Developments in Mathematics 23

Krishnaswami Alladi
Frank Garvan *Editors*

Partitions, *q*-Series, and Modular Forms

Springer

Developments in Mathematics

VOLUME 23

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ISSN 1389-2177
ISBN 978-1-4614-0027-1 e-ISBN 978-1-4614-0028-8
DOI 10.1007/978-1-4614-0028-8
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2011935455

Mathematics Subject Classification (2010): 11-XX, 05-XX, 33-XX

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

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Preface

In March 2008, an *International Conference on Partitions, q-Series, and Modular Forms* was held at the University of Florida. This conference was one of the highlights of the year-long Program in Algebra, Number Theory, and Combinatorics (ANTC) held in the Mathematics Department. The University of Florida Mathematics Department has been the venue of several conferences covering the theory of partitions and q-hypergeometric series. But what made this 2008 conference so special was that its outstanding success led to the start of year-long programs in ANTC in our department, with the 2007–2008 program being the first. The 2008 conference received generous support from the National Science Foundation, the Department of Mathematics, the College of Liberal Arts and Sciences, and the Office of Research and Graduate Programs of the University of Florida and for this we are most grateful. The organizers of this conference were Krishnaswami Alladi, Alexander Berkovich, and Frank Garvan of the University of Florida, and George Andrews of The Pennsylvania State University who is Distinguished Visiting Professor in Florida each year in the Spring Term. This volume is the outgrowth of the 2008 Gainesville conference on partitions, q-series, and modular forms, and contains major surveys and research papers related to some of the talks given at the conference. The papers have been arranged in the alphabetical order of the authors names.

Major MacMahon, a towering figure in the area of Combinatory Analysis, initiated several major lines of study, one of which was the subject of plane partitions. He created a calculational and analytic method for the purpose of determining the generating function for plane partitions, but it did not turn out to be what was he had intended, so he had to develop an alternative treatment in the next two decades. George Andrews and Peter Paule provide a charming account of the resurrection of MacMahon's dream of using partition analysis to treat plane partitions and show how the computer algebra package *Omega* has now played a decisive role in a successful treatment of plane partitions via partition analysis. Andrews and Paule point out that many essential features of this approach were known to MacMahon and so this solution is very much along the lines of MacMahon's original dream.

Srinivasa Ramanujan's discovery of congruences modulo 5, 7, and 11 for the partition function stunned the mathematical world and led to a deep study of congruences not just for partition functions but for coefficients of certain types of modular forms, an area that is intensely active even today. Ramanujan published three papers on congruences for the partition function $p(n)$, but there are several fascinating identities connected to congruences for partition functions in his "Lost Notebook." In particular, page 182 of Ramanujan's Lost Notebook is devoted to partitions and here Ramanujan has results on congruences of some general partition functions. Ramanujan demonstrates how some of these congruences follow by clever use of Jacobi's triple product identity for theta functions and Euler's pentagonal number theorem. Bruce Berndt, Chadwick Gugg, and Sun Kim closely investigate Ramanujan's elementary method and provide a detailed treatment of the entries on page 182 of Ramanujan's Lost Notebook. In doing so, they deduce some new results as well, one being a new congruence result for partition functions using r colors, and for this they employ a remarkable identity due to Winquist.

One of the sensational discoveries in recent years is the connection between mock theta functions of Ramanujan and the theory of harmonic Maass forms. Kathrin Bringmann and Ken Ono, two of the primary architects of this major development, have exploited this fundamental connection to explain many intriguing links between Borcherds products, values of modular L-functions, and Dyson's generating functions for ranks of partitions, to name a few. Here Bringmann and Ono study harmonic Maass forms with a certain bound on their weights and show that such forms can be described explicitly as linear combinations of Maass-Poincare series thereby extending the fundamental results of Rademacher and Zuckerman dating back to the 1930s.

Ever since Hardy and Ramanujan produced their remarkable asymptotic series for the partition function $p(n)$ by means of the powerful circle method in 1918, there has been detailed investigation on the asymptotic sizes of several partition functions by various analytic and elementary methods. In a charming paper, Rodney Canfield and Herb Wilf show that if the set of allowable parts S is infinite, then the function $p_S(n)$, which enumerates the number of partitions of n whose parts come from S, grows faster than any polynomial, no matter how sparse S is. They show how their results are best possible by explicitly constructing sparse sets S for which $p_S(n)$ grows faster than a polynomial but smaller than a prescribed function. They conclude their paper with some interesting open problems.

Jacobi's celebrated triple product identity for theta functions may be viewed as the beginning of a chain of identities, each member of the chain being of higher complexity than its predecessor. Thus the "next level" identity is the quintuple product identity for which several proofs are known. These identities built upward from the Jacobi triple product identity are viewed as special cases of the Macdonald identities. The next paper in this volume is by Zhu Cao who shows how the quintuple and septuple product identities can be proved in a simple manner by utilizing properties of the cubic and fifth roots of unity. Cao's method is a variation of the ideas used by Shaun Cooper in his 2006 survey of the proofs of the quintuple product identity.

In 1998, Bousquet-Melou and Kimmo Eriksson produced a startling refinement of Euler's rudimentary result connecting partitions into odd parts and distinct parts by means of the idea of Lecture Hall Partitions. This led to a flurry of activity on Lecture Hall-type identities where there is a constraint on the ratio between consecutive parts. Carla Savage, Sylvie Corteel, and Andrew Sills conduct a novel study here of Lecture Hall-type identities. They first extend the approach of Bousquet-Melou and Eriksson to encompass sequences of ratios in which the denominators are not monotone, and by doing so they derive new partition identities which are reminiscent of the classical partition theorems of Göllnitz.

The classical theta functions of Jacobi are considered among the most significant discoveries of the nineteenth century in the field of analytic functions. The concept of a theta function has been vastly generalized to include multivariable versions and to extend the domain of definition to Riemann surfaces. The famous Thomae problem deals with proportionalities between theta constants associated with certain singular curves called Hutchinson's curves, which define compact Riemann surfaces of genus 2. Hershel Farkas, a leading authority on Riemann surfaces and the study of theta functions and theta constants, discusses generalizations of Hutchinson's curves to higher genus values and the theta relations that can be deduced from such generalizations.

In the entire theory of partitions and q-series, the celebrated Rogers-Ramanujan identities are unmatched in simplicity of form, elegance, and depth. These identities and their generalizations arise in a variety of settings ranging from the study of vertex operators in the theory Lie algebras to conformal field theory in physics. Basil Gordon, one of the foremost authorities in the theory of Rogers-Ramanujan-type identities, studies the parity of the coefficients of the original two Rogers-Ramanujan identities. He shows, how in contrast to the partition function, the parity of these coefficients can be determined much more precisely.

Ramanujan's mock theta functions are considered among his deepest contributions. These intriguing functions, the discovery of which Ramanujan communicated in his last letter to Hardy from India in 1920 shortly before he died, continue to fascinate mathematicians to this day. We owe much to George Andrews and others for the present understanding of mock theta functions in the context of the theory of partitions and q-hypergeometric series. The recent work of Ono, Brigmann, Zwegers, and others connecting mock theta functions to Maass forms has led to a clearer understanding of how mock theta functions are connected to the theory of modular forms. This is a modern point of view, yet there is much significant work that has been done in recent years in the classical theory of mock theta functions. Basil Gordon has discovered new mock theta functions of order 8 and Richard McIntosh has conducted a systematic investigation on the asymptotics of the coefficients of mock theta functions, to give examples of significant work on the classical aspects of the subject. In this volume Gordon and McIntosh provide a fine comprehensive survey of the classical theory of the mock theta functions from Ramanujan's time to the present.

The Cauchy-Sylvester theorem on compound determinants is at the interface of q-series, algebraic combinatorics, special functions, and combinatorial representation theory. Masahiko Ito and Soichi Okada discuss the application of the Cauchy-Sylvester theorem to the evaluation of a certain multivariable integral of Jackson and discuss implications of the approach to determinant formulae for certain classical group characters.

Ramanujan's original notebooks contain hundreds of formulas that have inspired major lines of research in the twentieth century. The paper by Yasushi Kajihara in this volume contains a vast number of multiple series identities associated with root systems; most of these identities concern multiple series generalizations of q-series identities found in Ramanujan's notebooks.

In the last decade, one of the most important developments in the theory of theta function identities of Jacobi is the work of Steve Milne who provided multivariable versions and obtained in that process exact formulas for various sums of squares representations. In this volume Milne provides a comprehensive treatment of nonterminating Whipple transformations for basic hypergeometric series in $U(n)$. Among other things, classical work on very well-poised series on unitary groups is extended. It is expected that this approach will extend to a similar treatment of multiple basic hypergeometric series associated with the root system D_n.

In summary, the Conference on Partitions, q-Series, and Modular Forms held in Gainesville in 2008 was a meeting ground for the world's experts in these areas to interact and discuss the latest advances. This book contains survey and research papers by leading experts as outgrowths of that conference and covers a broad area of mathematics covering significant parts of number theory, combinatorics, and analysis. We are most thankful to Elizabeth Loew of Springer for including this volume in the Developments in Mathematics series.

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November 2010

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MacMahon's Dream

George E. Andrews[1] **and Peter Paule**[2]

Abstract We shall provide an account of MacMahon's development of a calculational, analytic method designed to produce the generating function for plane partitions. His efforts did not turn out as he had hoped, and he had to spend nearly 20 years finding an alternative treatment. This paper provides an account of our retrieval of MacMahon's original dream of using Partition Analysis to treat plane partitions in general.

Keywords Plane partitions • MacMahon's partition analysis • Generating functions

Mathematics Subject Classification: Primary: 05A17; Secondary: 05A15, 05E99, 11P81

1 Introduction

Major MacMahon's collected papers fill two large volumes [21] and [22]. Among these are seven lengthy works entitled, "Memoir on the theory of the partitions of numbers, I–VII."

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[1] Work of George E. Andrews was partially supported by NSF Grant DMS-0457003.
[2] Work of Peter Paule was supported SFB Grant F103 of the Austrian Science Foundation.

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K. Alladi and F. Garvan (eds.), *Partitions, q-Series, and Modular Forms*,
Developments in Mathematics 23, DOI 10.1007/978-1-4614-0028-8_1,

The first of these [23], also [21, pp. 1026–1080], appeared in 1895 when MacMahon was the President of the London Mathematical Society. It was 65 pages long and was mostly a leisurely account of what MacMahon termed partitions of multipartite numbers.

Multipartite numbers are in modern parlance n-tuples of nonnegative integers. For example, $(7,5,0,3)$ is a 4-partite number. Partitions of $(7,5,0,3)$ are direct sums of 4-tuples of nonnegative integers that add to $(7,5,0,3)$. For example,

$$(7,5,0,3) = (4,1,0,2) + (2,3,0,0) + (1,1,0,1),$$

or

$$= (3,3,0,3) + (4,2,0,0).$$

MacMahon considers a variety of combinatorial and geometrical aspects of such partitions. Of special interest is the classical representation of ordinary or unipartite partitions in "Sylvester graphs" (today called Ferrers graphs). For example, the unipartite partition of 29 given by

$$7+7+5+4+4+2$$

has the graphical representation

$$\begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & \bullet & & & \\
\bullet & \bullet & \bullet & \bullet & & & \\
\bullet & \bullet & & & & &
\end{array}$$

where each row of nodes represents the corresponding part of the partition.

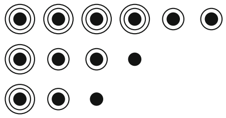
MacMahon then notes [21, p. 1058] that if one has a multipartite partition in which the Ferrers graph of each part contains the Ferrers graph of the next (called "the subjacent succession of lines" by him), then one may produce a three-dimensional analog of the Ferrers graph. Thus if we start with the "regularized" (i.e., the entries of the tuples involved are weakly decreasing) multipartite partition

$$\begin{aligned}(16,8,6) &= (6,4,3)+(6,3,2)+(4,1,1)\\ &= A+B+C,\end{aligned}$$

we may regard each of the parts as a unipartite partition, and the respective Ferrers graphs are

$$\begin{array}{ccc}
A & B & C \\
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & & &
\end{array} &
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & & & \\
\bullet & \bullet & & & &
\end{array} &
\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & & & \\
\bullet & & &
\end{array}
\end{array}$$

As MacMahon says [21, p. 1058], "it is clear that we may pile B upon A, and then C upon B & A, and thus form a three dimensional graph of the partition"



In subsequent papers, MacMahon refers to this three-dimensional graph as representing the plane partition of 30 given by

$$\begin{matrix} 6 & 4 & 3 \\ 6 & 3 & 2 \\ 4 & 1 & 1 \end{matrix}$$

He next determines that there are three such partitions of 2, six of 3 and 13 of 4. This leads him to the following conjecture [21, pp. 1064–1065]:

> The enumeration of the three-dimensional graphs that can be formed with a given number of nodes, corresponding to the regularised partitions of all multi-partite numbers of given content, is a weighty problem. I have verified to a high order that the generating function of the complete system is
>
> $$(1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-3}(1-x^4)^{-4}\cdots \textit{ad inf.},$$
>
> and, so far as my investigations have proceeded, everything tends to confirm the truth of this conjecture.

In Sect. 2 we shall look at MacMahon's efforts to develop a calculus (later to be named Partition Analysis) that he hoped would allow him to prove his conjecture. In Sect.3 we sketch our proof of all of MacMahon's conjectures. We conclude with a brief account of our discoveries made using our Mathematica implementation of Partition Analysis, the package Omega which is freely available at [25].

2 Partition Analysis: The Beginning

MacMahon [21, p. 1068] begins with some very simple problems. As an example, he considers plane partitions that only have 1s and 2s as parts and have only two columns. For example,

$$\begin{matrix} 2 & 2 \\ 2 & 2 \\ 2 & 1 \\ 1 & 1 \\ 1 & \\ 1 & \end{matrix} \tag{1}$$

The generating function for such partitions is

$$\sum_{\substack{n_1,n_2,m_1,m_2\geqq 0\\ n_1\geqq n_2\\ n_1+m_1\geqq n_2+m_2}} x^{2n_1+2n_2+m_1+m_2}.$$

Here n_1 counts the number of 2s in the first column, n_2 the number in the second; m_1 counts the number of 1s in the first column, m_2 the number in the second.

He then utilizes an idea (traceable back to Cayley in invariant theory [21, p. 1142]) of coding the inequalities on the indices by considering

$$\sum_{n_1,n_2,m_1,m_2\geqq 0} x^{2n_1+2n_2+m_1+m_2} a^{n_1+m_1-n_2-m_2} b^{n_1-n_2},$$

where all terms with negative exponents on either a or b will be thrown out and in all other terms a and b are set to 1. This device immediately allows all the series to be summed by the geometric series to

$$\frac{1}{(1-xa)\left(1-\frac{x}{a}\right)(1-abx^2)\left(1-\frac{x^2}{ab}\right)} = \frac{1}{(1-xa)\left(1-\frac{x}{a}\right)(1-x^4)} \times\left\{\frac{1}{1-abx^2}+\frac{\frac{x^2}{ab}}{1-\frac{x^2}{ab}}\right\}.$$

Now the second term inside $\{\ \}$ has only negative powers of b and so can be dropped from consideration. The first term has only positive powers of b, and so we may set $b=1$ in this term. Thus we have reduced the problem to considering

$$\begin{aligned}
&\frac{1}{(1-xa)\left(1-\frac{x}{a}\right)(1-x^4)(1-ax^2)}\\
&\quad=\frac{1}{(1-x^4)\left(1-\frac{x}{a}\right)(1-x)}\left\{\frac{1}{1-ax}-\frac{x}{1-ax^2}\right\}\\
&\quad=\frac{1}{(1-x^4)(1-x)}\times\left\{\frac{1}{1-x^2}\left(\frac{1}{1-ax}+\frac{\frac{x}{a}}{1-\frac{x}{a}}\right)-\frac{x}{1-x^3}\left(\frac{1}{1-ax^2}+\frac{\frac{x}{a}}{1-\frac{x}{a}}\right)\right\}.
\end{aligned}$$

As in the elimination of b, this reduces to

$$\frac{1}{(1-x^4)(1-x)}\left\{\frac{1}{(1-x^2)(1-x)}-\frac{x}{(1-x^3)(1-x^2)}\right\}=\frac{1}{(1-x)(1-x^2)^2(1-x^2)},$$

a result which, as MacMahon points out, is not obvious [21, p. 1068].

Now two things are clear. First, one must streamline this method which is cumbersome even in this simple example. Second, one must somehow introduce a simple notation for the deletion of terms with negative exponents on a and b. MacMahon turns his attention to these requirements in [24], and finally in his magnum opus [20, Vol. II, Sect. VIII], he has reduced the above treatment to the following. First he defines the omega operator [20, Vol. II, p. 92]

$$\underset{\geqq}{\Omega} \sum_{n_1,\ldots,n_j=-\infty} A(n_1,n_2,\ldots,n_j)\lambda_1^{n_1}\lambda_2^{n_2}\ldots\lambda_j^{n_j} = \sum_{n_1,n_2,\ldots,n_j\geqq 0} A(n_1,n_2,\ldots,n_j),$$

where $A(n_1,n_2,\ldots,n_j)$ is generally some rational function of variables like x, y, or z.

Then MacMahon prepares a list of valid omega evaluations [20, Vol. II, p. 102] including

$$\underset{\geqq}{\Omega} \frac{1}{(1-\lambda x)(1-\lambda y)(1-\frac{z}{\lambda})} = \frac{1-xyz}{(1-x)(1-y)(1-xz)(1-yz)}. \tag{2}$$

Hence

$$\begin{aligned}
\sum_{\substack{n_1,n_2,m_1,m_2\geqq 0\\ n_1\geqq n_2\\ n_1+m_1\geqq n_2+m_2}} x^{2n_1+2n_2+m_1+m_2} &= \underset{\geqq}{\Omega} \frac{1}{(1-x\lambda_2)\left(1-\frac{x}{\lambda_2}\right)(1-\lambda_1\lambda_2x^2)\left(1-\frac{x^2}{\lambda_1\lambda_2}\right)}\\
&= \underset{\geqq}{\Omega} \frac{1}{(1-x\lambda_2)\left(1-\frac{x}{\lambda_2}\right)(1-\lambda_2x^2)(1-x^4)}\\
&\quad \text{(by (2) with } x,\, y,\, z \text{ replaced by } \lambda_2x^2, 0, \tfrac{x^2}{\lambda_2}\text{, resp.)}\\
&= \frac{1}{(1-x^4)} \underset{\geqq}{\Omega} \frac{1}{(1-x\lambda_2)(1-x^2\lambda_2)\left(1-\frac{x}{\lambda_2}\right)}\\
&= \frac{1}{(1-x^4)} \cdot \frac{(1-x^4)}{(1-x)(1-x^2)(1-x^2)(1-x^3)}\\
&\quad \text{(by (2) with } x,\, y,\, z \text{ replaced by } x, x^2, x\text{, resp.)}\\
&= \frac{1}{(1-x)(1-x^2)^2(1-x^3)}.
\end{aligned}$$

MacMahon clearly hoped to hone this tool into one that could prove his conjectures on plane partitions. Clearly the problems can all be set up in the language of his Partition Analysis. However, he was unable to develop this machinery adequately. Sadly he sets up the general problem [20, Vol. II, p. 186], but is forced to conclude: "Our knowledge of the Ω operation is not sufficient to enable us to establish the final form of result."

In the next section, we describe the work in [5] where we have overcome MacMahon's difficulties.

3 Partition Analysis: The Dream

In our efforts to make MacMahon's dream come true the Omega package [25] has played a decisive role. Remarkably, MacMahon had already been aware of the algorithmic essence of Partition Analysis; see Sect. VIII of [20, Vol. 2, pp. 111–114] describing the "method of Elliott." However, 90 years before computer algebra systems emerged he was confined to use his technique essentially in the form of a table lookup method. After the first achievements of revitalizing Partition Analysis, [1] and [2], we have pursued the project of replacing MacMahon's transformation and elimination rules for his omega operator by a deterministic algorithmic procedure. Subsequently we have implemented these algorithms in the *Mathematica* system and called the corresponding package "Omega." For a description of this work and for a variety of new applications we refer to the articles [3–12].

As an illustration we show how the example discussed in Sect. 2 can be treated with Omega in a fully automatic fashion. We initialize by loading the package

```
In[1]:= <<Omega2.m
```

Following MacMahon's terminology, the first step is to compute the "crude generating function." To this end one has to only input the problem in a form which is very close to the usual mathematical syntax. (All summation parameters are assumed to be nonnegative, if not specified otherwise.)

In[2]:= Crude = OSum[$x^{2n_1+2n_2+m_1+m_2}, \{n_1 \geq n_2, n_1+m_1 \geq n_2+m_2\}, \lambda$]

Out[2]= $\underset{\lambda_1,\lambda_2}{\underset{\geq}{\Omega}} \frac{1}{(1-\frac{x}{\lambda_2})(1-\frac{x^2}{\lambda_1\lambda_2})(1-x\lambda_2)(1-x^2\lambda_1\lambda_2)}$

Finally the elimination of the λ variables is carried out by the procedure call

In[3]:= OR[Crude]
Eliminating λ_1...
Eliminating λ_2...

Out[3]= $\frac{1}{(1-x)(1-x^2)^2(1-x^3)}$

Note: During computation, the package tells the user in which order the elimination of the λ variables is carried out.

To present a brief account of how MacMahon's dream has come true we need a couple of definitions.

Given an $r \times c$ matrix $X = (x_{i,j})$ we define

$$p_{r,c}(X) := \sum_{(a_{i,j}) \in M_{r,c}} x_{1,1}^{a_{1,1}} \cdots x_{1,c}^{a_{1,c}} \cdots x_{r,1}^{a_{r,1}} \cdots x_{r,c}^{a_{r,c}},$$

where $M_{r,c}$ consists of all $r \times c$ matrices over nonnegative integers $a_{i,j}$ such that $a_{i,j} \geq a_{i,j+1}$ and $a_{i,j} \geq a_{i+1,j}$. Hence $p_{r,c}(X)$ is the generating function for all plane partitions with at most r rows and c columns. In $p_{r,c}(X)$, setting all the $x_{i,j}$ to x produces the corresponding enumerative generating function which we denote by $q_{r,c}(x)$. The limiting case $r, c \to \infty$ corresponds to MacMahon's original conjecture [21, pp. 1064–1065] cited in Sect. 1 namely

$$q_{\infty,\infty}(x) = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)^k}. \tag{3}$$

In [23] MacMahon also considered the case where r and c are set to concrete integers. His computations led him to conjecture that

$$q_{r,c}(x) = \sum_{n=0}^{\infty} P_{r,c}(n)x^n = \prod_{i=1}^{r}\prod_{j=1}^{c} \frac{1}{1-x^{i+j-1}}, \tag{4}$$

where $P_{r,c}(n)$ denotes the number of plane partitions of n with at most r rows and c columns. Obviously, for $r, c \to \infty$ this turns into (3).

In our attempt to give a possible explanation of why MacMahon had failed to prove (4) with his method, we first have to describe how Partition Analysis would work on such problems in principle.

The usual heuristics approach to prove (4) by means of Partition Analysis would be as follows: one tries to proceed by mathematical induction with respect to one of the free parameters, e.g., with respect to c with r fixed. To this end, one applies Partition Analysis to special instances of the problem in order to guess a pattern for the induction step from c to $c+1$. But in many applications it turns out that the enumerative generating function does not provide sufficient information into the mechanism of the induction. In such situations one often can overcome this problem by considering the full generating function, i.e., the generating function that constructs all the objects in question; see the various examples given in [3–12].

To illustrate this point let us consider $p_{r,c}(X)$ with $r = c = 3$. The case $q_{3,3}(x)$, where all the $x_{i,j}$ in $p_{3,3}(X)$ are set to x, causes no computational problem at all:

In[4]:=OSum[$x^{a_{11}+a_{12}+a_{13}+a_{21}+a_{22}+a_{23}+a_{31}+a_{32}+a_{33}}$,
$\{a_{11} \geq a_{12}, a_{12} \geq a_{13}, a_{21} \geq a_{22}, a_{22} \geq a_{23}, a_{31} \geq a_{32}, a_{32} \geq a_{33},$
$a_{11} \geq a_{21}, a_{21} \geq a_{31}, a_{12} \geq a_{22}, a_{22} \geq a_{32}, a_{13} \geq a_{23}, a_{23} \geq a_{33}\}, \lambda]$

Out[4]= $$\underset{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12}}{\underset{\geq}{\Omega}} \frac{1}{(1-x\lambda_1\lambda_7)\left(1-\frac{x\lambda_5}{\lambda_8}\right)\left(1-\frac{x\lambda_3\lambda_8}{\lambda_7}\right)} \times \frac{1}{\left(1-\frac{x\lambda_2\lambda_9}{\lambda_1}\right)\left(1-\frac{x\lambda_6}{\lambda_5\lambda_{10}}\right)\left(1-\frac{x\lambda_4\lambda_{10}}{\lambda_3\lambda_9}\right)\left(1-\frac{x\lambda_{11}}{\lambda_2}\right)\left(1-\frac{x}{\lambda_6\lambda_{12}}\right)\left(1-\frac{x\lambda_{12}}{\lambda_4\lambda_{11}}\right)}$$

In[5]:= OR[%4]

Out[5]= $$\frac{1}{(1-x)(1-x^2)^2(1-x^3)^3(1-x^4)^2(1-x^5)}$$

Despite the fact that the Omega package confirms (4) within a fraction of a second, a closer inspection shows that in order to exhibit an induction pattern for a Partition Analysis proof of (4) the setting $x_{i,j} = 0$ is too much restrictive.

So let us have a look at the full generating function $p_{3,3}(X)$. The crude generating function comes out in perfect analogy to Out [4] above:

$$\texttt{In[6]:=OSum}\big[x_{11}^{a_{11}}x_{12}^{a_{12}}x_{13}^{a_{13}}x_{21}^{a_{21}}x_{22}^{a_{22}}x_{23}^{a_{23}}x_{31}^{a_{31}}x_{32}^{a_{32}}x_{33}^{a_{33}},\\ \{a_{11}\geq a_{12},a_{12}\geq a_{13},a_{21}\geq a_{22},a_{22}\geq a_{23},a_{31}\geq a_{32},a_{32}\geq a_{33},\\ a_{11}\geq a_{21},a_{21}\geq a_{31},a_{12}\geq a_{22},a_{22}\geq a_{32},a_{13}\geq a_{23},a_{23}\geq a_{33}\},\lambda\big]$$

$$\texttt{Out[6]=} \underset{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12}}{\Omega_{\geq}} \frac{1}{(1-x_{11}\lambda_1\lambda_7)\left(1-\frac{x_{31}\lambda_5}{\lambda_8}\right)\left(1-\frac{x_{21}\lambda_3\lambda_8}{\lambda_7}\right)}\\ \times\frac{1}{\left(1-\frac{x_{12}\lambda_2\lambda_9}{\lambda_1}\right)\left(1-\frac{x_{32}\lambda_6}{\lambda_5\lambda_{10}}\right)\left(1-\frac{x_{22}\lambda_4\lambda_{10}}{\lambda_3\lambda_9}\right)\left(1-\frac{x_{13}\lambda_{11}}{\lambda_2}\right)\left(1-\frac{x_{33}}{\lambda_6\lambda_{12}}\right)\left(1-\frac{x_{23}\lambda_{12}}{\lambda_4\lambda_{11}}\right)}$$

The computation of the full generating function takes another couple of seconds:

```
In[7]:= OR[%6]
```

$$\texttt{Out[7]=}\ (1-x_{11}^2x_{12}x_{21}-x_{11}^2x_{12}x_{13}x_{21}-\cdots+x_{11}^{14}x_{12}^{12}x_{13}^7x_{21}^{12}x_{22}^7x_{23}^2x_{31}^7x_{32}^2)/\\ ((1-x_{11})(1-x_{11}x_{12})(1-x_{11}x_{12}x_{13})(1-x_{11}x_{21})(1-x_{11}x_{12}x_{21})\\ (1-x_{11}x_{12}x_{13}x_{21})(1-x_{11}x_{12}x_{21}x_{22})(1-x_{11}x_{12}x_{13}x_{21}x_{22})\\ (1-x_{11}x_{12}x_{13}x_{21}x_{22}x_{23})(1-x_{11}x_{21}x_{31})(1-x_{11}x_{12}x_{21}x_{31})(1-x_{11}x_{12}x_{13}x_{21}x_{31})\\ (1-x_{11}x_{12}x_{21}x_{22}x_{31})(1-x_{11}x_{12}x_{13}x_{21}x_{22}x_{31})(1-x_{11}x_{12}x_{13}x_{21}x_{22}x_{23}x_{31})\\ (1-x_{11}x_{12}x_{21}x_{22}x_{31}x_{32})(1-x_{11}x_{12}x_{13}x_{21}x_{22}x_{31}x_{32})\\ (1-x_{11}x_{12}x_{13}x_{21}x_{22}x_{23}x_{31}x_{32})(1-x_{11}x_{12}x_{13}x_{21}x_{22}x_{23}x_{31}x_{32}x_{33}))$$

However, the problem arising in this case consists in the complexity of the resulting rational function; namely, in order to display the numerator polynomial

$$1-x_{11}^2x_{12}x_{21}-x_{11}^2x_{12}x_{13}x_{21}-\cdots+x_{11}^{14}x_{12}^{12}x_{13}^7x_{21}^{12}x_{22}^7x_{23}^2x_{31}^7x_{32}^2$$

in fully explicit form, one would need more than 30 printed pages.

Summarizing, the coding of the full generating function $p_{r,c}(X)$ in terms of the omega operator is straight forward and has already been carried out by MacMahon [20, Vol. II, p. 92]. But without computer algebra he did not succeed in overcoming the computational difficulties when trying to obtain the beautiful product side of (3), resp. (4), with omega evaluation. Essentially the problem is this: when specifying all the $x_{i,j}$ to x, the underlying algebraic structure gets lost entirely. If all the $x_{i,j}$ are kept, the computational complexity soon gets out of hand.

Consequently, we used the Omega package in a heuristic search to find a substitution for the x_{ij} which, on the one side, provides more insight into the underlying Partition Analysis induction pattern than $q_{r,c}(x)$, and on the other side, for which the elimination of the λ_i results in a more feasible rational function than for the general $p_{r,c}(X)$.

Finally, after various attempts our strategy turned out to be successful. More precisely, we found that the substitution

$$x_{ij} \to z_{j-i} \tag{5}$$

has all the properties desired. First, the elimination of the λ_i results in a rational function that factors nicely for all choices of r and c. For instance, for $r = c = 3$,

In[8]:=OSum$[z_0^{a_{11}} z_1^{a_{12}} z_2^{a_{13}} z_{-1}^{a_{21}} z_0^{a_{22}} z_1^{a_{23}} z_{-2}^{a_{31}} z_{-1}^{a_{32}} z_0^{a_{33}},$
$\{a_{11} \geq a_{12}, a_{12} \geq a_{13}, a_{21} \geq a_{22}, a_{22} \geq a_{23}, a_{31} \geq a_{32}, a_{32} \geq a_{33},$
$a_{11} \geq a_{21}, a_{21} \geq a_{31}, a_{12} \geq a_{22}, a_{22} \geq a_{32}, a_{13} \geq a_{23}, a_{23} \geq a_{33}\}, \lambda]$

Out[8]= $$\underset{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12}}{\underset{\geq}{\Omega}} \frac{1}{(1-z_0\lambda_1\lambda_7)\left(1-\frac{z_{-2}\lambda_5}{\lambda_8}\right)\left(1-\frac{z_{-1}\lambda_3\lambda_8}{\lambda_7}\right)}$$
$$\times \frac{1}{\left(1-\frac{z_1\lambda_2\lambda_9}{\lambda_1}\right)\left(1-\frac{z_{-1}\lambda_6}{\lambda_5\lambda_{10}}\right)\left(1-\frac{z_0\lambda_4\lambda_{10}}{\lambda_3\lambda_9}\right)\left(1-\frac{z_2\lambda_{11}}{\lambda_2}\right)\left(1-\frac{z_0}{\lambda_6\lambda_{12}}\right)\left(1-\frac{z_1\lambda_{12}}{\lambda_4\lambda_{11}}\right)}$$

In[9]:=OR[%8]

Out[9]= $1/((1-z_0)(1-z_{-1}z_0)(1-z_{-2}z_{-1}z_0)(1-z_0z_1)(1-z_{-1}z_0z_1)$
$(1-z_{-2}z_{-1}z_0z_1)(1-z_0z_1z_2)(1-z_{-1}z_0z_1z_2)(1-z_{-2}z_{-1}z_0z_1z_2))$

Second, and more importantly, in this situation MacMahon's method of Partition Analysis works in a way that allows to set up an elementary induction proof for the corresponding plane partition result which originally is due to Emden Gansner. His theorem [16, Theorem 4.2] not only generalizes (4) but also Stanley's trace theorem [27, Theorem 2.2] which was also conjectured by MacMahon in [23]. In order to state Gansner's theorem we need a couple of definitions.

Let $\pi = (a_{i,j})$ be an $r \times c$ matrix over nonnegative integers $a_{i,j}$ such that $a_{i,j} \geq a_{i,j+1}$ and $a_{i,j} \geq a_{i+1,j}$; i.e., π represents a plane partition of $n := \Sigma_{ij} a_{ij}$ with at most r rows and c columns. For any integer k we define the k-trace t_k of π by $t_k := \Sigma a_{i,j}$ where the sum runs over all i, j such that $k = j - i$. For example, the traces of the plane partition of 30 in Sect. 1 are $t_{-2} = 4$, $t_{-1} = 7$, $t_0 = 10$, $t_1 = 6$, and $t_2 = 3$.

If $T_{r,c}(t_{-r+1}, \ldots, t_{-1}; t_0, \ldots, t_{c-1}; n)$ denotes the number of plane partitions of n with at most r rows and c columns, and with k-trace t_k, $-r+1 \leq k \leq c-1$, Gansner's theorem reads as follows:

$$\sum_{n=0}^{\infty} \sum_{t_{-r+1}=0}^{\infty} \cdots \sum_{t_{c-1}=0}^{\infty} T_{r,c}(t_{-r+1}, \ldots, t_{-1}; t_0, \ldots, t_{c-1}; n)\ z_{-r+1}^{t_{-r+1}} \cdots z_{-1}^{t_{-1}} z_0^{t_0} \cdots z_{c-1}^{t_{c-1}} x^n$$
$$= \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{1}{1 - z_{-i+1} z_{-i+2} \cdots z_{j-1} x^{i+j-1}}.$$

Obviously, $z_{-r+1} = z_{-r+2} = \ldots = z_{c-1} = 1$ gives (4); setting all $z_k = 1$, except z_0, gives Stanley's trace theorem [27, Theorem 2.2].

It is immediate that $p_{r,c}(X)$ for $X = (x_{i,j})$ with $x_{ij} := z_{j-i}$ can be rewritten as the multiple series in Gansner's theorem. Our Partition Analysis proof of the fact that it finds the product representation above can be found in [5].

Summarizing, we want to note that our proof in [5] uses only basic power series arithmetic; essentially it proceeds by complete induction involving recursively defined rational functions. So our Partition Analysis approach is completely different from Gansner's original proof which is based on a combinatorial bijection of Burge [14]. This bijection is one of those variations of the Schensted-Knuth correspondence which Burge derived in order to give combinatorial proofs for a collection of Schur function identities due to D.E. Littlewood.

4 Conclusion

The implementation of MacMahon's Partition Analysis in the Omega package has provided the exploratory tool for our dozen papers on this topic [1–12]. It is important to point out that there have been a number of parallel and complementing projects that can be viewed as having goals similar to MacMahon's. An incomplete list would include (1) LattE [15, 18], an implementation of the work of Barvinok and Pommersheim [13], (2) the MAPLE package designed by Stembridge [28] to implement the discoveries of R. Stanley [26] which in turn were based on another MacMahon paper [19]. Recently Xiu [29] has made contributions based on his work on partial fractions.

In the future we hope to explore further with Omega. Also we are modifying Omega to treat problems in which not only linear Diophantine inequalities are considered but also divisibility properties of the summands are treated. Toward this goal, we have developed an extension of Partition Analysis that allows us to treat the Göllnitz-Gordon partition functions.

It is perhaps fitting to close with Glaisher's evaluation [17] of [23] (printed with permission of the Royal Society):

> I don't fancy the paper very much, but it must be printed. I don't care much for a paper on very technical mathematics being published in the Phil. Trans. unless there is something very striking in it. However, it is one of a series, and they are in deep water now and cannot go on much farther. I have made my report because there is no more to be said than that it should be published (though the interesting results are the conjectural ones!), the balance being on that side.

How fortunate we are that Glaisher's lack of enthusiasm did not cause him to recommend against [23]. Also we can congratulate Glaisher on his recognition of the significance of MacMahon's conjectures.

Acknowledgements We thank Christian Krattenthaler for pointing out that substitution (5) we found with Partition Analysis leads us to rediscover Gansner's theorem.

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Ramanujan's Elementary Method in Partition Congruences

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Abstract Page 182 in Ramanujan's lost notebook corresponds to page 5 of an otherwise lost manuscript of Ramanujan closely related to his paper providing elementary proofs of his partition congruences $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$. The claims on page 182 are proved and discussed, and further results depending on Ramanujan's ideas are established.

Keywords Partitions • Congruences • Pentagonal number theorem • Jacobi's identity • Winquist's identity

Mathematics Subject Classification: Primary: 11P83; Secondary: 11P81, 05A17

1 Introduction

Ramanujan published three papers, [12–14], on congruences for the partition function $p(n)$. However, the second [13] is only a short announcement of results, and the third [14] was extracted by Hardy from a much longer handwritten manuscript after Ramanujan's death. The latter manuscript, concentrating on both $p(n)$ and Ramanujan's τ-function $\tau(n)$, was published for the first time with Ramanujan's

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K. Alladi and F. Garvan (eds.), *Partitions, q-Series, and Modular Forms*, Developments in Mathematics 23, DOI 10.1007/978-1-4614-0028-8_2,

lost notebook [17] in its original handwritten form. Later, this $p(n)/\tau(n)$ manuscript was prepared for journal publication, with amplification of details and extensive commentary, by the first author and Ono [7]. That article is reproduced in the book [1] by Andrews and Berndt, with the previous commentary considerably expanded.

Page 182 of [17] is also devoted to the theory of partitions. The number (5) is written in the upper right-hand corner of page 182 in [17], likely indicating that this is the fifth page of a handwritten manuscript. The first and second lines on this page are identical to the second and third lines of (11) in [12], [15, p. 211], where Ramanujan begins to relate his elementary proof of $p(5n+4) \equiv 0 \pmod{5}$. The tagged equation numbers on page 182 are (2.2)–(2.5), which clearly indicate that this page is in Sect. 2 of this manuscript. However, page 182 is not identical to any page or pages in [12]. Ramanujan's proof of $p(5n+4) \equiv 0 \pmod{5}$ here is considerably briefer than it is in [12]. Moreover, central to Ramanujan's thoughts is the more general partition function $p_r(n)$ defined by

$$\frac{1}{(q;q)_\infty^r} = \sum_{n=0}^{\infty} p_r(n) q^n, \qquad |q| < 1,$$

which is not discussed in [12]. This definition is actually not provided on page 182, but it is clear that it must have been given somewhere in the missing pages 1–4 of the manuscript. Of course, $p_1(n) = p(n)$. In a letter to Hardy written from Fitzroy House late in 1918 [8, pp. 192–193], Ramanujan writes, "I have considered more or less exhaustively about the congruency of $p(n)$ and in general that of $p_r(n)$ where

$$\sum p_r(n) x^n = \frac{1}{(x;x)_\infty^r},$$

by four different methods." This declaration appears to imply that he had established several results about $p_r(n)$, which quite likely were discussed in the manuscript for which we now unfortunately have only page 5.

Ramanujan deduces the congruence $p(5n-1) \equiv 0 \pmod{5}$ from the congruence $p_{-4}(5n-1) \equiv 0 \pmod{5}$, just as he does in [12] without using this notation. Ramanujan then remarks that, "Precisely in the same way we can show that

$$p_{-4}\left(n\varpi - \frac{\varpi+1}{6}\right) \equiv 0 \pmod{\varpi}, \tag{1.1}$$

where ϖ is a prime of the form $6\lambda - 1 \ldots$" He then states a more general theorem. It is therefore quite clear that Ramanujan's paper [12] was likely extracted from a greatly expanded longer manuscript, which was not a part of his $p(n)/\tau(n)$ manuscript and which, except for this single page, has been lost.

After Ramanujan, the function $p_r(n)$, in another notation $p_{-r}(n)$, was studied by, in particular, Newman [9], Ramanathan [10], and Atkin [3], who were also interested in its congruences. However, they confined themselves to congruences satisfied by a small set of primes and powers thereof, in contrast to Ramanujan's theorems on page 182 satisfying an infinite set of primes.

The keys to Ramanujan's elementary methods are the pentagonal number theorem

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}=(q;q)_\infty \tag{1.2}$$

and Jacobi's identity

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}=(q;q)_\infty^3. \tag{1.3}$$

It is well known that no one has been able to employ these identities to give an elementary proof of Ramanujan's congruence $p(11n+6)\equiv 0 \pmod{11}$. The only known elementary proof of this congruence is due to Winquist [18] and employs an identity that is now known by his name. At the end of Sect. 2, for the first time, we utilize Winquist's identity to derive new congruences for special instances of $p_r(n)$.

Ramanujan's elementary methods focusing on Jacobi's identity have been utilized and generalized by several authors. The most extensive applications of this method have been made by Andrews and Roy [2]; their paper contains several additional references. There are two further identities in the spirit of Jacobi's identity, both due to Ramanujan [16], and we employ them in Sect. 3, to derive two analogues of the general theorem of Andrews and Roy.

2 Page 182 in Ramanujan's Lost Notebook

A brief account of page 182 along with a proof of the first entry below has been given by Ramanathan [11].

Entry 2.1 (p. 182). *Let δ denote any integer, and let n denote a nonnegative integer. Suppose that ϖ is a prime of the form $6\lambda-1$. Then*

$$p_{\delta\varpi-4}\left(n\varpi-\frac{\varpi+1}{6}\right)\equiv 0 \pmod{\varpi}. \tag{2.1}$$

Proof. Consider

$$\begin{aligned}\sum_{n=0}^{\infty}p_{\delta\varpi-4}(n)q^{n+\lambda}&=(q;q)_\infty^{-\delta\varpi}(q;q)_\infty^3(q;q)_\infty q^\lambda\\ &\equiv (q^\varpi;q^\varpi)_\infty^{-\delta}\sum_{\mu=0}^{\infty}\sum_{\nu=-\infty}^{\infty}(-1)^{\mu+\nu}(2\mu+1)q^{\frac12\mu(\mu+1)+\frac12\nu(3\nu+1)+\lambda} \pmod{\varpi},\end{aligned} \tag{2.2}$$

upon the use of Euler's pentagonal number theorem (1.2) and Jacobi's identity (1.3). We want to examine those terms for which

$$\frac{1}{2}\mu(\mu+1)+\frac{1}{2}\nu(3\nu+1)+\frac{\varpi+1}{6}\equiv 0 \pmod{\varpi}. \tag{2.3}$$

Our goal is to prove that

$$\varpi \mid (2\mu+1). \tag{2.4}$$

Multiply (2.3) by 24 to obtain the equivalent congruence

$$12\mu(\mu+1)+12\nu(3\nu+1)+4\varpi+4\equiv 0 \pmod{\varpi},$$

or

$$3(2\mu+1)^2+(6\nu+1)^2\equiv 0 \pmod{\varpi}. \tag{2.5}$$

Using the fact that, for each prime p, the Legendre symbol $(\frac{-1}{p})=(-1)^{(p-1)/2}$, and the law of quadratic reciprocity, we find that

$$\left(\frac{-3}{\varpi}\right)=\left(\frac{\varpi}{3}\right)=\left(\frac{-1}{3}\right)=-1.$$

Thus, the only way that (2.5) can hold is for (2.4) to happen. But then, from the right-hand side of (2.2), we can conclude that

$$p_{\delta\varpi-4}\left(n\varpi-\frac{\varpi+1}{6}\right)\equiv 0 \pmod{\varpi}.$$

Thus, the proof is complete. □

Corollary 2.2 (p. 182). *For each positive integer n,*

$$p_6(5n-1)\equiv 0 \pmod{5},$$
$$p_7(11n-2)\equiv 0 \pmod{11}.$$

Proof. The first congruence arises from the case $\varpi=5$ and $\delta=2$, while the second arises from the case $\varpi=11$ and $\delta=1$ in Entry 2.1. □

Next, Ramanujan gives an elementary proof of the congruence $p(7n-2)\equiv 0 \pmod{7}$. He begins with the same first three lines of [12, (13)], [15, p. 212], and then argues in a somewhat more abbreviated fashion than he does in [12] to deduce the congruence

$$p_{-6}(7n-2)\equiv 0 \pmod{49}, \tag{2.6}$$

from which it follows that

$$p(7n-2)\equiv 0 \pmod{7}. \tag{2.7}$$

It should be remarked that the stronger congruence (2.6) is not mentioned by Ramanujan in [12], although it is implicit in his argument.

Unfortunately, the one-page manuscript ends with (2.7). It would seem that Ramanujan would have next offered a theorem analogous to Entry 2.1, and so we shall state and prove such a theorem here, but, of course, Ramanujan probably would have had much more to say to us, if his manuscript had survived.

Theorem 2.3. *For a prime ϖ with $4 \mid (\varpi+1)$, any integer δ, and any positive integer n,*

$$p_{\delta\varpi-6}\Big(n\varpi - \frac{\varpi+1}{4}\Big) \equiv 0 \pmod{\varpi}. \tag{2.8}$$

In the case $\delta = 0$ above, we can strengthen (2.8).

Entry 2.4 (p. 182). *We have*

$$p_{-6}\Big(n\varpi - \frac{\varpi+1}{4}\Big) \equiv 0 \pmod{\varpi^2}. \tag{2.9}$$

Observe that (2.6) is the special case $\varpi = 7$ of (2.9), and so, with slight exaggeration, we affixed "p. 182" to the entry above.

