22

[Kho96] D. Khoshnevisan. Lévy classes and self-normalization. *Electronic Journal of Probability*, 1, 1996. →**Chapter 2**. Quoted in page(s) 47

[KM94] F. Knight and B. Maisonneuve. A characterization of stopping times. *Ann. Probab.*, 22(3):1600–1606, 1994. →**Chapter 2**. Quoted in page(s) 41

[Kni73] F. Knight. Local variation of diffusion in local time. *Annals of Probability*, 1:1026–1034, 1973. →**Chapter 2**. Quoted in page(s) 47

[Kni78] F. Knight. On the sojourn times of killed Brownian motion. In *Séminaire de Probabilités, XII (Univ. Strasbourg, Strasbourg, 1976/1977)*, volume 649 of *Lecture Notes in Math.*, pages 428–445. Springer, Berlin, 1978. →**Chapter 1, 2**. Quoted in page(s) 122

[Kni91] F. Knight. Calculating the compensator: method and example. In *Seminar on Stochastic Processes, 1990 (Vancouver, BC, 1990)*, volume 24 of *Progr. Probab.*, pages 241–252. Birkhäuser Boston, Boston, MA, 1991. →**Chapter 0**. Quoted in page(s) 6

[Kni95] F. Knight. A remark on Walsh's Brownian motions. In *Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993)*, number Special Issue, pages 317–323, 1995. →**Chapter 6**. Quoted in page(s) 114

[KS91] I. Karatzas and S. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991. Quoted in page(s) 90

[Kun69] H. Kunita. Absolute continuity of Markov processes and generators. *Nagoya Math. J.*, 36:1–26, 1969. →**Chapter 0**. Quoted in page(s) 119

[Kun76] H. Kunita. Absolute continuity of Markov processes. In *Séminaire de Probabilités, X (Première partie, Univ. Strasbourg, Strasbourg, année universitaire 1974/1975)*, pages 44–77. Lecture Notes in Math., Vol. 511. Springer, Berlin, 1976. →**Chapter 0**. Quoted in page(s) 119

[Lam72] J. Lamperti. Semi-stable Markov processes. I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 22:205–225, 1972. →**Chapter 1**. Quoted in page(s) 21, 38

[Len77] E. Lenglart. Relation de domination entre deux processus. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 13(2):171–179, 1977. →**Chapter 2**. Quoted in page(s) 128

[Lév39] P. Lévy. Sur certains processus stochastiques homogènes. *Compositio Math.*, 7:283–339, 1939. →**Chapter 5**. Quoted in page(s) 88

[LNN03] J. León, R. Navarro, and D. Nualart. An anticipating calculus approach to the utility maximization of an insider. *Math. Finance*, 13(1):171–185, 2003. Conference on Applications of Malliavin Calculus in Finance (Rocquencourt, 2001), →**Chapter 1**. Quoted in page(s) 22

[Mal95] M. Malric. Transformation de Lévy et zéros du mouvement brownien. *Probab. Theory Related Fields*, 101(2):227–236, 1995. →**Chapter 6**. Quoted in page(s) 105

[Mal03] M. Malric. Densité des zéros des transformés de Lévy itérés d'un mouvement brownien. *C. R. Math. Acad. Sci. Paris*, 336(6):499–504, 2003. →**Chapter 6**. Quoted in page(s) 105

[McG84] P. McGill. Quelques propriétés du compensateur du processus des excursions browniennes. *C. R. Acad. Sci. Paris Sér. I Math.*, 298(15):357–359, 1984. →**Chapter 5**. Quoted in page(s) 93

[McG86a] P. McGill. Integral representation of martingales in the Brownian excursion filtration. In *Séminaire de Probabilités, XX, 1984/85*, volume 1204 of *Lecture Notes in Math.*, pages 465–502. Springer, Berlin, 1986. →**Chapter 5**. Quoted in page(s) 93

[McG86b] P. McGill. A martingale characterisation of the Brownian excursion compensator. *Probab. Theory Relat. Fields*, 71(1):117–128, 1986. →**Chapter 5**. Quoted in page(s) 93

[Mey73] P. A. Meyer. Le dual de "H^1" est "BMO" (cas continu). In *Séminaire de Probabilités, VII (Univ. Strasbourg, année universitaire 1971–1972)*, pages 136–145. Lecture Notes in Math., Vol. 321. Springer, Berlin, 1973. →**Chapter 0**. Quoted in page(s) 8