Corollary 2.5. *For each positive integer n,*

$$p_{3\delta-6}(3n-1) \equiv 0 \pmod{3}. \tag{2.10}$$

Proof. Set $\varpi = 3$ in Theorem 2.3. □

For the case $\delta = 3$ in (2.10), Baruah and Ojah [4], using more sophisticated means, obtained the stronger result

$$p_3(3n-1) \equiv 0 \pmod{3^2}.$$

Proof of Theorem 2.3. Consider, for $\lambda = (\varpi+1)/4$,

$$\begin{aligned}\sum_{n=0}^{\infty} p_{\delta\varpi-6}(n)q^{n+\lambda} &= (q;q)_\infty^{-\delta\varpi}(q;q)_\infty^6 q^\lambda \\ &\equiv (q^\varpi;q^\varpi)_\infty^{-\delta} \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}(-1)^{\mu+\nu}(2\mu+1)(2\nu+1)q^{\frac{1}{2}\mu(\mu+1)+\frac{1}{2}\nu(\nu+1)+\lambda} \pmod{\varpi},\end{aligned} \tag{2.11}$$

upon the use of Jacobi's identity (1.3). We need to show that if

$$\frac{1}{2}\mu(\mu+1)+\frac{1}{2}\nu(\nu+1)+\frac{\varpi+1}{4} \equiv 0 \pmod{\varpi}, \tag{2.12}$$

then

$$\varpi^2 \mid (2\mu+1)(2\nu+1). \tag{2.13}$$

The congruence (2.8) will then follow from (2.13) and (2.11). Multiply (2.12) by 8 to obtain

$$4\mu(\mu+1)+4\nu(\nu+1)+2\varpi+2 \equiv 0 \pmod{\varpi},$$

or

$$(2\mu+1)^2+(2\nu+1)^2 \equiv 0 \pmod{\varpi}.$$

Since

$$\left(\frac{-1}{\varpi}\right) = -1,$$

we conclude that

$$\varpi \mid (2\mu+1) \quad \text{and} \quad \varpi \mid (2\nu+1),$$

which completes the proof of (2.13). □

Observe that if $\delta = 0$, then the congruence in (2.11) can be replaced by an equality. Hence, in (2.8), the congruence modulo ϖ can be replaced by a congruence modulo ϖ^2 in view of (2.13). Entry 2.4 therefore follows.

Although Entry 2.1 and Theorem 2.3 are not special cases of the general theorem of Andrews and Roy [2], they would be instances of the general theorem envisioned by the authors in Sect. 5 of their paper [2].

Recall next that a corollary of Winquist's identity is given by [18]

$$\begin{aligned}48(q;q)_\infty^{10} = \sum_{m,n=-\infty}^{\infty} &(-1)^{m+n}((6m+3)^3(6n+1)-(6m+3)(6n+1)^3)\\ &\times q^{\frac{1}{2}(3m^2+3m+3n^2+n)}.\end{aligned} \tag{2.14}$$

Theorem 2.6. *For a prime ϖ with $12 \mid (\varpi+1)$, and any integer δ, we have*

$$p_{\delta\varpi-10}\left(n\varpi - \frac{5(\varpi+1)}{12}\right) \equiv 0 \pmod{\varpi}.$$

Proof. Let $\lambda = 5(\varpi+1)/12$, and from (2.14) consider

$$\begin{aligned}\sum_{n=0}^{\infty} p_{\delta\varpi-10}(n)q^{n+\lambda} &= (q;q)_\infty^{-\delta\varpi}(q;q)_\infty^{10}q^{\lambda}\\ &\equiv (q^{\varpi};q^{\varpi})_\infty^{-\delta}\frac{1}{48}\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}((6m+3)^3(6n+1)\\ &\quad -(6m+3)(6n+1)^3)q^{\frac{1}{2}(3m^2+3m+3n^2+n)+\lambda} \pmod{\varpi}.\end{aligned} \tag{2.15}$$

If

$$\frac{1}{2}(3m^2+3m+3n^2+n)+\lambda \equiv 0 \pmod{\varpi},$$

then upon multiplying both sides above by 24, we find that

$$12(3m^2+3m+3n^2+n)+10(\varpi+1) \equiv 0 \pmod{\varpi},$$

or

$$(6m+3)^2+(6n+1)^2 \equiv 0 \pmod{\varpi}.$$

Since

$$\left(\frac{-1}{\varpi}\right) = -1,$$

we see that

$$\varpi \mid (6m+3) \qquad \text{and} \qquad \varpi \mid (6n+1).$$

Using these observations in (2.15), we complete the proof. $\square$

We observe that in the special case $\delta = 0$, our proof yields a stronger result.

Corollary 2.7. *For a prime ϖ with $12 \mid (\varpi+1)$, we have*

$$p_{-10}\left(n\varpi - \frac{5(\varpi+1)}{12}\right) \equiv 0 \pmod{\varpi^4}.$$

3 Two Further General Congruences

Our analogues of the main theorem of Andrews and Roy [2] are dependent on the two identities

$$\sum_{j=-\infty}^{\infty}(6j+1)q^{3j^2+j} = (q^2;q^2)_\infty^3(q^2;q^4)_\infty^2 \tag{3.1}$$

and

$$\sum_{j=-\infty}^{\infty}(3j+1)q^{3j^2+2j} = (q^2;q^2)_\infty(q;q^2)_\infty^2(q^4;q^4)_\infty^2, \tag{3.2}$$

which arise from the quintuple product identity and which are found as Entries 8(ix), (x) in Chap. 17 of Ramanujan's second notebook [16]. For proofs, see [5, pp. 118–119] or [6, pp. 20–22]. More historical information can be found in [6, p. 25].

For each integer m, we shall denote by $\overline{m}$ the multiplicative inverse of $m \pmod{p}$. We prove the following theorems.

Theorem 3.1. *Suppose that p is a prime > 3, $0 < a < p$, and a and b are integers. Assume that $-6a$ is a quadratic nonresidue modulo p. Suppose that $\{\alpha_n\}_{n=-\infty}^{\infty} = \{\alpha_n(z_1,z_2,\dots,z_j)\}$ is a doubly infinite sequence of Laurent polynomials over $\mathbb{Z}$ with variables $z_1,\dots,z_j$ independent of q. Then there is an integer c such that the coefficients of $z_1^{m_1} z_2^{m_2} \cdots z_j^{m_j} q^{pN}$ in*

$$\frac{q^c \sum_{n=-\infty}^{\infty} \alpha_n q^{a(n^2-n)/2+bn}}{(q^2;q^2)_\infty^{p-3}(q^2;q^4)_\infty^{p-2}} \tag{3.3}$$

are divisible by p. The integer $c = c_p(a,b)$ may be chosen as the least positive integer congruent to $\overline{24}(3a\,(2b\overline{a}-1)^2+2) \pmod{p}$.

Proof. By (3.1) and the hypotheses of the theorem, we note that

$$\begin{aligned} \frac{q^c \sum_{n=-\infty}^{\infty} \alpha_n q^{a(n^2-n)/2+bn}}{(q^2;q^2)_\infty^{p-3}(q^2;q^4)_\infty^{p-2}} &= \frac{q^c \sum_{n=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}(6j+1)\alpha_n q^{a(n^2-n)/2+bn+3j^2+j}}{(q^2;q^2)_\infty^{p}(q^2;q^4)_\infty^{p}} \\ &\equiv \frac{q^c \sum_{n=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}(6j+1)\alpha_n q^{a(n^2-n)/2+bn+3j^2+j}}{(q^{2p};q^{2p})_\infty(q^{2p};q^{4p})_\infty} \pmod{p}. \end{aligned} \tag{3.4}$$

In the last expression, we see that the denominator is a function of q^p. Let us now examine the exponent of q in the numerator; for ease of computation, we multiply by 24 to achieve

$$\begin{aligned} 24c + 3a(4n^2-4n) + 24bn + 72j^2 + 24j \equiv{}& 3a(2n+2b\overline{a}-1)^2 \\ &+ 2(6j+1)^2 \pmod{p}. \end{aligned} \tag{3.5}$$

Now we observe that if $j \equiv (p-1)/6 \pmod{p}$ (i.e., $(6j+1) \equiv 0 \pmod{p}$,), then the last expression above is congruent to $0 \pmod{p}$ precisely when

$$n \equiv (1-2b\overline{a})\overline{2} \equiv \frac{p+1}{2} - b\overline{a} \pmod{p}.$$

If $j \not\equiv (p-1)/6 \pmod{p}$, then the last expression in (3.5) can never be congruent to $0 \pmod{p}$, because, by the assumption that $-6a$ is a quadratic nonresidue modulo p, exactly one of

$$-3a(2n+2b\overline{a}-1)^2 \quad \text{and} \quad 2(6j+1)^2$$

is a quadratic residue modulo p, and so they cannot be congruent to each other modulo p. Therefore the coefficients of q^{pN} in (3.4) are all linear combinations over $p\mathbb{Z}$ of various α_n. This completes the proof. □

Similarly, from (3.2), we can derive the next theorem. We forego the proof.

Theorem 3.2. *Suppose that p is a prime > 3, $0 < a < p$, and a and b are integers. Let $-6a$ be a quadratic nonresidue modulo p. Suppose $\{\alpha_n\}_{n=-\infty}^{\infty} = \{\alpha_n(z_1, z_2, \ldots, z_j)\}$ is a doubly infinite sequence of Laurent polynomials over $\mathbb{Z}$ with variables $z_1, \ldots, z_j$ independent of q. Then there exists an integer c such that the coefficients of $z_1^{m_1} z_2^{m_2} \cdots z_j^{m_j} q^{pN}$ in*

$$\frac{q^c \sum_{n=-\infty}^{\infty} \alpha_n q^{a(n^2-n)/2+bn}}{(q^2;q^2)_\infty^{p-1} (q;q^2)_\infty^{p-2} (q^4;q^4)_\infty^{p-2}} \tag{3.6}$$

are divisible by p. The integer $c = c_p(a,b)$ may be chosen as the least positive integer congruent to $\overline{24}(3a\,(2b\overline{a}-1)^2 + 8) \pmod{p}$.

Applications analogous to those made by Andrews and Roy in Sect. 4 of their paper [2] can be made here, but since they are easily deduced and no new ideas are involved, we do not proceed any further.

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Coefficients of Harmonic Maass Forms

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Abstract Harmonic Maass forms have recently been related to many different topics in number theory: Ramanujan's mock theta functions, Dyson's rank generating functions, Borcherds products, and central values and derivatives of quadratic twists of modular L-functions. Motivated by these connections, we obtain exact formulas for the coefficients of harmonic Maass forms of nonpositive weight, and we obtain a conditional result for such forms of weight $1/2$. This extends earlier work of Rademacher and Zuckerman in the case of weakly holomorphic modular forms of negative weight.

Keywords Harmonic maass forms • Fourier coefficients

Mathematics Subject Classification: 11F30

1 Introduction and Statement of Results

In work which gave birth to the "circle method," Hardy and Ramanujan [19, 20] derived their famous asymptotic formula for the partition function

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{2n/3}}. \tag{1.1}$$

[1] The authors thank the support of the NSF.
[2] The second author thanks the Manasse family and the Hilldale Foundation for their support.

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K. Alladi and F. Garvan (eds.), *Partitions, q-Series, and Modular Forms*,
Developments in Mathematics 23, DOI 10.1007/978-1-4614-0028-8_3,

In celebrated work, Rademacher perfected [30] the method to derive the exact formula

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right). \tag{1.2}$$

Here $I_\ell(x)$ is the I-Bessel function of order ℓ, and $A_k(n)$ is the Kloosterman sum

$$A_k(n) := \frac{1}{2}\sqrt{\frac{k}{12}} \sum_{\substack{x \pmod{24k} \\ x^2 \equiv -24n+1 \pmod{24k}}} \chi_{12}(x) \cdot \mathrm{e}\left(\frac{x}{12k}\right),$$

where $e(\alpha) := e^{2\pi i\alpha}$ and $\chi_{12}(x) := \left(\frac{12}{x}\right)$.

These works make use of the fact that

$$P(\tau) = \sum_{n=0}^{\infty} p(n)q^{n-\frac{1}{24}} = q^{-\frac{1}{24}} \prod_{n=1}^{\infty} \frac{1}{1-q^n},$$

where $q := e^{2\pi i\tau}$, is a weight $-1/2$ *weakly holomorphic modular form*, a meromorphic modular form whose poles (if any) are supported at cusps. Rademacher and Zuckerman [31, 38, 39] subsequently generalized (1.2) to obtain exact formulas for the coefficients of generic weakly holomorphic modular forms of negative weight.

We recall that the partition generating function also has the *Eulerian form*

$$\sum_{n=0}^{\infty} p(n)q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2}.$$

By changing signs, one obtains Ramanujan's mock theta function

$$f(q) = \sum_{n=0}^{\infty} a(n)q^n := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}. \tag{1.3}$$

The problem of obtaining an asymptotic formula for $a(n)$ is greatly complicated by the fact that $f(q)$ is not a modular form. In their doctoral theses (written under Rademacher), Andrews and Dragonette overcame this difficulty, and they confirmed a conjecture of Ramanujan by proving [1, 14] that

$$a(n) \sim \frac{(-1)^{n-1}}{2\sqrt{n-\frac{1}{24}}} \cdot \mathrm{e}^{\pi\sqrt{\frac{n}{6}-\frac{1}{144}}}.$$

Andrews and Dragonette conjectured an exact formula for $a(n)$ to accompany (1.2). However, without a suitable description of the modular transformation properties of $f(q)$, this conjecture seemed out of reach. Then in his 2002 Ph.D. thesis (written under Zagier), Zwegers [40, 41] provided this required theory. He related Ramanujan's mock theta functions to harmonic Maass forms. By combining his work with a lengthy argument, the authors proved the Andrews-Dragonette Conjecture [7].

Here we generalize this example to the general setting of *harmonic Maass forms* (see Sect. 2 for the definition), a class of automorphic forms which includes the weakly holomorphic modular forms. These results are of particular interest thanks to the recent appearance of harmonic Maass forms in a wide array of subjects: Ramanujan's mock theta functions [2,7–10,37,40,41], Borcherds products [3,4,11], derivatives and values of modular L-functions [13], probability theory [2, 6], and mathematical physics [22–25, 27, 28, 36].

Let $H_{2-k}(N,\chi)$ denote the space of weight $2-k$ harmonic Maass forms on $\Gamma_0(N)$ with Nebentypus character χ, where we assume that $\frac{3}{2} \le k \in \frac{1}{2}\mathbb{Z}$. The Fourier expansions of such forms are given in terms of the incomplete Gamma function

$$\Gamma(\alpha,x) := \int_x^\infty \mathrm{e}^{-t} t^{\alpha-1}\,\mathrm{d}t.$$

More precisely, suppose that $f(\tau) \in H_{2-k}(N,\chi)$. We then have that

$$f(\tau) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n)\Gamma(k-1, 4\pi|n|v) q^n, \tag{1.4}$$

where $\tau = u + iv \in \mathbb{H}$, with $u, v \in \mathbb{R}$. Obviously, each f is the sum of two disjoint pieces, the *holomorphic part of f*

$$f^+(\tau) := \sum_{n \gg -\infty} c_f^+(n) q^n, \tag{1.5}$$

and the *nonholomorphic part of f*

$$f^-(\tau) := \sum_{n<0} c_f^-(n)\Gamma(k-1, 4\pi|n|v) q^n. \tag{1.6}$$

Remark. Weakly holomorphic modular forms are those $f \in H_{2-k}(N,\chi)$ with $f^- = 0$.

Generalizing earlier work of Rademacher and Zuckerman for weakly holomorphic modular forms of nonpositive weight [30, 31], and work of Bruinier and Hejhal [11, 21] for the harmonic Maass forms, we determine exact formulas for the coefficients $c_f^+(n)$. The idea is simple. We shall obtain our results by explicitly constructing Maass-Poincaré series which have poles supported (to arbitrary order) at individual cusps. We then relate a generic harmonic Maass form f to that linear combination of Maass-Poincaré series which matches the divisor of f. If $2-k \le 0$, then it turns out that f equals this linear combination of Maass-Poincaré series.

We now define the functions which are required for these exact formulas. Throughout, we let $k \in \frac{1}{2}\mathbb{Z}$, and we let χ be a Dirichlet character modulo N, where $4 \mid N$ whenever $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. Using this character, for a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we let

$$\Psi_k(M) := \begin{cases} \chi(d) & \text{if } k \in \mathbb{Z}, \\ \chi(d)\left(\frac{c}{d}\right)\epsilon_d^{2k} & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, \end{cases} \tag{1.7}$$

where ϵ_d is defined by

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4, \end{cases} \tag{1.8}$$

and where $\left(\frac{c}{d}\right)$ denotes the extended Legendre symbol. In addition, if $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, then we let

$$\mu(T;\tau) := (c\tau + d)^{2-k}. \tag{1.9}$$

Moreover, for pairs of matrices $S, T \in \mathrm{SL}_2(\mathbb{Z})$, we let

$$\sigma(T,S) := \frac{\mu(T;S\tau)\mu(S;\tau)}{\mu(TS;\tau)}. \tag{1.10}$$

Using this notation, we now define certain generic Kloosterman sums which are naturally associated with cusps of $\Gamma_0(N)$.

Suppose that $\rho = \frac{a_\rho}{c_\rho} = L^{-1}\infty$, ($L \in \mathrm{SL}_2(\mathbb{Z})$) is a cusp of $\Gamma_0(N)$ with $c_\rho | N$ and $\gcd(a_\rho, N) = 1$. Let t_ρ and κ_ρ be the cusp width and parameter of ρ with respect to $\Gamma_0(N)$ (see (1.13)). Suppose that $c > 0$ with $c_\rho | c$ and $\frac{N}{c_\rho} \nmid c$. Then for integers n and m we have the Kloosterman sum

$$K_c(2-k,\rho,\chi,m,n) := \sum_{\substack{0<d<c \\ 0<a<ct \\ a_\rho a \equiv -\frac{c}{c_\rho} \left(\mathrm{mod}\ \frac{N}{c_\rho}\right) \\ (ad,c)=1}} \frac{\sigma(L^{-1},S)}{\Psi_k(L^{-1}S)} \cdot \exp\left(\frac{2\pi i}{c}\left(\frac{(m+\kappa_\rho)a}{t_\rho} + nd\right)\right), \tag{1.11}$$

where $S := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ is the unique matrix defined using the integers a, c, and d. Using properties of σ and Ψ_k, one can easily show that (1.11) is well defined. For similar Kloosterman sums we refer the reader to [32].

For convenience, we let $\mathcal{S}_N$ be a subset of $\mathrm{SL}_2(\mathbb{Z})$ with the property that $S_1^{-1}\infty$ and $S_2^{-1}\infty$ are inequivalent cusps in $\Gamma_0(N)$ whenever S_1 and S_2 are distinct elements of $\mathcal{S}_N$. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we define

$$f_M(\tau) := (c\tau + d)^{k-2} f\left(\frac{a\tau + b}{c\tau + d}\right), \tag{1.12}$$

where $\sqrt{\tau}$ is the principal branch of the holomorphic square root. Using this notation, we can speak of the Fourier expansion of a form f at a cusp ρ. More precisely, if $L \in \mathcal{S}_N$ with $\rho = L^{-1}\infty$, then we have

$$f_\rho(\tau) = \sum_{n \in \mathbb{Z}} a_\rho^+(n) q^{\frac{n+\kappa_\rho}{t_\rho}} + f_\rho^-(\tau). \tag{1.13}$$

We define the *principal part* of f at ρ by

$$P_{f,\rho}(\tau) := \sum_{m+\kappa_\rho<0} a_\rho^+(m) q^{\frac{m+\kappa_\rho}{t_\rho}}. \tag{1.14}$$

We shall use the principal parts of a form f to determine our exact formulas. To this end, we identify, for each cusp ρ, its contribution to the exact formula. To make this precise, let $M = L^{-1}$ and $\mu = L\infty$. For positive n, we then define

$$\begin{aligned}\mathcal{A}(N, 2-k, \chi, \rho, m, c; n) :&= -\frac{i^k 2\pi}{t_\mu} \left|\frac{(-m+\kappa_\mu)}{t_\mu n}\right|^{\frac{k-1}{2}} \\ &\times \sum_{\substack{c>0 \\ c_\mu | c,\, \frac{N}{c_\mu} \nmid c}} \frac{K_c(2-k, \mu, \chi, -m, -n)}{c} \\ &\times I_{k-1}\left(\frac{4\pi}{c}\sqrt{\frac{n|-m+\kappa_\mu|}{t_\mu}}\right).\end{aligned} \tag{1.15}$$

Here t_μ and κ_μ are the cusp parameters for μ as in the notation above.

Using this notation, we define the order $\mathcal{N}$ *Kloosterman approximation* of $c_f^+(n)$ by

$$\mathcal{C}(f, \mathcal{N}; n) := \sum_{L \in \mathcal{S}_N} \sum_{m+\kappa_\rho<0} a_\rho^+(m) \sum_{c=1}^{\mathcal{N}} \mathcal{A}(N, 2-k, \chi, \rho, m, c; n). \tag{1.16}$$

Moreover, we define $\mathcal{C}(f, \infty; n)$ in the obvious way.

Remark. We stress again that L and ρ are related (throughout this section) by the formula $\rho = L^{-1}\infty$.

Theorem 1.1. *If $f \in H_{2-k}(N, \chi)$ with $2 \leq k \in \frac{1}{2}\mathbb{Z}$, then for positive n we have*

$$c_f^+(n) = \mathcal{C}(f, \infty; n).$$

Two remarks.

1. Using the asymptotic behavior of I-Bessel functions, an inspection of the principal parts of f gives a minimal $\mathcal{N}$ for which

$$\mathcal{C}(f, \mathcal{N}; n) \sim c_f^+(n).$$

Moreover, it is not difficult to show that

$$c_f^+(n) = \mathcal{C}(f, \sqrt{n}; n) + O_f(n^\epsilon).$$

2. Theorem 1.1 gives the results of Rademacher and Zuckerman in the very special case of those $f \in H_{2-k}(N,\chi)$ for which $2-k<0$ and $f^{-}=0$.

For weight $2-k=\frac{1}{2}$, we have a conditional result. To make it precise, we say that a form $f \in H_{\frac{1}{2}}(N,\chi)$ is *good* if the Maass-Poincaré series corresponding to nontrivial terms in the principal parts of f are individually convergent.

Theorem 1.2. *If $f \in H_{\frac{1}{2}}(N,\chi)$ is good, then there is a finite set $S_\Theta(f)$ of complex numbers such that for positive n we have*

$$c_f^+(n) = \mathcal{C}(f,\infty;n) + \mu(n)$$

for some $\mu(n) \in S_\Theta(f)$. Moreover, if $n \neq dm^2$ for some $d \mid N$ and $m \in \mathbb{Z}^+$, then $\mu(n)=0$.

Four remarks.

1. Our method approximates each form as a linear combination of Poincaré series. In this paper, apart from the cases when the weight is 1/2, this exactly determines the form. When the weight is 1/2, this uniquely determines the form up to a holomorphic modular form, which must be a linear combination of theta functions. This contribution is given by $S_\Theta(f)$ and the numbers $\mu(n)$.
2. We believe that all $f \in H_{\frac{1}{2}}(N,\chi)$ are good. In earlier work we deduced convergence of such Maass-Poincaré series by making using of relationships between Kloosterman sums and Salié sums (see Sect. 4 of [7]) and by generalizing work of Goldfeld and Sarnak [18] on sums of Kloosterman sums (see [17]). It seems likely that a careful application of these ideas will prove that each such f is indeed good.
3. Theorems 1.1 and 1.2 give exact formulas for harmonic weak Maass forms on congruence groups of the form $\Gamma_1(N)$. This follows from the fact that

$$H_{2-k}(\Gamma_1(N)) = \oplus_\chi H_{2-k}(N,\chi),$$

   where the sum is over Dirichlet characters modulo N.
4. Apart from those harmonic Maass forms f which are holomorphic modular forms (which in this paper can only happen if $2-k=1/2$), the results above imply that the $c_f^+(n)$ are not bounded by any power of n. There are arithmetic progressions of n for which $c_f^+(n)$ grows subexponentially in n. We note that Bruinier and Funke [12] have a more general notion of a harmonic Maass form for which this claim is false. Indeed, Zagier's weight 3/2 Eisenstein series $g(\tau)$ [35], whose holomorphic part

$$g^+(\tau) = -1/12 + \sum_{n>0} h(-n)q^n$$

   is the generating function for Hurwitz class numbers, is such a form. Obviously, we have that $h(-n) = O(n^{\frac{1}{2}+\epsilon})$ which does not have subexponential growth.

As mentioned earlier, harmonic Maass forms have appeared in a wide variety of contexts in recent years. We conclude the introduction with one application of these results.

In his work on Ramanujan's partition congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}, \end{aligned}$$

Dyson defined [15] the *rank* of a partition to be its largest part minus its number of summands. If $0 \leq r < t$ are integers, then let $N(r,t;n)$ denote the number of partitions with rank congruent to $r \pmod{t}$. In a recent paper [5], the first author obtained asymptotic formulas for each $N(r,t;n)$ when t is odd. Theorem 1.2, combined with a generalization of Sect. 4 of [7], proves that these asymptotics can be extended to exact formulas.

Theorem 1.3. *If $0 \leq r < t$, where t is odd, then Theorem 1.2 gives an exact formula for $N(r,t;n)$.*

Remark. For brevity, we do not repeat the formulas from [5]. Theorem 1.2 proves that one obtains exact formulas, up to the coefficients of a linear combination of theta functions, by summing the first author's formulas to infinity.

This paper is organized as follows. In Sect. 2 we recall basic facts about harmonic Maass forms, and in Sect. 3 we construct the relevant Maass-Poincaré series. In Sect. 4 we prove Theorems 1.1 and 1.2.

2 Harmonic Maass Forms

We recall the notion of a harmonic Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$. Throughout let $\tau = u + iv \in \mathbb{H}$ with $u, v \in \mathbb{R}$, and we define the weight k hyperbolic Laplacian

$$\Delta_k := -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right). \tag{2.1}$$

For odd integers d, define ϵ_d by

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}, \\ i & \text{if } d \equiv 3 \pmod{4}. \end{cases} \tag{2.2}$$

Definition 2.1. If N is a positive integer (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$) and χ is a Dirichlet character modulo N, then a *weight k harmonic Maass form on $\Gamma_0(N)$ with Nebentypus χ* is any smooth function $M : \mathbb{H} \to \mathbb{C}$ satisfying the following:

(1) For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and all $\tau \in \mathbb{H}$, we have

$$M\left(\frac{a\tau+b}{c\tau+d}\right) = \begin{cases} \chi(d)(c\tau+d)^k M(\tau) & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} \chi(d)(c\tau+d)^k\, M(\tau) & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}. \end{cases} \tag{2.3}$$

(2) We have that $\Delta_k M = 0$.
(3) There is a polynomial $P_M = \sum_{n \leq 0} c^+(n) q^n \in \mathbb{C}[q^{-1}]$ such that

$$M(\tau) - P_M(\tau) = O(e^{-\epsilon v})$$

as $v \to +\infty$ for some $\epsilon > 0$. Analogous conditions are required at all cusps.

Remark. We call $P_M(\tau)$ the *principal part* of M at ∞, with analogous parts at others cusps.

Remark. Since holomorphic functions on $\mathbb{H}$ are harmonic, it follows that weakly holomorphic modular forms are harmonic Maass forms.

Harmonic Maass forms are related to classical modular forms thanks to the properties of differential operators. Here we require the differential operator

$$\xi_w := 2iv^w \cdot \overline{\frac{\partial}{\partial \overline{\tau}}}. \tag{2.4}$$

The following lemma,[2] which is a straightforward refinement of a proposition of Bruinier and Funke (see Proposition 3.2 of [12]), shall play a central role throughout this paper.

Lemma 2.2. *If $f \in H_{2-k}(N, \chi)$, then*

$$\xi_{2-k} : H_{2-k}(N, \chi) \longrightarrow S_k(N, \overline{\chi})$$

is a surjective map. Moreover, if

$$f(\tau) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|v) q^n,$$

then we have that

$$\xi_{2-k}(f) = -(4\pi)^{k-1} \sum_{n=1}^{\infty} \overline{c_f^-(-n)} n^{k-1} q^n.$$

We shall also require the following lemma.

[2] The formula for $\xi_{2-k}(f)$ corrects a typographical error in [12].

Lemma 2.3. *If $f \in H_{2-k}(N,\chi)$ has the property that $\xi_{2-k}(f) \neq 0$, then the principal part of f is nonconstant for at least one cusp.*

Proof. This lemma follows from the work of Bruinier and Funke [12]. Using their pairing $\{\bullet,\bullet\}$, one finds that $\{\xi_{2-k}f, f\} \neq 0$ thanks to its interpretation in terms of Petersson norms. On the other hand, Proposition 3.5 of [12] expresses this quantity in terms of the principal part of f and the coefficients of the cusp form $\xi_{2-k}(f)$. An inspection of this formula reveals that at least one principal part of f must be nonconstant. □

3 Maass-Poincaré Series

Here we use the method of Poincaré series to construct more general harmonic Maass forms with multiplier (i.e., generalizing the notion of Nebentypus). Such forms have been considered by Fay, Hejhal, and Niebur [16, 21, 29], and more recently by the authors and Bruinier [9, 11].

We closely follow the setup in Rankin's classic text [32]. Suppose that Γ is a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$ with $-I \in \Gamma$, $\frac{3}{2} \leq k \in \frac{1}{2}\mathbb{Z}$, and let $\nu(\bullet)$ be a multiplier system. Moreover, let $\rho := L^{-1}\infty$ be a cusp and let t and κ be its cusp width and parameter for Γ. Let $\hat{\Gamma}_\rho$ be the stabilizer of ρ in $\hat{\Gamma}$, the homogenization of Γ. For $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we let

$$\mu(T;\tau) := (c\tau + d)^{2-k}, \tag{3.1}$$

and for $T \in \Gamma$ we let

$$\nu(T;\tau) := \nu(T)\mu(T;\tau). \tag{3.2}$$

For $s \in \mathbb{C}$ and $y \in \mathbb{R} \setminus \{0\}$, let

$$\mathcal{M}_s(y) := |y|^{\frac{k}{2}-1} M_{\mathrm{sign}(y)(1-k/2),s-\frac{1}{2}}(|y|), \tag{3.3}$$

where $M_{\nu,\mu}(z)$ is the M-Whittaker function which is a solution to the differential equation

$$\frac{\partial^2 u}{\partial z^2} + \left(-\frac{1}{4} + \frac{\nu}{z} + \frac{\frac{1}{4} - \mu^2}{z^2} \right) u = 0.$$

Using this function we let

$$\phi_s(\tau) := \mathcal{M}_s(4\pi v) e(u),$$

where $\tau = u + iv$. If $m > 0$, then we have the Maass-Poincaré series

$$\mathcal{P}_L(\tau, m, \Gamma, 2-k, s, \nu) := \sum_{T \in \hat{\Gamma}_\rho \backslash \hat{\Gamma}} \frac{\phi_s\left(\frac{(-m+\kappa)}{t} LT\tau\right)}{\mu(L;T\tau)\nu(T;\tau)}. \tag{3.4}$$

It is easy to check that $\phi_s(\tau)$ is an eigenfunction of Δ_{2-k} with eigenvalue

$$s(1-s)+\frac{k^2-2k}{4}.$$

From this one can conclude, when the series converges absolutely, that $\mathcal{P}_L$ is an eigenfunction of Δ_{2-k}. Next define for $S \in \mathrm{SL}_2(\mathbb{Z})$

$$\Gamma^S := S^{-1}\Gamma S$$

and let ν^S be the multiplier defined on Γ^S by

$$\nu^S\left(S^{-1}TS\right) := \frac{\nu(T)\sigma(T,S)}{\sigma\left(S,S^{-1}TS\right)},$$

where

$$\sigma(T,S) := \frac{\mu(T;S\tau)\mu(S;\tau)}{\mu(TS;\tau)}. \tag{3.5}$$

We denote by $H_{2-k}(\Gamma,\nu)$ the space of harmonic Maass forms with multiplier ν. This is the space of forms satisfying the conditions in Definition 2.1 where (2.3) is replaced, for all $S \in \Gamma$, by

$$f(S\tau) = \frac{\mu(S;\tau)}{\sigma(L,S)} f(\tau). \tag{3.6}$$

The following lemma follows immediately from the properties described above for the functions in this construction.

Lemma 3.1. *If* $\mathrm{Re}(s) > 1$*, then the series in (3.4) is absolutely and uniformly convergent. Moreover, if* $k > 2$ *and* $s = k/2$*, then the series is in* $H_{2-k}(\Gamma,\nu)$*.*

Now we return to the setting in the introduction, where we consider forms with Nebentypus χ on $\Gamma_0(N)$. Recalling (1.7), we set

$$\mathcal{P}_L(\tau,m,N,2-k,\chi) := \frac{1}{\Gamma(k)}\mathcal{P}_L\left(\tau,m,\Gamma_0(N),2-k,\frac{k}{2},\Psi_k\right). \tag{3.7}$$

We can determine the Fourier expansion of these Poincaré series at all cusps. The next theorem gives the holomorphic parts (1.5) for these series.

Theorem 3.2. *If* $2 \leq k \in \frac{1}{2}\mathbb{Z}$*, then* $\mathcal{P}_L(\tau,m,N,2-k,\chi)$ *is in* $H_{2-k}(N,\chi)$*. Moreover, the following are true:*

(1) We have

$$\mathcal{P}_L^+(\tau,m,N,2-k,\chi) = \delta_{\infty,\rho}\cdot\frac{q^{-m}}{\Psi_k(L^{-1})\sigma(L,L^{-1})} + \sum_{n\geq 0} a^+(n)q^n,$$

where $\delta_{\infty,\rho} = 0$*, unless* $\infty \sim \rho$ *in* $\Gamma_0(N)$*, in which case it is* 1*. Moreover, if* $n > 0$*, then*

$$a^{+}(n) = -i^k 2\pi \left|\frac{(-m+\kappa)}{tn}\right|^{\frac{k-1}{2}} \frac{1}{t} \sum_{\substack{c>0 \\ c_\rho | c,\, \frac{N}{c_\rho} \nmid c}} \frac{K_c(2-k,\rho,\chi,-m,n)}{c}$$
$$\times I_{k-1}\left(\frac{4\pi}{c}\sqrt{\frac{n|-m+\kappa|}{t}}\right).$$

(2) The principal part of $\mathcal{P}_L(\tau,m,N,2-k,\chi)$ *at the cusp* $\mu = S\infty$ *is given by*

$$\delta_{L,S} \cdot q^{\frac{(-m+\kappa_\rho)}{t_\rho}},$$

where $\delta_{L,S} = 0$*, unless* $L = S$*, in which case it equals* 1*.*

Three remarks.

(1) Theorem 3.2 holds when $2-k = 1/2$, provided that one can guarantee convergence in the formulas for the Fourier coefficients.
(2) Although Theorem 3.2 (1) is about the coefficients of the holomorphic parts of these Poincaré series, we also give the Fourier coefficients of the nonholomorphic parts in the proof of the theorem.
(3) Two features of Theorem 3.2 are important for us. Obviously, the exact formulas for the coefficients are important. Secondly, the fact that the principal parts are distinguished by cusps is vital. This fact allows us to piece together such Maass forms using the collection of principal parts at cusps.

Proof of Theorem 3.2. We first consider the more general Poincaré series from above and assume $\mathrm{Re}(s) > 1$. We require more notation. Let t_1 (resp. t_2) be the cusp width of ∞ (resp. ρ) in Γ and κ_1 (resp. κ_2) the associated parameter. Define

$$\mathcal{W}_s(y) := |y|^{\frac{k}{2}-1} W_{\left(1-\frac{k}{2}\right)\mathrm{sign}(y),\, s-\frac{1}{2}}(|y|), \tag{3.8}$$

where $W_{\nu,\mu}(z)$ is the standard W-Whittaker function. Define the Kloosterman sum

$$K_c(\rho,\Gamma,\nu,m,n) := \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = S \in \mathcal{F}_L(c)} \frac{\sigma\left(L^{-1},S\right)}{\nu\left(L^{-1}S\right)} \exp\left(\frac{2\pi i}{c}\left(\frac{(m+\kappa_2)a}{t_2} + \frac{(n+\kappa_1)d}{t_1}\right)\right),$$

where $\mathcal{F}_L(c)$ consists of all matrices $S \in L\Gamma$ for which

$$0 \le d < ct_1 \qquad 0 \le a < ct_2.$$

One can compute the Fourier expansion of the Poincaré series using Poisson summation. Then the calculation boils down to computing integrals of the form

$$\int_{\mathbb{R}} \tau^{2-k} \exp\left(-2\pi i x(\kappa+n) - 2\pi i\lambda \mathrm{Re}\left(\frac{1}{\tau}\right)\right) \mathcal{M}_s\left(-4\pi\lambda \mathrm{Im}\left(\frac{1}{\tau}\right)\right) \mathrm{d}x.$$

This integral can be computed using pages 32–33 of [11]. This yields the following Fourier expansion:

$$\begin{aligned}\mathcal{P}_L(\tau,m,\Gamma,2-k,s,\nu) = \delta_{L,\Gamma}\cdot \frac{\mathcal{M}_s\left(\frac{4\pi(-m+\kappa_1)y}{t_1}\right)\mathrm{e}\left(\frac{(-m+\kappa_1)(x+r)}{t_1}\right)}{\nu\left(L^{-1}U^r\right)\sigma\left(L,L^{-1}\right)}\\ + \sum_{n\in\mathbb{Z}} a_y(n)\mathrm{e}\left(\frac{x}{t_1}(n+\kappa_1)\right),\end{aligned}$$

where $\delta_{L,\Gamma}=0$ unless $L^{-1}U^r\in\Gamma$ for some $r\in\mathbb{Z}$ with $U:=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ in which case it is equal to 1. In this case we have in particular that $t_1=t_2$ and $\kappa_1=\kappa_2$. Moreover the coefficients $a_y(n)$ are given as follows:

(1) If $n+\kappa_1<0$, then

$$\begin{aligned}a_y(n) = -i^k \frac{2\pi\Gamma(2s)}{\Gamma\left(s+\frac{k}{2}-1\right)}\left|\frac{t_1(-m+\kappa_2)}{t_2(n+\kappa_1)}\right|^{\frac{k-1}{2}} \mathcal{W}_s\left(\frac{4\pi(n+\kappa_1)y}{t_1}\right)\\ \times\frac{1}{t_1}\sum_{c>0}\frac{K_c(\rho,\Gamma,\nu,-m,n)}{c}\cdot J_{2s-1}\left(\frac{4\pi}{c}\sqrt{\frac{|-m+\kappa_2||n+\kappa_1|}{t_1t_2}}\right),\end{aligned}$$

where J_ℓ is the Bessel function of order ℓ.

(2) If $n+\kappa_1=0$, then

$$\begin{aligned}a_y(n) &= -i^k 2^k \pi^{\frac{k}{2}+s} y^{\frac{k}{2}-s} t_1^{-2} t_2^{-\frac{k}{2}+1-s}\\ &\quad\times\frac{\Gamma(2s)}{(2s-1)\Gamma\left(s-\frac{k}{2}+1\right)\Gamma\left(s+\frac{k}{2}-1\right)}|-m+\kappa_2|^{\frac{k}{2}+s-1}\\ &\quad\times\sum_{c>0}\frac{K_c(\rho,\Gamma,\nu,-m,0)}{c^{2s+1}}.\end{aligned}$$

(3) If $n+\kappa_1>0$, then

$$\begin{aligned}a_y(n) = -i^k \frac{2\pi\Gamma(2s)}{\Gamma\left(s-\frac{k}{2}+1\right)}\left|\frac{t_1(-m+\kappa_2)}{t_2(n+\kappa_1)}\right|^{\frac{k-1}{2}} \mathcal{W}_s\left(\frac{4\pi(n+\kappa_1)y}{t_1}\right)\\ \times\frac{1}{t_1}\sum_{c>0}\frac{K_c(\rho,\Gamma,\nu,-m,n)}{c}\cdot I_{2s-1}\left(\frac{4\pi}{c}\sqrt{\frac{|-m+\kappa_2||n+\kappa_1|}{t_1t_2}}\right).\end{aligned}$$

Using special values of Whittaker functions, we obtain

$$\begin{aligned}\mathcal{P}_L\left(\tau,m,\Gamma,2-k,\frac{k}{2},\nu\right) &= \delta_{L,\Gamma}\cdot\frac{\Gamma(k)\cdot\mathrm{e}\left(\frac{(-m+\kappa_1)r}{t_1}\right)}{\nu\left(L^{-1}U^r\right)\sigma\left(L,L^{-1}\right)}q^{\frac{(-m+\kappa_1)}{t_1}}\\ &\quad\times\left(1-\frac{\Gamma\left(k-1,\frac{4\pi(-m+\kappa_1)y}{t_1}\right)}{\Gamma(k-1)}\right)+\sum_{n+\kappa_1\geq 0}a(n)q^{\frac{n+\kappa_1}{t_1}}\\ &\quad+\sum_{n+\kappa_1<0}a(n)\Gamma\left(k-1,\frac{4\pi|n+k_1|y}{t_1}\right)q^{\frac{n+\kappa_1}{t_1}}.\end{aligned}$$

Here the coefficients $a(n)$ are given as follows:

(1) If $n+\kappa_1<0$, then

$$\begin{aligned}a(n) &= -i^k 2\pi(k-1)\left|\frac{t_1(-m+\kappa_2)}{t_2(n+\kappa_1)}\right|^{\frac{k-1}{2}}\\ &\quad\times\frac{1}{t_1}\sum_{c>0}\frac{K_c(\rho,\Gamma,\nu,-m,n)}{c}\cdot J_{k-1}\left(\frac{4\pi}{c}\sqrt{\frac{|-m+\kappa_2||n+\kappa_1|}{t_1t_2}}\right).\end{aligned}$$

(2) If $n+\kappa_1=0$, then

$$a(n)=-i^k(2\pi)^k t_1^{-2}t_2^{1-k}|-m+\kappa_2|^{k-1}\sum_{c>0}\frac{K_c(\rho,\Gamma,\nu,-m,0)}{c^{k+1}}.$$

(3) If $n+\kappa_1>0$, then

$$\begin{aligned}a(n) &= -i^k 2\pi\Gamma(k)\left|\frac{t_1(-m+\kappa_2)}{t_2(n+\kappa_1)}\right|^{\frac{k-1}{2}}\\ &\quad\times\frac{1}{t_1}\sum_{c>0}\frac{K_c(\rho,\Gamma,\nu,-m,n)}{c}\cdot I_{k-1}\left(\frac{4\pi}{c}\sqrt{\frac{|-m+\kappa_2||n+\kappa_1|}{t_1t_2}}\right).\end{aligned}$$

The proof of Theorem 3.2 follows easily for $2-k\leq 0$. One merely observes that the defining series are convergent. □

4 Proof of Theorems 1.1 and 1.2

Here we prove Theorems 1.1 and 1.2 simultaneously. Thanks to Theorem 3.2, we have an explicit linear combination of Maass-Poincaré series, say $\mathfrak{f}\in H_{2-k}(N,\chi)$, whose principal parts agree with the principal parts of f up to additive constants. There are three possibilities:

Case 1. We have that $f - \mathfrak{f}$ is a holomorphic modular form. It can only be nonzero when $2 - k = \frac{1}{2}$, in which case the Serre-Stark Basis Theorem implies that $f - \mathfrak{f}$ is a linear combination of theta functions. Either way, we obtain the relevant desired conclusions in Theorems 1.1 and 1.2.

Case 2. We have that $f - \mathfrak{f}$ is a weakly holomorphic modular form which is not a holomorphic modular form. Such a form must have a pole at a cusp. However, this cannot happen since we constructed $\mathfrak{f}$ so that the principal parts of $f - \mathfrak{f}$ are constant.

Case 3. We have that $f - \mathfrak{f}$ is a harmonic Maass form with a nontrivial nonholomorphic part. However, Lemma 2.3 shows that all such harmonic Maass forms have at least one principal part which is nonconstant. Therefore, this possibility never occurs.

This completes the proofs of the claimed exact formulas in Theorems 1.1 and 1.2.

Acknowledgments This paper was written while the first author was in residence at the Max-Planck Institute. She thanks the institute for providing a stimulating environment.

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On the Growth of Restricted Integer Partition Functions

E. Rodney Canfield and Herbert S. Wilf

Abstract We study the rate of growth of $p(n,S,M)$, the number of partitions of n whose parts all belong to S and whose multiplicities all belong to M, where S (resp. M) are given infinite sets of positive (resp. nonnegative) integers. We show that if M is all nonnegative integers then $p(n,S,M)$ cannot be of only polynomial growth and that no sharper statement can be made. We ask: if $p(n,S,M) > 0$ for all large enough n, can $p(n,S,M)$ be of polynomial growth in n?

Keywords Integer partitions • Asymptotic growth

Mathematics Subject Classification: Primary: 05A17

1 The Question

Let S be a set of positive integers, and let $p_S(n)$ denote the number of partitions of the integer n all of whose parts lie in S. For various sets S, the asymptotic growth rate of $p_S(n)$ is known, and the known rates lie in the range of polynomial growth to superpolynomial-but-subexponential rates.

For example, if S consists of all positive integers then the celebrated theorem of Hardy, Ramanujan, and Rademacher [3, 5] has given the complete asymptotic expansion, of which the first term is

$$p_S(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right). \tag{1}$$

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K. Alladi and F. Garvan (eds.), *Partitions, q-Series, and Modular Forms*, Developments in Mathematics 23, DOI 10.1007/978-1-4614-0028-8_4,

As an example of a sparse set of parts, take $S = \{1, 2, 2^2, 2^3, \ldots\}$, the case of binary partitions. Then de Bruijn [2] found several terms of the asymptotics of the logarithm, which begins as

$$\log p_S(2n) = \frac{1}{2\log 2}\left(\log\frac{n}{\log n}\right)^2 (1+o(1)). \tag{2}$$

For a final example, suppose the set S of allowable parts is finite. Then we have, say, $S = \{a_1 < \cdots < a_k\}$, and we are dealing with "the money changing problem," a.k.a. "the problem of Frobenius." A result of Schur [6] holds that in this case $p_S(n)$ is of polynomial growth.

Theorem 1 (Schur). *If $S = \{a_1 < \cdots < a_k\}$, and $gcd(S) = 1$, then*

$$p_S(n) \sim \frac{n^{k-1}}{(k-1)!a_1a_2\ldots a_k}, \tag{3}$$

and in particular, $p_S(n) > 0$ for all large enough n.

We show here that if the set of allowable parts is infinite, no matter how sparse, then the partition function $p_S(n)$ must grow faster than every polynomial. We show also that this result is best possible in the sense that if $\varepsilon(n)$ is any unbounded function of n then there exists an infinite set S of allowable parts such that $p_S(n) = O(n^{\varepsilon(n)})$.

We discuss also the situation in which we have an arbitrary set of allowable parts and an arbitrary set of allowable multiplicities.

2 Preliminaries

Lemma 1. *Let $S = a_1 < a_2 < a_3 < \ldots$ be a set of positive integers such that $gcd(S) = 1$. Then S contains a finite coprime subset.*

Proof. Let $g_n = \gcd(a_1, \ldots, a_n)$. Then $a_1 \geq g_1 \geq g_2 \geq \ldots$, so $\exists i_0$ such that $\forall i > i_0$: $g_i = 1$. Indeed, if not then $\exists i_0$ such that $\forall i > i_0$: $g_i = g > 1$. But then we would have $\gcd(S) = g > 1$, a contradiction. □

Lemma 2. *The following two properties of a set S of positive integers are equivalent:*

1. *For all sufficiently large integers n we have $p_S(n) > 0$.*
2. *$gcd(S) = 1$.*

Proof. If $\gcd(S) = 1$, then by Lemma 1, S contains a *finite* coprime subset $\overline{S}$. By Schur's theorem, $p_{\overline{S}}(n) > 0$ for all large enough n, hence so is $p_S(n)$, and conclusion 1 holds. On the other hand, if $\gcd(S) > 1$ then conclusion 1 is obviously false. □

We remark that Lemma 2, whose proof we have given in order to keep this paper self-contained, is a special case of a much more general result of Bateman and Erdős [1], who found the conditions on S under which, for a fixed $k \geq 0$, almost all values of the kth differences of $\{p_S(n)\}_{n=0}^{\infty}$ are strictly positive.

Next we will need a lemma that allows us to estimate the growth of $p_S(n)$ for arbitrary sets S of parts. We will in fact prove a more general result, in which not only the set S of allowable parts can be arbitrarily prescribed, but so can the set M of allowable multiplicities of those parts.

Hence, let S be a set of positive integers and let M be a set of nonnegative integers such that $0 \in M$. Let $M(x), S(x)$ denote the respective counting functions of M, S. That is $M(x) = |\{\mu \in M : \mu \leq x\}|$ and likewise for $S(x)$. Finally we denote by $p(n;S,M)$ the number of partitions of n whose parts all belong to S and the multiplicities of whose parts all belong to M.

Lemma 3. *For the general partition function $p(n;S,M)$ we have*

$$p(n;S,M) \leq \prod_{a_i \in S} M(n/a_i). \tag{4}$$

Further, there must exist at least one integer $r \leq n^2$ s.t.

$$p(r;S,M) \geq \frac{1}{n^2+1} \prod_{a_i \in S} M(n/a_i). \tag{5}$$

If also $p(n;S,M)$ is a nondecreasing function of n then we have the stronger statement that

$$p(n;S,M) \geq \frac{1}{n+1} \prod_{a_i \in S} M(\sqrt{n}/a_i).$$

Proof. Fix $n > 0$ and consider the form

$$\phi = m_1 a_1 + m_2 a_2 + m_3 a_3 + \cdots + m_n a_n.$$

Now allow each of the m_i to take any value that it wishes to take, subject to $m_i \in M$ and $m_i \leq n/a_i$. For each set of choices, the form ϕ is a partition of some integer $\leq n^2$, and *all* partitions of n occur.

The total number of values that the form takes, counting multiplicities, is

$$\prod_i M(n/a_i).$$

(Note that all terms with sufficiently large index i are $= 1$.) Since every partition of n occurs, we find that

$$p(n;S,M) \leq \prod_i M(n/a_i). \tag{6}$$

Furthermore, since the average number of occurrences of the integers $\leq n^2$ is

$$\frac{1}{n^2+1} \prod_i M(n/a_i),$$

the second conclusion of the lemma is proved. □

Let us test this with one or two examples. First take M to be all nonnegative integers and S to be all positive integers. then $M(x) = 1 + \lfloor x \rfloor$ and we find that $p(n;S,M) \leq \prod_i (1 + \lfloor n/i \rfloor)$. This is around $n^n/n!$, which is roughly e^n, whereas the correct growth is around $e^{C\sqrt{n}}$. The lower bound is about e^n/n^2, so there exists an integer $r \leq n^2$ s.t. $p(r;S,M) \geq \prod_i \lceil n/i \rceil / n^2$, which is about e^n/n^2. But indeed, if there is such an integer r, then since $p(n)$ is monotone, we can take $r = n^2$. Lemma 3 then says that $p(n^2;S,M) \geq e^n/n^2$, or

$$p(n;S,M) \geq e^{\sqrt{n}}/n, \tag{7}$$

which is reasonably sharp. For another example, in the case of binary partitions, the upper bound (6) yields the estimate

$$\log p_S(2n) \leq \log(2n+1)\log_2(2n) \sim \frac{(\log n)^2}{\log 2},$$

which can be compared with (2).

3 The Growth of $p_S(n)$

Theorem 2. *Let S be an infinite set of positive integers, and let $p_S(n)$ be the number of partitions of n whose parts belong to S. Then $p_S(n)$ is of superpolynomial growth, that is, for every fixed k the assertion $p_S(n) = O(n^k)$ is false. This result is best possible in the sense that if $\varepsilon(n)$ is any function of n that $\to \infty$, then we can find an infinite set S such that $p_S(n) = O(n^{\varepsilon(n)})$.*

Proof. Let $S = \{1 \leq a_1 < a_2 < \cdots\}$. Then $g = \gcd(S) \leq a_1 < \infty$, and the theorem is true for S iff it is true for S/g. Hence we can, and do, assume w.l.o.g. that $\gcd(S) = 1$.

Let $T \subseteq S$ be such a finite coprime subset, and put $k = |T|$. By Schur's theorem we have $p_S(n) \geq p_T(n) \sim Cn^{k-1}$. But we can make k arbitrarily large by adjoining elements of S to T since that adjunction preserves coprimality. Therefore $P_S(n)$ must grow superpolynomially.

For the second part of the theorem we use (4) with unconstrained multiplicities, i.e., with $M(x) = 1 + \lfloor x \rfloor$ for $x > 0$. If we write $A(n) = |\{i : a_i \leq n\}|$ then (4) reads as

$$\begin{aligned} p_S(n) &\leq \prod_{i \geq 1} \left(1 + \left\lfloor \frac{n}{a_i} \right\rfloor\right) \leq \prod_{a_i \leq n} \left(1 + \frac{n}{a_i}\right) \leq n^{A(n)} \prod_{a_i \leq n} \left(\frac{1}{n} + \frac{1}{a_i}\right) \\ &\leq n^{A(n)} \prod_{a_i \leq n} \left(1 + \frac{1}{a_i}\right) \leq n^{A(n)} \prod_{a_i \leq n} \mathrm{e}^{1/a_i} \leq n^{A(n)} \mathrm{e}^{H_n} = O\left(n^{A(n)+1}\right), \end{aligned}$$

in which H_n is the nth harmonic number. Evidently we can make this $O(n^{\varepsilon(n)})$ by taking the set S to be sufficiently sparse. □

4 A Partition Function That Grows Slowly

There are infinite sequences of allowable parts and multiplicities on which the partition function grows only polynomially fast; in fact, it can even grow subpolynomially.

One such example is the case where the allowable parts are the sequence $\{2^{2^j}\}_{j=0}^{\infty}$, and the allowable multiplicities are

$$\{0\} \cup \{2^{2^j}\}_{j=0}^{\infty}.$$

In this case we have, in the notation above, $M(x) = 1 + \lfloor \lg\lg x \rfloor$, for $x \geq 4$, where "lg" is the log to the base 2. Then by (6) we have

$$\begin{aligned} p(n;S,M) &\leq \prod_{2^{2^i} \leq n/4} \left(1 + \lfloor \lg\lg \frac{n}{2^{2^i}} \rfloor\right) \leq \prod_{2^{2^i} \leq n/4} \left(2 \lfloor \lg\lg \frac{n}{2^{2^i}} \rfloor\right) \\ &\leq (\lg n)(\lg\lg n)^{\lg\lg n}, \end{aligned}$$

which is of subpolynomial growth. This argument fails if the parts and multiplicities are all of the powers of 2.

The above argument can be generalized to give a fairly simple criterion, in terms of the sets of parts and multiplicities, for polynomial growth of the partition function.

5 Representing All Large Integers

The example above shows that if the allowable multiplicities and parts are thin enough, even though they both are infinite sets, then the partition function can grow very slowly. But the example has the property that some arbitrarily large integers are not represented at all. It may be that if we rule out such situations then the growth must be superpolynomial. We formulate this as

Unsolved problem 1. Let S, M be infinite sets of nonnegative integers with $0 \notin S$, and let $p(n;S,M)$ be the number of partitions of n whose parts all lie in S and the multiplicities of whose parts all lie in M. Suppose further that $p(n;S,M) > 0$ for all sufficiently large n. Must $p(n;S,M)$ then be of superpolynomial growth?

Unsolved problem 2. Find necessary and sufficient conditions on S, M in order that $p(n;S,M) > 0$ for all large enough n. Failing this, find as sharp as possible necessary conditions, and similarly sufficient conditions for this to happen.

Unsolved problem 3. Find necessary and sufficient conditions on S, M in order that $p(n;S,M)$ increase monotonically for all large enough n. Failing this, find as sharp as possible necessary conditions, and similarly sufficient conditions for this to happen.

6 Monotonicity of the Partition Function

With reference to unsolved problem 3, we consider the case where the set S of allowable parts is finite and all multiplicities are allowed, i.e., the problem of Frobenius.

Theorem 3. *Let $\{p(n)\}$ be generated by*

$$G(x) \stackrel{def}{=} \sum_{n \geq 0} p(n) x^n = \frac{1}{\prod_{i=1}^k (1 - x^{a_i})}, \tag{8}$$

where $gcd(a_1, \ldots, a_k) = 1$. The sequence $\{p(n)\}$ is strictly increasing for all sufficiently large n if there does not exist a prime p that divides all but one of the a_is, i.e., iff every $(k-1)$ subset of the a_is is coprime.

Proof. Evidently strict monotonicity holds from some point on iff

$$(1-x)G(x) = \frac{1-x}{\prod_{i=1}^k (1 - x^{a_i})}$$

has positive power series coefficients, from some point on. The partial fraction expansion of $(1-x)G(x)$ is of the form

$$(1-x)G(x) = \frac{A_0}{(1-x)^{k-1}} + \frac{A_1}{(1-x)^{k-2}} + \cdots + \frac{B_0}{(1-\omega x)^{k_1}} + \frac{B_1}{(1-\omega x)^{k_1 - 1}} + \cdots$$
$$+ \frac{C_0}{(1-\zeta x)^{k_2}} + \frac{C_1}{(1-\zeta x)^{k_2 - 1}} + \cdots. \tag{9}$$

In the above, ω, ζ, etc. run through the primitive pth roots of unity for each prime p that divides one or more of the a_is, and $k_1, k_2, \ldots$ are the number of a_is that each of these primes divides. If no prime divides all but one of the a_is then all of the k_is are $\leq k-2$. If in that case we take the coefficient of x^n on both sides of we have that

$$p(n) - p(n-1) = A_0 \binom{n+k-2}{n} + O(n^{k-3}),$$

which, since $A_0 > 0$, is positive for all large enough n, as claimed. □

7 A Refinement of the Lower Bound

Let us find a sharper lower bound for $p(n;S)$, when S is an infinite coprime set of admissible parts and all multiplicities are available.

Let $A = \{1 \leq a_1 < a_2 < \cdots < a_k\}$ be a finite coprime subset of S. If we put $r'(n;A) = \sum_{j \leq n} p(j;A)$, then an inequality due to Padberg [4] states that

$$r'(n;A) \geq \frac{(n+1)^k}{k!a_1 \ldots a_k}. \tag{10}$$

Now, for infinitely many n we have $p(n;A) = \max_{j \leq n} p(j;A)$. Hence for such n, $r'(n;A) \leq (n+1)p(n;A)$, and therefore

$$p(n;A) \geq \frac{(n+1)^{k-1}}{k!a_1 \ldots a_k}. \tag{11}$$

Next, extend the set A by adjoining to it the next h basis elements, to get a new coprime set

$$A_h = \{a_1, a_2, \ldots, a_k, a_{k+1}, a_{k+2}, \ldots, a_{k+h}\}.$$

If we apply (11) to A_h we find that

$$p(n;S) \geq p(n;A_h) \geq \frac{(n+1)^{k+h-1}}{(k+h)!a_1 a_2 \ldots a_{k+h}}.$$

Since h is arbitrary we can optimize this inequality by defining $j = j(n)$ to be the least integer such that $ja_j \geq n$.

Theorem 4. *Let S be an infinite coprime set, and let M consist of all nonnegative integers. Then for large enough n we will have*

$$p(n;S,M) \geq \frac{(n+1)^{j(n)-1}}{(j(n))!a_1 a_2 \ldots a_{j(n)}}. \tag{12}$$

For example if S consists of all positive integers we find for the classical partition function that $p(n) \geq e^{2\sqrt{n}}/(2\pi n^2)$ for all large enough n, which can be compared to the bound (7), obtained earlier.

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On Applications of Roots of Unity to Product Identities

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Abstract In this paper, we give simple proofs of the quintuple product identity and the septuple product identity using properties of cube and fifth roots of unity.

Keywords q-series • Quintuple product identity • Septuple product identity

Mathematics Subject Classification: Primary: 05A30; Secondary: 33D15, 14K25

1 Introduction

The quintuple product identity can be found in Ramanujan's lost notebook [11, p. 207]. Because Watson [12] gave the first published proof of the quintuple product identity, it is often called Watson's quintuple product identity. At least 29 proofs of the quintuple product identity have been given, and they are included in Cooper's comprehensive survey [4]. Readers can also refer to Berndt's book [1, p. 83] for the history of the quintuple product identity. The septuple (or septagonal) product identity was first discovered by Hirschhorn [9]. Unaware of Hirschhorn's work, Farkas and Kra [5] also found this identity later. Other proofs have been given by Garvan [7] and Foata and Han [6]. In [3], the author proved that many product identities for theta functions correspond to exact covering systems of $\mathbb{Z}^n$, including both of the quintuple and the septuple product identities.

In [10], Kongsiriwong and Liu gave new proofs of the quintuple product identity, the septuple product identity using basic properties of cube and fifth roots of unity. In this paper, we consider the minimal polynomial of roots of unity and obtain simple proofs of the quintuple and septuple product identities.

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K. Alladi and F. Garvan (eds.), *Partitions, q-Series, and Modular Forms*, Developments in Mathematics 23, DOI 10.1007/978-1-4614-0028-8_5,

We use the standard notation for q-products, defining

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \qquad |q|<1. \tag{1.1}$$

Jacobi's triple product identity is given by [2, p. 10]

$$\left(-qz;q^2\right)_\infty\left(-q/z;q^2\right)_\infty\left(q^2;q^2\right)_\infty = \sum_{n=-\infty}^{\infty} q^{n^2}z^n, \qquad |q|<1. \tag{1.2}$$

2 The Quintuple Product Identity

The quintuple product identity is given by [2, p. 18].

Theorem 2.1. *For any complex numbers z and q, with $z \neq 0$ and $|q|<1$,*

$$\begin{aligned}&(q^2;q^2)_\infty(zq;q^2)_\infty(z^{-1}q;q^2)_\infty(z^2;q^4)_\infty(z^{-2}q^4;q^4)_\infty\\ &= \sum_{n=-\infty}^{\infty}\left(q^{3n^2-2n}z^{3n}-q^{3n^2+2n}z^{3n+2}\right).\end{aligned} \tag{2.1}$$

Proof. Let $f(z)$ denote the left-hand side of (2.1). It is easy to see that $f(z)$ is analytic on $0<|z|<\infty$. We can write $f(z)$ in a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(q)z^n. \tag{2.2}$$

Then from the definition of $f(z)$,

$$f(z) = qz^3 f(q^2z). \tag{2.3}$$

Equating the coefficients of z^n on both sides of (2.3) implies

$$a_n(q) = q^{2n-5}a_{n-3}(q). \tag{2.4}$$

By iteration, we find that for any integer n,

$$\begin{aligned}a_{3n}(q) &= q^{3n^2-2n}a_0(q),\\ a_{3n+1}(q) &= q^{3n^2}a_1(q),\\ a_{3n+2}(q) &= q^{3n^2+2n}a_2(q).\end{aligned} \tag{2.5}$$

We give the proof of $a_{3n}(q) = q^{3n^2-2n}a_0(q)$ as an example.

Replacing n with $3n$ in (2.4), we have $a_{3n}(q) = q^{6n-5}a_{3n-3}(q)$.

For $n>0$, we can find that

$$a_{3n}(q) = q^{6n-5}a_{3n-3}(q) = \cdots = q^{(6n-5)+(6n-11)\cdots+1}a_0(q) = q^{3n^2-2n}a_0(q).$$

For $n<0$, from $a_{3n}(q)=q^{6n-5}a_{3n-3}(q)$, we can deduce that $a_{3n}(q)=q^{-6n-1}$ $a_{3n+3}(q)$. Since $\Sigma_{i=n}^{-1} i=-\dfrac{n(n-1)}{2}$ for $n<0$, we have

$$a_{3n}(q)=q^{-6n-1}a_{3n+3}(q)=\cdots=q^{(-6n-1)+(-6n-7)\cdots+5}a_0(q)=q^{3n^2-2n}a_0(q).$$

So $a_{3n}(q)=q^{3n^2-2n}a_0(q)$ holds for any integer n.

From (2.2) and (2.5),

$$f(z)=a_0(q)\sum_{n=-\infty}^{\infty}q^{3n^2-2n}z^{3n}+a_1(q)\sum_{n=-\infty}^{\infty}q^{3n^2}z^{3n+1}+a_2(q)\sum_{n=-\infty}^{\infty}q^{3n^2+2n}z^{3n+2}. \tag{2.6}$$

Let $\omega=\exp(2\pi i/3)$. For any complex number z,

$$(1-z)(1-z\omega)(1-z\omega^2)=1-z^3.$$

We have

$$(a;q)_\infty(a\omega;q)_\infty(a\omega^2;q)_\infty=(a^3;q^3)_\infty. \tag{2.7}$$

Setting $z=1$ in (2.6), we have

$$\begin{aligned}&[a_0(q)+a_2(q)](-q;q^6)_\infty(-q^5;q^6)_\infty(q^6;q^6)_\infty\\&+a_1(q)(-q^3;q^6)_\infty^2(q^6;q^6)_\infty=g(1)=0\end{aligned} \tag{2.8}$$

from the definition of $g(z)$.

Letting $z=\omega=\exp(2\pi i/3)$ in (2.6), we find that

$$\begin{aligned}&(1-\omega^2)(q^2;q^2)_\infty(q\omega;q^2)_\infty(q\omega^2;q^2)_\infty(q^4\omega^2;q^4)_\infty(q^4\omega;q^4)_\infty\\&=a_0(q)\sum_{n=-\infty}^{\infty}q^{3n^2-2n}+a_1(q)\sum_{n=-\infty}^{\infty}q^{3n^2}\omega+a_2(q)\sum_{n=-\infty}^{\infty}q^{3n^2+2n}\omega^2.\end{aligned} \tag{2.9}$$

Considering the left-hand side of (2.9), we have

$$\begin{aligned}&(q^2;q^2)_\infty(q\omega;q^2)_\infty(q\omega^2;q^2)_\infty(q^4\omega^2;q^4)_\infty(q^4\omega;q^4)_\infty\\&=\frac{(q^2;q^2)_\infty(q^3;q^6)_\infty(q^{12};q^{12})_\infty}{(q;q^2)_\infty(q^4;q^4)_\infty}=\frac{(q^2;q^4)_\infty(q^{12};q^{12})_\infty}{(q;q^6)_\infty(q^5;q^6)_\infty}\\&=\frac{(q^2;q^4)_\infty(-q;q^6)_\infty(-q^5;q^6)_\infty(q^6;q^6)_\infty}{(q^2;q^{12})_\infty(q^{10};q^{12})_\infty(q^6;q^{12})_\infty}=(-q;q^6)_\infty(-q^5;q^6)_\infty(q^6;q^6)_\infty.\end{aligned} \tag{2.10}$$

So by (2.8)–(2.10),

$$(1-\omega^2)(-q;q^6)_\infty(-q^5;q^6)_\infty(q^6;q^6)_\infty = [a_0(q)+a_2(q)\omega^2](-q;q^6)_\infty(-q^5;q^6)_\infty \times(q^6;q^6)_\infty - [a_0(q)+a_2(q)]\omega(-q;q^6)_\infty(-q^5;q^6)_\infty(q^6;q^6)_\infty.$$

After simplification, we have

$$[1-a_0(q)]+[a_0(q)+a_2(q)]\omega+[-1-a_2(q)]\omega^2 = 0. \tag{2.11}$$

Since the minimal polynomial of ω is x^2+x+1, we have $1-a_0(q) = a_0(q)+a_2(q) = -1-a_2(q)$. So $a_0(q) = 1$, $a_2(q) = -1$. From (2.8), $a_1(q) = 0$. We thus have completed the proof. □

For most proofs based on functional equations, the recurrence relation $a_n = -a_{-n+2}$ is needed to determine a_i $(i = 0,1,2)$. We do not need this in our proof.

3 The Septuple Product Identity

We cite the septuple product identity in [10].

Theorem 3.2. *For any complex numbers z and q, with $z \neq 0$ and $|q| < 1$,*

$$(q^4;q^{10})_\infty(q^6;q^{10})_\infty(q^{10};q^{10})_\infty \sum_{n=-\infty}^{\infty}(-1)^n(q^{5n^2+3n}z^{5n+3}+q^{5n^2-3n}z^{5n}) - (q^2;q^{10})_\infty(q^8;q^{10})_\infty(q^{10};q^{10})_\infty \sum_{n=-\infty}^{\infty}(-1)^n(q^{5n^2+n}z^{5n+2}+q^{5n^2-n}z^{5n+1}) = (q^2;q^2)_\infty^2(z;q^2)_\infty(z^{-1}q^2;q^2)_\infty(z^2;q^2)_\infty(z^{-2}q^2;q^2)_\infty. \tag{3.1}$$

Proof. Let $g(z)$ denote the right-hand side of (3.1). Since $g(z)$ is analytic on $0 < |z| < \infty$, we can write $g(z)$ as a Laurent series

$$g(z) = \sum_{n=-\infty}^{\infty} a_n(q)z^n. \tag{3.2}$$

Then from the definition of $g(z)$,

$$g(z) = -q^2z^5g(q^2z). \tag{3.3}$$

Equating the coefficients of z^n on both sides of (3.3) implies

$$a_n(q) = -q^{2n-8}a_{n-5}(q).$$

By iteration, we find that for any integer n,

$$\begin{aligned} a_{5n}(q) &= (-1)^n q^{5n^2-3n} a_0(q), \\ a_{5n+1}(q) &= (-1)^n q^{5n^2-n} a_1(q), \\ a_{5n+2}(q) &= (-1)^n q^{5n^2+n} a_2(q), \\ a_{5n+3}(q) &= (-1)^n q^{5n^2+3n} a_3(q), \\ a_{5n+4}(q) &= (-1)^n q^{5n^2+5n} a_4(q). \end{aligned} \tag{3.4}$$

The proof of (3.4) is similar to the proof of (2.5) and we omit it here.

So now we have

$$\begin{aligned} g(z) = a_0(q) &\sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2-3n} z^{5n} + a_1(q) \sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2-n} z^{5n+1} \\ &+ a_2(q) \sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2+n} z^{5n+2} + a_3(q) \sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2+3n} z^{5n+3} \\ &+ a_4(q) \sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2+5n} z^{5n+4} \\ = &\sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2+3n} \left(a_0(q) z^{5n} + a_3(q) z^{5n+3}\right) \\ &+ \sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2+n} \left(a_1(q) z^{5n+1} + a_2(q) z^{5n+2}\right) + \sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2+5n} a_4(q) z^{5n+4}. \end{aligned} \tag{3.5}$$

Let $\zeta = \exp(2\pi i/5)$. For any complex number z,

$$(1-z)(1-z\zeta)(1-z\zeta^2)(1-z\zeta^3)(1-z\zeta^4) = 1-z^5.$$

So

$$(a;q)_\infty (a\zeta;q)_\infty (a\zeta^2;q)_\infty (a\zeta^3;q)_\infty (a\zeta^4;q)_\infty = (a^5;q^5)_\infty. \tag{3.6}$$

From (3.6), by letting $z = \zeta$ in (3.5), we can obtain

$$\begin{aligned} (1-\zeta-\zeta^2+\zeta^3)(q^2;q^2)_\infty (q^{10};q^{10})_\infty &= [a_0(q) + a_3(q)\zeta^3](q^2;q^{10})_\infty (q^8;q^{10})_\infty \\ \times (q^{10};q^{10})_\infty + [a_1(q)\zeta + a_2(q)\zeta^2]&(q^4;q^{10})_\infty (q^6;q^{10})_\infty (q^{10};q^{10})_\infty. \end{aligned} \tag{3.7}$$

Since the minimal polynomial of ζ is $1+x+x^2+x^3+x^4$, the coefficients of the powers of ζ on both sides of (3.7) should be equal. So we obtain

$$a_0(q)=a_3(q)=\frac{(q^2;q^2)_\infty}{(q^8;q^{10})_\infty(q^2;q^{10})_\infty}=(q^4;q^{10})_\infty(q^6;q^{10})_\infty(q^{10};q^{10})_\infty,$$
$$a_1(q)=a_2(q)=-\frac{(q^2;q^2)_\infty}{(q^4;q^{10})_\infty(q^6;q^{10})_\infty}=-(q^2;q^{10})_\infty(q^8;q^{10})_\infty(q^{10};q^{10})_\infty. \quad (3.8)$$

Letting $z=-1$ in (3.5), we have $a_4(q)=0$. The proof is thus complete.

Similar to theproof of the quintuple product identity, the recurrence relation $a_n=a_{-n+3}$ is not needed to determine the constant terms in our proof of the septuple product identity. □

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Lecture Hall Sequences, q-Series, and Asymmetric Partition Identities

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Abstract We use generalized lecture hall partitions to discover a new pair of q-series identities. These identities are unusual in that they involve partitions into parts from asymmetric residue classes, much like the little Göllnitz partition theorems. We derive a two-parameter generalization of our identities that, surprisingly, gives new analytic counterparts of the little Göllnitz theorems. Finally, we show that the little Göllnitz theorems also involve "lecture hall sequences," that is, sequences constrained by the ratio of consecutive parts.

Keywords Lecture hall partitions • q-series identities • q-Gauss summation • Göllnitz partition theorems

Mathematics Subject Classification: 05A15 (05A17, 05A19, 05A30, 11P81, 11P82)

1 Introduction

In this paper we illustrate the role that can be played by sequences constrained by the *ratio* of consecutive parts in interpreting and discovering q-series identities.

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[1] Research supported in part by NSA Grant H98230-08-10072

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K. Alladi and F. Garvan (eds.), *Partitions, q-Series, and Modular Forms*,
Developments in Mathematics 23, DOI 10.1007/978-1-4614-0028-8_6,

Let $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$ and $(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$. We derive the identities

$$\sum_{j=0}^{\infty} q^{j(3j-1)/2} \frac{(q^2;q^6)_j}{(q;q)_{3j}} = \frac{1}{(q;q^3)_\infty (q^5;q^6)_\infty} \tag{1}$$

and

$$\sum_{j=0}^{\infty} q^{j(3j+1)/2} \frac{(q^4;q^6)_j}{(q;q)_{3j+1}} = \frac{1}{(q^2;q^3)_\infty (q;q^6)_\infty} \tag{2}$$

by showing that both sides of (1) count (by weight) the finite sequences of positive integers $\lambda_1, \lambda_2, \ldots$ satisfying

$$\frac{\lambda_1}{2} > \frac{\lambda_2}{1} > \frac{\lambda_3}{2} > \frac{\lambda_4}{1} > \cdots \tag{3}$$

and both sides of (2) count the finite sequences of positive integers $\lambda_1, \lambda_2, \ldots$ satisfying

$$\frac{\lambda_1}{1} > \frac{\lambda_2}{2} > \frac{\lambda_3}{1} > \frac{\lambda_4}{2} > \cdots . \tag{4}$$

Contrast these with Euler's odd-distinct partition identity

$$\sum_{j=0}^{\infty} q^{j(j+1)/2} \frac{1}{(q;q)_j} = \frac{1}{(q;q^2)_\infty}, \tag{5}$$

both sides of which count the finite sequences of positive integers satisfying

$$\frac{\lambda_1}{1} > \frac{\lambda_2}{1} > \frac{\lambda_3}{1} > \frac{\lambda_4}{1} > \cdots . \tag{6}$$

Our methods combine results on *lecture hall partitions* from [3], on *sequences constrained by the ratio of successive parts* from [5], and *combinatorial reciprocity* [13]. In Sect. 2 we use "lecture hall" methods to show that the right-hand sides of (1) and (2) count solutions to (3) and (4), respectively. In Sect. 3 we show that the left-hand sides of (1) and (2) also count solutions to (3) and (4), using results from [5].

In Sect. 4 we refine the counting arguments in Sects. 2 and 3 to derive a two-parameter q-series identity, $I(a,q)$, generalizing (1) and (2). We show in Sect. 5 that $I(a,q)$ can be obtained as a specialization of the q-Gauss summation [6].

Say that a set, $R = \{r_1, r_2, \ldots, r_k\}$, of residue classes modulo m, is *symmetric* if

$$R = \{m - r_1, m - r_2, \ldots, m - r_k\}.$$

It is noteworthy that the infinite products appearing in (1) and (2) are generating functions for partitions into parts from residue classes modulo 6 which are *not* symmetric.

Most well-known partition theorems involve symmetric residue classes, e.g., the Rogers-Ramanujan identities and the Gordon and Bressoud generalizations thereof [4, 8], Schur's 1926 partition theorem related to the modulus 6 [11], and the Göllnitz-Gordon identities [7, pp. 162–163, Satz 2.1 and 2.2], [9, p. 741, Theorems 2 and 3]. From a q-series perspective, this is a consequence of the fact that the relevant generating functions are modular forms (up to multiplication by a trivial factor).

Perhaps the best known partition identities involving *asymmetric* residue classes are a pair of identities known as "Göllnitz's little partition theorems" [7, pp. 166–167, Satz 2.3 and 2.4] and the "big" Göllnitz partition theorem related to the modulus 12 ([7, p. 175, Satz 4.1]; cf. [1, p. 37, Theorem 1]). In Sect. 6 we show that an appropriate specialization of $I(a,q)$ gives a different view of the infinite products appearing in (the analytic forms of) Göllnitz's little partition theorems. Furthermore, we show that the little Göllnitz theorems can be alternately viewed as statements about partitions constrained by the *ratio* of consecutive parts.

2 The "Lecture Hall" Approach

The purpose of this section is to show that the right-hand sides of identities (1) and (2) count solutions to the inequalities (3) and (4), respectively. We begin with a theorem of Bousquet-Mélou and Eriksson [3] about (k,l) *sequences*.

Given two integers k and l greater than one, the (k,l) *sequence* $\{a_n^{(k,l)}\}$ is defined in [3] for $n \geq 0$ by the following recurrence:

$$a_{2i}^{(k,l)} = l a_{2i-1}^{(k,l)} - a_{2i-2}^{(k,l)},$$
$$a_{2i+1}^{(k,l)} = k a_{2i}^{(k,l)} - a_{2i-1}^{(k,l)},$$

for $i \geq 1$, with initial conditions $a_0^{(k,l)} = 0$ and $a_1^{(k,l)} = 1$.

Let $L_n^{(k,l)}$ be the set of nonnegative integer sequences λ of length n satisfying

$$\frac{\lambda_1}{a_n^{(k,l)}} \geq \frac{\lambda_2}{a_{n-1}^{(k,l)}} \geq \ldots \geq \frac{\lambda_n}{a_1^{(k,l)}} \geq 0.$$

The following was shown in [3].

Theorem 1 (The *(k,l)*-Lecture Hall Theorem). *The generating function for $L_n^{(k,l)}$ is given by*

$$G_n^{(k,l)}(q) = \prod_{i=1}^{n} \frac{1}{1 - q^{a_i^{(k,l)} + a_{i-1}^{(l,k)}}} \quad \text{if } n \text{ is even},$$

$$G_n^{(k,l)}(q) = \prod_{i=1}^{n} \frac{1}{1 - q^{a_i^{(l,k)} + a_{i-1}^{(k,l)}}} \quad \text{if } n \text{ is odd}.$$

When $k \geq 2$ and $l \geq 2$, the sequence $\{a_n^{(k,l)}\}$ is strictly increasing. When $k = 1$ or $l = 1$, the sequence $\{a_n^{(k,l)}\}$ is not monotone and when $kl < 4$ some terms will be negative. Nevertheless, we make the following observation and prove it in Appendix 1.

Observation 1. *The (k,l)-Lecture Hall Theorem remains true when $k = 1$ or $l = 1$, as long as $kl \geq 4$.*

For our application, consider the sequences:

$$a^{(1,4)} = 0,\ 1,\ 4,\ 3,\ 8,\ 5,\ 12,\ 7,\ldots;$$
$$a^{(4,1)} = 0,\ 1,\ 1,\ 3,\ 2,\ 5,\ 3,\ 7,\ldots.$$

Then

$$a_{2i+1}^{(1,4)} = 2i+1; \qquad a_{2i}^{(1,4)} = 4i;$$
$$a_{2i+1}^{(4,1)} = 2i+1; \qquad a_{2i}^{(4,1)} = i.$$

So, by definition of $G_n^{(k,l)}(q)$,

$$G_{2k}^{(1,4)}(q) = \prod_{i=0}^{k-1} \frac{1}{\left(1 - q^{a_{2i+1}^{(1,4)} + a_{2i}^{(4,1)}}\right)\left(1 - q^{a_{2i+2}^{(1,4)} + a_{2i+1}^{(4,1)}}\right)} = \prod_{i=0}^{k-1} \frac{1}{(1-q^{3i+1})(1-q^{6i+5})}$$

and

$$\lim_{k\to\infty} G_{2k}^{(1,4)}(q) = \frac{1}{(q;q^3)_\infty (q^5;q^6)_\infty}, \tag{7}$$

giving the right-hand side of (1). On the other hand, by Theorem 1, $G_{2k}^{(1,4)}(q)$ is the generating function for $L_{2k}^{(1,4)}$, the set of sequences satisfying

$$L_{2k}^{(1,4)}: \quad \frac{\lambda_1}{4k} \geq \frac{\lambda_2}{2k-1} \geq \frac{\lambda_3}{4(k-1)} \geq \frac{\lambda_4}{2k-3} \geq \ldots \geq \frac{\lambda_{2k-1}}{4} \geq \frac{\lambda_{2k}}{1} \geq 0.$$

Note that

$$\lim_{k\to\infty} \frac{a_{2k}^{(1,4)}}{a_{2k-1}^{(1,4)}} = \frac{4k}{2k-1} = 2$$

and

$$\lim_{k\to\infty} \frac{a_{2k+1}^{(1,4)}}{a_{2k}^{(1,4)}} = \frac{2k+1}{4k} = \frac{1}{2}$$

so $\lim_{k\to\infty} L_{2k}^{(1,4)}$ is the set of sequences satisfying the constraints (3), whose generating function must therefore be (7).

Similarly, by definition of $G_n^{(k,l)}(q)$,

$$G_{2k+1}^{(1,4)}(q) = \frac{1}{1-q}\prod_{i=1}^{k}\frac{1}{\left(1-q^{a_{2i}^{(4,1)}+a_{2i-1}^{(1,4)}}\right)\left(1-q^{a_{2i+1}^{(4,1)}+a_{2i}^{(1,4)}}\right)}$$
$$= \frac{1}{1-q}\prod_{i=1}^{k}\frac{1}{(1-q^{3i-1})(1-q^{6i+1})}$$

and

$$\lim_{k\to\infty} G_{2k+1}^{(1,4)}(q) = \frac{1}{(q^2;q^3)_\infty (q;q^6)_\infty}, \tag{8}$$

giving the right-hand side of (2).

On the other hand, by Theorem 1, $G_{2k+1}^{(1,4)}(q)$ is the generating function for $L_{2k+1}^{(1,4)}$, the sequences satisfying

$$L_{2k+1}^{(1,4)}: \quad \frac{\lambda_1}{2k+1} \ge \frac{\lambda_2}{4k} \ge \frac{\lambda_3}{2k-1} \ge \frac{\lambda_4}{4(k-1)} \ge \cdots \ge \frac{\lambda_{2k}}{4} \ge \frac{\lambda_{2k+1}}{1} \ge 0.$$

As $k \to \infty$, $L_{2k+1}^{(1,4)}$ becomes the set of sequences satisfying the constraints (4), and thus their generating function is given by (8).

3 The "Enumerative Combinatorics" Approach

In this section, we use results from [5] to show that the left-hand sides of identities (1) and (2) count the integer solutions to the inequalities (3) and (4), respectively. Define $[n]_q$ by

$$[n]_q := (1-q^n)/(1-q).$$

The following is shown in [5] (we include a self-contained proof in Appendix 2).

Theorem 2. *Let $s_1, s_2, \ldots, s_k$ be a sequence of positive integers satisfying the condition $s_i = 1$ or $s_{i+1} = 1$ for $1 \le i \le k-1$. Then the generating for the nonnegative integer sequences λ satisfying*

$$\frac{\lambda_1}{s_1} \ge \frac{\lambda_2}{s_2} \ge \cdots \ge \frac{\lambda_k}{s_k} \ge 0$$

is

$$F(q) = \sum_{\lambda} q^{|\lambda|} = \frac{\prod_{i=2}^{k-1}\left(1+q^{b_i}\left([s_{i+1}]_q - 1\right)\right)}{\prod_{i=1}^{k}(1-q^{b_i})},$$

where $b_1 = 1$ *and* $b_i = s_1 + \ldots + s_i$ *for* $i > 1$.

Note that if $s_1 = s_2 = \cdots = s_k = 1$, then $F(q)$ is the generating function for ordinary partitions with at most k parts.

We first apply Theorem 2 to the sequence $s = (2,1,2,1,\ldots)$. Note that $b_{2j} = 3j$, $b_{2j+1} = 3j+2$, so

$$1+q^{b_{2j}}\left(\left[s_{2j+1}\right]_q - 1\right) = 1+q^{3j}\left([2]_q - 1\right) = 1+q^{3j+1},$$

$$1+q^{b_{2j+1}}\left(\left[s_{2j+2}\right]_q - 1\right) = 1+q^{3j+2}\left([1]_q - 1\right) = 1.$$

Thus, for $s = (2,1,2,1,\ldots)$, the generating function for the sequences satisfying

$$\frac{\lambda_1}{2} \geq \frac{\lambda_2}{1} \geq \frac{\lambda_3}{2} \geq \frac{\lambda_4}{1} \geq \cdots \geq \frac{\lambda_n}{s_n} \geq 0 \tag{9}$$

is

$$f_n(q) = \frac{\left(-q^4;q^3\right)_{\lfloor (n-1)/2 \rfloor}}{(1-q)\left(q^3;q^3\right)_{\lfloor n/2 \rfloor}\left(q^5;q^3\right)_{\lfloor (n+1)/2 \rfloor}} = \frac{(1+q)\left(-q^4;q^3\right)_{\lfloor (n-1)/2 \rfloor}}{\left(q^3;q^3\right)_{\lfloor n/2 \rfloor}\left(q^2;q^3\right)_{\lfloor (n+1)/2 \rfloor}}$$

$$= \frac{\left(-q;q^3\right)_{\lfloor (n+1)/2 \rfloor}}{\left(q^3;q^3\right)_{\lfloor n/2 \rfloor}\left(q^2;q^3\right)_{\lfloor (n+1)/2 \rfloor}} = \frac{\left(q^2;q^6\right)_{\lfloor (n+1)/2 \rfloor}}{(q;q)_{\lfloor (3n+1)/2 \rfloor}}.$$

Constraints (9) define a simplicial cone, so Stanley's reciprocity theorem [13] can be used to compute, from $f_n(q)$, the generating function for those integer points *interior* to the cone. Specifically, the generating function for those integer sequences λ satisfying the *strict* constraints

$$\frac{\lambda_1}{2} > \frac{\lambda_2}{1} > \frac{\lambda_3}{2} > \frac{\lambda_4}{1} > \cdots > \frac{\lambda_n}{s_n} > 0$$

is given by

$$h_n(q) = (-1)^n f_n(1/q) = \begin{cases} q^{\left(3n^2+10n\right)/8} f_n(q) & n \text{ even}, \\ q^{\left(3n^2+4n+1\right)/8} f_n(q) & n \text{ odd}. \end{cases} \tag{10}$$

Finally, the generating function for the integer sequences λ satisfying (3) can now be obtained by summing $h_n(q)$ in (10) over all $n \geq 0$:

$$\begin{aligned}
\sum_{n=0}^{\infty} h_n(q) &= 1+\sum_{k=1}^{\infty}(h_{2k-1}+h_{2k}) \\
&= 1+\sum_{k=1}^{\infty}\left(\frac{q^{\left(3(2k-1)^2+4(2k-1)+1\right)/8}\left(q^2;q^6\right)_k}{(q;q)_{3k-1}}+\frac{q^{\left(3(2k)^2+10(2k)\right)/8}\left(q^2;q^6\right)_k}{(q;q)_{3k}}\right) \\
&= 1+\sum_{k=1}^{\infty}\frac{q^{k(3k-1)/2}(q^2;q^6)_k}{(q;q)_{3k}},
\end{aligned}$$

which agrees with the left-hand side of (1).

We proceed similarly to find the generating function of the solutions of (4). In this case, we apply Theorem 2 to the sequence $s' = (1,2,1,2,...)$. For this sequence, $b_{2j} = 3j$, $b_{2j+1} = 3j+1$, so

$$\begin{aligned}
1+q^{b_{2j}}\left(\left[s'_{2j+1}\right]_q-1\right) &= 1+q^{3j}\left([1]_q-1\right)=1, \\
1+q^{b_{2j+1}}\left(\left[s'_{2j+2}\right]_q-1\right) &= 1+q^{3j+1}\left([2]_q-1\right)=1+q^{3j+2}.
\end{aligned}$$

Thus, for $s' = (1,2,1,2,...)$, the generating function for the sequences satisfying

$$\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \frac{\lambda_3}{1} \geq \frac{\lambda_4}{2} \geq \cdots \geq \frac{\lambda_n}{s'_n} \geq 0 \tag{11}$$

is

$$\begin{aligned}
f'_n(q) &= \frac{\left(-q^5;q^3\right)_{\lfloor (n-2)/2\rfloor}}{\left(q;q^3\right)_{\lfloor (n+1)/2\rfloor}\left(q^3;q^3\right)_{\lfloor (n-1)/2\rfloor}} \\
&= \frac{\left(q^{10};q^6\right)_{\lfloor (n-2)/2\rfloor}}{\left(q;q^3\right)_{\lfloor (n+1)/2\rfloor}\left(q^3;q^3\right)_{\lfloor (n-1)/2\rfloor}\left(q^5;q^3\right)_{\lfloor (n-2)/2\rfloor}} = \frac{\left(q^2;q^6\right)_{\lfloor n/2\rfloor}}{(q;q)_{\lfloor 3n/2\rfloor}}.
\end{aligned}$$

Again, by the reciprocity theorem [13], the generating function for those integer sequences λ satisfying the *strict* constraints

$$\frac{\lambda_1}{1} > \frac{\lambda_2}{2} > \frac{\lambda_3}{1} > \frac{\lambda_4}{2} > \cdots > \frac{\lambda_n}{s'_n} > 0$$

is given by

$$h'_n(q) = (-1)^n f'_n(1/q) = \begin{cases} q^{k(3k+1)/2} f'_{2k}(q) & n = 2k, \\ q^{(k+2)(3k+1)/2} f'_{2k+1}(q) & n = 2k+1. \end{cases} \tag{12}$$

Finally, the generating function for the integer sequences λ satisfying (4) can now be obtained by summing $h'_n(q)$ in (12) over all $n \geq 0$:

$$\begin{aligned}\sum_{n=0}^{\infty} h'_n(q) &= \sum_{k=0}^{\infty} \left(h'_{2k} + h'_{2k+1}\right) = \sum_{k=0}^{\infty} \left(\frac{q^{k(3k+1)/2}(q^4;q^6)_k}{(q;q)_{3k}} + \frac{q^{k(3k+1)/2}(q^2;q^6)_k}{(q;q)_{3k+1}} \right) \\ &= \sum_{k=0}^{\infty} \frac{q^{(k+1)(3k+2)/2}\left(q^4;q^6\right)_k}{(q;q)_{3k+1}}\end{aligned}$$

which agrees with the left-hand side of (2).