[Mey75] P. A. Meyer. Complément sur la dualité entre H^1 et BMO: "Le dual de "H^1" est "BMO" (cas continu)" (Séminaire de Probabilités, VII (Univ. Strasbourg, année universitaire 1971–1972), pp. 136–145, Lecture Notes in Math., Vol. 321, Springer, Berlin, 1973). In *Séminaire de Probabilités, IX (Seconde Partie, Univ. Strasbourg, Strasbourg, années universitaires 1973/1974 et 1974/1975)*, pages 237–238. Lecture Notes in Math., Vol. 465. Springer, Berlin, 1975. →**Chapter 0**. Quoted in page(s) 8

[Mey77] P. A. Meyer. Le dual de $H^1(\mathbf{R}^\nu)$: démonstrations probabilistes. In *Séminaire de Probabilités, XI (Univ. Strasbourg, Strasbourg, 1975/1976)*, pages 132–195. Lecture Notes in Math., Vol. 581. Springer, Berlin, 1977. →**Chapter 0**. Quoted in page(s) 8

[Mey89] P.-A. Meyer. Construction de solutions d' "équations de structure". In *Séminaire de Probabilités, XXIII*, volume 1372 of *Lecture Notes in Math.*, pages 142–145. Springer, Berlin, 1989. →**Chapter 4**. Quoted in page(s) 83

[MP03] M. Mensi and N. Privault. Conditional calculus on Poisson space and enlargement of filtration. *Stochastic Anal. Appl.*, 21(1):183–204, 2003. →**Chapter 1**. Quoted in page(s) 22

[MW91] T. Mortimer and D. Williams. Change of measure up to a random time: theory. *J. Appl. Probab.*, 28(4):914–918, 1991. →**Chapter 3**. Quoted in page(s) 55

[MY01] H. Matsumoto and M. Yor. A relationship between Brownian motions with opposite drifts via certain enlargements of the Brownian filtration. *Osaka J. Math.*, 38(2):383–398, 2001. →**Chapter 1**. Quoted in page(s) 27

[MY05a] R. Mansuy and M. Yor. *Aspects of Brownian motion*. Universitext. Springer-Verlag, 2005. Quoted in page(s) VII, 55

[MY05b] R. Mansuy and M. Yor. Harnesses, Lévy bridges and *Monsieur Jourdain*. *Stoch. Proc. and Applications*, 115(2):329–338, February 2005. → **Chapter 1**. Quoted in page(s) 20

[Nai47] M.A. Naimark. Extremal spectral functions of a symmetric operator. *Bull. Acad. Sci. URSS Sér. Math.*, 11:327–344, 1947. →**Chapter 4**. Quoted in page(s) 76

[Naj05] J. Najnudel. Pénalisations de l'araignée brownienne. Technical Report 993, Laboratoire de Probabilités et Modèles Aléatoires- Université Paris VI, June 2005. →**Chapter 6**. Quoted in page(s) 112

[Nev72] J. Neveu. *Martingales à temps discret*. Masson et Cie, éditeurs, Paris, 1972. Quoted in page(s) 45

[NP91] D. Nualart and E. Pardoux. Stochastic differential equations with boundary conditions. In *Stochastic analysis and applications (Lisbon, 1989)*, volume 26 of *Progr. Probab.*, pages 155–175. Birkhäuser Boston, Boston, MA, 1991. →**Chapter 1**. Quoted in page(s) 11

[NS00] D. Nualart and W. Schoutens. Chaotic and predictable representations for Lévy processes. *Stochastic Process. Appl.*, 90(1):109–122, 2000. →**Chapter 4**. Quoted in page(s) 76

[Nua93] D. Nualart. Anticipating stochastic differential equations. *Bull. Sci. Math.*, 117(1):49–62, 1993. →**Chapter 1**. Quoted in page(s) 11

[NY05a] A. Nikeghbali and M. Yor. A definition and some characteristic properties of pseudo-stopping times. *To appear in Annals of Probability*, 2005. →**Chapter 2**. Quoted in page(s) 43, 44

[NY05b] A. Nikeghbali and M. Yor. Doob's maximal identity, multiplicative decomposition and enlargements of filtrations. *To appear in Illinois Math. Journal*, 2005. →**Chapter 1**. Quoted in page(s) 16

[Obł05] J. Obłoj. A complete characterization of local martingales which are functions of Brownian motion and its maximum. *Submitted*, 2005. →**Chapter 1**. Quoted in page(s) 26, 123

[Ogu72] H. Ogura. Orthogonal functionals of the Poisson process. *IEEE Trans. Information Theory*, IT-18:473–481, 1972. →**Chapter 4**. Quoted in page(s) 73

[OY05] J. Obłoj and M. Yor. On local martingale and its supremum: harmonic functions and beyond, 2005. Festschrift for A. N. Shyriaev. Métabief→**Chapter 1**. Quoted in page(s) 26, 123

[Pic05a] J. Picard. Brownian excursions, stochastic integrals, and representation of Wiener functionals. *Preprint*, 2005. →**Chapter 5**. Quoted in page(s) 100

[Pic05b] J. Picard. Stochastic calculus and martingales on trees. *Ann. I. H. Poincaré*, 41:631–683, 2005. → **Chapter 6**. Quoted in page(s) 112

[Pit75] J. Pitman. One-dimensional Brownian motion and the three-dimensional Bessel process. *Advances in Appl. Probability*, 7(3):511–526, 1975. →**Chapter 5**. Quoted in page(s) 17

[PK96] I. Pikovsky and I. Karatzas. Anticipative portfolio optimization. *Adv. in Appl. Probab.*, 28(4):1095–1122, 1996. →**Chapter 1**. Quoted in page(s) 22

[PR88] E. J. Pauwels and L. C. G. Rogers. Skew-product decompositions of Brownian motions. In *Geometry of random motion (Ithaca, N.Y., 1987)*, volume 73 of *Contemp. Math.*, pages 237–262. Amer. Math. Soc., Providence, RI, 1988. →**Chapter 7**. Quoted in page(s) 131

[Pro04] P. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, second edition, 2004. Stochastic Modelling and Applied Probability. Quoted in page(s) 80

[PY92] J. Pitman and M. Yor. Arcsine laws and interval partitions derived from a stable subordinator. *Proc. London Math. Soc. (3)*, 65(2):326–356, 1992. →**Chapter 3**. Quoted in page(s) 88, 136

[Rai96] C. Rainer. Projection d'une diffusion sur sa filtration lente. In *Séminaire de Probabilités, XXX*, volume 1626 of *Lecture Notes in Math.*, pages 228–242. Springer, Berlin, 1996. →**Chapter 4**. Quoted in page(s) 82

[Rai97] C. Rainer. Errata: "Projection of a diffusion onto its slow filtration". In *Séminaire de Probabilités, XXXI*, volume 1655 of *Lecture Notes in Math.*, page 329. Springer, Berlin, 1997. →**Chapter 4**. Quoted in page(s) 82

[RdC84] J. Ruiz de Chávez. Le théorème de Paul Lévy pour des mesures signées. In *Seminar on probability, XVIII*, volume 1059 of *Lecture Notes in Math.*, pages 245–255. Springer, Berlin, 1984. →**Chapter 3**. Quoted in page(s) 69

[Rog87] L. C. G. Rogers. Continuity of martingales in the Brownian excursion filtration. *Probab. Theory Related Fields*, 76(3):291–298, 1987. →**Chapter 5**. Quoted in page(s) 93, 95

[RP81] L. C. G. Rogers and J. Pitman. Markov functions. *Ann. Probab.*, 9(4):573–582, 1981. →**Chapter 5**. Quoted in page(s) 134

[RSS84] L. M. Ricciardi, L. Sacerdote, and S. Sato. On an integral equation for first-passage-time probability densities. *J. Appl. Probab.*, 21(2):302–314, 1984. →**Chapter 2**. Quoted in page(s) 126

[RV95] B. Roynette and P. Vallois. Instabilité de certaines équations différentielles stochastiques non linéaires. *J. Funct. Anal.*, 130(2):477–523, 1995. →**Chapter 3**. Quoted in page(s) 64

[RVY05a] B. Roynette, P. Vallois, and M. Yor. Limiting laws associated with Brownian motion perturbated by its maximum, minimum and local time, II. *Studia Sci. Math. Hungarica*, 2005. →**Chapter 1**. Quoted in page(s) 24

[RVY05b] B. Roynette, P. Vallois, and M. Yor. *Penalizations of Brownian motion (Monography in preparation)*. A series of papers to appear in Studia Sci. Math. Hungarica, 2005. →**Chapter 5**. Quoted in page(s) 101

[RW91] L. C. G. Rogers and J. B. Walsh. Local time and stochastic area integrals. *Ann. Probab.*, 19(2):457–482, 1991. →**Chapter 5**. Quoted in page(s) 88, 93

[RY05] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 2005. Quoted in page(s) 5, 23, 55, 107, 121