4 A Refinement

Define the even and odd weight of a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ by

$$|\lambda|_o = \lambda_1 + \lambda_3 + \cdots; \quad |\lambda|_e = \lambda_2 + \lambda_4 + \cdots;$$

and define

$$G_n^{(k,l)}(x,y) = \sum_{\lambda \in L_n^{(k,l)}} x^{|\lambda|_o} y^{|\lambda|_e}.$$

As we indicate in Appendix 1, what Bousquet-Mélou and Eriksson proved in [3] was the following: The generating function for $L_n^{(k,l)}$ is given by

$$G_n^{(k,l)}(x,y) = \prod_{i=1}^{n} \frac{1}{1 - x^{a_i^{(k,l)}} y^{a_{i-1}^{(l,k)}}} \quad \text{if } n \text{ is even,}$$

$$G_n^{(k,l)}(x,y) = \prod_{i=1}^{n} \frac{1}{1 - x^{a_i^{(l,k)}} y^{a_{i-1}^{(k,l)}}} \quad \text{if } n \text{ is odd.}$$

So

$$\lim_{k\to\infty} G_{2k}^{(1,4)}(x,y) = \frac{1}{(x;x^2y)_\infty (x^4y;x^4y^2)_\infty} = \frac{\left(-x;x^2y\right)_\infty}{(x^2;x^2y)_\infty}. \qquad (13)$$

Similarly, the counting method of Sect. 3 also admits an x,y-refinement. From the bijective proof Theorem 2 that appears in Appendix 2, it can be checked that the 2-variable version of the generating function for the sequences λ satisfying (9) is

$$f_n(x,y) = \sum_{\lambda} x^{|\lambda|_o} y^{|\lambda|_e} = \frac{\left(-x;x^2y\right)_{\lfloor (n+1)/2 \rfloor}}{\left(x^2y;x^2y\right)_{\lfloor n/2 \rfloor} \left(x^2;x^2y\right)_{\lfloor (n+1)/2 \rfloor}}.$$

Proceeding as in Sect. 3 using reciprocity,

$$h_n(x,y) = (-1)^n f_n(1/x, 1/y),$$

and summing over all n, gives another expression for the generating function of (3):

$$\sum_{n=0}^{\infty} h_n(x,y) = 1 + \sum_{k=1}^{\infty} (h_{2k-1}(x,y) + h_{2k}(x,y)) = \sum_{j=0}^{\infty} \frac{x^{j^2} y^{j(j-1)/2} (-x;x^2y)_j}{(x^2;x^2y)_j (x^2y;x^2y)_j}. \quad (14)$$

Since both (13) and (14) count (3), we have the following.

Theorem 3.

$$\sum_{j=0}^{\infty} \frac{x^{j^2} y^{j(j-1)/2} \left(-x;x^2y\right)_j}{\left(x^2;x^2y\right)_j \left(x^2y;x^2y\right)_j} = \sum_{\lambda} x^{\lambda_1+\lambda_3+\cdots} y^{\lambda_2+\lambda_4+\cdots} = \frac{\left(-x;x^2y\right)_\infty}{\left(x^2;x^2y\right)_\infty}$$

where the second sum is over all positive integer sequences λ *satisyfing*

$$\frac{\lambda_1}{2} > \frac{\lambda_2}{1} > \frac{\lambda_3}{2} > \frac{\lambda_4}{1} > \cdots. \quad (15)$$

Setting $x = -a$ *and* $y = q/a^2$ *gives the following identity, which we refer to as* $I(a,q)$.

Corollary 1.

$$I(a,q) \; := \; \sum_{n=0}^{\infty} \frac{(a;q)_n (-a)^n q^{\binom{n}{2}}}{(a^2;q)_n (q;q)_n} = \frac{(a;q)_\infty}{(a^2;q)_\infty}. \quad (16)$$

As an alternative to reciprocity, we could explain the sum side of (14) combinatorially.

5 Deriving the Identities from the q-Gauss Summation

Recall Heine's q-Gauss summation [6, (II.8)]:

$$H(a,b,c;q) := \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} \left(\frac{c}{ab}\right)^n = \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/(ab);q)_\infty}. \quad (17)$$

Note that as $b \to \infty$, we have $(b;q)_n / b^n \to (-1)^n q^{\binom{n}{2}}$ and $(x/b;q)_\infty \to 1$, so (17) becomes

$$H(a,\infty,c;q) = \sum_{n=0}^{\infty} \frac{(a;q)_n (-c/a)^n q^{\binom{n}{2}}}{(c;q)_n (q;q)_n} = \frac{(c/a;q)_\infty}{(c;q)_\infty}. \quad (18)$$

Thus

$$H(a,\infty,a^2;q) = \sum_{n=0}^{\infty} \frac{(a;q)_n(-a)^n q^{\binom{n}{2}}}{(a^2;q)_n(q;q)_n} = \frac{(a;q)_\infty}{(a^2;q)_\infty},$$

which is (16). So

$$I(a,q) = H(a,\infty,a^2;q).$$

Then,

$$I\left(-q,q^3\right) = \sum_{n=0}^{\infty} \frac{\left(-q;q^3\right)_n q^{n(3n-1)/2}}{\left(q^2;q^3\right)_n \left(q^3;q^3\right)_n} = \frac{\left(-q;q^3\right)_\infty}{\left(q^2;q^3\right)_\infty},$$

which is equivalent to (1).

On the other hand, identity (2) is equivalent to

$$\frac{I\left(-q^2,q^3\right)}{1-q} = \sum_{n=0}^{\infty} \frac{\left(-q^2;q^3\right)_n q^{n(3n+1)/2}}{\left(q;q^3\right)_{n+1} \left(q^3;q^3\right)_n} = \frac{\left(-q^2;q^3\right)_\infty}{\left(q;q^3\right)_\infty}.$$

It is interesting to note that (5) follows in a similar way from (18):

$$H\left(\infty,\infty,q;q^2\right) = \sum_{n=0}^{\infty} \frac{q^{n(2n-1)}}{(q;q)_{2n}} = \frac{1}{(q;q^2)_\infty}. \tag{19}$$

Equation (19) is in fact equivalent to an identity appearing in Slater's compendium of Rogers-Ramanujan type identities [12, p. 157, (52)].

Observe that

$$\frac{q^{n(2n-1)}}{(q;q)_{2n}} = \frac{q^{(2n-1)(2n)/2}}{(q;q)_{2n-1}} + \frac{q^{(2n)(2n+1)/2}}{(q;q)_{2n}},$$

so that each term of the sum in (19) is the sum of two successive terms of the sum in (5).

6 Lebesgue's Identity and a New View of Göllnitz's "Little" Partition Theorems

The infinite products appearing in (1) and (2) enumerate partitions whose parts belong to the residue classes $\{1,4,5\}$ modulo 6 and $\{1,2,5\}$ modulo 6, respectively. It is noteworthy that these residue classes are *not* symmetric modulo 6, since most well-known partition theorems involve symmetric residue classes.

Two of the best known partition identities involving *asymmetric* residue classes are known as "Göllnitz's little partition theorems" [7, pp. 166–167, Satz 2.3 and 2.4].

Theorem 4 (Göllnitz). *The number of partitions of N into parts differing by at least* 2 *and no consecutive odd parts equals the number of partitions of N into parts congruent to* 1, 5 *or* 6 *modulo* 8.

Theorem 5 (Göllnitz). *The number of partitions of N into parts differing by at least* 2*, no consecutive odd parts, and no ones equals the number of partitions of N into parts congruent to* 2, 3 *or* 7 *modulo* 8.

It is well known that the analytic counterparts to Theorems 4 and 5 are special cases of an identity due to Lebesgue ([10]; cf. [2, p. 21, Cor. 2.7]):

$$L(a,q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(a;q)_n}{(q;q)_n} = \frac{\left(aq;q^2\right)_\infty}{(q;q^2)_\infty}. \tag{20}$$

The analytic counterpart to Theorem 4 is [7, (2.22)]

$$L(-q^{-1},q^2) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q^{-1};q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q;q^4)_\infty(q^6;q^8)_\infty}, \tag{21}$$

while that of Theorem 5 is [7, (2.24)]

$$L\left(-q,q^2\right) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}\left(-q;q^2\right)_n}{(q^2;q^2)_n} = \frac{1}{(q^2;q^8)_\infty\,(q^3;q^4)_\infty}. \tag{22}$$

However, it may not have been observed previously that *the infinite products in* (21) *and* (22) *also arise as special cases of the q-Gauss sum* (17), via $I(a,q)$. By appropriate specialization, we obtain

$$I(-q,q^4) = \sum_{n=0}^{\infty} \frac{q^{2n^2-n}(-q;q^4)_n}{(q^2;q^2)_{2n}} = \frac{1}{(q;q^4)_\infty(q^6;q^8)_\infty} \tag{23}$$

and

$$\frac{I\left(-q^3,q^4\right)}{1-q^2} = \sum_{n=0}^{\infty} \frac{q^{2n^2+n}\left(-q^3;q^4\right)_n}{(q^2;q^2)_{2n+1}} = \frac{1}{(q^2;q^8)_\infty\,(q^3;q^4)_\infty}. \tag{24}$$

Finally, we observe that Göllnitz's little partition theorems can be alternately viewed as theorems about partitions constrained by the ratio of consecutive parts and give a combinatorial interpretation of (23).

Observation 2. *The set of partitions of N into parts differing by at least* 2 *and no consecutive odd parts is the same as the set of finite sequences of positive integers* $\lambda_1, \lambda_2, \ldots$ *of weight N satisfying*

$$\left\lfloor \frac{\lambda_i}{2} \right\rfloor > \left\lceil \frac{\lambda_{i+1}}{2} \right\rceil.$$

7 Suggestions for Further Study

Can we derive other series-product identities from the lecture hall approach, via Theorem 1 and Observation 1? Although these tools produce a "product side" for any positive integers (k, j) with $kl \geq 4$, deriving a "sum side" from the ratio characterization is more difficult. As shown in [3], the limiting form of Theorem 1 gives rise to the following ratios between consecutive parts: $(kl + \sqrt{kl(kl-1)})/(2k)$ and $(kl + \sqrt{kl(kl-1)})/(2l)$. These ratios are rational only if either $\{k, l\} = \{1, 4\}$ (the case considered in this paper) or $k = l = 2$ (giving ratio 1, the familiar case of distinct parts).

Are there other classical partition theorems that can be reinterpreted as statements about partitions constrained by the ratio of consecutive parts? For example, Gordon's combinatorial interpretation [9, p. 741, Theorems 2 and 3] of the the Göllnitz-Gordon identities involves partitions into parts differing by at least 2 and no consecutive even parts. Such partitions can be alternatively characterized as the set of finite sequences of positive integers $\lambda_1, \lambda_2, \ldots$ satisfying, for each i,

$$\left\lfloor \frac{\lambda_i + 1}{2} \right\rfloor > \left\lceil \frac{\lambda_{i+1} + 1}{2} \right\rceil.$$

We expect to find other examples. What can be learned from these reinterpretations?

Appendix 1: Proof of Observation 1

To verify that the (k,l)-Lecture Hall Theorem remains true for all *positive* k, l satisfying $kl \geq 4$, we first observe that these conditions are necessary and sufficient to guarantee that $a_n^{(k,l)}$ is positive for all $n \geq 1$. When $kl \geq 4$, each of the sequences $\{a_{2i}^{(k,l)}\}_{i \geq 0}$, $\{a_{2i+1}^{(k,l)}\}_{i \geq 0}$ satisfies the recurrence

$$w_i = (kl - 2)w_{i-1} - w_{i-2},$$

and, with their respective initial conditions, are strictly increasing. On the other hand, it can be checked that negative elements appear in the sequence when $(k,l) \in \{(1,1), (1,2), (2,1), (1,3), (3,1)\}$.

We then outline the clever combinatorial/algebraic approach of Bousquet-Mélou and Eriksson in [3] to illustrate that in order for Theorem 1 to hold, it is not necessary that $a_1^{(k,l)},\ldots,a_n^{(k,l)}$ be weakly increasing, rather only that all terms are positive.

Define the even and odd weight of a sequence $\lambda=(\lambda_1,\lambda_2,\ldots)$ by

$$|\Lambda|_o=\lambda_1+\lambda_3+\cdots; \quad |\lambda|_e=\lambda_2+\lambda_4+\cdots;$$

and define

$$G_n^{(k,l)}(x,y)=\sum_{\lambda\in L_n^{(k,l)}} x^{|\lambda|_o}y^{|\lambda|_e}.$$

The strategy is to show that the following recurrence from [3] holds for all *positive* k,l satisfying $kl\geq 4$.

$$G_n^{(k,l)}(x,y)=\begin{cases} G_{n-1}^{(k,l)}(x^l y,x^{-1})/(1-x) & \text{if } n \text{ is even,}\\ G_{n-1}^{(k,l)}(x^k y,x^{-1})/(1-x) & \text{if } n \text{ is odd,}\end{cases} \tag{25}$$

with initial condition $G_0^{(k,l)}(x,y)=1$. Using the recursive definition of $a_n^{(k,l)}$ and the fact that $a_{2i+1}^{(k,l)}= a_{2i+1}^{(l,k)}$, solving this recurrence gives Theorem 1.

To simplify notation in what follows, let $a_n=a_n^{(k,l)}$. To derive the recurrence (25), define a function

$$\Upsilon_n: \ L_{n-1}^{(k,l)}\times\mathbb{N} \ \rightarrow \ L_n^{(k,l)}$$

by $\Upsilon_n(\lambda,s)=\mu$, where

$$\mu_1 \leftarrow \left\lceil \frac{a_n\lambda_1}{a_{n-1}}\right\rceil+s;$$

$$\mu_{2t} \leftarrow \lambda_{2t-1}, \quad 1\leq t\leq n/2;$$

$$\mu_{2t+1} \leftarrow \begin{cases} \left\lceil \frac{a_{n-2t}\lambda_{2t+1}}{a_{n-2t-1}}\right\rceil+\left\lfloor \frac{a_{n-2t}\lambda_{2t-1}}{a_{n-2t+1}}\right\rfloor-\lambda_{2t}, & 1\leq t<(n-1)/2,\\ \left\lfloor \frac{a_{n-2t}\lambda_{2t-1}}{a_{n-2t+1}}\right\rfloor-\lambda_{2t}, & t=(n-1)/2.\end{cases}$$

The key is to use the properties of the (k,l) sequence to prove that $\mu\in L_n^{(k,l)}$, that Υ_n is a bijection, and that

$$|\mu|_e=|\lambda|_o; \tag{26}$$

$$|\mu|_o=\begin{cases} l|\lambda|_o-|\lambda|_e+s & \text{if } n \text{ is even,}\\ k|\lambda|_o-|\lambda|_e+s & \text{if } n \text{ is odd.}\end{cases} \tag{27}$$

For then this implies that when n is even,

$$L_n^{(k,l)}(x,y) \triangleq \sum_{\mu \in L_n^{(k,l)}} x^{|\mu|_o} y^{|\mu|_e} = \sum_{\lambda \in L_{n-1}^{(k,l)}} \sum_{s=0}^{\infty} x^{l|\lambda|_o - |\lambda|_e + s} y^{|\lambda|_o}$$

$$= \frac{1}{1-x} \sum_{\lambda \in L_{n-1}^{(k,l)}} (x^l y)^{|\lambda|_o} (1/x)^{|\lambda|_e} = \frac{L_{n-1}^{(k,l)}(x^l y, x^{-1})}{1-x},$$

giving the even case of recurrence (25) and the case for odd n is similar.

It remains to prove that Υ_n is a bijection satisfying (26) and (27). First observe that since a is a (k,l) sequence, for any $m \geq 0$

$$\left\lceil \frac{a_{i+1}}{a_i} m \right\rceil + \left\lfloor \frac{a_{i-1}}{a_i} m \right\rfloor = \begin{cases} km & \text{if } i \text{ even,} \\ lm & \text{if } i \text{ odd.} \end{cases}$$

Thus, when n is even,

$$|\mu|_o = \mu_1 + \mu_3 + \ldots = s + l(\lambda_1 + \lambda_3 + \ldots) - (\lambda_2 + \lambda_4 + \ldots) = l|\lambda|_o - |\lambda|_e + s,$$

and similarly, for n odd, proving (27). Condition (26) is easy to check.

To show that $\mu \in L_n^{(k,l)}$, note that consecutive parts $\lambda_{2t-1}, \lambda_{2t}, \lambda_{2t+1}$ in λ map to the consecutive parts of μ:

$$\mu_{2t} = \lambda_{2t-1},$$
$$\mu_{2t+1} = \left\lceil \frac{a_{n-2t}\lambda_{2t+1}}{a_{n-2t-1}} \right\rceil + \left\lfloor \frac{a_{n-2t}\lambda_{2t-1}}{a_{n-2t+1}} \right\rfloor - \lambda_{2t},$$
$$\mu_{2t+2} = \lambda_{2t+1}.$$

We need to show that

$$\mu_{2t} \geq \frac{a_{n-2t+1}}{a_{n-2t}} \mu_{2t+1}; \quad \mu_{2t+1} \geq \frac{a_{n-2t}}{a_{n-2t-1}} \mu_{2t+2}.$$

As $\lambda \in L_{n-1}^{(k,l)}$,

$$\lambda_{2t-1} \geq \frac{a_{n-2t+1}}{a_{n-2t}} \lambda_{2t}; \quad \lambda_{2t} \geq \frac{a_{n-2t}}{a_{n-2t-1}} \lambda_{2t+1},$$

so

$$\mu_{2t+1} \geq \left\lceil \frac{a_{n-2t}\lambda_{2t+1}}{a_{n-2t-1}} \right\rceil = \left\lceil \frac{a_{n-2t}\mu_{2t+2}}{a_{n-2t-1}} \right\rceil$$

and

$$\mu_{2t+1} \leq \left\lfloor \frac{a_{n-2t}\lambda_{2t-1}}{a_{n-2t+1}} \right\rfloor = \left\lfloor \frac{a_{n-2t}\mu_{2t}}{a_{n-2t+1}} \right\rfloor.$$

Note that this did not require that the sequence $\{a_n\}$ be nondecreasing.

Finally, (λ, s) can be recovered from μ by:

$$s \leftarrow \mu_1 - \left\lceil \frac{a_n \mu_2}{a_{n-1}} \right\rceil;$$

$$\lambda_{2t-1} \leftarrow \mu_{2t}, \quad 1 \le t \le n/2;$$

$$\lambda_{2t} \leftarrow \begin{cases} \left\lceil \frac{a_{n-2t}\mu_{2t+2}}{a_{n-2t-1}} \right\rceil + \left\lfloor \frac{a_{n-2t}\mu_{2t}}{a_{n-2t+1}} \right\rfloor - \mu_{2t+1}, & 1 \le t < (n-1)/2, \\ \left\lfloor \frac{a_{n-2t}\mu_{2t}}{a_{n-2t+1}} \right\rfloor - \mu_{2t+1}, & t = (n-1)/2. \end{cases}$$

Appendix 2: Bijective Proof of Theorem 2

Let $s_1, s_2, \ldots, s_k$ be a sequence of positive integers satisfying the condition $s_i = 1$ or $s_{i+1} = 1$ for $1 \le i \le k-1$. Recall that $b_1 = 1$ and $b_i = s_1 + \ldots + s_i$ for $i > 0$.

In the numerator of $F(q)$ in Theorem 2, write

$$1 + q^{b_i}([s_{i+1}]_q - 1) = q^{b_i+1} + q^{b_i+2} + \ldots + q^{b_i+s_{i+1}-1} = q^{b_i+1} + q^{b_i+2} + \ldots + q^{b_{i+1}-1}.$$

So, each positive integer in the set $\{b_1\} \cup \{b_2, b_2+1, b_2+3, \ldots, b_k\}$ occurs exactly once in $F(q)$ as an exponent of q, either in the numerator or in the denominator. We can interpret $F(q)$ as the generating function for the set of partitions of an integer into parts from the set $\{1, b_2, b_2+1, \ldots, b_k\}$ in which parts in the set $\cup_{i=2}^{k-1}\{b_i + 1, b_i+2, \ldots, b_{i+1}-1\}$ can occur at most once.

To prove Theorem 2, we give a weight-preserving bijection from the set of sequences $\lambda = \lambda_1, \ldots, \lambda_k$ satisfying

$$\frac{\lambda_1}{s_1} \ge \frac{\lambda_2}{s_2} \ge \cdots \ge \frac{\lambda_k}{s_k} \ge 0$$

to the set of pairs of partitions (μ, ν), where μ is a partition into parts in $\{b_1, \ldots, b_k\}$, and where ν is a partition into distinct parts from $\cup_{i=2}^{k-1}\{b_i+1, b_i+2, \ldots, b_{i+1}-1\}$.

```
Given λ, construct (μ, ν) as follows:
For i from k down to 2 do
        While λ_i/s_i ≥ 1 do
                For j from 1 to i do
                        λ_j ← λ_j − s_j
                μ ← μ ∪ b_i
        If λ_i > 0 then
                For j from 1 to i − 1 do
                        λ_j ← λ_j − s_j
                ν ← ν ∪ (b_{i−1} + λ_i)
μ ← μ ∪ b_1^{λ_1}
```

Inside the main loop, if $\lambda_i \geq s_i$ and

$$\frac{\lambda_1}{s_1} \geq \frac{\lambda_2}{s_2} \geq \cdots \geq \frac{\lambda_i}{s_i} \geq 0$$

then

$$\frac{\lambda_1 - s_1}{s_1} \geq \frac{\lambda_2 - s_2}{s_2} \geq \cdots \geq \frac{\lambda_i - s_i}{s_i} \geq 0$$

and another part $s_1 + \ldots + s_i = b_i$ is added to μ. If $0 < \lambda_i < s_i$, then $s_i \geq 2$. By definition of s, then $s_{i-1} = 1$, so $\lambda_{i-1} \geq s_{i-1}$ and

$$\frac{\lambda_1 - s_1}{s_1} \geq \frac{\lambda_2 - s_2}{s_2} \geq \cdots \geq \frac{\lambda_i - s_{i-1}}{s_{i-1}} \geq 0$$

and, for the first and only time, part $s_1 + \ldots + s_{i-1} + \lambda_i = b_{i-1} + \lambda_i < b_i$ is added to ν.

The reverse is straightforward.

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Generalizations of Hutchinson's Curve and the Thomae Formulae

Hershel M. Farkas

Abstract In this note we derive Thomae formulae for the family of z_n curves defined by

$$w^n = (z-\lambda_0)(z-\lambda_1)(z-\lambda)^{n-1}.$$

These may be viewed as either giving rise to a generalization of the classical λ-function of elliptic function theory or a special class of singular z_n curves.

Keywords Compact Riemann surface • z_n curve • Nonspecial integral divisor • Theta functions • Abel-Jacobi map • λ-function

Mathematics Subject Classification: 14H55, 30F10, 32J15, 32G15

1 Introduction

Hutchinson's curve [H] is the Riemann surface defined by

$$w^3 = (z-\lambda_0)(z-\lambda_1)(z-\lambda)^2,$$

a compact Riemann surface of genus 2, thus hyperelliptic. In this note we treat the family of Riemann surfaces parameterized by

$$w^n = (z-\lambda_0)(z-\lambda_1)(z-\lambda)^{n-1}.$$

This is a one parameter family of compact surfaces, X, of genus $g = n-1$ which are hyperelliptic. They all have an automorphism of period n, which we shall denote

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K. Alladi and F. Garvan (eds.), *Partitions, q-Series, and Modular Forms*,
Developments in Mathematics 23, DOI 10.1007/978-1-4614-0028-8_7,

by T, which has precisely four fixed points and such that the quotient surface is the sphere. In other words these surfaces have a realization as an n-sheeted cover of the sphere branched over four points,

$$\lambda_0, \lambda_1, \lambda, \infty.$$

Hutchinson's curve is an example of a singular z_n curve with $n = 3$. We recall that a curve of the form

$$w^n = \prod_{i=1}^{i=rn-1} (z - \lambda_i)$$

with $\lambda_i \neq \lambda_j$ for $i \neq j$ is called a nonsingular z_n curve, while if the polynomial has multiple roots it is called a singular z_n curve. This curve has a representation as an n-sheeted cover of the sphere branched over the points

$$\lambda_1, \ldots, \lambda_{nr-1}, \infty$$

with each branch point of maximal order $n-1$. Our generalization of Hutchinson's curve has precisely four branch points,

$$\lambda_0, \lambda_1, \lambda, \infty.$$

The Thomae problem [B,R 1], [B,R 2], [E,F], [E,G] is concerned with writing a set of proportionalities between theta constants associated with these curves and polynomials in the variables λ_i. The characteristics of the theta constants are derived from the set of branch points using the Abel Jacobi map [F,K] and this note treats the simplest case of four branch points. The genus of the z_n curve in this case is $n-1$ and all these curves are hyperelliptic. The hyperelliptic case was already treated by Thomae, see [E,F] for a new proof.

It is clear from the above that if we denote the point over λ_i by P_i and the point over ∞ by P_∞, then denoting the Abel Jacobi map with base point P_i by ϕ_{P_i}, we have $\Phi_{P_j}(P_i)$ as a point of order n. We shall soon see that the vector of Riemann constants K_{P_i} is a point of order $2n$. Moreover it is easy to see that a basis for the holomorphic differentials on this surface is given by

$$\frac{\mathrm{d}z}{w}, (z-\lambda)\frac{\mathrm{d}z}{w^2}, \ldots, (z-\lambda)^{n-2}\frac{\mathrm{d}z}{w^{n-1}}$$

and that the respective divisors are

$$(P_0P_1)^{n-2}, (P_0P_1)^{n-3}(P_\lambda P_\infty), \ldots, (P_0P_1)(P_\lambda P_\infty)^{n-3}, (P_\lambda P_\infty)^{n-2}.$$

We wish to consider now the integral divisors Δ, of degree $g = n-1$ whose support is in the set $P_1 P_\lambda, P_\infty$ which are nonspecial. This means those with the property $i(\Delta) = 0$, i.e., those for which there is no holomorphic differential whose divisor is a multiple of Δ.

It is clear that the integral divisors are those which belong to the set

$$S = \left(P_1^{n-1}, P_\lambda^{n-1}, P_\infty^{n-1}, P_1 P_\lambda^{n-2}, P_1^2 P_\lambda^{n-3}, ..., P_1^{n-2} P_\lambda, P_1 P_\infty^{n-2}, ..., P_1^{n-2} P_\infty\right).$$

We recall [F,K] that the vector K_{P_i} has the following property. An integral divisor ζ of degree $2g-2$ is the divisor of a holomorphic differential on the surface if and only if

$$\phi_{P_i}(\zeta) = -2K_{P_i}.$$

2 Properties of the Surface X

It is clear from the basis given above for the holomorphic differentials on X that X is hyperelliptic and in addition that if we denote the hyperelliptic involution by H then $H(P_0) = P_1, H(P_\lambda) = P_\infty$. Moreover, the $2n$ Weierstrass points are in two fibers above two specific points on the sphere which we can compute but will not. We begin with

Lemma 2.1. *Let $k+l = n-1$ with $k = -1, 0, ..., n-2$.*

$$\phi_{P_0}\left(P_1^k P_\lambda^l\right) + K_{P_0} = -\left(\phi_{P_0}(P_1^{n-2-k} P_\lambda^{n-l}) + K_{P_0}\right),$$
$$\phi_{P_0}\left(P_1^k P_\infty^l\right) + K_{P_0} = -\left(\phi_{P_0}(P_1^{n-2-k} P_\infty^{n-l}) + KK_{P_0}\right).$$

In particular we have

$$\phi_{P_0}\left(P_1^{n-1}\right) + K_{P_0}$$

is always a point of order 2, and when n is even

$$\phi_{P_0}\left(P_1^{\frac{n-2}{2}} P_\lambda^{\frac{n}{2}}\right) + K_{P_0}, \phi_{P_0}\left(P_1^{\frac{n-2}{2}} P_\infty^{\frac{n}{2}}\right) + K_{P_0}$$

are also points of order 2.

Proof. Consider

$$\phi_{P_0}\left(P_1^k P_1^{n-2-k} P_\lambda^l P_\lambda^{n-l}\right) + 2K_{P_0}$$

or

$$\phi_{P_0}\left(P_1^k P_1^{n-2-k} P_\infty^l P_\infty^{n-l}\right) + 2K_{P_0}.$$

In either case since $\phi_{P_0}(P_\lambda^n) = \phi_{P_0}(P_\infty^n) = 0$ we get the result $\phi_{P_0}(P_1^{n-2}) + 2K_{P_0}$. We have already seen that

$$-2K_{P_0} = \phi_{P_0}\left(P_0^{n-2} P_1^{n-2}\right) = \phi_{P_0}\left(P_1^{n-2}\right)$$

since the first divisor is the divisor of a holomorphic differential. Hence the sum we obtained is 0 and this proves the first statement. Setting $k=-1$ and the observation that $\phi_{P_0}(P_1^{-1})=\phi_{P_0}(P_1^{n-1})$ gives the second statement while the third follows from choosing$k=\frac{n-2}{2}$. □

Since we shall be interested in theta constants, $\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0,\Pi)$ and they satisfy $\theta\begin{bmatrix}-\epsilon\\ -\epsilon'\end{bmatrix}(0,\Pi)=\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0,\Pi)$ and since

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\zeta,\Pi)=\exp(l(\zeta)\theta\left(\zeta+\frac{1}{2}(\Pi\epsilon+I\epsilon'),\Pi\right),$$

where $l(\zeta)$ is a linear function of ζ, and finally since

$$\phi_{P_0}(\Delta)+K_{P_0}=\frac{1}{2}\left(\Pi\left(\frac{2\epsilon+\delta}{n}\right)+I\left(\frac{2\epsilon'+\delta'}{n}\right)\right)$$

we can write $\theta[\phi_{P_0}(\Delta)+K_{P_0}](\zeta,\Pi)$ to represent $\theta\begin{bmatrix}\frac{2\epsilon+\delta}{n}\\ \frac{2\epsilon'+\delta'}{n}\end{bmatrix}(\zeta,\Pi)$. As a consequence of the previous remarks the characteristics or the integral divisors of degree g that we shall be interested in are

$$\left(\phi_{P_0}\left(P_1^{n-1}\right)+K_{P_0},\phi_{P_0}\left(P_\lambda^{n-1}\right)+K_{P_0},\phi_{P_0}\left(P_1P_\lambda^{n-2}\right)+K_{P_0},..,\phi_{P_0}\left(P_1^{\frac{n-3}{2}}P_\lambda^{\frac{n+1}{2}}\right)+K_{P_0},\right.$$
$$\left.\phi_{P_0}\left(P_\infty^{n-1}\right)+K_{P_0},\phi_{P_0}\left(P_1P_\infty^{n-2}\right)+K_{P_0},..,\phi_{P_0}\left(P_1^{\frac{n-3}{2}}P_\infty^{\frac{n+1}{2}}\right)+K_{P_0}\right),$$

when n is odd and

$$\left(\phi_{P_0}(P_1^{n-1})+K_{P_0},\phi_{P_0}(P_\lambda^{n-1})+K_{P_0},\phi_{P_0}(P_1P_\lambda^{n-2})+K_{P_0},..,\phi_{P_0}\left(P_1^{\frac{n-2}{2}}P_\lambda^{\frac{n}{2}}\right)+K_{P_0},\right.$$
$$\left.\phi_{P_0}(P_\infty^{n-1})+K_{P_0},\phi_{P_0}(P_1P_\infty^{n-2})+K_{P_0},..,\phi_{P_0}\left(P_1^{\frac{n-2}{2}}P_\infty^{\frac{n}{2}}\right)+K_{P_0}\right),$$

when n is even.

3 Even *n*

In this section we shall consider the case of even n. In this case we begin by studying the sequence of functions

$$f_i(P) = \frac{\theta^{n^2}[\phi_{P_0}(P_1^{i-1}P_\lambda^{n-i}) + K_{P_0}](\phi_{P_0}(P), \Pi)}{\theta^{n^2}[\phi_{P_0}(P_1^{i}P_\lambda^{n-i-1}) + K_{P_0}](\phi_{P_0}(P), \Pi)} \qquad i = 0, \ldots, \frac{n-2}{2}.$$

The first observation is that $f_0(P) = \frac{\theta^{n^2}[\phi_{P_0}(P_1^{n-1})+K_{P_0}](\phi_{P_0}(P),\Pi)}{\theta^{n^2}[\phi_{P_0}(P_\lambda^{n-1})+K_{P_0}](\phi_{P_0}(P),\Pi)}$.

The second observation is that $f_i(P) = c_i(\frac{z-\lambda_1}{z-\lambda})^n$. These observations follow from Lemma 2.1, from the elementary properties of the theta functions, and from the fact that $\phi_{P_0}(P_1^n) = 0$.

If we now substitute $P = P_0$, we find c_i. We then substitute $P = P_\infty$, use the elementary properties of theta functions in addition to the fact that $\Phi_{P_0}(P_0P_1) = \phi_{P_0}(P_1) = \phi_{P_0}(P_\lambda P_\infty)$, and find that for $i = 0, .., \frac{n-2}{2}$

$$\begin{aligned} D_i &= \frac{\theta^{2n^2}[\phi_{P_0}(P_1^{i}P_\lambda^{n-i-1}) + K_{P_0}](0, \Pi)}{\theta^{n^2}[\phi_{P_0}(P_1^{i+1}P_\lambda^{n-i-2}) + K_{P_0}](0, \Pi)\theta^{n^2}[\phi_{P_0}(P_1^{i-1}P_\lambda^{n-i}) + K_{P_0}](0, \Pi)} \\ &= \left(\frac{\lambda_0 - \lambda}{\lambda_0 - \lambda_1}\right)^n. \end{aligned}$$

We check that

$$D_{\frac{n-2}{2}} = \frac{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-2}{2}}P_\lambda^{\frac{n}{2}}\right) + K_{P_0}\right](0, \Pi)}{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-4}{2}}P_\lambda^{\frac{n+2}{2}}\right) + K_{P_0}\right](0, \Pi)} = \left(\frac{\lambda_0 - \lambda}{\lambda_0 - \lambda_1}\right)^n.$$

We then continue to compute

$$D_{\frac{n-4}{2}}D_{\frac{n-2}{2}} = \frac{\theta^{n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-2}{2}}P_\lambda^{\frac{n}{2}}\right) + K_{P_0}\right](0, \Pi)}{\theta^{n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-6}{2}}P_\lambda^{\frac{n+4}{2}}\right) + K_{P_0}\right](0, \Pi)} = \left(\frac{\lambda_0 - \lambda}{\lambda_0 - \lambda_1}\right)^{2n}$$

and conclude therefore

$$\frac{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-2}{2}}P_\lambda^{\frac{n}{2}}\right) + K_{P_0}\right](0, \Pi)}{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-6}{2}}P_\lambda^{\frac{n+4}{2}}\right) + K_{P_0}\right](0, \Pi)} = \left(\frac{\lambda_0 - \lambda}{\lambda_0 - \lambda_1}\right)^{4n}.$$

Continuing in this fashion to compute

$$D_{\frac{n-6}{2}}D_{\frac{n-4}{2}}, D_{\frac{n-8}{2}}D_{\frac{n-6}{2}}, ..D_0D_1,$$

we find

Theorem 3.1. *For $k = 1,..,\frac{n-2}{2}$ we have*

$$\frac{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-2k}{2}}P_\lambda^{\frac{n+2k-2}{2}}\right)+K_{P_0}\right](0,\Pi)}{(\lambda_0-\lambda)^{4kn}} = \frac{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-2k-4}{2}}P_\lambda^{\frac{n+2k+2}{2}}\right)+K_{P_0}\right](0,\Pi)}{(\lambda_0-\lambda_1)^{4kn}}.$$

If we were to begin again with the sequence

$$g_i(P) = \frac{\theta^{n^2}[\phi_{P_0}(P_1^{i-1}P_\infty^{n-i})+K_{P_0}](\phi_{P_0}(P),\Pi)}{\theta^{n^2}[\phi_{P_0}(P_1^{i}P_\infty^{n-i-1})+K_{P_0}](\phi_{P_0}(P),\Pi)}$$

we would find that it equals $\tilde{c}_i(z-\lambda_1)^n$ and a repeat of what we have done would give the following result.

Theorem 3.2. *For $k = 1,..,\frac{n-2}{2}$ we have*

$$\frac{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-2k}{2}}P_\infty^{\frac{n+2k-2}{2}}\right)+K_{P_0}\right](0,\Pi)}{(\lambda-\lambda_1)^{4kn}} = \frac{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-2k-4}{2}}P_\infty^{\frac{n+2k+2}{2}}\right)+K_{P_0}\right](0,\Pi)}{(\lambda_0-\lambda_1)^{4kn}}.$$

Before we explore the consequences of these theorems we do an example. The theorems for even n give us the following identities: (alternatively, the algebraic operations explained above when applied to the following example with $n = 10$ give the following identities:)

$$(4)\qquad \frac{\theta^{200}[\phi_{P_0}(P_1^4P_\lambda^5)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda)^{40}} = \frac{\theta^{200}[\phi_{P_0}(P_1^2P_\lambda^7)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{40}},$$

$$(3)\qquad \frac{\theta^{200}[\phi_{P_0}(P_1^3P_\lambda^6)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda)^{80}} = \frac{\theta^{200}[\phi_{P_0}(P_1P_\lambda^8)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{80}},$$

$$(2)\qquad \frac{\theta^{200}[\phi_{P_0}(P_1^2P_\lambda^7)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda)^{120}} = \frac{\theta^{200}[\phi_{P_0}(P_\lambda^9)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{120}},$$

$$(1)\qquad \frac{\theta^{200}[\phi_{P_0}(P_1P_\lambda^8)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda)^{160}} = \frac{\theta^{200}[\phi_{P_0}(P_1^9)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{160}}.$$

In addition we have

$$(0)\qquad \frac{\theta^{200}[\phi_{P_0}(P_1^4P_\lambda^5)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda)^{10}}=\frac{\theta^{200}[\phi_{P_0}(P_1^3P_\lambda^6)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{10}}.$$

We now do the following arithmetic operations:

We multiply (4) by $\frac{1}{(\lambda_0-\lambda)^{120}}$, (3) by $\frac{1}{(\lambda_0-\lambda)^{160}}$, (2) by $\frac{1}{(\lambda_0-\lambda_1)^{40}}$, and finally (1) by $\frac{1}{(\lambda_0-\lambda_1)^{80}}$.

We obtain from these multiplications

$$\begin{aligned}\frac{\theta^{200}[\phi_{P_0}(P_1^9)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{240}}&=\frac{\theta^{200}[\phi_{P_0}(P_1P_\lambda^8)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{80}(\lambda_0-\lambda)^{160}}\\&=\frac{\theta^{200}[\phi_{P_0}(P_1^3P_\lambda^6)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda)^{240}}\end{aligned}$$

and

$$\begin{aligned}\frac{\theta^{200}[\phi_{P_0}(P_\lambda^9)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{160}}&=\frac{\theta^{200}[\phi_{P_0}(P_1^2P_\lambda^7)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{40}(\lambda_0-\lambda)^{120}}\\&=\frac{\theta^{200}[\phi_{P_0}(P_1^4P_\lambda^5)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda)^{160}}.\end{aligned}$$

In order to obtain one identity we need to multiply (0) by $\frac{1}{(\lambda_0-\lambda)^{240}}$. It is then clear after multiplying the first of the two identities by $\frac{1}{(\lambda_0-\lambda_1)^{10}}$ and the second by $\frac{1}{(\lambda_0-\lambda)^{90}}$ that in fact

$$\begin{aligned}\frac{\theta^{200}[\phi_{P_0}(P_1^9)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{250}}&=\frac{\theta^{200}[\phi_{P_0}(P_\lambda^9)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{160}(\lambda_0-\lambda)^{90}}\\&=\frac{\theta^{200}[\phi_{P_0}(P_1P_\lambda^8)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{90}(\lambda_0-\lambda)^{160}}\\&=\frac{\theta^{200}[\phi_{P_0}(P_1^2P_\lambda^7)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{40}(\lambda_0-\lambda)^{210}}\\&=\frac{\theta^{200}[\phi_{P_0}(P_1^3P_\lambda^6)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{10}(\lambda_0-\lambda)^{240}}\\&=\frac{\theta^{200}[\phi_{P_0}(P_1^4P_\lambda^5)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda)^{250}}.\end{aligned}$$

If we used the sequence $g_i(P)$ rather than the sequence $f_i(P)$ we would have derived in place of the above the formulae

$$\frac{\theta^{200}[\phi_{P_0}(P_1^9)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{250}} = \frac{\theta^{200}[\phi_{P_0}(P_\infty^9)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{160}(\lambda_1-\lambda)^{90}}$$
$$= \frac{\theta^{200}[\phi_{P_0}(P_1P_\infty^8)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{90}(\lambda_1-\lambda)^{160}}$$
$$= \frac{\theta^{200}[\phi_{P_0}(P_1^2P_\infty^7)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{40}(\lambda_1-\lambda)^{210}}$$
$$= \frac{\theta^{200}[\phi_{P_0}(P_1^3P_\infty^6)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{10}(\lambda_1-\lambda)^{240}}$$
$$= \frac{\theta^{200}[\phi_{P_0}(P_1^4P_\infty^5)+K_{P_0}](0,\Pi)}{(\lambda_1-\lambda)^{250}}.$$

One can do the same computation for any even value of n and we come to the following conclusion (not formally proved but quite evident).

Theorem 3.3. *Let $k+l=n-1$ for $k=-1,0,..,\frac{n-2}{2}$. Then*

$$\frac{\theta^{2n^2}[\phi_{P_0}(P_1^kP_\lambda^l)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{(\frac{n}{2}-k-1)^2n}(\lambda_0-\lambda)^{\frac{n^3}{4}-(\frac{n}{2}-k-1)^2n}} = constant$$
$$\frac{\theta^{2n^2}[\phi_{P_0}(P_1^kP_\infty^l)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{(\frac{n}{2}-k-1)^2n}(\lambda_1-\lambda)^{\frac{n^3}{4}-(\frac{n}{2}-k-1)^2n}} = constant$$

and of course the constants are equal since the first terms are.

The above are the Thomae formulae for even n for the Riemann surface

$$w^n=(z-\lambda_0)(z-\lambda_1)(z-\lambda)^{n-1}.$$

If we normalize and let $\lambda_0=0,\lambda_1=1$ then the term $\lambda_0-\lambda_1$ disappears and we only obtain powers of λ and $(1-\lambda)$.

4 Odd *n*

The case of odd n is a bit simpler. In this case we begin once again with a sequence of functions with $i=0,..,\frac{n-3}{2}$:

$$f_i(P)=\frac{\theta^{2n^2}[\phi_{P_0}(P_1^{i-1}P_\lambda^{n-i})+K_{P_0}](\phi_{P_0}(P),\Pi)}{\theta^{2n^2}[\phi_{P_0}(P_1^iP_\lambda^{n-i-1}+K_{P_0}](\phi_{P_0}(P),\Pi)} = c_i\left(\frac{z-\lambda_1}{z-\lambda}\right)^{2n}.$$

Setting first $P = P_0$ in the above allows us to solve for c_i and then setting $P = P_\infty$ and using the fact that $\phi_{P_0}(P_1) = \phi_{P_0}(P_\lambda P_\infty)$ yields

$$D_i = \frac{\theta^{4n^2}[\phi_{P_0}(P_1^i P_\lambda^{n-i-1}) + K_{P_0}](0,\Pi)}{\theta^{2n^2}[\phi_{P_0}(P_1^{i+1} P_\lambda^{n-i-2}) + K_{P_0}](0,\Pi)\theta^{2n^2}[\phi_{P_0}(P_1^{i-1} P_\lambda^{n-i}) + K_{P_0}](0,\Pi)}$$
$$= \left(\frac{\lambda_0 - \lambda}{\lambda_0 - \lambda_1}\right)^{2n}.$$

Till now the only change from even n has been the fact that we have taken the square of the function we took before. We observe that

$$D_{\frac{n-3}{2}} = \frac{\theta^{4n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-3}{2}} P_\lambda^{\frac{n+1}{2}}\right) + K_{P_0}\right](0,\Pi)}{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-1}{2}} P_\lambda^{\frac{n-1}{2}}\right) + K_{P_0}\right](0,\Pi)\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-5}{2}} P_\lambda^{\frac{n+3}{2}}\right) + K_{P_0}\right](0,\Pi)}$$
$$= \frac{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-3}{2}} P_\lambda^{\frac{n+1}{2}}\right) + K_{P_0}\right](0,\Pi)}{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-5}{2}} P_\lambda^{\frac{n+3}{2}}\right) + K_{P_0}\right](0,\Pi)} = \left(\frac{\lambda_0 - \lambda}{\lambda_0 - \lambda_1}\right)^{2n}.$$

The above follows from Lemma 2.1.

We now compute

$$D_{\frac{n-5}{2}} D_{\frac{n-3}{2}} = \frac{\theta^{4n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-5}{2}} P_\lambda^{\frac{n+3}{2}}\right) + K_{P_0}\right](0,\Pi)}{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-3}{2}} P_\lambda^{\frac{n+1}{2}}\right) + K_{P_0}\right](0,\Pi)\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-7}{2}} P_\lambda^{\frac{n+5}{2}}\right) + K_{P_0}\right](0,\Pi)}$$
$$\times \frac{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-3}{2}} P_\lambda^{\frac{n+1}{2}}\right) + K_{P_0}\right](0,\Pi)}{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-5}{2}} P_\lambda^{\frac{n+3}{2}}\right) + K_{P_0}\right](0,\Pi)}$$
$$= \frac{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-5}{2}} P_\lambda^{\frac{n+3}{2}}\right) + K_{P_0}\right](0,\Pi)}{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-7}{2}} P_\lambda^{\frac{n+5}{2}}\right) + K_{P_0}\right](0,\Pi)} = \left(\frac{\lambda_0 - \lambda}{\lambda_0 - \lambda_1}\right)^{4n}.$$

We now compute

$$D_{\frac{n-7}{2}} D_{\frac{n-5}{2}} = \frac{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-7}{2}} P_\lambda^{\frac{n+5}{2}}\right) + K_{P_0}\right](0,\Pi)}{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-9}{2}} P_\lambda^{\frac{n+7}{2}}\right) + K_{P_0}\right](0,\Pi)} \cdot \frac{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-5}{2}} P_\lambda^{\frac{n+3}{2}}\right) + K_{P_0}\right](0,\Pi)}{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-3}{2}} P_\lambda^{\frac{n+1}{2}}\right) + K_{P_0}\right](0,\Pi)}.$$

The above on the one hand equals $(\frac{\lambda_0-\lambda}{\lambda_0-\lambda_1})^{4n}$ and on the other hand equals

$$\frac{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-7}{2}}P_\lambda^{\frac{n+5}{2}}\right)+K_{P_0}\right](0,\Pi)}{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-9}{2}}P_\lambda^{\frac{n+7}{2}}\right)+K_{P_0}\right](0,\Pi)}\cdot\left(\frac{\lambda_0-\lambda_1}{\lambda_0-\lambda}\right)^{2n}$$

from which we conclude

$$\frac{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-7}{2}}P_\lambda^{\frac{n+5}{2}}\right)+K_{P_0}\right](0,\Pi)}{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-9}{2}}P_\lambda^{\frac{n+7}{2}}\right)+K_{P_0}\right](0,\Pi)}=\left(\frac{\lambda_0-\lambda}{\lambda_0-\lambda_1}\right)^{6n}.$$

Continuing in this fashion leads us to the following result.

Theorem 4.1. *For k = 1,.., $\frac{n-1}{2}$ we have*

$$\frac{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-(2k+1)}{2}}P_\lambda^{\frac{n+2k-1}{2}}\right)+K_{P_0}\right](0,\Pi)}{(\lambda_0-\lambda)^{2kn}}$$

$$=\frac{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-(2k+3)}{2}}P_\lambda^{\frac{n+2k+1}{2}}\right)+K_{P_0}\right](0,\Pi)}{(\lambda_0-\lambda_1)^{2kn}}.$$

If we were to begin again with the sequence

$$g_i(P)=\frac{\theta^{2n^2}[\phi_{P_0}(P_1^{i-1}P_\infty^{n-i})+K_{P_0}](\phi_{P_0}(P),\Pi)}{\theta^{2n^2}[\phi_{P_0}(P_1^{i}P_\infty^{n-i-1})+K_{P_0}](\phi_{P_0}(P),\Pi)}=\tilde{c}_i(z-\lambda_1)^{2n}$$

a repeat of the arguments given would lead to the following result.

Theorem 4.2. *For* $k=1,..,\frac{n-1}{2}$ *we have*

$$\frac{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-(2k+1)}{2}}P_\infty^{\frac{n+2k-1}{2}}\right)+K_{P_0}\right](0,\Pi)}{(\lambda_1-\lambda)^{2kn}}$$

$$=\frac{\theta^{2n^2}\left[\phi_{P_0}\left(P_1^{\frac{n-(2k+3)}{2}}P_\infty^{\frac{n+2k+1}{2}}\right)+K_{P_0}\right](0,\Pi)}{(\lambda_0-\lambda_1)^{2kn}}.$$

As in the case of even n we do an example before exploring the consequences of our results. The theorems derived yield in the case $n=9$ the following:

$$(3)\quad \frac{\theta^{162}[\phi_{P_0}(P_1^3P_\lambda^5)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda)^{18}}=\frac{\theta^{162}[\phi_{P_0}(P_1^2P_\lambda^6)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{18}},$$

$$(2)\quad \frac{\theta^{162}[\phi_{P_0}(P_1^2P_\lambda^6)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda)^{36}}=\frac{\theta^{162}[\phi_{P_0}(P_1P_\lambda^7)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{36}},$$

$$(1)\quad \frac{\theta^{162}[\phi_{P_0}(P_1P_\lambda^7)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda)^{54}}=\frac{\theta^{162}[\phi_{P_0}(P_\lambda^8)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{54}},$$

$$(0)\quad \frac{\theta^{162}[\phi_{P_0}(P_\lambda^8)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda)^{72}}=\frac{\theta^{162}[\phi_{P_0}(P_1^8)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{72}}.$$

We now multiply (3) by $\frac{1}{(\lambda_0-\lambda)^{162}}$, (2) by $\frac{1}{(\lambda_0-\lambda_1)^{18}(\lambda_0-\lambda)^{126}}$, (1) by $\frac{1}{(\lambda_0-\lambda_1)^{54}(\lambda_0-\lambda)^{72}}$, and finally (0) by $\frac{1}{(\lambda_0-\lambda_1)^{108}}$ we get the following Thomae formulae:

$$\begin{aligned}\frac{\theta^{162}[\phi_{P_0}(P_1^8)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{180}}&=\frac{\theta^{162}[\phi_{P_0}(P_\lambda^8)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{108}(\lambda_0-\lambda)^{72}}\\&=\frac{\theta^{162}[\phi_{P_0}(P_1P_\lambda^7)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{54}(\lambda_0-\lambda)^{126}}\\&=\frac{\theta^{162}[\phi_{P_0}(P_1^2P_\lambda^6)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{18}(\lambda_0-\lambda)^{162}}\\&=\frac{\theta^{162}[\phi_{P_0}(P_1^3P_\lambda)^5)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda)^{180}}.\end{aligned}$$

If we used the sequence $g_i(P)$ rather than $f_i(P)$ we would have obtained in place of the above formula

$$\begin{aligned}\frac{\theta^{162}[\phi_{P_0}(P_1^8)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{180}}&=\frac{\theta^{162}[\phi_{P_0}(P_\infty^8)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{108}(\lambda_1-\lambda)^{72}}\\&=\frac{\theta^{162}[\phi_{P_0}(P_1P_\infty^7)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{54}(\lambda_1-\lambda)^{126}}\\&=\frac{\theta^{162}[\phi_{P_0}(P_1^2P_\infty^6)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{18}(\lambda_1-\lambda)^{162}}\\&=\frac{\theta^{162}[\phi_{P_0}(P_1^3P_\infty)^5)+K_{P_0}](0,\Pi)}{(\lambda_1-\lambda)^{180}}.\end{aligned}$$

One can make this same computation for any odd n and conclude (once again not formally proved but rather evident) the following.

Theorem 4.3. *Let $k+l=n-1$ for $k=-1,0,..,\frac{n-3}{2}$. Then*

$$\frac{\theta^{2n^2}[\phi_{P_0}(P_1^k P_\lambda^l)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{\frac{(n-2k-1)(n-2k-3)}{4}n}(\lambda_0-\lambda)^{\frac{n^2-1}{4}n-\frac{(n-2k-1)(n-2k-3)}{4}n}} = constant$$

$$\frac{\theta^{2n^2}[\phi_{P_0}(P_1^k P_\infty^l)+K_{P_0}](0,\Pi)}{(\lambda_0-\lambda_1)^{\frac{(n-2k-1)(n-2k-3)}{4}n}(\lambda_1-\lambda)^{\frac{n^2-1}{4}n-\frac{(n-2k-1)(n-2k-3)}{4}n}} = constant$$

and of course the constants are equal since the first terms are.

5 Concluding Remarks

We have titled this note as a generalization of Hutchinson's curve. It may also be viewed as giving an extension or generalization of the "lambda" function of elliptic function theory. The identities derived in the case $n=2$ are

$$\frac{\theta^8[\phi_{P_0}(P_\infty)+K_{P_\infty}](0,\Pi)}{(1-\lambda)^2} = \frac{\theta^8[\phi_{P_0}(P_1)+K_{P_0}](0,\Pi)}{1} = \frac{\theta^8[\phi_{P_0}(P_\lambda)+K_{P_0}](0,\Pi)}{\lambda^2}.$$

The above identity defines the square of the "lambda" function, and in fact gives Jacobi's famous formula as well. If we write the identity when $n=3$ we obtain

$$\frac{\theta^{18}[\phi_{P_0}(P_\infty^2)+K_{P_0}](0,\Pi)}{(1-\lambda)^6} = \frac{\theta^{18}[\phi_{P_0}(P_1^2)+K_{P_0}](0,\Pi)}{1} = \frac{\theta^{18}[\phi_{P_0}(P_\lambda^2)+K_{P_0}](0,\Pi)}{\lambda^6}.$$

Once again we have a definition of a "lambda" function to a power and once again obtain an identity.

When n grows we get more terms in the basic proportionalities and get associated powers of a "lambda" function. Inthese extensions though the characteristics of the associate theta functions do not remain integral characteristics but rather become rational characteristics. These give more identities and may shed some more light on the celebrated Scottky relation.

6 Added in Proof

Some progress has been made since the presentation of this paper at the conference. In particular we cite [E,F1] where the Thomae formulae have been derived for nonsingular z_n curves for all $n \geq 2$ with at least $2n$ branch points. In addition we cite the forthcoming book [F,Z] where among other things Thomae formulae have been derived for a class of singular z_n curves which include the curves discussed here. The result we present here is however a bit stronger.

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On the Parity of the Rogers-Ramanujan Coefficients

Basil Gordon

Abstract The parity of $g(n)$ and $h(n)$, the enumerators of restricted partitions of n appearing in the Rogers-Ramanujan identities, is studied. The parity of $g(n)$ for odd n and that of $h(n)$ for even n are completely determined. It is shown that these numbers are almost always even.

Keywords Partitions • Congruences • Rogers-Ramanujan

Mathematics Subject Classification: Primary 11P83, 11F11 and 11E12

1 Introduction

The infinite products

$$G(x) = \prod_{k=1}^{\infty} \frac{1}{(1-x^{5k-4})(1-x^{5k-1})}$$

and

$$H(x) = \prod_{k=1}^{\infty} \frac{1}{(1-x^{5k-3})(1-x^{5k-2})}$$

converge absolutely for $|x| < 1$ and are holomorphic there. The Rogers-Ramanujan identities assert that

$$G(x) = \sum_{n=0}^{\infty} \frac{x^{n^2}}{(x)_n}, \tag{1}$$

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K. Alladi and F. Garvan (eds.), *Partitions, q-Series, and Modular Forms*, Developments in Mathematics 23, DOI 10.1007/978-1-4614-0028-8_8,

$$H(x) = \sum_{n=0}^{\infty} \frac{x^{n(n+1)}}{(x)_n}, \tag{2}$$

where $(x)_n = \prod_{k=1}^{n}(1-x^k)$.

Identity (1) can be interpreted combinatorially as saying that the partitions of any positive integer n into parts $\equiv \pm 1 \pmod 5$ are equinumerous with partitions $n = n_1 + n_2 + \cdots + n_r$ with minimal difference 2, i.e., satisfying $n_i \geq n_{i+1} + 2$ for $1 \leq i < r$. Identity (2) says that those partitions of n having parts $\equiv \pm 2 \pmod 5$ are equinumerous with partitions of minimal difference 2 with $n_r \neq 1$.

Proofs of all the above can be found in [1, pp. 104–113].

Let

$$G(x) = \sum_{n=0}^{\infty} g(n)x^n$$

and

$$H(x) = \sum_{n=0}^{\infty} h(n)x^n$$

be the Taylor series expansions of $G(x)$ and $H(x)$. If $x = e^{2\pi i\tau}$ with $\mathrm{Im}(\tau) > 0$, then $x^{\frac{-1}{60}}G(x)$ and $x^{\frac{11}{60}}H(x)$ are modular forms, and hence the same is true of

$$x^{-1}G(x^{60}) = \sum_{n=0}^{\infty} g(n)x^{60n-1}$$

and

$$x^{11}H(x^{60}) = \sum_{n=0}^{\infty} h(n)x^{60n+11}.$$

This helps explain why the congruences for $g(n)$ and $h(n)$ obtained here depend on arithmetic properties of $60n-1$ and $60n+11$, respectively.

2 The Parity of *g(n)* for *n* Odd

It is remarkable that this parity can be completely determined, in sharp contrast to the problem of finding the parity of the unrestricted partition function $p(n)$. The result is as follows:

Theorem 1. *Suppose $n \equiv 1 \pmod 2$. Then $g(n) \equiv 1 \pmod 2$ if and only if $60n - 1 = p^{4a+1}m^2$, where p is a prime not dividing m.*

For example,

$$\begin{array}{lll} g(1)=1 & 59=\text{prime} & (a=0, m=1) \\ g(3)=1 & 179=\text{prime} & \\ g(5)=2 & 299=13\cdot 23 & \\ g(7)=3 & 419=\text{prime} & \end{array}$$

Proof of Theorem 1. If

$$A(x)=\sum_{n=0}^{\infty} a(n)x^n \qquad \text{and} \qquad B(x)=\sum_{n=0}^{\infty} b(n)x^n$$

are power series with integer coefficients, we write $A(x)\equiv B(x) \pmod m$ to mean that $a(n)\equiv b(n) \pmod m$ for all n.

First of all,

$$\begin{aligned} G(x) &= \prod_{k=1}^{\infty} \frac{1}{(1-x^{5k-4})(1-x^{5k-1})} \cdot \frac{(1-x^{5k-3})(1-x^{5k-2})(1-x^{5k})}{(1-x^{5k-3})(1-x^{5k-2})(1-x^{5k})} \\ &= \prod_{k=1}^{\infty} \frac{(1-x^{5k-3})(1-x^{5k-2})(1-x^{5k})}{1-x^k}. \end{aligned}$$

Since

$$\prod_{k=1}^{\infty} \frac{(1+x^{5k-3})(1+x^{5k-2})(1+x^{5k-4})(1+x^{5k-1})(1+x^{5k})}{1+x^k} = 1,$$

we have

$$G(x)=\prod_{k=1}^{\infty} \frac{(1-x^{10k-6})(1-x^{10k-4})(1+x^{5k-4})(1+x^{5k-1})(1-x^{10k})}{1-x^{2k}}. \tag{3}$$

The product of the last three factors in the numerator of (3) can be written as

$$\prod_{k=1}^{\infty}(1+x^{10k-6})(1+x^{10k-4})(1+x^{10k-9})(1+x^{10k-1})(1-x^{10k}).$$

Hence

$$G(x)=\prod_{k=1}^{\infty} \frac{(1-x^{20k-12})(1-x^{20k-8})(1-x^{10k})(1+x^{10k-9})(1+x^{10k-1})}{1-x^{2k}}. \tag{4}$$

We now apply Jacobi's triple product identity ([1, p.21])

$$\prod_{k=1}^{\infty}(1-q^{2k})(1+q^{2k-1}t)(1+q^{2k-1}t^{-1}) = \sum_{r=-\infty}^{\infty} q^{r^2}t^r \qquad (|q|<1\,,\ t\neq 0) \tag{5}$$

with $q=x^5$, $t=x^{-4}$ to obtain

$$\prod_{k=1}^{\infty}(1-x^{10k})(1+x^{10k-9})(1+x^{10k-1}) = \sum_{r=-\infty}^{\infty} x^{5r^2-4r}. \tag{6}$$

The exponent $5r^2-4r$ is even or odd according as $r=2s$ or $r=2s+1$. Hence the terms of (6) with odd exponent have sum

$$\sum_{s=-\infty}^{\infty} x^{5(2s+1)^2-4(2s+1)} = x\sum_{s=-\infty}^{\infty} x^{20s^2+12s}.$$

By Jacobi's identity applied in reverse, this is equal to

$$\begin{aligned} &x\prod_{k=1}^{\infty}(1-x^{40k})(1+x^{40k-32})(1+x^{40k-8}) \\ &\quad\equiv x\prod_{k=1}^{\infty}(1-x^{20k})^2(1-x^{20k-16})^2(1-x^{20k-4})^2 \pmod 2. \end{aligned}$$

Substituting this into (4), we obtain

$$\sum_{n \text{ odd}} g(n)x^n \equiv x\prod_{k=1}^{\infty}\frac{(1-x^{20k})(1-x^{20k-16})(1-x^{20k-4})(1-x^{4k})}{1-x^{2k}} \pmod 2.$$

Another application of Jacobi's identity, with $q=x^{10}$ and $t=x^6$, shows that

$$\begin{aligned} \sum_{n \text{ odd}} g(n)x^n &\equiv x\sum_{s=-\infty}^{\infty} x^{10s^2+6s}\prod_{k=1}^{\infty}(1+x^{2k}) \pmod 2 \\ &\equiv x\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} x^{3r^2+r+10s^2+6s} \pmod 2, \end{aligned} \tag{7}$$

using Euler's identity for $\prod_{k=1}^{\infty}(1-x^{2k})$.

Replacing x by x^{60} in (7) and then multiplying both sides by x^{-1}, we obtain

$$\begin{aligned} \sum_{n \text{ odd}} g(n)x^{60n-1} &\equiv x^{59}\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} x^{180r^2+60r+600s^2+360s} \pmod 2 \\ &= \sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} x^{5(6r+1)^2+6(10s+3)^2}. \end{aligned}$$

Now consider the quadratic form $Q_1(u,v) = 5u^2 + 6v^2$. If $Q_1(u,v) = 60n-1$, then $u^2 \equiv 1 \pmod 6$, so $u = \pm(6r+1)$. Thus $5u^2 \equiv 1 \pmod 4$, which implies that v is odd. Since $v^2 \equiv 6v^2 \equiv -1 \pmod 5$, it follows that $v \equiv \pm(10s+3)$. Therefore $\frac{1}{4}$ of the solutions of $5u^2 + 6v^2 = 60n-1$ are of the form $u = 6r+1$, $v = 10s+3$. In other words, if $r_Q(N)$ is the number of representations of N by the quadratic form Q, then for n odd,

$$g(n) \equiv \frac{1}{4} r_{Q_1}(60n-1) \pmod 2.$$

There are four primitive reduced quadratic forms of discriminant -120, namely,

$$\begin{aligned} Q_1(u,v) &= 5u^2 + 6v^2, \\ Q_2(u,v) &= 3u^2 + 10v^2, \\ Q_3(u,v) &= 2u^2 + 15v^2, \\ Q_4(u,v) &= u^2 + 30v^2. \end{aligned}$$

The equations $Q_j(u,v) = 60n-1$ are impossible (mod 5) for $j = 2$ and 3, while $Q_4(u,v) = 60n-1$ is impossible (mod 3). Hence $r_{Q_j}(60n-1) = 0$ for $j > 1$, so that for n odd,

$$g(n) \equiv \frac{1}{4} \sum_{j=1}^{4} r_{Q_j}(60n-1) \pmod 2.$$

By the classical theory of quadratic forms,

$$\sum_{j=1}^{4} r_{Q_j}(N) = 2 \sum_{d|N} \left(\frac{-120}{N} \right) = 2 \sum_{d|N} \left(\frac{-30}{N} \right),$$

where $\left(\frac{a}{b}\right)$ is the Kronecker symbol. Hence for n odd, we have

$$g(n) \equiv \frac{1}{2} \sum_{d|60n-1} \left(\frac{-30}{d} \right) \pmod 2.$$

If the prime factorization of $60n-1$ is

$$60n-1 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t},$$

then the multiplicativity of the Kronecker symbol implies that

$$\sum_{d|60n-1} \left(\frac{-30}{d} \right) = \prod_{i=1}^{t} \sum_{d_i|p_i^{\alpha_i}} \left(\frac{-30}{d_i} \right).$$

The inner sum is even or odd according as α_i is odd or even. Therefore if at least two of the α_i are odd, we have $g(n) \equiv 0 \pmod 2$. Since $60n-1 \equiv -1 \pmod 3$, it is not a square. Hence $t-1$ of the exponents α_i are even, and one of them is odd. Thus for n odd, $g(n) \equiv 1 \pmod 2$ implies that $60n-1 = p^{\alpha} m^2$, where p is a prime not dividing m, and α is odd. Let $\alpha = 2\beta+1$ with $\beta \geq 0$. Then $p^{2\beta+1}m^2 = 60n-1 = 120\nu+59$ since n is odd. Since p and m are odd, we have $p \equiv 3 \pmod 8$ and $p \equiv 2 \pmod 3$. Moreover, $p \equiv \pm 1 \pmod 5$, so $(\frac{-2}{p}) = (\frac{3}{p}) = (\frac{5}{p}) = 1$. This yields $(\frac{-30}{p}) = 1$ and therefore

$$\frac{1}{2}\sum_{\alpha|p^\alpha}\left(\frac{-30}{d}\right) = \frac{1}{2}(\alpha+1).$$

This is odd if and only if $\alpha = 4a+1$, and the proof is complete.

3 The Parity of *h(n)* for *n* Even

Theorem 2. *Suppose* $n \equiv 0 \pmod 2$. *Then* $h(n) \equiv 1 \pmod 2$ *if and only if* $60n+11 = p^{4a+1}m^2$, *where p is a prime not dividing m.*

For example,

$$\begin{array}{ll} h(0)=1 & 11 = \text{prime} \\ h(2)=1 & 131 = \text{prime} \\ h(4)=1 & 251 = \text{prime} \\ h(6)=2 & 371 = 7\cdot 53 \end{array}$$

Proof of Theorem 2. The proof resembles that of Theorem 1. We have

$$\begin{aligned} H(x) &= \prod_{k=1}^{\infty}\frac{1}{(1-x^{5k-3})(1-x^{5k-2})}\cdot\frac{(1-x^{5k-4})(1-x^{5k-1})(1-x^{5k})}{(1-x^{5k-4})(1-x^{5k-1})(1-x^{5k})} \\ &= \prod_{k=1}^{\infty}\frac{(1-x^{5k-4})(1-x^{5k-1})(1-x^{5k})}{1-x^k}. \end{aligned}$$

Since

$$\prod_{k=1}^{\infty}\frac{(1+x^{5k-4})(1+x^{5k-1})(1+x^{5k-3})(1+x^{5k-2})(1+x^{5k})}{1+x^k} = 1,$$

we have

$$H(x) = \prod_{k=1}^{\infty}\frac{(1-x^{10k-8})(1-x^{10k-2})(1+x^{5k-3})(1+x^{5k-2})(1-x^{10k})}{1-x^{2k}}. \tag{8}$$

The product of the last three factors in the numerator of (8) can be written as

$$\prod_{k=1}^{\infty}(1+x^{10k-8})(1+x^{10k-2})(1+x^{10k-7})(1+x^{10k-3})(1-x^{10k}).$$

Hence

$$H(x)=\prod_{k=1}^{\infty}\frac{(1-x^{20k-16})(1-x^{20k-4})(1+x^{10k-7})(1+x^{10k-3})(1-x^{10k})}{1-x^{2k}}. \tag{9}$$

By the Jacobi identity (5) with $q=x^5,\ t=x^{-2}$, we have

$$\prod_{k=1}^{\infty}(1-x^{10k})(1+x^{10k-7})(1+x^{10k-3})=\sum_{r=-\infty}^{\infty}x^{5r^2-2r}. \tag{10}$$

The exponent $5r^2-2r$ is even or odd according as $r=2s$ or $r=2s+1$. Hence the terms of (10) with even exponent have sum

$$\sum_{s=-\infty}^{\infty}x^{5(2s)^2-2(2s)}=x\sum_{s=-\infty}^{\infty}x^{20s^2-4s}.$$

By Jacobi's identity in reverse, this is equal to

$$\prod_{k=1}^{\infty}(1-x^{40k})(1+x^{40k-24})(1+x^{40k-16})$$
$$\equiv\prod_{k=1}^{\infty}(1-x^{20k})^2(1-x^{20k-12})^2(1-x^{20k-8})^2 \pmod 2.$$

Substituting this into (9) and using the fact that

$$\prod_{k=1}^{\infty}(1-x^{20k})(1-x^{20k-16})(1-x^{20k-12})(1-x^{20k-8})(1-x^{20k-4})=\prod_{k=1}^{\infty}(1-x^{4k}),$$

we obtain

$$\sum_{n\text{ even}}h(n)x^n\equiv x\prod_{k=1}^{\infty}\frac{(1-x^{20k})(1-x^{20k-12})(1-x^{20k-8})(1-x^{4k})}{1-x^{2k}} \pmod 2.$$

By Jacobi's identity applied to

$$\prod_{k=1}^{\infty}(1-x^{20k})(1+x^{20k-12})(1-x^{20k-8})$$

and Euler's identity applied to

$$\prod_{k=1}^{\infty} \frac{1-x^{4k}}{1-x^{2k}} \equiv \prod_{k=1}^{\infty} (1-x^{2k}) \pmod{2},$$

we have

$$\sum_{n \text{ even}} h(n)x^n \equiv \sum_{r=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} x^{3r^2+r+10s^2+2s} \pmod{2}. \tag{11}$$

Replacing x by x^{60} in (11) and then multiplying both sides by x^{11}, we obtain

$$\begin{aligned}\sum_{n \text{ even}} h(n)x^{60n+11} &\equiv \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} x^{180r^2+60r+600s^2+120s+11} \pmod{2} \\ &= \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} x^{5(6r+1)^2+6(10s+1)^2}.\end{aligned}$$

If $Q_1(u,v) = 5u^2+6v^2 = 60n+11$, then $u^2 \equiv 1 \pmod{6}$, so $u = \pm(6r+1)$. Thus $5u^2 \equiv 1 \pmod{4}$, so v is odd. Since $v^2 \equiv 6v^2 \equiv 11 \pmod{5}$, it follows that $v \equiv \pm(10s+1)$. Therefore $\frac{1}{4}$ of the solutions of $5u^2+6v^2 = 60n+11$ are of the form $u = 6r+1,\ v = 10s+1$. The rest of the proof is identical with the corresponding part of the proof of Theorem 1.

4 Linear Zero Congruences (mod 2) for *g(n)* and *h(n)*

Let $c(n)$ be an integer-valued arithmetical function. A *linear zero congruence* (LZC) for $c(n)$ is one of the form

$$c(n) \equiv 0 \pmod{m} \text{ for all } n \text{ with } an \equiv b \pmod{M}.$$

Familiar examples are the Ramanujan congruences

$$p(n) \equiv 0 \pmod{m} \text{ when } 24n \equiv 1 \pmod{m},$$

where $p(n)$ is the partition function and $m = 5$, 7, or 11. In this section we present three examples of LZCs (mod 2) for $g(n)$ and $h(n)$ which follow from Theorems 1 and 2.

Example 1. Let the prime factorization of $60n-1$ be $60n-1 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$. By Theorem 1, if both n and $g(n)$ are odd, then one of the α_i is $\equiv 1 \pmod{4}$ and the others are all even. Therefore if n is odd and $\alpha_i \equiv 3 \pmod{4}$ for some i, then $g(n) \equiv 0$

(mod 2). This situation is achieved if $n \equiv 1 \pmod 2$ and $60n - 1 \equiv cp^3 \pmod{p^4}$, where $p > 5$ is a prime and $c \not\equiv 0 \pmod p$. The two conditions on n are equivalent to a single congruence of the form $an \equiv b \pmod{2p^4}$.

Similarly, Theorem 2 implies that $h(n) \equiv 0 \pmod 2$ if $n \equiv 0 \pmod 2$ and $60n + 11 \equiv cp^3 \pmod{p^4}$, where p and c are as above.

Example 2. It follows from Theorem 1 that if n and $g(n)$ are both odd, then $60n - 1 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where exactly one α_i is odd. Therefore if $n \equiv 1 \pmod 2$ and two of the α_i are odd, then $g(n) \equiv 0 \pmod 2$. This situation occurs if $n \equiv 1 \pmod 2$ and $60n - 1 \equiv c_1 p \pmod{p^2}$, $60n - 1 \equiv c_2 q \pmod{q^2}$, where $p \neq q$ are primes > 5, with $c_1 \not\equiv 0 \pmod p$ and $c_2 \not\equiv 0 \pmod q$.

Similarly, Theorem 2 implies that $h(n) \equiv 0 \pmod 2$ if $n \equiv 0 \pmod 2$ and $60n + 11 \equiv cpq \pmod{p^2 q^2}$, where p, q, c are as above.

Example 3. It follows from Theorem 1 that if n and $g(n)$ are both odd, then $60n - 1 = p^{4a+1} m^2$, where p is prime. Let $p > 5$ be a prime with $p \not\equiv 3 \pmod 8$, and suppose that $n \equiv 1 \pmod 2$ and $60n - 1 \equiv cp \pmod{p^2}$ with $c \not\equiv 0 \pmod p$. Then $p \parallel 60n - 1$, and since n is odd, we have $60n - 1 \equiv 3 \pmod 8$. Since $p \not\equiv 3 \pmod 8$, it follows that $\frac{60n-1}{p} \not\equiv 1 \pmod 8$, and therefore $\frac{60n-1}{p}$ is not a square. Hence $g(n) \equiv 0 \pmod 2$.

Alternatively, p can be chosen with $\left(\frac{p}{3}\right) = 1$ or with $\left(\frac{5}{p}\right) = -1$.

The same reasoning, using Theorem 2, shows that if $n \equiv 0 \pmod 2$ and $n \equiv cp \pmod{p^2}$ with p and c as above, then $h(n) \equiv 0 \pmod 2$.

5 Arithmetic Densities

Theorem 3. *Let $\gamma(N)$ be the number of odd integers n with $1 \leq n \leq N$ and $g(n) \equiv 1$ (mod 2), and let $\delta(N)$ be the number of even n with $0 \leq n \leq N$ and $h(n) \equiv 1$ (mod 2). Then there are positive constants A and B such that*

$$A \frac{N}{\log N} < \gamma(N),\ \delta(N) < B \frac{N}{\log N}$$

for sufficiently large N.

Proof of Theorem 3. By Theorem 1, $g(n) \equiv 1 \pmod 2$ if $60n - 1$ is prime. By Chebychev's inequality for primes in an arithmetic progression, the number of such primes $\leq 60N - 1$ exceeds

$$A \frac{60N - 1}{\log(60N - 1)} > \frac{AN}{\log N}$$

for some positive constant A and all large enough N. This implies in particular that $\gamma(N) > \frac{AN}{\log N}$ for large enough N. The proof for $\delta(N)$ is similar.

We now obtain the upper bound $\gamma(N) < B\,\frac{N}{\log N}$ for large N; a bound of the form $\delta(N) < B\,\frac{N}{\log N}$ for large N can be similarly obtained.

Put $T = 60N - 1$ and define

$$\begin{aligned} R &= \{n \le N \mid 60n-1 = rm^2,\ \ r \text{ prime}\} \\ S_\alpha &= \{n \le N \mid 60n-1 = s^\alpha m^2\}. \end{aligned}$$

Here s is not required to be prime; r and s may divide m. Since the exponent $4a+1 = \alpha$ in Theorem 1 is either 1 or ≥ 5, the theorem gives the estimate

$$\gamma(N) \le |R| + \sum_{\substack{\alpha \ge 5 \\ 2^\alpha \le T}} |S_\alpha| \,.$$

If $\pi(x)$ is the number of primes $\le x$, we have

$$\gamma(N) \le \sum_{m \ge 1} \pi\left(\frac{T}{m^2}\right) + \sum_{\alpha=5}^{\lfloor \frac{\log T}{\log 2} \rfloor} \sum_{m^2 \le T} \left(\frac{T}{m^2}\right)^{\frac{1}{\alpha}} . \tag{12}$$

Now

$$m^{-\beta} < \int_{m-1}^{m} x^{-\beta} \mathrm{d}x \ \text{ for } \ m \ge 2 \ \text{ and } \ \beta > 0, \ \text{ so}$$

$$\sum_{1 \le m^2 \le T} m^{-\beta} \ < \ 1 + \int_1^{T^{\frac{1}{2}}} x^{-\beta} \mathrm{d}x \ = \ 1 + \frac{x^{1-\beta}}{1-\beta}\,\Bigg|_1^{T^{\frac{1}{2}}} = 1 \ + \ \frac{T^{(1-\beta)/2}}{1-\beta} - \frac{1}{1-\beta}.$$

Hence

$$\sum_{m^2 \le T} \left(\frac{T}{m^2}\right)^{\frac{1}{\alpha}} \ \le \ T^{\frac{1}{\alpha}} \left(1 \ + \ \frac{T^{\frac{1-2/\alpha}{2}}}{1-\frac{2}{\alpha}} \ - \ \frac{1}{1-\frac{2}{\alpha}}\right).$$

Next,

$$\begin{aligned} 1 - \frac{1}{1-\frac{2}{\alpha}} \ &= \ 1 - \frac{\alpha}{\alpha-2} \ < \ 0 \ \ \text{ and} \\ \frac{1}{1-\frac{2}{\alpha}} \ &= \ \frac{\alpha}{\alpha-2} \ < \ 2 \quad \text{for } \alpha \ge 5, \text{ so} \end{aligned}$$

$$\sum_{m^2 \le T} \left(\frac{T}{m^2}\right)^{\frac{1}{\alpha}} \ < \ T^{\frac{1}{\alpha}} \cdot 2T^{\left(\frac{1}{2}-\frac{1}{\alpha}\right)} \ = \ 2T^{\frac{1}{2}}.$$

Therefore the second term of (12) satisfies

$$\sum_{\alpha=5}^{\lfloor \frac{\log T}{\log 2} \rfloor} \sum_{m^2 \le T} \left(\frac{T}{m^2}\right)^{\frac{1}{\alpha}} = O(T^{\frac{1}{2}} \log T) \quad \text{as } T \longrightarrow \infty. \tag{13}$$

To estimate the other term $\sum_{m \ge 1} \pi\left(\frac{T}{m^2}\right)$ in (12) we make use of Chebychev's inequality for $\pi(x)$. First take $T \ge 4$, so that $T^{\frac{1}{2}} \le \frac{1}{2}$. There is a positive constant C so that

$$\pi\left(\frac{T}{m^2}\right) < C\left(\frac{T}{m^2}\right) \Big/ \log\left(\frac{T}{m^2}\right) < \frac{2C}{m^2} \cdot \frac{T}{\log T}$$

for $1 \le m \le T^{\frac{1}{4}}$. Since $\pi(x) < x$ for all $x > 0$, we have

$$\begin{aligned}\sum_{m \ge 1} \pi\left(\frac{T}{m^2}\right) &< \frac{2CT}{\log T} \sum_{m \ge T^{\frac{1}{4}}} m^{-2} + T \sum_{T^{\frac{1}{4}} < m < \left(\frac{T}{2}\right)^{\frac{1}{2}}} m^{-2} \\ &= O\left(\frac{T}{\log T}\right) + T \cdot O(T^{-\frac{1}{4}}).\end{aligned} \tag{14}$$

Substituting (13) and (14) into (12), we find that

$$\begin{aligned}\gamma(N) &= O(T^{\frac{1}{2}} \log T) + O\left(\frac{T}{\log T}\right) + O(T^{\frac{3}{4}}) \\ &= O\left(\frac{T}{\log T}\right) = O\left(\frac{N}{\log N}\right),\end{aligned}$$

since $T = 60N - 1$. This completes the proof.

Theorem 3 immediately implies the following.

Theorem 4. *In the sense of arithmetic density,*

(i) $g(n)$ *is almost always even for odd* n
(ii) $h(n)$ *is almost always even for even* n
(iii) $g(n)h(n)$ *is almost always even*

Reference

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A Survey of Classical Mock Theta Functions

Basil Gordon and Richard J. McIntosh[1]

Dedicated to George E. Andrews

Abstract In his last letter to Hardy, Ramanujan defined 17 functions $M(q)$, $|q| < 1$, which he called mock θ-functions. He observed that as q radially approaches any root of unity ζ at which $M(q)$ has an exponential singularity, there is a θ-function $T_\zeta(q)$ with $M(q) - T_\zeta(q) = O(1)$. Since then, other functions have been found which possess this property. We list various linear relations between these functions and develop their transformation laws under the modular group. We show that each mock θ-function is related to a member of a universal family (mock θ-conjectures). In recent years the subject has received new impetus and importance through a strong connection with the theory of Maass forms. The final section of this survey provides some brief remarks concerning these new developments.

Keywords Mock theta functions • q-series • Modular forms

Mathematics Subject Classification: Primary: 33D15, 11F27, 11F37

[1] Support of the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

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K. Alladi and F. Garvan (eds.), *Partitions, q-Series, and Modular Forms*, Developments in Mathematics 23, DOI 10.1007/978-1-4614-0028-8_9,

1 Introduction

We begin with the partition generating function $P(q) = (q)_\infty^{-1}$, where as usual

$$(q)_0 = 1\,, \quad (q)_n = \prod_{m=1}^{n}(1-q^m), \quad \text{and} \quad (q)_\infty = \prod_{m=1}^{\infty}(1-q^m)\,, \quad |q| < 1.$$

More generally, we put

$$(a\,;q^k)_0 = 1\,, \quad (a\,;q^k)_n = \prod_{m=0}^{n-1}(1-aq^{mk}), \quad \text{and} \quad (a\,;q^k)_\infty = \prod_{m=0}^{\infty}(1-aq^{mk}),$$

so that $(q)_n = (q\,;q)_n$ and $(q)_\infty = (q\,;q)_\infty$. We have

$$(a\,;q^k)_n = \frac{(a\,;q^k)_\infty}{(aq^{nk};q^k)_\infty}$$

for $n \geq 0$, and for other real n, we take this as the definition of $(a\,;q^k)_n$.

$P(q)$ satisfies the Euler and Durfee identities

$$P(q) = \sum_{n=0}^{\infty}\frac{q^n}{(q)_n} = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n^2}. \tag{1.1}$$

These express $P(q)$ in what S. Ramanujan, in his last letter to Hardy [R1, 354–355], [R2, 127–131], [W1, 56–61], called *transformed Eulerian form*. Other examples are provided by the Rogers-Ramanujan identities

$$\left.\begin{aligned} G(q) &= \prod_{m=1}^{\infty}\frac{1}{(1-q^{5m-4})(1-q^{5m-1})} = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n}, \\ H(q) &= \prod_{m=1}^{\infty}\frac{1}{(1-q^{5m-3})(1-q^{5m-2})} = \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q)_n}. \end{aligned}\right\} \tag{1.2}$$

In his letter, Ramanujan remarked that as q tends radially to exponential singularities at roots of unity, the functions $P(q)$ and $G(q)$ have asymptotic approximations involving "closed exponential factors." To describe these approximations, he introduced a complex variable α with $\mathrm{Re}(\alpha) > 0$ and put $q = \mathrm{e}^{-\alpha}$. Then, for example, if α is real and $\alpha \to 0^+$, we have

$$\left.\begin{aligned} P(q) &= \sqrt{\frac{\alpha}{2\pi}}\exp\left(\frac{\pi^2}{6\alpha} - \frac{\alpha}{24}\right) + o(1), \\ G(q) &= \sqrt{\frac{2}{5-\sqrt{5}}}\exp\left(\frac{\pi^2}{15\alpha} - \frac{\alpha}{60}\right) + o(1), \end{aligned}\right\} \tag{1.3}$$

with similar approximations near exponential singularities at other roots of unity. Ramanujan noted that for other q-series in Eulerian form, approximations analogous to (1.3) may or may not exist. He stated that if

$$F(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)}}{(q)_n^2}$$

and $q = \mathrm{e}^{-\alpha}$ with $\alpha \to 0^+$, then for each positive integer p we have

$$F(q) = \sqrt{\frac{\alpha}{2\pi\sqrt{5}}} \exp\left(\frac{\pi^2}{5\alpha} + \frac{\alpha}{8\sqrt{5}} + c_2\alpha^2 + \cdots + c_p\alpha^p + O\left(\alpha^{p+1}\right)\right) \tag{1.4}$$

with infinitely many $c_j \neq 0$. Ramanujan said that in this case "the exponential factor does not close," but an actual proof has not yet been found. An example of a q-series in Eulerian form having an approximation with an unclosed exponential factor is given by

$$\sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)+rn}}{(q)_n} = \prod_{m=0}^{\infty} (1-q^{m+r}),$$

where $0 < r < 1$, $r \neq \frac{1}{2}$. (To obtain a function holomorphic for $|q| < 1$, take $r = \frac{a}{b}$ rational and replace q by q^b.) A proof is given in [M4].

At this point we need to clarify what Ramanujan meant by a θ-function. For this purpose, we recall the definition of the Jacobi triple product

$$j(x,q) = (x;q)_\infty \left(x^{-1}q;q\right)_\infty (q;q)_\infty \tag{1.5}$$

and the identity

$$j(x,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} x^n.$$

Following Hickerson [Hi1], we define a θ-*product* to be an expression of the form

$$Cq^e x_1^{f_1} \cdots x_r^{f_r} L_1^{g_1} \cdots L_s^{g_s},$$

where C is a complex number, e, f_i, g_j are integers, and each L_j has the form

$$j(Dq^h x_1^{k_1} \cdots x_r^{k_r}, \pm q^m)$$

for some complex number D (usually $D = \pm 1$) and integers h, k_i, m with $m \geq 1$. A θ-*function* is a finite sum of θ-products. Thus $(q)_\infty = j(q,q^3)$ is a θ-function, even though it lacks the factor $q^{\frac{1}{24}}$ needed to make it a modular form.

In this survey, θ-functions are typically denoted by $T(x_1,x_2,\ldots,x_r,q)$, or simply by $T(q)$ if $r = 0$.

Every θ-function with an exponential singularity at a root of unity ζ has an asymptotic approximation under radial approach to ζ consisting of one or more terms like (1.3), each with $c_j = 0$ for all $j \geq 2$, and where $o(1)$ may be $O(1)$. As an example of an approximation with more than one term, Ramanujan gave $(q)_\infty^{-120}$. A simpler example is provided by $(q)_\infty^{-48}$. By the functional equation of the Dedekind η-function (see for example [Ap, 48]), we have

$$q^{\frac{1}{24}}(q)_\infty = \sqrt{\frac{2\pi}{\alpha}}\, q_1^{\frac{1}{6}}(q_1^4;q_1^4)_\infty,$$

where $q = \mathrm{e}^{-\alpha}$ and $q_1 = \mathrm{e}^{-\frac{\pi^2}{\alpha}}$. Hence as $\alpha \to 0^+$,

$$\begin{aligned}
(q)_\infty^{-48} &= q^2[q^{\frac{1}{24}}(q)_\infty]^{-48} \\
&= q^2\left[\sqrt{\frac{2\pi}{\alpha}}\, q_1^{\frac{1}{6}}(q_1^4;q_1^4)_\infty\right]^{-48} \\
&= \frac{\alpha^{24}}{(2\pi)^{24}} q^2 q_1^{-8}[1 - q_1^4 + O(q_1^8)]^{-48} \\
&= \frac{\alpha^{24}}{(2\pi)^{24}} q^2 q_1^{-8}[1 + 48q_1^4 + O(q_1^8)] \\
&= \frac{\alpha^{24}}{(2\pi)^{24}} q^2 q_1^{-8} + \frac{48\alpha^{24}}{(2\pi)^{24}} q^2 q_1^{-4} + o(1) \\
&= \frac{\alpha^{24}}{(2\pi)^{24}} \exp\left(\frac{8\pi^2}{\alpha} - 2\alpha\right) + \frac{48\alpha^{24}}{(2\pi)^{24}} \exp\left(\frac{4\pi^2}{\alpha} - 2\alpha\right) + o(1).
\end{aligned}$$

A *mock θ-function* is a function $M(q)$, holomorphic for $|q| < 1$, such that

(i) $M(q)$ has infinitely many exponential singularities at roots of unity
(ii) Under radial approach to every such singularity, $M(q)$ has an approximation consisting of a finite sum of terms with closed exponential factors, and an error term $O(1)$
(iii) There is no θ-function $T(q)$ which differs from $M(q)$ by a "trivial function", i.e., a function bounded under radial approach to every root of unity

If $L(q)$ satisfies (i)–(iii) and has an expansion

$$L(q) = \sum_{n=0}^{\infty} a_n q^{\frac{n}{c}},$$

convergent for $|q| < 1$, where c is a positive integer, then $M(q) = L(q^c)$ is a mock θ-function. Such a function $L(q)$ will be called a mock theta function in the wide sense. The theory of such functions is not incorporated in this survey; their transformation laws are found in [BO2] and [BOR]. In this survey, we do not require $M(q)$ to be an Eulerian q-series.

In his letter, Ramanujan next introduced the function

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} \tag{1.6}$$

and the θ-function

$$T(q) = \frac{(q)_\infty^3}{(q^2;q^2)_\infty^2}.$$

He stated that if ζ is a primitive νth root of unity and $q \to \zeta$ radially, then

$$f(q) = \begin{cases} O(1), & \nu \text{ odd}, \\ -T(q) + O(1), & \nu \equiv 2 \pmod 4, \\ T(q) + O(1), & \nu \equiv 0 \pmod 4). \end{cases}$$

Thus $M(q) = f(q)$ satisfies:

(iv) At each root of unity ζ, there is a θ-function $T_\zeta(q)$ such that $M(q) = T_\zeta(q) + O(1)$ as $q \to \zeta$ radially

This implies (ii). We call (iv) the *strong approximation property*. Andrews and Hickerson [AH] actually take (i), (iii), and (iv) as the definition of a mock θ-function. Such functions will be called *strong mock θ-functions*.

Ramanujan wrote "it is inconceivable that a single θ-function could be found to cut out the singularities of $f(q)$." Thus he suggested that $f(q)$ satisfies property (iii) and is therefore a strong mock θ-function. This remains an open conjecture. Henceforth when we speak of mock θ-functions, it is with the understanding that they have not yet been shown to possess property (iii).

From now on we will use the abbreviations θf and mf for θ-functions and mock θ-functions, respectively. The notation mf$_n$ stands for an mf of order n.

Ramanujan listed 17 mfs to which he assigned orders 3, 5, and 7. (The order is somewhat analogous to the level of a modular form.) Watson [W1] found three further mf$_3$s, and in constructing transformation laws for them, Gordon and McIntosh [GM2, 204] found two more. Still other mfs, to which orders 2, 6, 8, and 10 have been assigned, are discussed in [A2], [M3], [Hk2], [AH], [BC], [M8], [GM1], and [C1]–[C4].

The work described above belongs to the era that we have chosen to call "classical." Our aim has been to provide a comprehensive description of the results achieved by classical means. Detailed calculations have been avoided where possible.

Around the year 2000 the theory of mock θ-functions took a new turn with the work of Zagier [Za1], Zwegers [Zw1]–[Zw3], Bringmann–Ono [BO1]–[BO4], and others, who found important connections with Maass forms (Sect. 9 provides more details). These developments are described in a forthcoming paper by Ono [On].

2 The Watson–Selberg Era

The above title was first used by Andrews [A4] in discussing the groundbreaking work on mfs done in the 1930s. This work dealt with the 17 functions of orders 3, 5, and 7 defined in Ramanujan's letter. They are the following:

Order 3

$$\left.\begin{aligned} f(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}, & \phi(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}, \\ \psi(q) &= \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}, & \chi(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q)_n}{(-q^3;q^3)_n}, \end{aligned}\right\} \tag{2.1}$$

Order 5

$$\left.\begin{aligned} f_0(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n}, & f_1(q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q)_n}, \\ F_0(q) &= \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n}, & F_1(q) &= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}}, \\ \phi_0(q) &= \sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_n, & \phi_1(q) &= \sum_{n=0}^{\infty} q^{(n+1)^2}(-q;q^2)_n, \\ \psi_0(q) &= \sum_{n=0}^{\infty} q^{\frac{1}{2}(n+1)(n+2)}(-q;q)_n, & \psi_1(q) &= \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)}(-q;q)_n, \\ \chi_0(q) &= \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_n}, & \chi_1(q) &= \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_{n+1}}, \end{aligned}\right\} \tag{2.2}$$

Order 7

$$\mathcal{F}_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1};q)_n}, \quad \mathcal{F}_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q^{n+1};q)_{n+1}}, \quad \mathcal{F}_2(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^{n+1};q)_{n+1}}. \tag{2.3}$$

Watson wrote two papers [W1] and [W2] dealing with (2.1) and (2.2), respectively, while Selberg [Se1], [Se2] dealt with (2.3). In [W1], Watson showed that Ramanujan's functions (2.1), together with the three further ones

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}, \quad \upsilon(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}, \quad \rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(q;q^2)_{n+1}}{(q^3;q^6)_{n+1}}, \tag{2.4}$$

have the strong property (iv). This was accomplished by obtaining modular transformation laws for all but $\chi(q)$ and $\rho(q)$. The laws for $\chi(q)$ and $\rho(q)$, proved in [GM2, 204], involve two more mf$_3$s:

$$\xi(q) = 1 + 2\sum_{n=1}^{\infty} \frac{q^{6n(n-1)}}{(q;q^6)_n(q^5;q^6)_n}, \qquad \sigma(q) = \sum_{n=1}^{\infty} \frac{q^{3n(n-1)}}{(-q;q^3)_n(-q^2;q^3)_n}. \tag{2.5}$$

A detailed account of transformation theory and a list of the transformation laws for the mf$_3$s are given in Sect. 4.