[Sat99] K. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University

Press, Cambridge, 1999. Translated from the 1990 Japanese original, Revised by the author. Quoted in page(s) 119

[Sch00] W. Schoutens. *Stochastic processes and orthogonal polynomials*, volume 146 of *Lecture Notes in Statistics*. Springer-Verlag, New York, 2000. →**Chapter 4**. Quoted in page(s) 73

[Shi96] Z. Shi. Brownian motion normalized by maximum local time. *Stochastic Processes and their Applications*, 65(2):217–231, 1996. →**Chapter 2**. Quoted in page(s) 47

[Sko75] A. Skorohod. On a generalization of the stochastic integral. *Teor. Verojatnost. i Primenen.*, 20(2):223–238, 1975. →**Chapter 1**. Quoted in page(s)

[ST90] Y. Saisho and H. Tanemura. Pitman type theorem for one-dimensional diffusion processes. *Tokyo J. Math.*, 13(2):429–440, 1990. →**Chapter 5**. Quoted in page(s) 17, 101

[Str93] D. Stroock. *Probability theory, an analytic view.* Cambridge University Press, Cambridge, 1993. →**Chapter 2**. Quoted in page(s) 51

[SY80] D. W. Stroock and M. Yor. On extremal solutions of martingale problems. *Ann. Sci. École Norm. Sup. (4)*, 13(1):95–164, 1980. →**Chapter 4**. Quoted in page(s) 84, 107

[SY87] S. Song and M. Yor. Inégalités pour les processus self-similaires arrêtés à un temps quelconque. In *Séminaire de Probabilités, XXI*, volume 1247 of *Lecture Notes in Math.*, pages 230–245. Springer, Berlin, 1987. →**Chapter 2**. Quoted in page(s) 127

[Tak97] K. Takaoka. On the martingales obtained by an extension due to Saisho, Tanemura and Yor of Pitman's theorem. In *Séminaire de Probabilités, XXXI*, volume 1655 of *Lecture Notes in Math.*, pages 256–265. Springer, Berlin, 1997. →**Chapter 1**. Quoted in page(s) 18

[Tan89] H. Tanaka. Time reversal of random walks in one-dimension. *Tokyo J. Math.*, 12(1):159–174, 1989. →**Chapter 1**. Quoted in page(s) 18

[Tan90] H. Tanaka. Time reversal of random walks in $\mathbf{R}^d$. *Tokyo J. Math.*, 13(2):375–389, 1990. →**Chapter 1**. Quoted in page(s) 18

[Tsi75] B. Tsirel'son. An example of a stochastic differential equation that has no strong solution. *Teor. Verojatnost. i Primenen.*, 20(2):427–430, 1975. →**Chapter 6**. Quoted in page(s) 107

[Tsi97] B. Tsirel'son. Triple points: from non-Brownian filtrations to harmonic measures. *Geom. Funct. Anal.*, 7(6):1096–1142, 1997. →**Chapter 6**. Quoted in page(s) 110, 111

[Tsi98] B. Tsirel'son. Within and beyond the reach of Brownian innovation. In *Proceedings of the International Congress of Mathematicians, Vol. III (Berlin, 1998)*, number Extra Vol. III, pages 311–320 (electronic), 1998. →**Chapter 6**. Quoted in page(s) 111

[Val95] P. Vallois. Decomposing the Brownian path via the range process. *Stochastic Process. Appl.*, 55(2):211–226, 1995. →**Chapter 3**. Quoted in page(s) 133

[Wal78a] J.B. Walsh. A diffusion with a discontinuous local time. *Astérisque (Temps locaux)*, (52-53):37–45, 1978. →**Chapter 6**. Quoted in page(s) 109

[Wal78b] J.B. Walsh. Excursions and local time. *Astérisque (Temps locaux)*, (52-53):159–192, 1978. →**Chapter 5**. Quoted in page(s) 88, 93

[War99] J. Warren. On the joining of sticky Brownian motion. In *Séminaire de Probabilités, XXXIII*, volume 1709 of *Lecture Notes in Math.*, pages 257–266. Springer, Berlin, 1999. →**Chapter 6**. Quoted in page(s) 115

[Wat95] S. Watanabe. Generalized arc-sine laws for one-dimensional diffusion processes and random walks. In *Stochastic analysis (Ithaca, NY, 1993)*, volume 57 of *Proc. Sympos. Pure Math.*, pages 157–172. Amer. Math. Soc., Providence, RI, 1995. →**Chapter 5**. Quoted in page(s) 89