Watson also proved a number of linear relations connecting the functions (2.1), such as

$$4\chi(q) - f(q) = 3\theta_4^2(0,q^3)(q)_\infty^{-1}, \tag{2.6}$$

where

$$\theta_4(0,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = j(q,q^2) = \frac{(q)_\infty^2}{(q^2;q^2)_\infty} = (q)_\infty(q;q^2)_\infty. \tag{2.7}$$

In [W2], Watson went on to consider the ten 5th order functions (2.2). He was unable to obtain transformation laws for these, so proceeded differently. He first proved a number of linear relations stated by Ramanujan:

$$\left.\begin{aligned} \phi_0(-q) + \chi_0(q) &= 2F_0(q), \\ f_0(-q) + 2F_0(q^2) - 2 &= \phi_0(-q^2) + \psi_0(-q) \\ &= 2\phi_0(-q^2) - f_0(q) \\ &= \theta_4(0,q)G(q), \\ \psi_0(q) - F_0(q^2) + 1 &= q\psi(q^2)H(q^4), \end{aligned}\right\}, \tag{2.8}$$

where $\psi(q)$ is the Gauss function

$$\psi(q) = \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)} = \frac{1}{2}j(-q,q) = j(-q,q^4) = \frac{(q^2;q^2)_\infty^2}{(q)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{2.9}$$

Watson also found and proved similar relations connecting $f_1(q)$, $\phi_1(q)$, $\psi_1(q)$, $F_1(q)$, $\chi_1(q)$, not stated by Ramanujan. Each relation has three terms, one of which is a θf, while the other two are of the form $cq^r\mu(\pm q^k)$, with functions $\mu(q)$ appearing in (2.2). Next Watson determined, directly from their definitions, which of the functions (2.2) are bounded under radial approach to certain roots of unity. Using the linear relations he then obtained strong approximations for all the functions (2.2) with singularities at these and other roots of unity.

A striking consequence of (2.8) and other identities found by Watson is that the bilateral sums associated with some mf$_5$s are actually θfs. For example, the bilateral analogue of

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n}$$

is

$$\sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{(-q;q)_n} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n} + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q;q)_{-n}} = f_0(q) + 2\psi_0(q),$$

and this turns out to be the θf

$$\frac{1}{2}\left[\theta_3(0,q)G(-q) + \theta_4(0,q)G(q)\right] + 3q\psi\left(q^2\right)H\left(q^4\right).$$

Here

$$\theta_3(0,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = j\left(-q,q^2\right) = \left(-q;q^2\right)_\infty^2 \left(q^2;q^2\right)_\infty, \tag{2.10}$$

and $\psi(q)$ is the Gauss function. The bilateral analogues of $F_0(q)$ and $F_1(q)$ are

$$\sum_{n=-\infty}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n} + \sum_{n=1}^{\infty} \frac{q^{2n^2}}{(q;q^2)_{-n}} = F_0(q) + \phi_0(-q) - 1$$

and

$$\sum_{n=-\infty}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}} + \sum_{n=1}^{\infty} \frac{q^{2n(n-1)}}{(q;q^2)_{-n+1}} = F_1(q) - \frac{\phi_1(-q)}{q},$$

respectively. Using his identities, Watson [W2, 290] proved that

$$\left.\begin{aligned} F_0\left(q^2\right) + \phi_0\left(-q^2\right) - 1 &= \frac{1}{2}\left[\theta_3(0,q)G(-q) + \theta_4(0,q)G(q)\right], \\ q^2F_1\left(q^2\right) - \phi_1\left(-q^2\right) &= \frac{1}{2}q\left[\theta_3(0,q)H(-q) - \theta_4(0,q)H(q)\right]. \end{aligned}\right\} \tag{2.11}$$

(A misprint in the second identity of (2.11) has been corrected here.) Further relations between bilateral hypergeometric sums and mfs are found in [C5].

Finally, Watson related the mf$_5$s to the Lerch sums studied in [L1], [L2].

In [Se2], Selberg proved that Ramanujan's functions (2.3) are strong mfs. The transformation theory was not yet available, and in contrast to orders 3 and 5, there are no known linear relations between the $\mathcal{F}_i$. Thus a new approach was required. To deal with $\mathcal{F}_0$, for example, Selberg obtained an identity of the form

$$\mathcal{F}_0(q) = \mathcal{A}(q) + \mathcal{B}(q)\phi(q) + \mathcal{C}(q), \tag{2.12}$$

where $\mathcal{A}(q)$ and $\mathcal{B}(q)$ are θfs, and $\phi(q)$ is the 3rd order function listed in (2.1). He then proved that $\mathcal{C}(q)$ is bounded under radial approach to every root of unity ζ. Since $\phi(q)$ can be strongly approximated at ζ, (2.12) provides the required approximation to $\mathcal{F}_0(q)$ there. Similar identities for $\mathcal{F}_1(q)$ and $\mathcal{F}_2(q)$ show that they are also strong mfs.

3 The Andrews–Hickerson Era

The next major advances were made starting in the 1950s. As noted above, Watson's paper [W1] showed that the mf3s (2.1), (2.4) could be strongly approximated by θfs at every root of unity. This raised the possibility of applying the circle method to obtain convergent or asymptotic series expansions for the Taylor coefficients of these mfs. Such an asymptotic expansion for the partition function $p(n)$, whose generating function

$$\sum_{n=0}^{\infty} p(n)q^n = (q)_\infty^{-1}$$

is a θf, had been found earlier by Hardy and Ramanujan [R1]. Subsequently, Rademacher [Rd] improved their result by obtaining a convergent series expansion of $p(n)$. Work on mf3s was begun by Dragonette [D], who selected the function

$$f(q) = \sum_{n=0}^{\infty} a(n)q^n$$

of (2.1) for detailed study. Watson's paper [W1] gave only the transformation laws for $f(q)$ under the generators $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$ of the modular group $\Gamma(1) = \mathrm{PSL}_2(\mathbf{Z})$ (where $q = \mathrm{e}^{\pi i \tau}$), and Dragonette first needed to determine laws under all the transformations $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ of $\Gamma(1)$. After doing so, she used Cauchy's formula

$$a(n) = \frac{1}{2\pi i}\int_C \frac{f(q)}{q^{n+1}}\,\mathrm{d}q\,,$$

taking for C the circle $|q| = \mathrm{e}^{-\frac{\pi}{n}}$. The next step was to divide C into Farey arcs of order $N = \lfloor n^{\frac{1}{2}} \rfloor$. With the aid of the transformation laws, in each arc, $f(q)$ was replaced by another mf3 plus an "error term" (a Mordell integral [Mo]; a more detailed account of these integrals is given in Sect. 4). This error term was then estimated. Evaluation of the resulting integrals over the arcs with centers $\mathrm{e}^{-\frac{\pi}{n}+i\frac{\pi h}{k}}$ (k fixed) lead to an exponential sum $\lambda(k) = \lambda(k,n)$ involving some unevaluated roots of unity $\epsilon_{h,k}$. The final result was the expansion

$$a(n) = \sum_{k=1}^{\lfloor n^{\frac{1}{2}} \rfloor} \frac{\lambda(k)\exp\left(\pi(n-\frac{1}{24})^{\frac{1}{2}}/k\sqrt{6}\right)}{k^{\frac{1}{2}}(n-\frac{1}{24})^{\frac{1}{2}}} + O(n^{\frac{1}{2}}\log n)\,. \tag{3.1}$$

In [A1], Andrews made a substantial improvement in evaluating both the "error terms" and $\epsilon_{h,k}$. This enabled him to express $\lambda(k)$ in terms of the exponential sum $A_k(n)$ appearing in the Hardy-Ramanujan series for $p(n)$ [R1, 284–285].

The improved result is that for every $\epsilon > 0$, the term $O(n^{\frac{1}{2}}\log n)$ in (3.1) can be replaced by $O(n^{\epsilon})$, and that

$$\lambda(k) = \begin{cases} \frac{1}{2}(-1)^{\frac{1}{2}(k+1)}A_{2k}(n), & k \text{ odd}, \\ \frac{1}{2}(-1)^{\frac{1}{2}k}A_{2k}\Big(n-\frac{1}{2}k\Big), & k \text{ even}. \end{cases}$$

Andrews conjectured that if $\exp x$ is replaced by $2\sinh x$ in (3.1), and the resulting series is extended to infinity, it converges to $a(n)$. This was later proved by Bringmann and Ono [BO1]. They went on [BO3] to obtain convergent series for the coefficients of all harmonic weak Maass forms of weight $\leq \frac{1}{2}$. Here the circle method was replaced by considering the principal parts of the Fourier expansions of these forms at all cusps.

In 1976 Andrews discovered, in the mathematics library of Trinity College, Cambridge, a notebook written by Ramanujan toward the end of his life. This important work has come to be known as the Lost Notebook [R2]. In it, Ramanujan defined further mfs and stated linear relations between them. We will discuss this in Sect. 5, dealing with mfs of even order.

The Lost Notebook also lists ten identities satisfied by the mf_5s (2.2), which have come to be known as the Mock Theta Conjectures. Each of them involves an mf_5, a θf, and the function

$$g_3(x,q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(x;q)_{n+1}(x^{-1}q;q)_{n+1}}. \tag{3.2}$$

For later reference we remark that this series converges absolutely if $|q| < 1$ and x is neither 0 nor an integral power of q. The Watson-Whipple transformation of basic hypergeometric series [GR, 242, (III.17)] (see also [GM2, 196–198]) can be applied to show that

$$(q)_\infty g_3(x,q) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{3}{2}n(n+1)}}{1-xq^n}. \tag{3.3}$$

The Mock Theta Conjectures are

$$\left.\begin{aligned} f_0(q) &= -2q^2 g_3(q^2,q^{10}) + \theta_4(0,q^5)G(q), \\ F_0(q) - 1 &= q g_3(q,q^5) - q\psi(q^5)H(q^2), \\ \phi_0(-q) &= -q g_3(q,q^5) + j(-q^2,q^5)G(q^2), \\ \psi_0(q) &= q^2 g_3(q^2,q^{10}) + q j(q,q^{10})H(q), \\ \chi_0(q) - 2 &= 3q g_3(q,q^5) - j(q^2,q^5)G(q)^2, \end{aligned}\right\} \tag{3.4}$$

$$\left.\begin{aligned} f_1(q) &= -2q^3 g_3(q^4,q^{10}) + \theta_4(0,q^5)H(q)\,, \\ F_1(q) &= q g_3(q^2,q^5) + \psi(q^5)G(q^2)\,, \\ \phi_1(-q) &= q^2 g_3(q^2,q^5) - q j(-q,q^5)H(q^2)\,, \\ \psi_1(q) &= q^3 g_3(q^4,q^{10}) + j(q^3,q^{10})G(q)\,, \\ \chi_1(q) &= 3q g_3(q^2,q^5) + j(q,q^5)H(q)^2, \end{aligned}\right\}, \tag{3.5}$$

where $G(q)$, $H(q)$, $\theta_4(0,q)$, and $\psi(q)$ are the functions defined in (1.2), (2.7), and (2.9). These identities were proved by Hickerson [Hi1]. Before describing his groundbreaking work, we mention that it opened the floodgates to the discovery and proof of analogous identities involving other mfs. For convenience, we will nonetheless refer to these in the sequel as mock theta "conjectures."

The first step in proving (3.4) and (3.5) was taken by Andrews and Garvan [AG], who showed that the identities (3.4) are all equivalent, as are all of (3.5). This reduced the problem to proving the identities for $f_0(q)$ and $f_1(q)$.

In [He], Hecke obtained identities of the form

$$T(q) = \sum_{\substack{m,n=-\infty \\ Q(m,n)\geq 0}}^{\infty} q^{Q(m,n)+L(m,n)}\,, \tag{3.6}$$

where $T(q)$ is a θf, $Q(m,n)$ is an indefinite quadratic form, and $L(m,n)$ is a linear form. In studying characters of affine Lie algebras, Kac and Peterson [KP] found more such identities. These involve q-series related to modular forms. Later, Kac and Wakimoto [KW1], [KW2], [Wm] extended this investigation to Lie superalgebras. They encountered q-series which they thought might be similarly related. It was shown by Bringmann and Ono [BO4] that these series are in fact the holomorphic parts of nonholomorphic Maass forms of weight $k = 0$. (For definitions see Sect. 9.)

In [A3], Andrews generalized identities like (3.6) to one of the form

$$T(z,q) = \sum_{\substack{m,n=-\infty \\ Q(m,n)\geq 0}}^{\infty} (-1)^{c_{m,n}} q^{Q(m,n)+L(m,n)} z^n\,,$$

and in [A5] he applied his theory of Bailey chains to produce such identities for Ramanujan's mf$_5$s and mf$_7$s. In particular,

$$\left.\begin{aligned} (q)_\infty f_0(q) &= \sum_{\substack{m,n=-\infty \\ |m|\leq n}}^{\infty} (-1)^m q^{\frac{1}{2}n(5n+1)-m^2}(1-q^{4n+2})\,, \\ (q)_\infty f_1(q) &= \sum_{\substack{m,n=-\infty \\ |m|\leq n}}^{\infty} (-1)^m q^{\frac{1}{2}n(5n+3)-m^2}(1-q^{2n+1}). \end{aligned}\right\} \tag{3.7}$$

In [A7], Andrews showed that such identities could be used to express the mf5s $f_\nu(q)$, $F_\nu(q)$, $\phi_\nu(q)$, and $\psi_\nu(q)$ of (2.2) as constant terms in the z-Laurent expansions of θfs $T(z,q)$. In particular, if

$$\Theta(z,q) = (z;q)_\infty (z^{-1}q;q)_\infty\,, \tag{3.8}$$

then

$$f_0(q) = \text{ coefficient of } z^0 \text{ in } \frac{(q^3;q^3)_\infty (q^5;q^5)_\infty^2 \Theta(-zq^4,q^3)\Theta(z,q^5)\Theta(q4,q^5)}{\Theta(-z^{-1}q^2,q^5)\Theta(-zq^2,q^5)\Theta(-q^2,q^5)}, \tag{3.9}$$

and

$$f_1(q) = \text{ coefficient of } z^0 \text{ in } \frac{-q(q^3;q^3)_\infty (q^5;q^5)_\infty^2 \Theta(-zq^4,q^3)\Theta(z,q^5)\Theta(q^2,q^5)}{(q)_\infty \Theta(-z^{-1}q,q^5)\Theta(-zq,q^5)\Theta(-q,q^5)}. \tag{3.10}$$

The stage was now set for Hickerson [Hi1] to prove the Mock Theta Conjectures for $f_0(q)$ and $f_1(q)$. This is one of the fundamental results of the theory of mfs, and the proof is a tour de force. In this survey we will have to be content with highlighting some of the key steps.

(i) First of all, using (3.3), it is shown that $g_3(x,q)$ is the coefficient of z^0 in the z-Laurent expansion of

$$A(x,z,q) = \frac{(q)_\infty^3 j(xz,q) j(z,q^3)}{j(x,q) j(z,q)}\,.$$

(ii) Next, the auxiliary functions

$$L(z) = \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\frac{3}{2}r(r+1)} z^{3r+1} z^{r+1}}{1-q^{3r+1}z}$$

and

$$M(z) = \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\frac{3}{2}r(r+3)} z^{-3r-1} z^{-r-1}}{1-q^{3r+1}z^{-1}}$$

are introduced, and the identity

$$A(z,x,q) = j(x^3z,q^3) g_3(x,q) - L(z) - M(z)$$

is established.

(iii) The next step is to define the function

$$B(z,q) = \frac{z^2\left(q^2;q^2\right)_\infty j(-z,q)\, j\left(z,q^3\right)}{j\left(z,q^2\right)}$$

and prove that

$$B(z,q) = qf_0(q)\left[zj(q^6z^5,q^{30}) + z^4 j\left(q^{24}z^5,q^{30}\right)\right] + f_1(q)\left[z^2 j\left(q^{12}z^5,q^{30}\right)\right.$$
$$\left. + z^3 j\left(q^{18}z^5,q^{30}\right)\right] + L^*(z,q) + M^*(z,q),$$

where

$$L^*(z,q) = 2\sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{15r^2+15r+3} z^{5r+5} z^{r+1}}{1-q^{6r+2}z}$$

an

$$M^*(z,q) = 2\sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{15r^2+15r+3} z^{-5r} z^{r+1}}{1-q^{6r+2}z^{-1}}.$$

An important role in proving the above identities is played by [AS, Lemma 2].

(iv) Finally, a careful study is made of the function $B(z,q)$ and its z-Laurent expansion to establish the Mock Theta Conjectures for $f_0(q)$ and $f_1(q)$.

Key ingredients of the proof are the identities (3.7) and the technique for deriving (3.9) and (3.10) from them.

Folsom [F1] gave a proof of the Mock Theta Conjectures using Maass forms.

Hickerson [Hi2] went on to formulate and prove analogous identities for the mf_7s. These are

$$\left.\begin{aligned} \mathcal{F}_0(q) - 2 &= 2qg_3(q,q^7) - j(q^3,q^7)^2(q)_\infty^{-1}, \\ \mathcal{F}_1(q) &= 2q^2g_3(q^2,q^7) + qj(q,q^7)^2(q)_\infty^{-1}, \\ \mathcal{F}_2(q) &= 2q^2g_3(q^3,q^7) + j(q^2,q^7)^2(q)_\infty^{-1}. \end{aligned}\right\} \qquad (3.11)$$

In view of (3.4), (3.5), and (3.11), it is natural to ask if there are analogous expressions for the mf_3s $\kappa(q)$ listed in (2.1), (2.4), and (2.5). It turns out [GM4] that for each of them, either $\kappa(q)$ or $\kappa(-q)$ has the form $Aq^c g_3(q^a,q^b) + T(q)$, where a, b, c are nonnegative integers and $T(q)$ is a θf. More precisely,

$$\left.\begin{aligned}
f(-q) &= -4qg_3\left(q,q^4\right) + \frac{\left(q^2;q^2\right)_\infty^7}{(q)_\infty^3\left(q^4;q^4\right)_\infty^3}\,,\\
\phi(q) &= -2qg_3\left(q,q^4\right) + \frac{\left(q^2;q^2\right)_\infty^7}{(q)_\infty^3\left(q^4;q^4\right)_\infty^3}\,,\\
\psi(q) &= qg_3\left(q,q^4\right)\,,\\
\chi(-q) &= -qg_3\left(q,q^4\right) + \frac{\left(q^4;q^4\right)_\infty^3\left(q^6;q^6\right)_\infty^3}{\left(q^2;q^2\right)_\infty^2\left(q^3;q^3\right)_\infty\left(q^{12};q^{12}\right)_\infty^2}\,,\\
\omega(q) &= g_3\left(q,q^2\right)\,,\\
\upsilon(q) &= -qg_3\left(q^2,q^4\right) + \frac{\left(q^4;q^4\right)_\infty^3}{\left(q^2;q^2\right)_\infty^2}\,,\\
\rho(q) &= -\frac{g_3\left(q,q^2\right)}{2} + \frac{3\left(q^6;q^6\right)_\infty^4}{2\left(q^2;q^2\right)_\infty\left(q^3;q^3\right)_\infty^2}\,,\\
\xi(q) &= 1+2qg_3\left(q,q^6\right)\\
&= q^2g_3\left(q^3,q^6\right) + \frac{\left(q^2;q^2\right)_\infty^4}{(q)_\infty^2\left(q^6;q^6\right)_\infty}\,,\\
\sigma(-q) &= q^2g_3\left(q^3,q^{12}\right) + \frac{\left(q^2;q^2\right)_\infty^3\left(q^{12};q^{12}\right)_\infty^3}{(q)_\infty\left(q^4;q^4\right)_\infty^2\left(q^6;q^6\right)_\infty^2}\,.
\end{aligned}\right\} \tag{3.12}$$

These identities can be viewed as 3rd order mock theta "conjectures."

4 Transformation Theory

In discussing the approximation of mfs near roots of unity, we have adhered to the notation $q = \mathrm{e}^{-\alpha}$, employed by Ramanujan and his early successors. This maps the right half-plane $\mathrm{Re}(\alpha) > 0$ onto the punctured disc $0 < |q| < 1$. In the classical theory of θfs, as expounded for example in [TM] and [WW], it is customary to write instead $q = \mathrm{e}^{\pi i \tau}$ with $\mathrm{Im}(\tau) > 0$. Thus $\alpha = -\pi i \tau$. Starting around 1950, it became increasingly common to write instead $q = \mathrm{e}^{2\pi i \tau}$. We have retained the original notation because it is more appropriate when discussing θfs. The reader should have little difficulty with this, since we are primarily using Ramanujan's α and β. In Sect. 9 we will use the more modern notation $q = \mathrm{e}^{2\pi i \tau}$.

The variable τ is subjected to the transformations

$$\tau \mapsto A\tau = \frac{a\tau + b}{c\tau + d},$$

where a, b, c, d are integers with $ad - bc = 1$. These transformations form the modular group $\Gamma(1)$; it is generated by

$$T\tau = \tau + 1, \qquad S\tau = -1/\tau = \tau_1 .$$

These generators map $q = \mathrm{e}^{\pi i\tau}$ to $-q$ and to $q_1 = \mathrm{e}^{\pi i\tau_1} = \mathrm{e}^{-\pi i/\tau}$, respectively. Equivalently, $q_1 = \mathrm{e}^{-\beta}$, where $\alpha\beta = \pi^2$.

In connection with θfs and mfs, it is appropriate to consider the subgroup

$$\Gamma_\theta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : b \equiv c (\mathrm{mod}\ 2) \right\}.$$

This subgroup is generated by S and T^2. Under the action of Γ_θ, the functions $f(q)$ and $f(-q)$ are inequivalent. Therefore to construct a complete transformation theory it is necessary to obtain the transforms of $f(q)$ and $f(-q)$ under S.

When a function $F(q)$ is being considered as a function of τ, it is customary to denote it by $F(\tau)$. The transformation laws for an mf $M(\tau)$ express $M(A\tau)$ (where $A \in \Gamma(1)$) in terms of another mf $M^*(\tau)$ and a Mordell integral. In this survey the term Mordell integral is extended to include linear combinations of the original integrals $\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ax^2+bx}}{\mathrm{e}^{cx}+d}\,\mathrm{d}x$ and of those arising from them by changes of variable. Since Watson's fundamental paper [W1], it has become standard to write two laws for each $M(q)$. One of these expresses $M(q)$ in terms of $M^*(\pm q_1^r)$ (where r is a rational number) and a Mordell integral; the other does the same for $M(-q)$. For example, the transformation laws for the mf$_3$ $f(q)$ in (2.1) are given by

$$\left.\begin{aligned} q^{-\frac{1}{24}} f(q) &= \sqrt{\frac{8\pi}{\alpha}}\, q_1^{\frac{4}{3}}\, \omega(q_1^2) + \sqrt{\frac{24\alpha}{\pi}} \int_0^\infty \mathrm{e}^{-\frac{3}{2}\alpha x^2} \frac{\sinh \alpha x}{\sinh \frac{3}{2}\alpha x}\,\mathrm{d}x, \\ q^{-\frac{1}{24}} f(-q) &= -\sqrt{\frac{\pi}{\alpha}}\, q_1^{-\frac{1}{24}} f(-q_1) + \sqrt{\frac{24\alpha}{\pi}} \int_0^\infty \mathrm{e}^{-\frac{3}{2}\alpha x^2} \frac{\cosh \frac{5}{2}\alpha x + \cosh \frac{1}{2}\alpha x}{\cosh 3\alpha x}\,\mathrm{d}x, \end{aligned}\right\}. \tag{4.1}$$

where $\omega(q)$ is the mf$_3$ in (2.4). (An alternate notation, more customary in the general theory of modular forms $F(\tau)$, has $F(A\tau)$ on the left and $F^*(\tau)$ on the right.) When r is not an integer this leads to mfs in the wide sense as defined in Sect. 1. Therefore a complete transformation theory requires inclusion of the behavior of mfs in the wide sense. This phenomenon is dealt with in [BO2] and [BOR].

Using the mock theta "conjectures" the laws (4.1), as well as those for all the other mf3s, mf5s and mf7s (given at the end of this section), can be obtained from those for the function $g_3(x,q)$ defined in (3.2). One transformation law for $g_3(q^r,q)$ is [GM2]

$$q^{\frac{3}{2}r(1-r)-\frac{1}{24}}g_3(q^r,q) = \sqrt{\frac{\pi}{2\alpha}}\csc(\pi r)\,q_1^{-\frac{1}{6}}h_3(\mathrm{e}^{2\pi ir},q_1^4) - \sqrt{\frac{3\alpha}{2\pi}}\int_0^\infty \mathrm{e}^{-\frac{3}{2}\alpha x^2}\frac{\cosh(3r-1)\alpha x+\cosh(3r-2)\alpha x}{\cosh\frac{3}{2}\alpha x}\,\mathrm{d}x, \tag{4.2}$$

where

$$h_3(y,q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(yq;q)_n(y^{-1}q;q)_n}. \tag{4.3}$$

The law for $g_3((-q)^a,(-q)^b)$ involves the parity of a and b.

As noted by Watson [W1], it is desirable to supplement the transformation laws by rules governing the behavior of Mordell integrals such as

$$W_c(r,\alpha) = \int_0^\infty \mathrm{e}^{-\alpha x^2}\frac{\cosh r\alpha x}{\cosh\alpha x}\,\mathrm{d}x \quad \text{and} \quad W_s(r,\alpha) = \int_0^\infty \mathrm{e}^{-\alpha x^2}\frac{\sinh r\alpha x}{\sinh\alpha x}\,\mathrm{d}x$$

under the map $\alpha \mapsto \beta = \pi^2/\alpha$ (and thus $q \mapsto q_1$). These laws are [M7]

$$\left.\begin{aligned} \sqrt{\frac{\alpha^3}{\pi^3}}W_c(r,\alpha) &= 2\cos\left(\frac{\pi r}{2}\right)\int_0^\infty \mathrm{e}^{-\beta x^2}\frac{\cosh\beta x}{\cos\pi r+\cosh 2\beta x}\,\mathrm{d}x, \\ \sqrt{\frac{\alpha^3}{\pi^3}}W_s(r,\alpha) &= \sin(\pi r)\int_0^\infty \frac{\mathrm{e}^{-\beta x^2}}{\cos\pi r+\cosh 2\beta x}\,\mathrm{d}x \end{aligned}\right\} \tag{4.4}$$

for $|r|<1$.

We now outline a proof of (4.2). It is more convenient to work with the functions

$$g_3(q^r,q) = \frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{\frac{3}{2}n(n+1)}}{1-q^{n+r}} \tag{4.5}$$

and

$$h_3(\mathrm{e}^{2\pi ir},q) = \frac{4\sin^2\pi r}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{\frac{1}{2}n(3n+1)}}{(1-\mathrm{e}^{2\pi ir}q^n)(1-\mathrm{e}^{-2\pi ir}q^n)}, \tag{4.6}$$

where $r = a/b$. The series here are called *generalized Lambert series*. As in [GM2, 196–198], (4.5) and (4.6) are obtained from the Watson-Whipple transformation [GR, 242, (III.17)]. The transformation law (4.2) is obtained by starting from (4.5), then using contour integration and the saddle-point method. This technique can be applied more generally to the series

$$u_k(x,q) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{1-xq^n}, \quad k \in \mathbf{Z}_{>0}, \tag{4.7}$$

where $x = q^r$. Observe that

$$g_3(x,q) = \frac{1}{j(q,q^3)} u_3(x,q).$$

The corresponding normalizations of u_k with $k > 1$ are

$$\frac{1}{j(q^h,q^k)} u_k(x,q),$$

where h is an integer with $0 < h < k$. Application of the saddle-point method to the integral

$$\frac{1}{2\pi i}\left(\int_{-\infty-\epsilon i}^{+\infty-\epsilon i} + \int_{+\infty+\epsilon i}^{-\infty+\epsilon i}\right) \frac{\pi}{\sin \pi z}\,\frac{\mathrm{e}^{-\frac{1}{2}k\alpha z(z+1)}}{1-\mathrm{e}^{-\alpha(z+r)}}\,\mathrm{d}z$$

leads to the transformation law

$$\begin{aligned} q^{\frac{1}{2}kr(1-r)} u_k(q^r,q) &= \frac{4\pi}{\alpha} \sin(\pi r)\, v_k(\mathrm{e}^{2\pi i r}, q_1^4) \\ &\quad - \sqrt{\frac{k\alpha}{2\pi}} \sum_{m=1}^{k-1} q^{\frac{(k-2m)^2}{8k}} j(q^m,q^k) \int_0^\infty \mathrm{e}^{-\frac{1}{2}k\alpha x^2} \frac{\cosh(kr-m)\alpha x}{\cosh \frac{1}{2}k\alpha x}\,\mathrm{d}x, \end{aligned} \tag{4.8}$$

where

$$v_k(y,q) = \frac{1}{1-y^{-1}} \sum_{n=-\infty}^{\infty} \frac{(-1)^{kn} q^{\frac{1}{2}n(kn+1)}}{1-yq^n} \tag{4.9}$$

and $j(x,q)$ is defined in (1.5). From this it follows that the normalized functions

$$\frac{1}{j(q^h,q^k)} u_k(q^r,q), \quad r \in \mathbf{Q},\ r \notin \mathbf{Z},$$

are mfs in the wide sense. It is easily seen that

$$u_k(x,q) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{(1-xq^n)(1-x^{-1}q^{n+1})}, \quad v_k(y,q) = \sum_{n=-\infty}^{\infty} \frac{(-1)^{kn} q^{\frac{1}{2}n(kn+1)}}{(1-yq^n)(1-y^{-1}q^n)}.$$

For details see [M5].

When $k = 1$, 2, or 3, the Watson-Whipple transformation can be used to express $u_k(x,q)$ and $v_k(y,q)$ as follows:

$$\left.\begin{aligned} u_1(x,q) &= \frac{(q)_\infty^3}{j(x,q)} = \frac{(q)_\infty^2}{(x;q)_\infty(x^{-1}q;q)_\infty}\,, \\ v_1(y,q) &= \frac{(q)_\infty^2}{(y;q)_\infty(y^{-1};q)_\infty}\,, \\ u_2(x,q) &= j(q,q^2)g_2(x,q) = \theta_4(0,q)g_2(x,q)\,, \\ v_2(y,q^2) &= \frac{\psi(q)h_2(y,q)}{(1-y)(1-y^{-1})}\,, \\ u_3(x,q) &= j(q,q^3)g_3(x,q) = (q)_\infty g_3(x,q)\,, \\ v_3(y,q) &= \frac{(q)_\infty h_3(y,q)}{(1-y)(1-y^{-1})}\,. \end{aligned}\right\} \tag{4.10}$$

Here $\psi(q)$ is the Gauss function (2.9), the normalized u_2 is

$$g_2(x,q) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)}(-q;q)_n}{(x;q)_{n+1}(x^{-1}q;q)_{n+1}}\,, \tag{4.11}$$

and

$$h_2(y,q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(yq^2;q^2)_n(y^{-1}q^2;q^2)_n}. \tag{4.12}$$

A more detailed study of the functions g_2 and h_2 is found in [M5] and a complete transformation theory for these functions is presented in [BOR].

When $k = 1$, there is no Mordell integral in (4.8), which reduces to

$$q^{\frac{1}{2}r(1-r)}u_1(q^r,q) = \frac{4\pi}{\alpha}\sin(\pi r)\,v_1(\mathrm{e}^{2\pi i r},q_1^4)\,.$$

In view of (4.10), this asserts that

$$\frac{q^{\frac{1}{2}r(1-r)}(q)_\infty^2}{(q^r;q)_\infty(q^{1-r};q)_\infty} = \frac{4\pi}{\alpha}\sin(\pi r)\frac{(q_1^4;q_1^4)_\infty^2}{(\mathrm{e}^{2\pi i r};q_1^4)_\infty(\mathrm{e}^{-2\pi i r};q_1^4)_\infty}\,,$$

a transformation law for a θf.

When $k = 2$, (4.8) simplifies to

$$q^{r(1-r)}u_2(q^r,q) = \frac{4\pi}{\alpha}\sin(\pi r)\,v_2(\mathrm{e}^{2\pi i r},q_1^4) - \sqrt{\frac{\alpha}{\pi}}\,\theta_4(0,q)\int_0^\infty \mathrm{e}^{-\alpha x^2}\frac{\cosh(2r-1)\alpha x}{\cosh\alpha x}\,\mathrm{d}x\,. \tag{4.13}$$

By the functional equation of the Dedekind η-function we have

$$q^{\frac{1}{24}}(q)_\infty = \sqrt{\frac{2\pi}{\alpha}}\, q_1^{\frac{1}{6}}(q_1^4;q_1^4)_\infty. \tag{4.14}$$

Hence

$$\theta_4(0,q) = \sqrt{\frac{4\pi}{\alpha}}\, q_1^{\frac{1}{4}}\psi(q_1^2). \tag{4.15}$$

Dividing (4.13) by (4.15) and using (4.10), we obtain

$$\begin{aligned} q^{r(1-r)}g_2(q^r,q) = {} & \sqrt{\frac{4\pi}{\alpha}}\sin(\pi r)\frac{q_1^{-\frac{1}{4}}h_2(\mathrm{e}^{2\pi ir},q_1^2)}{(1-\mathrm{e}^{2\pi ir})(1-\mathrm{e}^{-2\pi ir})} \\ & -\sqrt{\frac{\alpha}{\pi}}\int_0^\infty \mathrm{e}^{-\alpha x^2}\frac{\cosh(2r-1)\alpha x}{\cosh\alpha x}\,\mathrm{d}x\,. \end{aligned}$$

Since $(1-\mathrm{e}^{2\pi ir})(1-\mathrm{e}^{-2\pi ir}) = 2-2\cos 2\pi r = 4\sin^2\pi r$, this simplifies to

$$q^{r(1-r)}g_2\left(q^r,q\right) = \sqrt{\frac{\pi}{4\alpha}}\csc(\pi r)\, q_1^{-\frac{1}{4}}h_2\left(\mathrm{e}^{2\pi ir},q_1^2\right) - \sqrt{\frac{\alpha}{\pi}}\int_0^\infty \mathrm{e}^{-\alpha x^2}\frac{\cosh(2r-1)\alpha x}{\cosh\alpha x}\,\mathrm{d}x. \tag{4.16}$$

The transformation law for $g_2((-q)^a,(-q)^c)$ depends on the parity of a and c and is given in [M5].

When $k=3$, (4.8) becomes

$$\begin{aligned} q^{\frac{3}{2}r(1-r)}u_3(q^r,q) = {} & \frac{4\pi}{\alpha}\sin(\pi r)\, v_3(\mathrm{e}^{2\pi ir},q_1^4) \\ & -\sqrt{\frac{3\alpha}{2\pi}}\, q^{\frac{1}{24}}(q)_\infty \int_0^\infty \mathrm{e}^{-\frac{3}{2}\alpha x^2}\frac{\cosh(3r-1)\alpha x + \cosh(3r-2)\alpha x}{\cosh\frac{3}{2}\alpha x}\,\mathrm{d}x, \end{aligned}$$

since $j(q,q^3) = j(q^2,q^3) = (q)_\infty$. Dividing by (4.14) and using (4.10), we obtain

$$\begin{aligned} q^{\frac{3}{2}r(1-r)-\frac{1}{24}}g_3(q^r,q) = {} & \sqrt{\frac{8\pi}{\alpha}}\sin(\pi r)\frac{q_1^{-\frac{1}{6}}h_3(\mathrm{e}^{2\pi ir},q_1^4)}{(1-\mathrm{e}^{2\pi ir})(1-\mathrm{e}^{-2\pi ir})} \\ & -\sqrt{\frac{3\alpha}{2\pi}}\int_0^\infty \mathrm{e}^{-\frac{3}{2}\alpha x^2}\frac{\cosh(3r-1)\alpha x + \cosh(3r-2)\alpha x}{\cosh\frac{3}{2}\alpha x}\,\mathrm{d}x \\ = {} & \sqrt{\frac{\pi}{2\alpha}}\csc(\pi r)\, q_1^{-\frac{1}{6}}h_3(\mathrm{e}^{2\pi ir},q_1^4) \\ & -\sqrt{\frac{3\alpha}{2\pi}}\int_0^\infty \mathrm{e}^{-\frac{3}{2}\alpha x^2}\frac{\cosh(3r-1)\alpha x + \cosh(3r-2)\alpha x}{\cosh\frac{3}{2}\alpha x}\,\mathrm{d}x, \end{aligned}$$

which is (4.2). A complete transformation theory for g_3 and h_3 is found in [BO2].

The transformation laws for the mf$_3$s (2.1), (2.4), (2.5) are [W1, 78–79], [GM2, 204]

$$q^{-\frac{1}{24}} f(q) = \sqrt{\frac{8\pi}{\alpha}}\, q_1^{\frac{4}{3}}\, \omega(q_1^2) + \sqrt{\frac{24\alpha}{\pi}} W_1(\alpha),$$

$$q^{-\frac{1}{24}} f(-q) = -\sqrt{\frac{\pi}{\alpha}}\, q_1^{-\frac{1}{24}} f(-q_1) + \sqrt{\frac{24\alpha}{\pi}} W(\alpha),$$

$$q^{\frac{2}{3}} \omega(q) = \sqrt{\frac{\pi}{4\alpha}}\, q_1^{-\frac{1}{12}} f(q_1^2) - \sqrt{\frac{3\alpha}{\pi}} W_2\left(\frac{\alpha}{2}\right),$$

$$q^{\frac{2}{3}} \omega(-q) = -\sqrt{\frac{\pi}{\alpha}}\, q_1^{\frac{2}{3}} \omega(-q_1) + \sqrt{\frac{12\alpha}{\pi}} W_3(\alpha),$$

$$q^{-\frac{1}{24}} \phi(q) = \sqrt{\frac{4\pi}{\alpha}}\, q_1^{-\frac{1}{24}} \psi(q_1) + \sqrt{\frac{6\alpha}{\pi}} W(\alpha),$$

$$q^{-\frac{1}{24}} \phi(-q) = \sqrt{\frac{2\pi}{\alpha}}\, q_1^{\frac{1}{3}} \upsilon(-q_1) + \sqrt{\frac{6\alpha}{\pi}} W_1(\alpha),$$

$$q^{-\frac{1}{24}} \psi(q) = \sqrt{\frac{\pi}{4\alpha}}\, q_1^{-\frac{1}{24}} \phi(q_1) - \sqrt{\frac{3\alpha}{2\pi}} W(\alpha),$$

$$q^{-\frac{1}{24}} \psi(-q) = \sqrt{\frac{\pi}{2\alpha}}\, q_1^{\frac{1}{3}} \upsilon(q_1) - \sqrt{\frac{3\alpha}{2\pi}} W_1(\alpha),$$

$$q^{\frac{1}{3}} \upsilon(q) = \sqrt{\frac{2\pi}{\alpha}}\, q_1^{-\frac{1}{24}} \psi(-q_1) + \sqrt{\frac{6\alpha}{\pi}} W_2(\alpha),$$

$$q^{\frac{1}{3}} \upsilon(-q) = \sqrt{\frac{\pi}{2\alpha}}\, q_1^{-\frac{1}{24}} \phi(-q_1) - \sqrt{\frac{6\alpha}{\pi}} W_2(\alpha),$$

$$q^{-\frac{1}{24}} \chi(q) = \sqrt{\frac{\pi}{2\alpha}}\, \xi\left(q_1^{\frac{2}{3}}\right) + \sqrt{\frac{3\alpha}{2\pi}} W_1(\alpha),$$

$$q^{-\frac{1}{24}} \chi(-q) = \sqrt{\frac{\pi}{\alpha}} q_1^{\frac{7}{24}} \sigma\left(-q_1^{\frac{1}{3}}\right) + \sqrt{\frac{3\alpha}{2\pi}} W(\alpha),$$

$$q^{\frac{2}{3}}\rho(q) = -\sqrt{\frac{\pi}{\alpha}}q_1^{\frac{7}{12}}\sigma\left(q_1^{\frac{2}{3}}\right) + \sqrt{\frac{3\alpha}{4\pi}}W_2\left(\frac{\alpha}{2}\right),$$

$$q^{\frac{2}{3}}\rho(-q) = \sqrt{\frac{\pi}{4\alpha}}\xi\left(-q_1^{\frac{1}{3}}\right) - \sqrt{\frac{3\alpha}{\pi}}W_3(\alpha),$$

$$\xi(q) = \sqrt{\frac{4\pi}{3\alpha}}q_1^{-\frac{1}{36}}\chi\left(q_1^{\frac{2}{3}}\right) - \sqrt{\frac{9\alpha}{\pi}}W_2\left(\frac{3\alpha}{2}\right),$$

$$\xi(-q) = \sqrt{\frac{4\pi}{3\alpha}}q_1^{\frac{2}{9}}\rho\left(-q_1^{\frac{1}{3}}\right) + \sqrt{\frac{36\alpha}{\pi}}W_3(3\alpha),$$

$$q^{\frac{7}{8}}\sigma(q) = -\sqrt{\frac{2\pi}{3\alpha}}q_1^{\frac{4}{9}}\rho\left(q_1^{\frac{2}{3}}\right) + \sqrt{\frac{9\alpha}{2\pi}}W_1(3\alpha),$$

$$q^{\frac{7}{8}}\sigma(-q) = \sqrt{\frac{\pi}{3\alpha}}q_1^{-\frac{1}{72}}\chi\left(-q_1^{\frac{1}{3}}\right) - \sqrt{\frac{9\alpha}{2\pi}}W(3\alpha),$$

where the Mordell integrals W, W_1, W_2, W_3 are defined in [W1] by

$$W(\alpha) = \int_0^\infty e^{-\frac{3}{2}\alpha x^2}\frac{\cosh\frac{5}{2}\alpha x + \cosh\frac{1}{2}\alpha x}{\cosh 3\alpha x}\,dx,$$

$$W_1(\alpha) = \int_0^\infty e^{-\frac{3}{2}\alpha x^2}\frac{\sinh\alpha x}{\sinh\frac{3}{2}\alpha x}\,dx,$$

$$W_2(\alpha) = \int_0^\infty e^{-\frac{3}{2}\alpha x^2}\frac{\cosh\alpha x}{\cosh 3\alpha x}\,dx,$$

$$W_3(\alpha) = \frac{1}{4}W_2\left(\frac{\alpha}{8}\right) - W_1(2\alpha) = \int_0^\infty e^{-3\alpha x^2}\frac{\sinh\alpha x}{\sinh 3\alpha x}\,dx.$$

They satisfy the inversion rules

$$W(\beta) = \sqrt{\frac{\alpha^3}{\pi^3}}W(\alpha), \quad W_1(\beta) = \sqrt{\frac{2\alpha^3}{\pi^3}}W_2(\alpha),$$

$$W_2(\beta) = \sqrt{\frac{\alpha^3}{2\pi^3}}W_1(\alpha), \quad W_3(\beta) = \sqrt{\frac{\alpha^3}{\pi^3}}W_3(\alpha).$$

The last eight of these transformation laws involve mf$_3$s in the wide sense, which may account for their omission in [W1]. The transformation laws for the mf$_5$s (2.2) are [GM2, 207–209]

$$q^{-\frac{1}{60}} f_0(q) = \sqrt{\frac{2\pi(5-\sqrt{5})}{5\alpha}} q_1^{-\frac{1}{60}} (F_0(q_1^2)-1) + \sqrt{\frac{2\pi(5+\sqrt{5})}{5\alpha}} q_1^{\frac{71}{60}} F_1(q_1^2)$$

$$+\sqrt{\frac{60\alpha}{\pi}} L\left(\frac{1}{5}, 10\alpha\right),$$

$$q^{\frac{11}{60}} f_1(q) = \sqrt{\frac{2\pi(5+\sqrt{5})}{5\alpha}} q_1^{-\frac{1}{60}} (F_0(q_1^2)-1) - \sqrt{\frac{2\pi(5-\sqrt{5})}{5\alpha}} q_1^{\frac{71}{60}} F_1(q_1^2)$$

$$+\sqrt{\frac{60\alpha}{\pi}} L\left(\frac{2}{5}, 10\alpha\right),$$

$$q^{-\frac{1}{60}} f_0(-q) = -\sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}} q_1^{-\frac{1}{60}} f_0(-q_1) + \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}} q_1^{\frac{11}{60}} f_1(-q_1)$$

$$+\sqrt{\frac{60\alpha}{\pi}} L\left(\frac{1}{5}, 10\alpha\right),$$

$$q^{\frac{11}{60}} f_1(-q) = \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}} q_1^{-\frac{1}{60}} f_0(-q_1) + \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}} q_1^{\frac{11}{60}} f_1(-q_1)$$

$$-\sqrt{\frac{60\alpha}{\pi}} L\left(\frac{2}{5}, 10\alpha\right),$$

$$q^{-\frac{1}{120}} (F_0(q)-1) = \sqrt{\frac{\pi(5-\sqrt{5})}{20\alpha}} q_1^{-\frac{1}{30}} f_0(q_1^2) + \sqrt{\frac{\pi(5+\sqrt{5})}{20\alpha}} q_1^{\frac{11}{30}} f_1(q_1^2)$$

$$-\sqrt{\frac{15\alpha}{2\pi}} L\left(\frac{1}{5}, 5\alpha\right),$$

$$q^{\frac{71}{120}} F_1(q) = \sqrt{\frac{\pi(5+\sqrt{5})}{20\alpha}} q_1^{-\frac{1}{30}} f_0(q_1^2) - \sqrt{\frac{\pi(5-\sqrt{5})}{20\alpha}} q_1^{\frac{11}{30}} f_1(q_1^2)$$

$$-\sqrt{\frac{15\alpha}{2\pi}} L\left(\frac{2}{5}, 5\alpha\right),$$

$$\begin{aligned}
q^{-\frac{1}{120}}(F_0(-q)-1) &= \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}q_1^{-\frac{1}{120}}(F_0(-q_1)-1)+\sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}q_1^{\frac{71}{120}}F_1(-q_1)\\
&\quad+\sqrt{\frac{15\alpha}{2\pi}}L_1\left(\frac{4}{5},5\alpha\right),\\
q^{\frac{71}{120}}F_1(-q) &= \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}q_1^{-\frac{1}{120}}(F_0(-q_1)-1)-\sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}q_1^{\frac{71}{120}}F_1(-q_1)\\
&\quad+\sqrt{\frac{15\alpha}{2\pi}}L_1\left(\frac{2}{5},5\alpha\right),\\
q^{-\frac{1}{120}}\phi_0(q) &= \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}q_1^{-\frac{1}{120}}\phi_0(q_1)+\sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}q_1^{-\frac{49}{120}}\phi_1(q_1)\\
&\quad-\sqrt{\frac{15\alpha}{2\pi}}L_1\left(\frac{4}{5},5\alpha\right),\\
q^{-\frac{49}{120}}\phi_1(q) &= \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}q_1^{-\frac{1}{120}}\phi_0(q_1)-\sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}q_1^{-\frac{49}{120}}\phi_1(q_1)\\
&\quad-\sqrt{\frac{15\alpha}{2\pi}}L_1\left(\frac{2}{5},5\alpha\right),\\
q^{-\frac{1}{120}}\phi_0(-q) &= \sqrt{\frac{\pi(5-\sqrt{5})}{5\alpha}}q_1^{-\frac{1}{30}}\psi_0(q_1^2)+\sqrt{\frac{\pi(5+\sqrt{5})}{5\alpha}}q_1^{\frac{11}{30}}\psi_1(q_1^2)\\
&\quad+\sqrt{\frac{15\alpha}{2\pi}}L\left(\frac{1}{5},5\alpha\right),\\
q^{-\frac{49}{120}}\phi_1(-q) &= -\sqrt{\frac{\pi(5+\sqrt{5})}{5\alpha}}q_1^{-\frac{1}{30}}\psi_0(q_1^2)+\sqrt{\frac{\pi(5-\sqrt{5})}{5\alpha}}q_1^{\frac{11}{30}}\psi_1(q_1^2)\\
&\quad-\sqrt{\frac{15\alpha}{2\pi}}L\left(\frac{2}{5},5\alpha\right),\\
q^{-\frac{1}{60}}\psi_0(q) &= \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}q_1^{-\frac{1}{60}}\phi_0(-q_1^2)-\sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}q_1^{-\frac{49}{60}}\phi_1(-q_1^2)\\
&\quad-\sqrt{\frac{15\alpha}{\pi}}L\left(\frac{1}{5},10\alpha\right),\\
q^{\frac{11}{60}}\psi_1(q) &= \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}q_1^{-\frac{1}{60}}\phi_0(-q_1^2)+\sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}q_1^{-\frac{49}{60}}\phi_1(-q_1^2)\\
&\quad-\sqrt{\frac{15\alpha}{\pi}}L\left(\frac{2}{5},10\alpha\right),
\end{aligned}$$

$$
\begin{aligned}
q^{-\frac{1}{60}}\psi_0(-q) &= -\sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}\,q_1^{-\frac{1}{60}}\psi_0(-q_1)+\sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}\,q_1^{\frac{11}{60}}\psi_1(-q_1)\\
&\quad -\sqrt{\frac{15\alpha}{\pi}}L\Big(\frac{1}{5},10\alpha\Big),\\
q^{\frac{11}{60}}\psi_1(-q) &= \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}\,q_1^{-\frac{1}{60}}\psi_0(-q_1)+\sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}\,q_1^{\frac{11}{60}}\psi_1(-q_1)\\
&\quad +\sqrt{\frac{15\alpha}{\pi}}L\Big(\frac{2}{5},10\alpha\Big),\\
q^{-\frac{1}{120}}(\chi_0(q)-2) &= -\sqrt{\frac{\pi(5-\sqrt{5})}{5\alpha}}\,q_1^{-\frac{1}{30}}(\chi_0(q_1^4)-2)-\sqrt{\frac{\pi(5+\sqrt{5})}{5\alpha}}\,q_1^{\frac{71}{30}}\chi_1(q_1^4)\\
&\quad -\sqrt{\frac{135\alpha}{2\pi}}L\Big(\frac{1}{5},5\alpha\Big),\\
q^{\frac{71}{120}}\chi_1(q) &= -\sqrt{\frac{\pi(5+\sqrt{5})}{5\alpha}}\,q_1^{-\frac{1}{30}}(\chi_0(q_1^4)-2)+\sqrt{\frac{\pi(5-\sqrt{5})}{5\alpha}}\,q_1^{\frac{71}{30}}\chi_1(q_1^4)\\
&\quad -\sqrt{\frac{135\alpha}{2\pi}}L\Big(\frac{2}{5},5\alpha\Big),\\
q^{-\frac{1}{120}}(\chi_0(-q)-2) &= \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}\,q_1^{-\frac{1}{120}}(\chi_0(-q_1)-2)+\sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}\,q_1^{\frac{71}{120}}\chi_1(-q_1)\\
&\quad +\sqrt{\frac{135\alpha}{2\pi}}L_1\Big(\frac{4}{5},5\alpha\Big),\\
q^{\frac{71}{120}}\chi_1(-q) &= \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}\,q_1^{-\frac{1}{120}}(\chi_0(-q_1)-2)-\sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}\,q_1^{\frac{71}{120}}\chi_1(-q_1)\\
&\quad +\sqrt{\frac{135\alpha}{2\pi}}L_1\Big(\frac{2}{5},5\alpha\Big),
\end{aligned}
$$

where

$$
L(r,\alpha)=\int_0^\infty \mathrm{e}^{-\frac{3}{2}\alpha x^2}\,\frac{\cosh(3r-2)\alpha x+\cosh(3r-1)\alpha x}{\cosh\frac{3}{2}\alpha x}\,\mathrm{d}x
$$

and

$$
\begin{aligned}
L_1(r,\alpha)=\int_0^\infty \mathrm{e}^{-\frac{3}{2}\alpha x^2}\Big\{&\cosh\Big(3r-\frac{7}{2}\Big)\alpha x+\cosh\Big(3r-\frac{5}{2}\Big)\alpha x\\
&+\cosh\Big(3r-\frac{1}{2}\Big)\alpha x-\cosh\Big(3r+\frac{1}{2}\Big)\alpha x\Big\}\Big/\cosh 3\alpha x\,\mathrm{d}x.
\end{aligned}
$$

The transformation laws for the mf$_7$s (2.3) are [GM2, 218–219]

$$
\begin{aligned}
q^{-\frac{1}{168}}(\mathcal{F}_0(q)-2) &= \sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{\pi}{7}\right)q_1^{-\frac{1}{42}}\mathcal{F}_0(q_1^4)+\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{2\pi}{7}\right)q_1^{-\frac{25}{42}}\mathcal{F}_1(q_1^4)\\
&\quad +\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{3\pi}{7}\right)q_1^{\frac{47}{42}}\mathcal{F}_2(q_1^4)-\sqrt{\frac{42\alpha}{\pi}}L\left(\frac{1}{7},7\alpha\right),\\
q^{-\frac{25}{168}}\mathcal{F}_1(q) &= \sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{2\pi}{7}\right)q_1^{-\frac{1}{42}}\mathcal{F}_0(q_1^4)-\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{3\pi}{7}\right)q_1^{-\frac{25}{42}}\mathcal{F}_1(q_1^4)\\
&\quad +\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{\pi}{7}\right)q_1^{\frac{47}{42}}\mathcal{F}_2(q_1^4)-\sqrt{\frac{42\alpha}{\pi}}L\left(\frac{2}{7},7\alpha\right),\\
q^{\frac{47}{168}}\mathcal{F}_2(q) &= \sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{3\pi}{7}\right)q_1^{-\frac{1}{42}}\mathcal{F}_0(q_1^4)+\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{\pi}{7}\right)q_1^{-\frac{25}{42}}\mathcal{F}_1(q_1^4)\\
&\quad -\sqrt{\frac{8\pi}{7\alpha}}\sin\left(\frac{2\pi}{7}\right)q_1^{\frac{47}{42}}\mathcal{F}_2(q_1^4)-\sqrt{\frac{42\alpha}{\pi}}L\left(\frac{3}{7},7\alpha\right),\\
q^{-\frac{1}{168}}(\mathcal{F}_0(-q)-2) &= -\sqrt{\frac{4\pi}{7\alpha}}\sin\left(\frac{3\pi}{7}\right)q_1^{-\frac{1}{168}}\mathcal{F}_0(-q_1)+\sqrt{\frac{4\pi}{7\alpha}}\sin\left(\frac{\pi}{7}\right)q_1^{-\frac{25}{168}}\mathcal{F}_1(-q_1)\\
&\quad +\sqrt{\frac{4\pi}{7\alpha}}\sin\left(\frac{2\pi}{7}\right)q_1^{\frac{47}{168}}\mathcal{F}_2(-q_1)+\sqrt{\frac{42\alpha}{\pi}}L_1\left(\frac{6}{7},7\alpha\right),\\
q^{-\frac{25}{168}}\mathcal{F}_1(-q) &= \sqrt{\frac{4\pi}{7\alpha}}\sin\left(\frac{\pi}{7}\right)q_1^{-\frac{1}{168}}\mathcal{F}_0(-q_1)-\sqrt{\frac{4\pi}{7\alpha}}\sin\left(\frac{2\pi}{7}\right)q_1^{-\frac{25}{168}}\mathcal{F}_1(-q_1)\\
&\quad +\sqrt{\frac{4\pi}{7\alpha}}\sin\left(\frac{3\pi}{7}\right)q_1^{\frac{47}{168}}\mathcal{F}_2(-q_1)-\sqrt{\frac{42\alpha}{\pi}}L_1\left(\frac{2}{7},7\alpha\right),\\
q^{\frac{47}{168}}\mathcal{F}_2(-q) &= \sqrt{\frac{4\pi}{7\alpha}}\sin\left(\frac{2\pi}{7}\right)q_1^{-\frac{1}{168}}\mathcal{F}_0(-q_1)+\sqrt{\frac{4\pi}{7\alpha}}\sin\left(\frac{3\pi}{7}\right)q_1^{-\frac{25}{168}}\mathcal{F}_1(-q_1)\\
&\quad +\sqrt{\frac{4\pi}{7\alpha}}\sin\left(\frac{\pi}{7}\right)q_1^{\frac{47}{168}}\mathcal{F}_2(-q_1)-\sqrt{\frac{42\alpha}{\pi}}L_1\left(\frac{4}{7},7\alpha\right).
\end{aligned}
$$

5 Mock Theta Functions of Even Order

In Sect. 3 we observed that the mfs of odd order are related to the function $g_3(x,q)$ of (3.3). It turns out that the mfs of even order are similarly related to the function

$$g_2(x,q)=\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}n(n+1)}(-q;q)_n}{(x;q)_{n+1}(x^{-1}q;q)_{n+1}}$$

of (4.11).

We begin with the mf$_2$s

$$\left.\begin{aligned} A(q) &= \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}\left(-q;q^2\right)_n}{(q;q^2)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^{n+1}\left(-q^2;q^2\right)_n}{(q;q^2)_{n+1}}, \\ B(q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}\left(-q^2;q^2\right)_n}{(q;q^2)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^{n}\left(-q;q^2\right)_n}{(q;q^2)_{n+1}}, \\ \mu(q) &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}\left(q;q^2\right)_n}{(-q^2;q^2)_n^2}. \end{aligned}\right\} \tag{5.1}$$

The function $\mu(q)$ appears several times in the Lost Notebook [R2, (3.1), (3.4), (3.8), (3.9), (3.11), (3.13) with $a = 1$]. It is related to $A(q)$ by the identity [A2, (3.28)]

$$\mu(q) + 4A(-q) = \frac{(q)_\infty^5}{(q^2;q^2)_\infty^4}.$$

The mock theta "conjectures" of order 2 are [GM4]

$$\left.\begin{aligned} &A(q^2) = qg_2(q,q^4) - q(-q^2;q^2)_\infty(-q^4;q^4)_\infty^2(q^8;q^8)_\infty, \\ &B(q) \;\; = g_2(q,q^2), \\ &\mu(q^4) = -2qg_2(q,q^2) + \frac{(q^2;q^2)_\infty(q^4;q^4)_\infty^3(q^8;q^8)_\infty}{(q)_\infty^2(q^{16};q^{16})_\infty^2}. \end{aligned}\right\} \tag{5.2}$$

These are not needed to prove the transformation laws, which are [A2, (4.7)–(4.10)], [M3, 285],

$$\left.\begin{aligned} q^{-\frac{1}{8}}A(q) &= \sqrt{\frac{\pi}{16\alpha}}\, q_1^{-\frac{1}{8}}\mu(-q_1) - \sqrt{\frac{\alpha}{2\pi}}K(\alpha), \\ q^{-\frac{1}{8}}A(-q) &= \sqrt{\frac{\pi}{2\alpha}}\, q_1^{\frac{1}{2}}B(-q_1) - \sqrt{\frac{\alpha}{8\pi}}J\left(\frac{\alpha}{2}\right), \\ q^{\frac{1}{2}}B(q) &= \sqrt{\frac{\pi}{8\alpha}}\, q_1^{-\frac{1}{8}}\mu(q_1) - \sqrt{\frac{2\alpha}{\pi}}J(2\alpha), \\ q^{\frac{1}{2}}B(-q) &= \sqrt{\frac{2\pi}{\alpha}}\, q_1^{-\frac{1}{8}}A(-q_1) + \sqrt{\frac{2\alpha}{\pi}}J(2\alpha), \\ q^{-\frac{1}{8}}\mu(q) &= \sqrt{\frac{8\pi}{\alpha}}\, q_1^{\frac{1}{2}}B(q_1) + \sqrt{\frac{2\alpha}{\pi}}J\left(\frac{\alpha}{2}\right), \\ q^{-\frac{1}{8}}\mu(-q) &= \sqrt{\frac{16\pi}{\alpha}}\, q_1^{-\frac{1}{8}}A(q_1) + \sqrt{\frac{8\alpha}{\pi}}K(\alpha), \end{aligned}\right\} \tag{5.3}$$

where as usual $q = \mathrm{e}^{-\alpha}$ and $q_1 = \mathrm{e}^{-\beta}$ with $\alpha\beta = \pi^2$. The Mordell integrals J, K, and their inversions are

$$J(\alpha) = \int_0^\infty \frac{\mathrm{e}^{-\alpha x^2}}{\cosh \alpha x}\,\mathrm{d}x, \qquad J(\beta) = \sqrt{\frac{\alpha^3}{\pi^3}}J(\alpha),$$

$$K(\alpha) = \int_0^\infty \mathrm{e}^{-\frac{1}{2}\alpha x^2}\,\frac{\cosh \frac{1}{2}\alpha x}{\cosh \alpha x}\,\mathrm{d}x, \qquad K(\beta) = \sqrt{\frac{\alpha^3}{\pi^3}}K(\alpha).$$

Also appearing in the Lost Notebook and studied in [AH], [BC], [M8] are mfs to which order 6 has been assigned. These are the following:

$$\beta(q) = \sum_{n=0}^\infty \frac{q^{3n^2+3n+1}}{(q;q^3)_{n+1}(q^2;q^3)_{n+1}} = q g_3(q,q^3), \tag{5.4}$$

$$\gamma(q) = \sum_{n=0}^\infty \frac{q^{n^2}(q)_n}{(q^3;q^3)_n} = h_3(\mathrm{e}^{\frac{2\pi i}{3}},q), \tag{5.5}$$

$$\left.\begin{aligned}
\phi(q) &= \sum_{n=0}^\infty \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q)_{2n}}, & \psi(q) &= \sum_{n=0}^\infty \frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q)_{2n+1}},\\
\rho(q) &= \sum_{n=0}^\infty \frac{q^{\frac{1}{2}n(n+1)}(-q;q)_n}{(q;q^2)_{n+1}}, & \sigma(q) &= \sum_{n=0}^\infty \frac{q^{\frac{1}{2}(n+1)(n+2)}(-q;q)_n}{(q;q^2)_{n+1}},\\
\lambda(q) &= \sum_{n=0}^\infty \frac{(-1)^n q^n(q;q^2)_n}{(-q;q)_n}, & \mu(q) &= \sum_{n=0}^\infty \frac{(-1)^n(q;q^2)_n}{(-q;q)_n},
\end{aligned}\right\} \tag{5.6}$$

$$\nu(q) = \sum_{n=0}^\infty \frac{q^{n+1}(-q;q)_{2n+1}}{(q;q^2)_{n+1}}, \quad \xi(q) = \sum_{n=0}^\infty \frac{q^{n+1}(-q;q)_{2n}}{(q;q^2)_{n+1}}. \tag{5.7}$$

The series defining $\mu(q)$ in (5.6) converges in the Cesàro (C,1) sense. In fact the sequence of its even partial sums converges, as does the sequence of its odd partial sums; $\mu(q)$ is the average of their limits. Functions (5.5) and (5.6) are in the Lost Notebook, while (5.4) and (5.7) arise in the modular transformation laws [M8]. In view of their expressions in terms of $g_3(x,q)$ and $h_3(y,q)$, a case can be made for designating $\beta(q)$ and $\gamma(q)$ as mf$_3$s.

Ramanujan listed five linear relations connecting mf$_6$s:

$$\left.\begin{aligned}
q^{-1}\psi(q^2) + \rho(q) &= (-q;q^2)^2_\infty j(-q,q^6),\\
\phi(q^2) + 2\sigma(q) &= (-q;q^2)^2_\infty j(-q^3,q^6),\\
2\phi(q^2) - 2\mu(-q) &= (-q;q^2)^2_\infty j(-q^3,q^6),\\
2q^{-1}\psi(q^2) + \lambda(-q) &= (-q;q^2)^2_\infty j(-q,q^6),\\
3\phi(q) - 2\gamma(q) &= \frac{j(q,q^2)^2}{j(-q,q^3)}.
\end{aligned}\right\} \tag{5.8}$$

Additional relations include [M8]

$$\left.\begin{aligned}\nu(q^2)-\sigma(-q) &= \frac{q(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^2(q^6;q^6)_\infty}\,,\\ 2q^{-1}\xi(q^2)+\rho(-q) &= \frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^3(q^{12};q^{12})_\infty}\,,\\ \phi(q^3)+2q^{-1}\psi(q^3)+2\beta(q) &= \frac{(q^2;q^2)_\infty(q^3;q^3)_\infty^5}{(q)_\infty^2(q^6;q^6)_\infty^3}\,,\end{aligned}\right\} \tag{5.9}$$

and the mock theta "conjectures" [GM4]

$$\left.\begin{aligned}\phi(q^4) &= \frac{(q^2;q^2)_\infty^3(q^3;q^3)_\infty^2(q^{12};q^{12})_\infty^3}{(q)_\infty^2(q^6;q^6)_\infty^3(q^8;q^8)_\infty(q^{24};q^{24})_\infty}-2qg_2(q,q^6)\,,\\ \psi(q^4) &= \frac{q^3(q^2;q^2)_\infty^2(q^4;q^4)_\infty(q^{24};q^{24})_\infty^2}{(q)_\infty(q^3;q^3)_\infty(q^8;q^8)_\infty^2}-q^3g_2(q^3,q^6)\,.\end{aligned}\right\} \tag{5.10}$$

With the aid of (5.8)–(5.10), all the mf$_6$s (5.4)–(5.7) can be expressed in terms of $g_2(x,q)$ and θfs.

The transformation laws for $\beta(q)$ and $\gamma(q)$ are

$$\begin{aligned}q^{-\frac{1}{8}}\beta(q) &= \sqrt{\frac{2\pi}{9\alpha}}\,q_1^{-\frac{1}{18}}\gamma\left(q_1^{\frac{4}{3}}\right)-\sqrt{\frac{81\alpha}{8\pi}}\left[J\left(\frac{9\alpha}{2}\right)+\frac{1}{9}J\left(\frac{\alpha}{2}\right)\right],\\ q^{-\frac{1}{8}}\beta(-q) &= -\sqrt{\frac{\pi}{9\alpha}}\,q_1^{-\frac{1}{72}}\gamma\left(-q_1^{\frac{1}{3}}\right)+\sqrt{\frac{81\alpha}{2\pi}}\left[K(9\alpha)-\frac{1}{9}K(\alpha)\right],\\ q^{-\frac{1}{24}}\gamma(q) &= \sqrt{\frac{6\pi}{\alpha}}\,q_1^{-\frac{1}{6}}\beta\left(q_1^{\frac{4}{3}}\right)+\sqrt{\frac{27\alpha}{2\pi}}J_1\left(\frac{3\alpha}{2}\right),\\ q^{-\frac{1}{24}}\gamma(-q) &= -\sqrt{\frac{3\pi}{\alpha}}\,q_1^{-\frac{1}{24}}\beta\left(-q_1^{\frac{1}{3}}\right)+\sqrt{\frac{27\alpha}{2\pi}}K_1(3\alpha),\end{aligned}$$

where

$$J_1(\alpha)=\frac{1}{2}J(\alpha)+\frac{1}{6}J\left(\frac{\alpha}{9}\right)=\int_0^\infty \mathrm{e}^{-\alpha x^2}\frac{\cosh\frac{2}{3}\alpha x}{\cosh\alpha x}\,\mathrm{d}x$$

and

$$K_1(\alpha)=\frac{1}{3}K\left(\frac{\alpha}{9}\right)-K(\alpha)=\int_0^\infty \mathrm{e}^{-\frac{1}{2}\alpha x^2}\frac{\cosh\frac{5}{6}\alpha x-\cosh\frac{1}{6}\alpha x}{\cosh\alpha x}\,\mathrm{d}x.$$

The transformation laws for (5.6) and (5.7) are more complex than those for $\beta(q)$ and $\gamma(q)$. They are

$$q^{-\frac{1}{36}}\phi(q^{\frac{2}{3}}) = \sqrt{\frac{4\pi}{\alpha}}\,q_1^{-\frac{1}{4}}[q_1\rho(q_1^3)+\sigma(q_1^3)] + \sqrt{\frac{4\alpha}{\pi}}\,J_1(\alpha),$$

$$q^{-\frac{1}{72}}\phi(-q^{\frac{1}{3}}) = \sqrt{\frac{\pi}{\alpha}}\,q_1^{-\frac{9}{8}}[q_1\phi(-q_1^3)-2\psi(-q_1^3)] + \sqrt{\frac{2\alpha}{\pi}}\,K_1(\alpha),$$

$$q^{-\frac{1}{4}}\psi(q^{\frac{2}{3}}) = -\sqrt{\frac{\pi}{\alpha}}\,q_1^{-\frac{1}{4}}[q_1\rho(q_1^3)-2\sigma(q_1^3)] + \sqrt{\frac{\alpha}{\pi}}\,J(\alpha),$$

$$q^{-\frac{1}{8}}\psi(-q^{\frac{1}{3}}) = -\sqrt{\frac{\pi}{\alpha}}\,q_1^{-\frac{9}{8}}[q_1\phi(-q_1^3)+\psi(-q_1^3)] + \sqrt{\frac{2\alpha}{\pi}}\,K(\alpha),$$

$$q^{\frac{1}{6}}\rho(q^{\frac{2}{3}}) = \sqrt{\frac{\pi}{2\alpha}}\,q_1^{-\frac{9}{8}}[q_1\phi(q_1^3)-\psi(q_1^3)] - \sqrt{\frac{2\alpha}{\pi}}\,J(2\alpha),$$

$$q^{\frac{1}{12}}\rho(-q^{\frac{1}{3}}) = \sqrt{\frac{\pi}{\alpha}}\,q_1^{-\frac{1}{4}}[q_1\rho(-q_1^3)+2\sigma(-q_1^3)] + \sqrt{\frac{\alpha}{\pi}}\,J(\alpha),$$

$$q^{-\frac{1}{18}}\sigma(q^{\frac{2}{3}}) = \sqrt{\frac{\pi}{8\alpha}}\,q_1^{-\frac{9}{8}}[q_1\phi(q_1^3)+2\psi(q_1^3)] - \sqrt{\frac{2\alpha}{\pi}}\,J_1(2\alpha),$$

$$q^{-\frac{1}{36}}\sigma(-q^{\frac{1}{3}}) = \sqrt{\frac{\pi}{\alpha}}\,q_1^{-\frac{1}{4}}[q_1\rho(-q_1^3)-\sigma(-q_1^3)] - \sqrt{\frac{\alpha}{\pi}}\,J_1(\alpha),$$

$$q^{\frac{1}{6}}\lambda(q^{\frac{2}{3}}) = \sqrt{\frac{8\pi}{\alpha}}\,q_1^{-\frac{9}{8}}[q_1\nu(q_1^3)-\xi(q_1^3)] + \sqrt{\frac{8\alpha}{\pi}}\,J(2\alpha),$$

$$q^{\frac{1}{12}}\lambda(-q^{\frac{1}{3}}) = \sqrt{\frac{\pi}{\alpha}}\,q_1^{-\frac{1}{4}}[q_1\lambda(-q_1^3)+2\mu(-q_1^3)] - \sqrt{\frac{4\alpha}{\pi}}\,J(\alpha),$$

$$q^{-\frac{1}{18}}\mu(q^{\frac{2}{3}}) = \sqrt{\frac{2\pi}{\alpha}}\,q_1^{-\frac{9}{8}}[q_1\nu(q_1^3)+2\xi(q_1^3)] + \sqrt{\frac{8\alpha}{\pi}}\,J_1(2\alpha),$$

$$q^{-\frac{1}{36}}\mu(-q^{\frac{1}{3}}) = \sqrt{\frac{\pi}{\alpha}}\,q_1^{-\frac{1}{4}}[q_1\lambda(-q_1^3)-\mu(-q_1^3)] + \sqrt{\frac{4\alpha}{\pi}}\,J_1(\alpha),$$

$$q^{-\frac{1}{36}}\nu(q^{\frac{2}{3}}) = \sqrt{\frac{\pi}{4\alpha}}\,q_1^{-\frac{1}{4}}[q_1\lambda(q_1^3)+\mu(q_1^3)] - \sqrt{\frac{\alpha}{\pi}}\,J_1(\alpha),$$

$$q^{-\frac{1}{72}}\nu(-q^{\frac{1}{3}}) = \sqrt{\frac{\pi}{\alpha}}\,q_1^{-\frac{9}{8}}[q_1\nu(-q_1^3)-2\xi(-q_1^3)] - \sqrt{\frac{\alpha}{2\pi}}\,K_1(\alpha),$$

$$q^{-\frac{1}{4}}\xi(q^{\frac{2}{3}}) = -\sqrt{\frac{\pi}{16\alpha}}\,q_1^{-\frac{1}{4}}[q_1\lambda(q_1^3)-2\mu(q_1^3)] - \sqrt{\frac{\alpha}{4\pi}}\,J(\alpha),$$

$$q^{-\frac{1}{8}}\xi(-q^{\frac{1}{3}}) = -\sqrt{\frac{\pi}{\alpha}}\,q_1^{-\frac{9}{8}}[q_1\nu(-q_1^3)+\xi(-q_1^3)] - \sqrt{\frac{\alpha}{2\pi}}\,K(\alpha).$$

Before continuing on to the 8th order mfs, we call attention to the *half-shift* method, introduced in [M2, 421–424] and further developed in [GM1, 328–330]. This is a procedure for obtaining mfs from θfs. Generally speaking, it starts with a series

$$\sum_{n=0}^{\infty} a_n \,,$$

where a_n is defined for all real n. One then forms the series

$$\sum_{n=0}^{\infty} b_n \,,$$

where $b_n = a_{n-\frac{1}{2}}$ or $b_n = a_{n+\frac{1}{2}}$. Application of this method to the series in the Rogers-Ramanujan identities (1.2) gives rise to the mf_5s $F_1(q)$, $F_0(q)$ in (2.2). The mf_7s in (2.3) can be obtained by half-shifting the Selberg functions $A(q), B(q), C(q)$ in [Se2, p. 5].

The mf_8s were discovered by applying this method to the series in the Göllnitz-Gordon identities [Gö], [G], [Sl, (34), (36)]:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty},$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q^3;q^8)_\infty(q^4;q^8)_\infty(q^5;q^8)_\infty}.$$

The resulting mf_8s are

$$\left.\begin{aligned} S_0(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n}, & S_1(q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n}, \\ T_0(q) &= \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}}, & T_1(q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}}. \end{aligned}\right\} \tag{5.11}$$

They satisfy the linear relations [GM1] [GM2, p. 222].

$$\left.\begin{aligned} S_0(q^2)+2T_0(q^2) &= \sum_{n=-\infty}^{\infty} \frac{q^{2n^2}(-q^2;q^4)_n}{(-q^4;q^4)_n} = \frac{1}{2}\Big[(-q;q^2)_\infty^3 + (q;q^2)_\infty^3\Big]\theta_4(0,q^2), \\ S_1(q^2)+2T_1(q^2) &= \sum_{n=-\infty}^{\infty} \frac{q^{2n(n+2)}(-q^2;q^4)_n}{(-q^4;q^4)_n} = \frac{1}{2}q^{-1}\Big[(-q;q^2)_\infty^3 - (q;q^2)_\infty^3\Big]\theta_4(0,q^2), \end{aligned}\right\} \tag{5.12}$$

and the mock theta "conjectures" [GM4]

$$S_0(-q^2) = \frac{j(-q,q^2)j(q^6,q^{16})}{j(q^2;q^8)} - 2qg_2(q,q^8),$$

$$S_1(-q^2) = \frac{j(-q,q^2)j(q^2,q^{16})}{j(q^2,q^8)} - 2qg_2(q^3,q^8).$$

Relations (5.12) are 8th order analogues of Watson's 5th order relations (2.11).

The transformation laws for (5.11) involve the mfs

$$U_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n} = S_0(q^2) + qS_1(q^2),$$

$$U_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}} = T_0(q^2) + qT_1(q^2),$$

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n},$$

$$V_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}},$$

the last two of which are obtained by half-shifting the series in identities [Sl, (8), (12)].

The complete set of laws reads as follows [GM1]:

$$q^{-\frac{1}{16}}S_0(q) = \sqrt{\frac{\pi}{4\alpha}}V_0(q_1) + \sqrt{\frac{2\pi}{\alpha}}q_1^{-\frac{1}{4}}V_1(q_1) + \sqrt{\frac{4\alpha}{\pi}}K_3(\alpha),$$

$$q^{\frac{7}{16}}S_1(q) = \sqrt{\frac{\pi}{4\alpha}}V_0(q_1) - \sqrt{\frac{2\pi}{\alpha}}q_1^{-\frac{1}{4}}V_1(q_1) - \sqrt{\frac{4\alpha}{\pi}}K_2(\alpha),$$

$$q^{-\frac{1}{16}}T_0(q) = \sqrt{\frac{\pi}{16\alpha}}V_0(-q_1) - \sqrt{\frac{\pi}{2\alpha}}q_1^{-\frac{1}{4}}V_1(-q_1) - \sqrt{\frac{\alpha}{\pi}}K_3(\alpha),$$

$$q^{\frac{7}{16}}T_1(q) = \sqrt{\frac{\pi}{16\alpha}}V_0(-q_1) + \sqrt{\frac{\pi}{2\alpha}}q_1^{-\frac{1}{4}}V_1(-q_1) + \sqrt{\frac{\alpha}{\pi}}K_2(\alpha),$$

$$q^{-\frac{1}{16}}S_0(-q) = \sqrt{\frac{\pi(2-\sqrt{2})}{\alpha}}q_1^{-\frac{1}{16}}T_0(-q_1) + \sqrt{\frac{\pi(2+\sqrt{2})}{\alpha}}q_1^{\frac{7}{16}}T_1(-q_1) + \sqrt{\frac{4\alpha}{\pi}}J_3(\alpha),$$

$$q^{\frac{7}{16}}S_1(-q) = \sqrt{\frac{\pi(2+\sqrt{2})}{\alpha}}q_1^{-\frac{1}{16}}T_0(-q_1) - \sqrt{\frac{\pi(2-\sqrt{2})}{\alpha}}q_1^{\frac{7}{16}}T_1(-q_1) + \sqrt{\frac{4\alpha}{\pi}}J_2(\alpha),$$

$$q^{-\frac{1}{16}}T_0(-q) = \sqrt{\frac{\pi(2-\sqrt{2})}{16\alpha}}q_1^{-\frac{1}{16}}S_0(-q_1) + \sqrt{\frac{\pi(2+\sqrt{2})}{16\alpha}}q_1^{\frac{7}{16}}S_1(-q_1) - \sqrt{\frac{\alpha}{\pi}}J_3(\alpha),$$

$$q^{\frac{7}{16}}T_1(-q) = \sqrt{\frac{\pi(2+\sqrt{2})}{16\alpha}}q_1^{-\frac{1}{16}}S_0(-q_1) - \sqrt{\frac{\pi(2-\sqrt{2})}{16\alpha}}q_1^{\frac{7}{16}}S_1(-q_1) - \sqrt{\frac{\alpha}{\pi}}J_2(\alpha),$$

$$q^{-\frac{1}{8}}U_0(q) = \sqrt{\frac{\pi}{2\alpha}}V_0\left(q_1^{\frac{1}{2}}\right) + \sqrt{\frac{\alpha}{2\pi}}J\left(\frac{\alpha}{2}\right),$$

$$q^{-\frac{1}{8}}U_1(q) = \sqrt{\frac{\pi}{8\alpha}}V_0\left(-q_1^{\frac{1}{2}}\right) - \sqrt{\frac{\alpha}{8\pi}}J\left(\frac{\alpha}{2}\right),$$

$$V_0(q) = \sqrt{\frac{\pi}{\alpha}}q_1^{-\frac{1}{16}}U_0\left(q_1^{\frac{1}{2}}\right) - \sqrt{\frac{16\alpha}{\pi}}J(4\alpha),$$

$$q^{-\frac{1}{4}}V_1(q) = \sqrt{\frac{\pi}{8\alpha}}q_1^{-\frac{1}{16}}U_0\left(-q_1^{\frac{1}{2}}\right) - \sqrt{\frac{\alpha}{\pi}}K(2\alpha),$$

$$q^{-\frac{1}{8}}U_0(-q) = \sqrt{\frac{4\pi}{\alpha}}q_1^{-\frac{1}{8}}V_1\left(q_1^{\frac{1}{2}}\right) + \sqrt{\frac{2\alpha}{\pi}}K(\alpha),$$

$$q^{-\frac{1}{8}}U_1(-q) = -\sqrt{\frac{\pi}{\alpha}}q_1^{-\frac{1}{8}}V_1\left(-q_1^{\frac{1}{2}}\right) - \sqrt{\frac{\alpha}{2\pi}}K(\alpha),$$

$$V_0(-q) = \sqrt{\frac{4\pi}{\alpha}}q_1^{-\frac{1}{16}}U_1\left(q_1^{\frac{1}{2}}\right) + \sqrt{\frac{16\alpha}{\pi}}J(4\alpha),$$

$$q^{-\frac{1}{4}}V_1(-q) = -\sqrt{\frac{\pi}{2\alpha}}q_1^{-\frac{1}{16}}U_1\left(-q_1^{\frac{1}{2}}\right) - \sqrt{\frac{\alpha}{\pi}}K(2\alpha).$$

Here the Mordell integrals J_2, J_3, K_2, K_3 are

$$J_2(\alpha) = \int_0^\infty \mathrm{e}^{-\alpha x^2}\frac{\cosh\frac{1}{2}\alpha x}{\cosh 2\alpha x}\,\mathrm{d}x, \qquad J_3(\alpha) = \int_0^\infty \mathrm{e}^{-\alpha x^2}\frac{\cosh\frac{3}{2}\alpha x}{\cosh 2\alpha x}\,\mathrm{d}x,$$
$$K_2(\alpha) = \int_0^\infty \mathrm{e}^{-\alpha x^2}\frac{\sinh\frac{1}{2}\alpha x}{\sinh 2\alpha x}\,\mathrm{d}x, \qquad K_3(\alpha) = \int_0^\infty \mathrm{e}^{-\alpha x^2}\frac{\sinh\frac{3}{2}\alpha x}{\sinh 2\alpha x}\,\mathrm{d}x.$$

In view of the last eight transformation laws, a case can be made for regarding $U_0(q)$, $U_1(q)$, $V_0(q)$, $V_1(q)$ as mf$_2$s. Indeed, the relevant Mordell integrals are the same as those in (5.3).

In [M3] it is shown that $V_1(q)$ is equal to the function

$$\lambda(q) = \sum_{n=0}^{\infty}\frac{q^{n+1}(-q;q)_{2n}}{(-q^2;q^4)_{n+1}},$$

found on page 8 of the Lost Notebook (see also [A2, (3.21)] and [AB, (12.5.3)]).

The functions $U_i(q)$ and $V_i(q)$ satisfy the linear relations [GM1], [M3]

$$\left.\begin{aligned} U_0(q)+2U_1(q) &= (q)_\infty(-q;q^2)_\infty^4\,,\\ V_0(q)+V_0(-q) &= 2(-q^2;q^4)_\infty^4(q^8;q^8)_\infty\,,\\ V_1(q)-V_1(-q) &= 2q(-q^2;q^2)_\infty(-q^4;q^4)_\infty^2(q^8;q^8)_\infty\,, \end{aligned}\right\} \tag{5.13}$$

and are connected to the mf$_2$s (5.1) by

$$\left.\begin{aligned} U_0(q)-2U_1(q) &= \mu(q)\,,\\ V_0(q)-V_0(-q) &= 4qB(q^2)\,,\\ V_1(q)+V_1(-q) &= 2A(q^2)\,, \end{aligned}\right\} \tag{5.14}$$

proved in [M3]. Combining (5.13) and (5.14), we obtain

$$\left.\begin{aligned} 2U_0(q) &= (q)_\infty(-q;q^2)_\infty^4+\mu(q)\,,\\ 4U_1(q) &= (q)_\infty(-q;q^2)_\infty^4-\mu(q)\,,\\ V_0(q) &= (-q^2;q^4)_\infty^4(q^8;q^8)_\infty+2qB(q^2)\,,\\ V_1(q) &= q(-q^2;q^2)_\infty(-q^4;q^4)_\infty^2(q^8;q^8)_\infty+A(q^2)\,. \end{aligned}\right\} \tag{5.15}$$

Identities (5.15), together with (5.2), yield mock theta "conjectures" for $U_i(q)$ and $V_i(q)$. For example, $V_1(q)=qg_2(q,q^4)$, proved in [GM1, 322–324].

We turn finally to the mf$_{10}$s, which appear on page 9 of the Lost Notebook:

$$\left.\begin{aligned} \phi(q) &= \sum_{n=0}^\infty \frac{q^{\frac12 n(n+1)}}{(q;q^2)_{n+1}}\,, & \psi(q) &= \sum_{n=0}^\infty \frac{q^{\frac12 (n+1)(n+2)}}{(q;q^2)_{n+1}}\,,\\ X(q) &= \sum_{n=0}^\infty \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}}\,, & \chi(q) &= \sum_{n=0}^\infty \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}\,. \end{aligned}\right\} \tag{5.16}$$

Srivastava [Sr1] showed that these functions can be obtained by half-shifting:

$$\left.\begin{aligned} \phi(q^2)+1 \text{ arises from } & (q^4;q^4)_\infty\sum_{n=0}^\infty \frac{q^{n^2}}{(q^4;q^4)_n} = (q^2;q^2)_\infty G(q)\,,\\ \psi(q^2)+1 \text{ arises from } & (q^4;q^4)_\infty\sum_{n=0}^\infty \frac{q^{n(n+2)}}{(q^4;q^4)_n} = (q^2;q^2)_\infty H(q)\,,\\ X(q) \text{ arises from } & \sum_{n=0}^\infty \frac{(-1)^n q^{n(n+1)}}{(-q;q)_{2n+1}} = \sum_{n=0}^\infty q^{\frac12 n(5n+3)}(1-q^{2n+1})\,,\\ \chi(q) \text{ arises from } & \sum_{n=0}^\infty \frac{(-1)^n q^{n(n+3)}}{(-q;q)_{2n+2}} = \sum_{n=0}^\infty q^{\frac12 n(5n+9)}(1-q^{n+1})\,. \end{aligned}\right\} \tag{5.17}$$

The first two of these half-shifts were earlier applied in [M2, 423]. They, together with the transformation laws, make it plausible to assign order 5, rather than order 10, to the functions (5.16).

The first two identities of (5.17) appear in [Sl, (16), (20)] and the third in [Ro, (9.7)] (see also [A6, 36, 58, 92]). The fourth identity is easily deduced from

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)}}{(-q;q)_{2n}}=\sum_{n=0}^{\infty}q^{\frac{1}{2}n(5n+1)}(1-q^{4n+2})=\left(\sum_{n\geq 0}-\sum_{n<0}\right)q^{\frac{1}{2}n(5n+1)},$$

which is part of [Ro, (9.7)].

Ramanujan gave eight linear relations connecting the mf_{10}s, which were proved by Choi [C1]–[C4]. In the notation of this survey, they are

$$q^2\phi(q^9)-\frac{\psi(\omega q)-\psi(\omega^2 q)}{\omega-\omega^2}=-q\frac{(q)_\infty j(q,q^6)j(q^3,q^{15})}{(q^3;q^3)_\infty^2},$$

$$q^{-2}\psi(q^9)+\frac{\omega\phi(\omega q)-\omega^2\phi(\omega^2 q)}{\omega-\omega^2}=\frac{(q)_\infty j(q,q^6)j(q^6,q^{15})}{(q^3;q^3)_\infty^2},$$

$$X(q^9)-\frac{\omega\chi(\omega q)-\omega^2\chi(\omega^2 q)}{\omega-\omega^2}=\frac{(q^2;q^2)_\infty^2(q^3;q^3)_\infty^2 j(q^{12},q^{30})}{(q)_\infty(q^6;q^6)_\infty^3},$$

$$\chi(q^9)+q^2\frac{X(\omega q)-X(\omega^2 q)}{\omega-\omega^2}=-q^3\frac{(q^2;q^2)_\infty^2(q^3;q^3)_\infty^2 j(q^6,q^{30})}{(q)_\infty(q^6;q^6)_\infty^3},$$

$$\phi(q)-q^{-1}\psi(-q^4)+q^{-2}\chi(q^8)=\frac{j(-q,q^2)j(-q^2,-q^{10})}{j(q^2,q^8)},$$

$$\psi(q)+q\phi(-q^4)+X(q^8)=\frac{j(-q,q^2)j(q^4,-q^{10})}{j(q^2,q^8)},$$

$$\int_0^\infty\frac{\mathrm{e}^{-\pi nx^2}}{\cosh\frac{2\pi x}{\sqrt{5}}+\frac{1+\sqrt{5}}{4}}\,\mathrm{d}x+\frac{1}{\sqrt{n}}\mathrm{e}^{\frac{\pi}{5n}}\psi(-\mathrm{e}^{-\frac{\pi}{n}})$$
$$=\sqrt{\frac{5+\sqrt{5}}{2}}\mathrm{e}^{-\frac{\pi n}{5}}\phi(-\mathrm{e}^{-\pi n})-\frac{\sqrt{5}+1}{2\sqrt{n}}\mathrm{e}^{-\frac{\pi}{5n}}\phi(-\mathrm{e}^{-\frac{\pi}{n}}),$$

$$\int_0^\infty\frac{\mathrm{e}^{-\pi nx^2}}{\cosh\frac{2\pi x}{\sqrt{5}}+\frac{1-\sqrt{5}}{4}}\,\mathrm{d}x+\frac{1}{\sqrt{n}}\mathrm{e}^{\frac{\pi}{5n}}\psi(-\mathrm{e}^{-\frac{\pi}{n}})$$
$$=-\sqrt{\frac{5-\sqrt{5}}{2}}\mathrm{e}^{\frac{\pi n}{5}}\psi(-\mathrm{e}^{-\pi n})+\frac{\sqrt{5}-1}{2\sqrt{n}}\mathrm{e}^{-\frac{\pi}{5n}}\phi(-\mathrm{e}^{-\frac{\pi}{n}}),$$

where ω is a primitive cube root of unity and n is a positive real number.

The mock theta "conjectures" of order 10 are

$$\left.\begin{aligned}
\phi(q) &= \frac{(q^{10};q^{10})_\infty^2 j(-q^2,q^5)}{(q^5;q^5)_\infty j(q^2,q^{10})} + 2qg_2(q^2,q^5),\\
\psi(q) &= -\frac{q(q^{10};q^{10})_\infty^2 j(-q,q^5)}{(q^5;q^5)_\infty j(q^4,q^{10})} + 2qg_2(q,q^5),\\
X(-q^2) &= \frac{(q^4;q^4)_\infty^2 \left(j(-q^2,q^{20})^2 j(q^{12},q^{40}) + 2q(q^{40};q^{40})_\infty^3\right)}{(q^2;q^2)_\infty (q^{20};q^{20})_\infty (q^{40};q^{40})_\infty j(q^8,q^{40})}\\
&\quad -2qg_2(q,q^{20}) + 2q^5 g_2(q^9,q^{20}),\\
\chi(-q^2) &= \frac{q^2(q^4;q^4)_\infty^2 \left(2q(q^{40};q^{40})_\infty^3 - j(-q^6,q^{20})^2 j(q^4,q^{40})\right)}{(q^2;q^2)_\infty (q^{20};q^{20})_\infty (q^{40};q^{40})_\infty j(q^{16},q^{40})}\\
&\quad -2q^3 g_2(q^3,q^{20}) - 2q^5 g_2(q^7,q^{20}).
\end{aligned}\right\} \tag{5.18}$$

The first two are proved in [C1, 533–534], and the last two in [GM4].