[Wat99] S. Watanabe. The existence of a multiple spider martingale in the natural filtration of a certain diffusion in the plane. In *Séminaire de Probabilités, XXXIII*, volume 1709 of *Lecture Notes in Math.*, pages 277–290. Springer, Berlin, 1999. →**Chapter 6**. Quoted in page(s) 112, 115

[Wil69] David Williams. Markov properties of Brownian local time. *Bull. Amer. Math. Soc.*, 75:1035–1036, 1969. →**Chapter 5**. Quoted in page(s) 88

[Wil74] D. Williams. Path decomposition and continuity of local time for one-dimensional diffusions. I. *Proc. London Math. Soc. (3)*, 28:738–768, 1974. →**Chapter 3**. Quoted in page(s) 53

[Wil79] D. Williams. Conditional excursion theory. In *Séminaire de Probabilités, XIII*, volume 721 of *Lecture Notes in Math.*, pages 490–494. Springer, Berlin, 1979. →**Chapter 5**. Quoted in page(s) 93

[Wil02] D. Williams. A “non-stopping” time with the optional-stopping property. *Bull. London Math. Soc.*, 34(5):610–612, 2002. →**Chapter 2**. Quoted in page(s) 42

[Yan02] L. Yan. Maximal inequalities for a one-dimensional diffusion process. *Math. J. Toyama Univ.*, 25:39–51, 2002. →**Chapter 2**. Quoted in page(s) 22, 49

[Yoe76] C. Yoeurp. Décompositions des martingales locales et formules exponentielles. In *Séminaire de Probabilités, X (Seconde partie: Théorie des intégrales stochastiques, Univ. Strasbourg, Strasbourg, année universitaire 1974/1975)*, pages 432–480. Lecture Notes in Math., Vol. 511. Springer, Berlin, 1976. →**Chapter 0**. Quoted in page(s) 9

[Yoe85] C. Yoeurp. *Théorème de Girsanov généralisé et grossissement d'une filtration*, volume Grossissements de filtrations: exemples et applications, pages 172–196. Springer, 1985. →**Chapter 1**. Quoted in page(s) 11

[Yor78a] M. Yor. Grossissement d'une filtration et semi-martingales: théorèmes généraux. In *Séminaire de Probabilités, XII (Univ. Strasbourg, Strasbourg, 1976/1977)*, volume 649 of *Lecture Notes in Math.*, pages 61–69. Springer, Berlin, 1978. →**Chapter 1**. Quoted in page(s) 14

[Yor78b] M. Yor. Sous-espaces denses dans L^1 ou H^1 et représentation des martingales. In *Séminaire de Probabilités, XII (Univ. Strasbourg, Strasbourg, 1976/1977)*, volume 649 of *Lecture Notes in Math.*, pages 265–309. Springer, Berlin, 1978. Avec un appendice de l'auteur et J. de Sam Lazaro →**Chapter 4**. Quoted in page(s) 76, 77

[Yor85] M. Yor. *Inégalités de martingales continues arrêtées à un temps quelconque I et II*, volume Grossissements de filtrations: exemples et applications, pages 110–171. Springer, 1985. →**Chapter 2**. Quoted in page(s) 46

[Yor92a] M. Yor. *Some aspects of Brownian motion. Part I.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1992. Some special functionals. Quoted in page(s) VII, 105

[Yor92b] M. Yor. Tsirel'son's equation in discrete time. *Probab. Theory Related Fields*, 91(2):135–152, 1992. →**Chapter 6**. Quoted in page(s) 108

[Yor95] M. Yor. *Local times and Excursions for Brownian Motion: a concise introduction*, volume 1 of *Lecciones en Matematicas Postgrado de Matematicas*. Facultad de Ciencias, Universidad Central de Venezuela, 1995. Quoted in page(s) 55, 136

[Yor97a] M. Yor, editor. *Exponential functionals and principal values related to Brownian motion*. Biblioteca de la Revista Matemática Iberoamericana. [Library of the Revista Matemática Iberoamericana]. Revista Matemática Iberoamericana, Madrid, 1997. A collection of research papers, →**Chapter 3**. Quoted in page(s) 69

[Yor97b] M. Yor. *Some aspects of Brownian motion. Part II.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1997. Some recent martingale problems. Quoted in page(s) VII, 82

[Zvo74] A. Zvonkin. A transformation of the phase space of a diffusion process that will remove the drift. *Mat. Sb. (N.S.)*, 93(135):129–149, 152, 1974. →**Chapter 6**. Quoted in page(s) 106

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