The transformation laws for (5.16) are

$$q^{\frac{1}{5}}\phi(q) = \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}q_1^{-\frac{1}{20}}X(q_1^2) - \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}q_1^{-\frac{9}{20}}\chi(q_1^2) - \sqrt{\frac{20\alpha}{\pi}}J_4(\alpha),$$

$$q^{\frac{1}{5}}\phi(-q) = \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}q_1^{\frac{1}{5}}\phi(-q_1) + \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}q_1^{-\frac{1}{5}}\psi(-q_1) + \sqrt{\frac{20\alpha}{\pi}}K_4(\alpha),$$

$$q^{-\frac{1}{5}}\psi(q) = \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}q_1^{-\frac{1}{20}}X(q_1^2) + \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}q_1^{-\frac{9}{20}}\chi(q_1^2) - \sqrt{\frac{20\alpha}{\pi}}J_5(\alpha),$$

$$q^{-\frac{1}{5}}\psi(-q) = \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}q_1^{\frac{1}{5}}\phi(-q_1) - \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}q_1^{-\frac{1}{5}}\psi(-q_1) - \sqrt{\frac{20\alpha}{\pi}}K_6(\alpha),$$

$$q^{-\frac{1}{40}}X(q) = \sqrt{\frac{\pi(5+\sqrt{5})}{5\alpha}}q_1^{\frac{2}{5}}\phi(q_1^2) + \sqrt{\frac{\pi(5-\sqrt{5})}{5\alpha}}q_1^{-\frac{2}{5}}\psi(q_1^2) + \sqrt{\frac{10\alpha}{\pi}}K_7\left(\frac{\alpha}{2}\right),$$

$$q^{-\frac{1}{40}}X(-q) = \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}q_1^{-\frac{1}{40}}X(-q_1) - \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}q_1^{-\frac{9}{40}}\chi(-q_1) + \sqrt{\frac{40\alpha}{\pi}}J_6(\alpha),$$

$$q^{-\frac{9}{40}}\chi(q) = -\sqrt{\frac{\pi(5-\sqrt{5})}{5\alpha}}q_1^{\frac{2}{5}}\phi(q_1^2) + \sqrt{\frac{\pi(5+\sqrt{5})}{5\alpha}}q_1^{-\frac{2}{5}}\psi(q_1^2) + \sqrt{\frac{10\alpha}{\pi}}K_5\left(\frac{\alpha}{2}\right),$$

$$q^{-\frac{9}{40}}\chi(-q) = -\sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}q_1^{-\frac{1}{40}}X(-q_1) - \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}q_1^{-\frac{9}{40}}\chi(-q_1) + \sqrt{\frac{40\alpha}{\pi}}J_7(\alpha),$$

where as usual $q = \mathrm{e}^{-\alpha}$ and $q_1 = \mathrm{e}^{-\beta}$ with $\alpha\beta = \pi^2$. The Mordell integrals J_n and K_n $(n = 4,5,6,7)$ are

$$J_4(\alpha) = \int_0^\infty \mathrm{e}^{-5\alpha x^2} \frac{\cosh \alpha x}{\cosh 5\alpha x}\,\mathrm{d}x, \qquad J_5(\alpha) = \int_0^\infty \mathrm{e}^{-5\alpha x^2} \frac{\cosh 3\alpha x}{\cosh 5\alpha x}\,\mathrm{d}x,$$

$$J_6(\alpha) = \int_0^\infty \mathrm{e}^{-10\alpha x^2} \frac{\cosh 9\alpha x - \cosh \alpha x}{\cosh 10\alpha x}\,\mathrm{d}x,$$

$$J_7(\alpha) = \int_0^\infty \mathrm{e}^{-10\alpha x^2} \frac{\cosh 7\alpha x + \cosh 3\alpha x}{\cosh 10\alpha x}\,\mathrm{d}x,$$

$$K_4(\alpha) = \int_0^\infty \mathrm{e}^{-5\alpha x^2} \frac{\sinh \alpha x}{\sinh 5\alpha x}\,\mathrm{d}x, \qquad K_5(\alpha) = \int_0^\infty \mathrm{e}^{-5\alpha x^2} \frac{\sinh 2\alpha x}{\sinh 5\alpha x}\,\mathrm{d}x,$$

$$K_6(\alpha) = \int_0^\infty \mathrm{e}^{-5\alpha x^2} \frac{\sinh 3\alpha x}{\sinh 5\alpha x}\,\mathrm{d}x, \qquad K_7(\alpha) = \int_0^\infty \mathrm{e}^{-5\alpha x^2} \frac{\sinh 4\alpha x}{\sinh 5\alpha x}\,\mathrm{d}x.$$

6 General Relations Between Mock Theta Functions

In this section we consider how linear relations between mfs can be found. One method is to compare the Mordell integrals in their transformation laws. If two mfs transform with the same Mordell integral, their difference may well be a θf. If this difference is a θ-product, a proposed expression for it can be found by computer algebra. Sometimes the difference is not itself a θ-product, but the even and odd parts of its q-series are. This phenomenon underlies the last two mock theta "conjectures" in (5.18).

Comparison of the Mordell integrals in (4.2) and (4.16) suggests that

$$g_3(q^{4r},q^4) - q^{1-2r} g_2(q^{6r+1},q^6) - q^{2r-1} g_2(q^{6r-1},q^6)$$

is a θf. Computer algebra leads to the conjecture that

$$\begin{aligned} g_3(x^4,q^4) = {} & \frac{q g_2(x^6 q,q^6)}{x^2} + \frac{x^2 g_2(x^6 q^{-1},q^6)}{q} \\ & - \frac{x^2 (q^2;q^2)_\infty^3 (q^{12};q^{12})_\infty j(x^2 q,q^2) j(x^{12} q^6,q^{12})}{q (q^4;q^4)_\infty (q^6;q^6)_\infty^2 j(x^4,q^2) j(x^6 q^{-1},q^2)}, \end{aligned} \tag{6.1}$$

which is proved in [GM3]. Further development along the lines of (6.1) leads to identities involving Appell functions and Zweger's μ-function ([M5], followed by [BO2], [BOR], [K]). For more details see Sect. 9.

It was noted in Sect. 3 that the mfs of odd order are related to $g_3(x,q)$, and in Sect. 5 that the mfs of even order are related to $g_2(x,q)$. By (6.1) and its limiting cases discussed in Sect. 7, we can express all of the classical mfs in terms of $g_2(x,q)$. For this reason we can regard $g_2(x,q)$ as a *universal mock θ-function*.

Identity (6.1) has a broad generalization which we now develop. Recall the generalized Lambert series

$$u_k(x,q) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{1-xq^n} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{(1-xq^n)(1-x^{-1}q^{n+1})},$$

$$v_k(y,q) = \frac{1}{1-y^{-1}} \sum_{n=-\infty}^{\infty} \frac{(-1)^{kn} q^{\frac{1}{2}n(kn+1)}}{1-yq^n} = \sum_{n=-\infty}^{\infty} \frac{(-1)^{kn} q^{\frac{1}{2}n(kn+1)}}{(1-yq^n)(1-y^{-1}q^n)}$$

of (4.7) and (4.9). Hence

$$u_k(x,q) = u_k(x^{-1}q,q), \qquad v_k(y,q) = v_k(y^{-1},q).$$

From the definition of $u_k(x,q)$ it is easily seen that

$$u_{2k}(x,q) + u_{2k}(-x,q) = 2u_k(x^2,q^2).$$

Somewhat more difficult to prove [GM3] are the functional equation

$$u_k(xq,q) = -x^k u_k(x,q) - \sum_{m=1}^{k-1} x^m j(q^m,q^k), \tag{6.2}$$

and for odd k the identity

$$(1-x)v_k(x,q) = -x^{\frac{1}{2}(k+1)} u_k(x,q) - \sum_{m=1}^{\frac{1}{2}(k-1)} x^{\frac{1}{2}(k+1)-m} j(q^m,q^k). \tag{6.3}$$

When $k=3$, (6.3) says that

$$(1-x)v_3(x,q) = -x^2 u_3(x,q) - x(q)_\infty,$$

which is equivalent to

$$\frac{h_3(x,q)}{1-x} = xg_3(x,q) + 1 \tag{6.4}$$

by (4.10). For even k the situation is more complicated. A proof that

$$\frac{h_2(-x^2q^{-1},q^2)}{1+x^2q^{-1}} = \frac{(q^2;q^2)_\infty^3}{j(x,q)j(-x^2q^{-1},q^4)} - \frac{qg_2(x,q)}{x}\,. \tag{6.5}$$

is given in [GM3].

Another useful identity [GM3], valid for all positive integers k, is

$$\sum_{n=-\infty}^{\infty}{}' \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{1-q^n} = \sum_{m=1}^{k-1} \frac{1-j(q^m,q^k)}{2}\,, \tag{6.6}$$

where the dash indicates that the term with $n=0$ is to be omitted.

We can now state the general identities of which (6.11) is the special case $k=3$. These identities express $u_k(x,q)$ in terms of $u_2(x,q)$ (and hence in terms of the universal mf $g_2(x,q)$). They are as follows [GM3]:

$$\begin{aligned} u_k(x^4,q^4) &= -\frac{x^2(q^2;q^2)_\infty^3 j(x^{2k-4}q,q^2)j(x^{4k}q^{2k},q^{4k})}{qj(x^4,q^2)j(x^{2k}q^{-1},q^2)j(q^{2k},q^{4k})} \\ &\quad + \sum_{m=1}^{k-1} \frac{q^{k-2m}j(q^{4m},q^{4k})}{x^{2k-4m}j(q^{2k},q^{4k})} u_2(x^{2k}q^{k-2m},q^{2k})\,, \quad k \text{ odd}\,, \end{aligned} \tag{6.7}$$

$$\begin{aligned} u_k(x^4,q^4) &= -\frac{x^4(q^2;q^2)_\infty^3 j(x^{2k-4},q^2)j(x^{4k}q^{2k},q^{4k})}{q^2j(x^4,q^2)j(x^{2k}q^{-2},q^2)j(q^{2k},q^{4k})} \\ &\quad + \sum_{m=1}^{k-1} \frac{q^{k-2m}j(q^{4m},q^{4k})}{x^{2k-4m}j(q^{2k},q^{4k})} u_2(x^{2k}q^{k-2m},q^{2k})\,, \quad k \text{ even}\,. \end{aligned} \tag{6.8}$$

These identities are proved by showing that both sides satisfy the same functional equation and that their difference has only removable singularities for q fixed and $x \neq 0$.

When $k=3$, (6.7) becomes

$$\begin{aligned} u_3(x^4,q^4) &= \frac{qj(q^4,q^{12})}{x^2j(q^6,q^{12})} u_2(x^6q,q^6) + \frac{x^2j(q^4,q^{12})}{qj(q^6,q^{12})} u_2(x^6q^{-1},q^6) \\ &\quad - \frac{x^2(q^2;q^2)_\infty^3 j(x^2q,q^2)j(x^{12}q^6,q^{12})}{qj(x^4,q^2)j(x^6q^{-1},q^2)j(q^6,q^{12})}\,, \end{aligned} \tag{6.9}$$

since $j(q^8,q^{12}) = j(q^4,q^{12})$. It follows from (4.10) and (4.14) that

$$u_2(x^6q^{\pm 1},q^6) = \frac{(q^6;q^6)_\infty^2}{(q^{12};q^{12})_\infty} g_2(x^6q^{\pm 1},q^6) = j(q^6,q^{12})g_2(x^6q^{\pm 1},q^6)$$

and

$$u_3(x^4,q^4) = (q^4;q^4)_\infty g_3(x^4,q^4) = j(q^4,q^{12})g_3(x^4,q^4).$$

Substituting these expressions for u_2 and u_3 into (6.9), we obtain (6.1).

7 Singular Cases of the General Relations

As already remarked, identities (6.7) and (6.8) hold whenever all the terms are defined. These terms, regarded as functions of x with q fixed, are meromorphic for $x \neq 0$. At their poles, which are all simple, (6.7) and (6.8) become identities between Laurent series. By equating the constant terms in the Laurent series of the two sides, we obtain a set of identities which we call *singular cases*.

The left sides of (6.7) and (6.8) are defined when $x \neq 0$ and $x^4q^{4n} \neq 1$ for any $n \in \mathbf{Z}$. For such x, the right sides of (6.7) and (6.8) are undefined when $x = \mu q^{\frac{m_0}{k}-\frac{1}{2}+n}$, where $\mu^{2k} = 1$, $1 \leq m_0 \leq k-1$ and $n \in \mathbf{Z}$. In this case the product on the right side of either (6.7) or (6.7) and the mth term of the sum have simple poles of equal residues. The constant terms of their Laurent series can be determined. For the mth term of the sum this is done using (6.6), and for the product, by logarithmic differentiation. This results in identities of the form

$$u_k(x^4,q^4) = T(x,q) + \sum_{\substack{m=1\\ m\neq m_0}}^{k-1} \frac{q^{k-2m} j(q^{4m},q^{4k})}{x^{2k-4m} j(q^{2k},q^{4k})} u_2(x^{2k}q^{k-2m},q^{2k}), \tag{7.1}$$

where $x = \mu q^{\frac{m_0}{k}-\frac{1}{2}+n}$, and $T(x,q)$ is a θf [M6].

For example, when $k=3$ and $m_0 = 2$, identity (7.1) becomes

$$u_3(q^{\frac{2}{3}},q^4) = \frac{(q^4;q^4)_\infty(q^6;q^6)_\infty(q^{12};q^{12})_\infty}{(q^2;q^2)_\infty} + \frac{(q^4;q^4)_\infty^2(q^6;q^6)_\infty^4}{2q^{\frac{2}{3}}(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty^2} - \frac{(q^4;q^4)_\infty}{2q^{\frac{2}{3}}}$$
$$+ \frac{q^{\frac{2}{3}} j(q^4,q^{12})}{j(q^6,q^{12})} u_2(q^2,q^6)$$

or equivalently

$$g_3(q,q^6) = \frac{(q^9;q^9)_\infty(q^{18};q^{18})_\infty}{(q^3;q^3)_\infty} + \frac{(q^6;q^6)_\infty(q^9;q^9)_\infty^4}{2q(q^3;q^3)_\infty^2(q^{18};q^{18})_\infty^2} - \frac{1}{2q} + qg_2(q^3,q^9). \tag{7.2}$$

Ramanujan's letter includes the identity (2.6),

$$4h_3(\mathrm{e}^{\frac{\pi i}{3}},q) - h_3(-1,q) = 3\theta_4^2(0,q^3)(q)_\infty^{-1},$$

proved by Watson [W1, p. 63]. After modular transformation, this becomes

$$1+2qg_3(q,q^6)-q^2g_3(q^3,q^6)=\frac{(q^2;q^2)_\infty^4}{(q)_\infty^2(q^6;q^6)_\infty}\,. \tag{7.3}$$

Using (6.1) with q replaced by $q^{\frac{3}{2}}$ and $x=q^{\frac{3}{4}}$, (7.3) can be written as

$$g_3(q,q^6)=-\frac{q(q^{18};q^{18})_\infty^4}{2(q^6;q^6)_\infty(q^9;q^9)_\infty^2}+\frac{(q^2;q^2)_\infty^4}{2q(q)_\infty^2(q^6;q^6)_\infty}-\frac{1}{2q}+qg_2(q^3,q^9)\,. \tag{7.4}$$

Equality of the θfs in (7.2) and (7.4) is not hard to prove. This alternate derivation of (7.2) does not extend to a proof of (7.1) for $k>3$.

Another family of singular cases is obtained from (6.7) and (6.8) when $x=\mu q^{-n}$, where $\mu^4=1$ and $n\in\mathbf{Z}$. Using (6.6) and logarithmic differentiation to calculate the constant terms of their Laurent series, we get identities of the form

$$\sum_{m=1}^{k-1}\frac{q^{k-2m}j(q^{4m},q^{4k})}{x^{2k-4m}j(q^{2k},q^{4k})}\,u_2(x^{2k}q^{k-2m},q^{2k}) = T(x,q)\,,\quad k\text{ odd},$$

$$\sum_{\substack{m=1\\ m\neq k/2}}^{k-1}\frac{q^{k-2m}j(q^{4m},q^{4k})}{x^{2k-4m}j(q^{2k},q^{4k})}\,u_2(x^{2k}q^{k-2m},q^{2k}) = T(x,q)\,,\quad k\text{ even},$$

where $T(x,q)$ is a θf. These identities can also be proved using (6.2). More precisely, repeated application of (6.2) shows that

$$\frac{q^{k-2m}}{x^{2k-4m}}\,u_2(x^{2k}q^{k-2m},q^{2k})+\frac{q^{k-2(k-m)}}{x^{2k-4(k-m)}}\,u_2(x^{2k}q^{k-2(k-m)},q^{2k})$$
$$=\frac{q^{k-2m}}{x^{2k-4m}}\,u_2(x^{2k}q^{k-2m},q^{2k})+\frac{x^{2k-4m}}{q^{k-2m}}\,u_2(x^{2k}q^{2m-k},q^{2k})$$

is a θf whenever $x=\mu q^{-n}$.

8 Related Results

(A) Several authors, including Andrews, Garvan, and Agarwal [AG], [AA], [Ag1], [Ag2], [Ag3], have found combinatorial interpretations of the coefficients $a(n)$ in the q-series expansions

$$M(q)=\sum_{n=0}^{\infty}a(n)q^n$$

of mfs $M(q)$. These involve ranks of partitions of n (largest part minus number of parts) as well as partitions in which each part k can occur in as many as k different "colors." For example, if

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} = \sum_{n=0}^{\infty} a(n)q^n$$

is the mf$_3$ discussed in Sects. 1 and 3, $a(n)$ is the number of partitions of n with even rank minus the number of those with odd rank.

(B) Srivastava [Sr2] has obtained analogues for mf$_6$s and mf$_8$s of Andrews's identities (3.8) and (3.9) (which express mf$_5$s as constant terms of the z-Laurent expansions of θfs $T(z,q)$). These include, in the notation of (3.8), (5.6), and (5.11)

Order 6

$$j(q,q^4)\phi(q) = \text{ coefficient of } z^0 \text{ in}$$
$$\frac{q^2(q^4;q^4)_\infty(q^6;q^6)_\infty^2\Theta(-zq^5,q^4)\Theta(z,q^6)\Theta(q^4,q^6)}{\Theta(z^{-1}q^2,q^6)\Theta(zq^2,q^6)\Theta(q^2,q^6)},$$

Order 8

$$j(q,q^4)S_0(q) = \text{ coefficient of } z^0 \text{ in}$$
$$\frac{-q^3(q^4;q^4)_\infty(q^8;q^8)_\infty^2\Theta(-zq^6,q^4)\Theta(z,q^8)\Theta(q^6,q^4)}{\Theta(-z^{-1}q^3,q^8)\Theta(-zq^3,q^8)\Theta(-q^3,q^8)}.$$

(C) In his work on mathematical physics and the quantum invariant of a three-manifold, Hikami [Hk1], [Hk2] encountered the q-series

$$D_5(q) = \sum_{n=0}^{\infty} \frac{q^n(-q;q)_n}{(q;q^2)_{n+1}}.$$

He noted that

$$D_5(q) = 2h_1(q) - (-q;q)_\infty^2\omega(q), \tag{8.1}$$

where

$$h_1(q) = \sum_{n=0}^{\infty} \frac{q^n(-q;q)_{2n}}{(q;q^2)_{n+1}^2} = \frac{1}{\theta_4(0,q)} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+2)}}{1-q^{2n+1}}$$
$$= \frac{1}{2\theta_4(0,q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+2)}}{1-q^{2n+1}},$$

and $\omega(q)$ is the mf$_3$ in (2.4). (A misprint in the original definition of $h_1(q)$ has been corrected here.) Equation (8.1) is the special case $a = 1$ of Ramanujan's identity [R2, 1]:

$$\left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty}\frac{q^n(-q;q)_{2n}}{(aq;q^2)_{n+1}(a^{-1}q;q^2)_{n+1}} = \frac{(-q;q)_\infty}{(aq;q^2)_\infty}\sum_{n=0}^{\infty}\frac{a^{-n-1}q^{2n(n+1)}}{(q;q^2)_{n+1}(a^{-1}q;q^2)_{n+1}} + \sum_{n=0}^{\infty}\frac{q^n(-q;q)_n}{(aq;q^2)_{n+1}}$$

for all $a \neq 0$. In [A2, 42], it is shown that

$$B(q) = \frac{1}{\theta_4(0,q^2)}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n(n+1)}}{1-q^{2n+1}},$$

where $B(q)$ is the mf$_2$ in (5.1). From this it follows that

$$h_1(q^2) = \frac{B(q)-B(-q)}{4q}.$$

The even part of $B(q)$ is the θf

$$\frac{B(q)+B(-q)}{2} = \frac{1}{\theta_4(0,q^2)}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n(n+1)}}{1-q^{4n+2}} = (q^4;q^4)_\infty(-q^2;q^2)_\infty^4.$$

Hence the odd part of $B(q)$ is an mf$_2$. Thus $D_5(q)$ is a linear combination of mfs of different orders.

9 Concluding Remarks: The Maass Form Era

The aim of this survey has been to document aspects of the theory of mock theta functions which were discovered and proved using classical methods. As noted in the introduction, the twenty first century saw a dramatic infusion of new insights and achievements which largely transformed the subject. A comprehensive, lucid account of these developments is contained in Ono's memoir [On]. Our purpose here is to provide some remarks which may help smooth the transition to the newer approach.

Central to this approach is the theory of *Maass forms*, initiated in [Ma]. These are real-analytic, complex-valued functions $F(z)$, where $z = x + iy$ is in the upper half plane $\mathbf{H}$. Their defining properties are

(i) For every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in a given subgroup Γ of the modular group $\Gamma(1)$, we have

$$F\left(\frac{az+b}{cz+d}\right) = F(z)$$

(ii) $F(x+iy)$ is an eigenfunction of the hyperbolic Laplacian

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

(iii) For some $N > 0$,

$$F(x+iy) = O(y^N) \quad \text{as} \quad y \to \infty$$

A *Maass cusp form* is one which satisfies the additional condition

(iv) $\int_0^1 F(z+x)\,\mathrm{d}x = 0$.

Also needed is the concept of a *weakly harmonic* Maass form of weight k. Here Γ is either $\Gamma_0(N)$ or $\Gamma_1(N)$, and k is an integer or half an odd integer; in the latter case, $4|N$. Property (i) is replaced by

(i′)

$$F\left(\frac{az+b}{cz+d}\right) = \begin{cases} (cz+d)^k F(z) & \text{if } k \in \mathbf{Z}, \\ \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} (cz+d)^k F(z) & \text{otherwise},. \end{cases}$$

where $\left(\frac{c}{d}\right)$ is the Kronecker symbol, $\sqrt{cz+d}$ is the principal branch of the holomorphic square root, and

$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

Property (ii) is replaced by

(ii′)

$$\Delta_k F = 0,$$

where

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Property (iii) is replaced by the existence of a polynomial

$$P_F = \sum_{n \le 0} c_F^+(n) q^n \in \mathbf{C}[q^{-1}]$$

such that

(iii′)

$$F(z) - P_F(z) = O(\mathrm{e}^{-\epsilon y}) \quad \text{as} \quad y \to \infty$$

for some $\epsilon > 0$, with analogous requirements at the other cusps of Γ. This is a more precise formulation of the requirement that $F(z)$ have at most linear exponential growth at the cusps.

Both θfs and mfs are related to weakly harmonic Maass forms $F(z)$ of weight $2-k$ on the group $\Gamma_1(N)$, where $k > 1$ is an integer or a half-integer. The connection involves the Fourier series of $F(z)$, i.e., its expansion in powers of the uniformizing variable $q = \mathrm{e}^{2\pi iz}$. This series has the form

$$F(z) = \sum_{n>>-\infty} c^+(n)q^n + \sum_{n\leq 0} c^-(n)\Gamma(k-1, 4\pi|n|y)q^n, \tag{9.1}$$

where $z = x+iy \in \mathbf{H}$ and $\Gamma(s,t)$ is the incomplete Γ-function

$$\Gamma(s,t) = \int_t^\infty \mathrm{e}^{-u}u^{s-1}\,\mathrm{d}u\,.$$

The first series in (9.1) is denoted by $F^+(z)$ and is called the *holomorphic part* of $F(z)$. The second series is denoted by $F^-(z)$ and called the *nonholomorphic part* of $F(z)$.

It has been established (see [On] for details) that an extensive class of θfs $T(q)$ and mfs $M(q)$ are of the form $F^+(q)$ for some harmonic Maass form $F(z)$. When $T(q) = F^+(z)$, the nonholomorphic $F^-(z)$ is a period integral, and when $M(q) = F^+(z)$ the part $F^-(z)$ is a Mordell integral. Thus harmonic Maass forms constitute a fundamental link between θfs and mfs, quite apart from the defining properties (i), (ii), (iii) or (i), (iii), (iv) stated in the introduction.

Throughout the rest of this section we will use the modern notation $q = \mathrm{e}^{2\pi i\tau}$. For $l > 0$ the Appell function of level l (not to be confused with the level of a modular form) is defined by

$$A_l(u,v;\tau) = a^{\frac{l}{2}} \sum_{n=-\infty}^{\infty} \frac{(-1)^{ln} q^{\frac{1}{2}ln(n+1)} b^n}{1-aq^n}\,,$$

where $a = \mathrm{e}^{2\pi iu}$, $b = \mathrm{e}^{2\pi iv}$. These functions are related to Lerch sums [L1], [L2]. Zagier and Zwegers (see [Zw3]) showed that

$$A_l(u,v;\tau) = \sum_{m=0}^{l-1} a^m A_1(lu, v+m\tau+(l-1)/2; l\tau)\,.$$

It follows that u_2, u_3, and hence g_2, g_3, can be expressed in terms of A_1. In his ground-breaking work on mock theta functions, Zwegers [Zw2] constructed a nonholomorphic Jacobi form $\hat{\mu}(u,v;\tau)$ which is a "completion" of the normalized level 1 Appell function:

$$\mu(u,v;\tau)=\frac{1}{\vartheta(v;\tau)}A_1(u,v;\tau),$$

where

$$\vartheta(v;\tau)=i\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2}b^{n+\frac{1}{2}}=-ib^{-\frac{1}{2}}q^{\frac{1}{8}}j(b,q).$$

Zwegers established laws governing the behavior of $\hat{\mu}(u,v;\tau)$ under transformation of the elliptic variables u, v and the modular variable τ. Using these, he constructed weight 1/2 weakly harmonic Maass forms which include completions of many of the classical mfs. Zwegers' work reveals fundamental properties of these functions.

The transformation laws for g_2 and μ can be combined to eliminate the Mordell integrals. This resulting transformation law is

$$\begin{aligned}&q^{-\frac{1}{2}r^2}\left(q^{\frac{1}{8}}g_2(a^{\frac{1}{2}}b^{-\frac{1}{2}}q^{\frac{1}{4}},q^{\frac{1}{2}})+i\mu(u,v;\tau)\right)\\&\quad=\frac{1}{\sqrt{-i\tau}}\left(\frac{1}{2}\sec(\pi r)\,q_1^{-\frac{1}{8}}h_2(-q_1^{v-u},q_1)-i\mu\left(\frac{u}{\tau},\frac{v}{\tau};-\frac{1}{\tau}\right)\right),\end{aligned}$$

where $q=e^{2\pi i\tau}$, $q_1=e^{-2\pi i/\tau}$, $a=e^{2\pi iu}$, $b=e^{2\pi iv}$ and $r=(u-v)/\tau$. Hence

$$i\mu(u,v;\tau)+q^{\frac{1}{8}}g_2(a^{\frac{1}{2}}b^{-\frac{1}{2}}q^{\frac{1}{4}},q^{\frac{1}{2}}) \tag{9.2}$$

and

$$i\mu(u,v;\tau)-\frac{a^{\frac{1}{2}}b^{-\frac{1}{2}}q^{-\frac{1}{8}}h_2(-ab^{-1},q)}{1+ab^{-1}} \tag{9.3}$$

are Jacobi forms; they behave like θfs.

A proof that (9.2) vanishes when $u+v=\tau/2$ and (9.3) vanishes when $u+v=1/2$ is given in [GM3]. Therefore

$$g_2(a,q)=-iq^{-\frac{1}{4}}\mu(u,\tau-u;2\tau) \tag{9.4}$$

and

$$h_2(a,q)=(a^{\frac{1}{2}}-a^{-\frac{1}{2}})q^{\frac{1}{8}}\mu\left(\frac{u}{2},\frac{1-u}{2};\tau\right)=2i\sin(\pi u)q^{\frac{1}{8}}\mu\left(\frac{u}{2},\frac{1-u}{2};\tau\right). \tag{9.5}$$

In an important paper, S.-Y. Kang [K] further connected classical mfs with the work of Zwegers and Bringmann-Ono. In particular, she found the expressions for g_2 and g_3 in terms of μ:

$$iag_2(a,q) = \frac{\eta^4(2\tau)}{\eta^2(\tau)\vartheta(2u;2\tau)} + aq^{-\frac{1}{4}}\mu(2u,\tau;2\tau)\,,$$

$$ia^{\frac{3}{2}}q^{-\frac{1}{24}}g_3(a,q) = \frac{\eta^3(3\tau)}{\eta(\tau)\vartheta(3u;3\tau)} + aq^{-\frac{1}{6}}\mu(3u,\tau;3\tau) + a^2q^{-\frac{2}{3}}\mu(3u,2\tau;3\tau)\,,$$

where $a = e^{2\pi i u}$, $q = e^{2\pi i\tau}$ and $\eta(\tau) = q^{\frac{1}{24}}(q)_\infty$ is the Dedekind η-function.

Choi [C5] proved that

$$\sum_{n=-\infty}^{\infty} \frac{a^n b^n q^{n^2}}{(aq;q)_n(bq;q)_n} = \frac{-a(a^{-1};q)_\infty}{(q;q)_\infty(bq;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n+1)} b^n}{1-aq^n}$$
$$= -ia^{\frac{1}{2}}b^{\frac{1}{2}}q^{\frac{1}{8}}(a^{-1};q)_\infty(b^{-1};q)_\infty\mu(u,v;\tau)\,,$$

where a and b are not equal to integral powers of q. This provides an expansion of μ as a bilateral hypergeometric sum. We immediately see that μ is symmetric in the variables u and v. Since

$$\sum_{n=-\infty}^{\infty} \frac{(ab)^n q^{n^2}}{(aq;q)_n(bq;q)_n} = \sum_{n=0}^{\infty} \frac{(ab)^n q^{n^2}}{(aq;q)_n(bq;q)_n} + \sum_{n=1}^{\infty} q^n(a^{-1};q)_n(b^{-1};q)_n\,,$$

it follows that μ can be expressed as a combination of normalized Eulerian series.

In 1987, F.J. Dyson [Dy] wrote as follows:

> The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi. This remains a challenge for the future. My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include not only theta-functions but mock theta-functions.

A host of talented mathematicians and physicists, both young and more senior, are now engaged in just such an effort. In particular, the work of Bringmann, Choi, Folsom, Kang, Ono, Rhoades, Zagier and Zwegers has shown that the theory of Maass forms is a cornerstone of the structure envisioned by Dyson.

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An Application of Cauchy–Sylvester's Theorem on Compound Determinants to a BC_n-Type Jackson Integral

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Abstract A determinant formed by multiple ${}_{2r}\psi_{2r}$ basic hypergeometric series is evaluated as a product of q-gamma functions. Its simple and direct proof is presented herein as an application of Cauchy–Sylvester's theorem on compound determinants, which also provides a very simple proof of determinant formulae for classical group characters given in [M. Ito and K. Koike, A generalization of Weyl's denominator formulas for the classical groups, J. Algebra 302 (2006), 817–825].

Keywords Cauchy–Sylvester's compound determinant • Vandermonde determinant • BC_n-type Jackson integral • Schur functions

Mathematics Subject Classification: Primary: 15A15, 33D67

1 Introduction

The aim of this paper is to give a simple and direct proof to a determinant identity involving BC_n-type Jackson integrals, by applying Cauchy–Sylvester's theorem on compound determinants. BC_n-type Jackson integrals are a multiple generalization of the basic hypergeometric series in a class of what is called *very-well-poised-balanced* ${}_{2r}\psi_{2r}$. A key reason to consider the BC_n-type Jackson

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K. Alladi and F. Garvan (eds.), *Partitions, q-Series, and Modular Forms*, Developments in Mathematics 23, DOI 10.1007/978-1-4614-0028-8_10,

integrals, which admit Weyl group symmetry, is to provide an explanation and an extension of these hypergeometric series with respect to the Weyl group symmetry and the q-difference equations of the BC_n-type Jackson integrals with respect to their parameters. For instance, the formula called Slater's transformation for a very-well-poised-balanced ${}_{2r}\psi_{2r}$ series can be regarded as a connection formula [19] for the solutions of q-difference equations [17] of the BC_1-type Jackson integral. In [16], one of the authors extended the connection formula to that for a BC_n-type Jackson integral, which is defined in [3–5]. As an application of the connection formula, it can be proven that a determinant of matrix formed by the BC_n-type Jackson integrals is expressed as a product of q-gamma functions.

To state our determinant identity, we review some terminology (see Sect. 2 for details). For a point $\xi = (\xi_1, \dots, \xi_n) \in (\mathbb{C}^*)^n$ and a holomorphic function $\varphi(z)$ on $(\mathbb{C}^*)^n$ that is invariant under the Weyl group action, consider the function $\langle \varphi, \xi \rangle$ defined using the Jackson integral, which is the sum over the lattice $\mathbb{Z}^n$:

$$\langle \varphi, \xi \rangle := \iint \cdots \int_0^{\xi\infty} \varphi(z)\Phi(z)\Delta(z) \frac{\mathrm{d}_q z_1}{z_1} \wedge \cdots \wedge \frac{\mathrm{d}_q z_n}{z_n}.$$

Here, $\Phi(z)$ and $\Delta(z)$ are functions in $z = (z_1, z_2, \dots, z_n) \in (\mathbb{C}^*)^n$ defined by

$$\Phi(z) := \prod_{i=1}^{n} \prod_{m=1}^{2s+2} z_i^{1/2-\alpha_m} \frac{(qa_m^{-1} z_i)_\infty}{(a_m z_i)_\infty},$$

$$\Delta(z) := \prod_{i=1}^{n} \frac{1-z_i^2}{z_i} \prod_{1 \le j < k \le n} \frac{(1-z_j/z_k)(1-z_j z_k)}{z_j},$$

where s is an arbitrary integer satisfying $s \ge n$, and $q^{\alpha_m} = a_m$ $(1 \le m \le 2s+2)$ are parameters. The function $\Delta(z)$ is the Weyl denominator of type C_n. The sum $\langle \varphi, \xi \rangle$ is called a *Jackson integral of type* BC_n. If we regard $\langle \varphi, z \rangle$ as a function of $z \in (\mathbb{C}^*)^n$, there exists a holomorphic function $\langle\langle \varphi, z \rangle\rangle$ on $(\mathbb{C}^*)^n$ such that

$$\langle \varphi, z \rangle = \langle\langle \varphi, z \rangle\rangle\, \Theta(z),$$

where $\Theta(z)$ is defined by

$$\Theta(z) := \prod_{i=1}^{n} \frac{z_i^s \theta(z_i^2)}{\prod_{m=1}^{2s+2} z_i^{\alpha_m} \theta(a_m z_i)} \prod_{1 \le j < k \le n} \frac{\theta(z_j z_k)\theta(z_j/z_k)}{z_j}$$

in terms of theta functions $\theta(z)$. We call $\langle\langle \varphi, z \rangle\rangle$ the *regularized Jackson integral* of type BC_n.

The rows and columns of our determinant are indexed by integer sequences in

$$Z = \{\mu = (\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{Z}^n\,;\, 1 \le \mu_1 < \mu_2 < \cdots < \mu_n \le s\}$$

and

$$B = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n \,;\, s-n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\},$$

respectively. For a partition $\lambda \in B$, the corresponding symplectic Schur function $\chi_\lambda(z_1, \ldots, z_n)$ is defined by

$$\chi_\lambda(z_1, \ldots, z_n) = \frac{\det\left(z_j^{\lambda_k+n-k+1} - z_j^{-\lambda_k-n+k-1}\right)_{1\leq j,k\leq n}}{\det\left(z_j^{n-k+1} - z_j^{-n+k-1}\right)_{1\leq j,k\leq n}}.$$

Also, for $x = (x_1, x_2, \ldots, x_s) \in (\mathbb{C}^*)^s$ and $\mu \in Z$, we put

$$x_{(\mu)} := (x_{\mu_1}, x_{\mu_2}, \ldots, x_{\mu_n}) \in (\mathbb{C}^*)^n.$$

The determinant formula, which was proven in [16], is now stated as follows:

Theorem 1.1.

$$\det\left(\langle\langle \chi_\lambda, x_{(\mu)}\rangle\rangle\right)_{\substack{\lambda\in B\\ \mu\in Z}} = \{(1-q)(q)_\infty\}^{n\binom{s}{n}}$$
$$\times \left[\frac{\prod_{1\leq i<j\leq 2s+2}(qa_i^{-1}a_j^{-1})_\infty}{(qa_1^{-1}a_2^{-1}\ldots a_{2s+2}^{-1})_\infty}\right]^{\binom{s-1}{n-1}} \left[\prod_{1\leq j<k\leq s}\frac{\theta(x_j/x_k)\theta(x_jx_k)}{x_j}\right]^{\binom{s-2}{n-1}}, \qquad (1)$$

where the rows $\lambda \in B$ are arranged in decreasing order of the lexicographic ordering, and the columns $\mu \in Z$ are arranged in increasing order of the lexicographic ordering.

The matrix $(\langle\langle \chi_\lambda, x_{(\mu)}\rangle\rangle)_{\lambda,\mu}$ can be regarded as the pairing between the nth cohomology and the nth homology associated with $\Phi(z)$ (see [1,2]). Thus, Theorem 1.1 enables us to determine explicitly when the paring is nondegenerate.

In this paper we give a simple proof of Theorem 1.1, by applying Cauchy–Sylvester's theorem on compound determinants. Given an $s\times s$ matrix A, the nth compound matrix $A^{(n)}$ is the $\binom{s}{n}\times\binom{s}{n}$ matrix whose entries are the minors of A of order n. Cauchy–Sylvester's theorem (Proposition 3.1) says that $\det A^{(n)}$ is equal to the power of $\det A$ with exponent $\binom{s-1}{n-1}$. Cauchy–Sylvester's theorem also provides a very simple proof of determinant formulae for classical group characters given in [18]. See Proposition 3.2.

This paper is organized as follows. In Sect. 2 we review Jackson integral of type BC_n and its regularization. After giving Cauchy–Sylvester's theorem in Sect. 3 we give a proof of Theorem 1.1 in Sect. 4.

We should mention a similar determinant identity [6, Theorem 1.3] involving other BC_n-type Jackson integrals. Taking the limit $q \to 0$ in this determinant identity, we can deduce other determinant formulae for classical group characters, which are similar to those in Proposition 3.2. The limiting case can be obtained by specializing another formula for compound determinant recently found in [12].

2 Definition of Jackson Integral of Type BC_n and Its Regularization

Throughout this paper, we assume that $0 < q < 1$ and use the standard notation for q-shifted factorials $(x)_\infty := \prod_{i=0}^{\infty}(1-q^i x)$ and $(x)_N := (x)_\infty/(q^N x)_\infty$.

2.1 Weyl Group and Symplectic Schur Functions

Let W be the Weyl group of type C_n, which is isomorphic to the semidirect product $(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathcal{S}_n$, where $\mathcal{S}_n$ is the symmetric group of n letters. This group W acts on the torus $(\mathbb{C}^*)^n$, where $\mathcal{S}_n$ acts by permuting the coordinates and $(\mathbb{Z}/2\mathbb{Z})^n$ acts by inverting the coordinates. This action induces an action of W on functions $f(z)$ on $(\mathbb{C}^*)^n$ by

$$(wf)(z) := f(w^{-1}(z)) \quad \text{for} \quad w \in W.$$

A function $f(z)$ on $(\mathbb{C}^*)^n$ is said to be W-*symmetric* if $(wf)(z) = f(z)$ for all $w \in W$.

For an n-tuple of integers $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{Z}^n$, we define a function $A_\beta(z)$ in $z = (z_1, z_2, \ldots, z_n) \in (\mathbb{C}^*)^n$ by putting

$$A_\beta(z) := \det\left(z_j^{\beta_k} - z_j^{-\beta_k}\right)_{1\le j,k\le n}.$$

For example, if $\beta = \rho = (n, n-1, \ldots, 2, 1) \in \mathbb{Z}^n$, then we have

$$A_\rho(z) = \prod_{i=1}^{n}(z_i - z_i^{-1}) \prod_{1\le j<k\le n} \frac{(z_k - z_j)(1 - z_j z_k)}{z_j z_k}, \tag{2}$$

which is Weyl's denominator formula for the symplectic group. For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length at most n, we put

$$\chi_\lambda(z) := \frac{A_{\lambda+\rho}(z)}{A_\rho(z)} = \frac{\det\left(z_j^{\lambda_k+n-k+1} - z_j^{-\lambda_k-n+k-1}\right)_{1\le j,k\le n}}{\det\left(z_j^{n-k+1} - z_j^{-n+k-1}\right)_{1\le j,k\le n}},$$

where $\lambda+\rho=(\lambda_1+n-1,\lambda_2+n-2,\dots,\lambda_n)$ and call it the *symplectic Schur function*. This function χ_λ is the irreducible character of the symplectic group evaluated on the maximal torus (see [8]). It is easy to see that $\chi_\lambda(z_1,\dots,z_n)$ is a W-symmetric Laurent polynomial in $z_1,\dots,z_n$.

2.2 BC_n-Type Jackson Integral

In this section, the definition of the BC_n-type Jackson integral is reviewed following [3].

Associated to a lattice point $\nu=(\nu_1,\nu_2,\dots,\nu_n)\in\mathbb{Z}^n$, we define a q-shift

$$(\mathbb{C}^*)^n \ni z=(z_1,z_2,\dots,z_n)\longmapsto q^\nu z:=(q^{\nu_1}z_1,q^{\nu_2}z_2,\dots,q^{\nu_n}z_n)\in(\mathbb{C}^*)^n.$$

Definition 2.1. For a point $\xi\in(\mathbb{C}^*)^n$ and a function $f(z)$ on $(\mathbb{C}^*)^n$, we write

$$\iint\cdots\int_0^{\xi\infty} f(z)\frac{d_qz_1}{z_1}\wedge\cdots\wedge\frac{d_qz_n}{z_n}:=(1-q)^n\sum_{\nu\in\mathbb{Z}^n} f(q^\nu\xi),$$

which is referred to as the *Jackson integral* if it converges.

Let s be an arbitrary integer satisfying $s\ge n$, and let $\alpha_1,\alpha_2,\dots,\alpha_{2s+2}$ be parameters and put $a_m=q^{\alpha_m}$ $(1\le m\le 2s+2)$. Let $\Phi(z)$ be the function of $z=(z_1,z_2,\dots,z_n)\in(\mathbb{C}^*)^n$ defined by

$$\Phi(z):=\prod_{i=1}^n\prod_{m=1}^{2s+2} z_i^{1/2-\alpha_m}\frac{(qa_m^{-1}z_i)_\infty}{(a_mz_i)_\infty}. \tag{3}$$

By definition, the following holds for $\Phi(z)$.

Lemma 2.2. *For $w\in W$, we put $U_w(z):=(w\Phi)(z)/\Phi(z)$. Then $U_w(z)$ is invariant under the shifts $z\to q^\nu z$ for all $\nu\in\mathbb{Z}^n$.*

Next, we set

$$\Delta(z):=\prod_{i=1}^n\frac{1-z_i^2}{z_i}\prod_{1\le j<k\le n}\frac{(1-z_j/z_k)(1-z_jz_k)}{z_j}. \tag{4}$$

Comparing with (2), we have

$$\Delta(z)=(-1)^nA_\rho(z),$$

where $\rho=(n,n-1,\dots,2,1)$.

Definition 2.3. For a point $\xi \in (\mathbb{C}^*)^n$ and an arbitrary W-symmetric function $\varphi(z)$, the *BC_n-type Jackson integral* is defined by

$$\langle \varphi, \xi \rangle := \iint \cdots \int_0^{\xi\infty} \varphi(z)\Phi(z)\Delta(z)\frac{d_q z_1}{z_1} \wedge \cdots \wedge \frac{d_q z_n}{z_n}.$$

By definition, the sum $\langle \varphi, \xi \rangle$ viewed as a function in ξ is invariant under the shifts $\xi \to q^\nu \xi$ for $\nu \in \mathbb{Z}^n$. From the relations $(w\varphi)(z) = \varphi(z)$, $(w\Delta)(z) = (\operatorname{sgn} w)\Delta(z)$, and Lemma 2.2, it follows that

$$w\langle \varphi, \xi \rangle = (\operatorname{sgn} w)\, U_w(\xi)\, \langle \varphi, \xi \rangle \quad (w \in W) \tag{5}$$

as functions in ξ.

In what follows, we assume that the parameters $a_1, a_2, \ldots, a_{2s+2}$ satisfy the condition

$$|a_1 a_2 \cdots a_{2s+2}| > q^{s-n+1}.$$

Under this condition, if $a_m \xi_i \notin \{q^l\,;\, l \in \mathbb{Z}\}$ for $1 \le i \le n$ and $1 \le m \le 2s+2$, the convergence of $\langle 1, \xi \rangle$ can be confirmed in the same manner as [13, p. 158, Theorem 4]. In addition, we assume that the parameters $a_1, a_2, \ldots, a_{2s+2}$ are all generic.

2.3 Regularization

Let $\Theta(z)$ be the function on $(\mathbb{C}^*)^n$ defined by

$$\Theta(z) := \prod_{i=1}^{n} \frac{z_i^s \theta(z_i^2)}{\prod_{m=1}^{2s+2} z_i^{\alpha_m} \theta(a_m z_i)} \prod_{1 \le j < k \le n} \frac{\theta(z_j z_k)\theta(z_j/z_k)}{z_j}, \tag{6}$$

where $\theta(x)$ denotes the theta function $(x)_\infty (q/x)_\infty$, which satisfies

$$\theta(x) = \theta(q/x), \quad \theta(qx) = -\theta(x)/x. \tag{7}$$

By definition, we have

$$(w\Theta)(z) = (\operatorname{sgn} w)\, U_w(z)\, \Theta(z) \quad (w \in W). \tag{8}$$

Proposition 2.4. *Suppose that the parameters $a_1, a_2, \ldots, a_{2s+2}$ are generic. If $\varphi(z)$ is a W-symmetric holomorphic function on $(\mathbb{C}^*)^n$, then there exists a holomorphic function $f(z)$ on $(\mathbb{C}^*)^n$ such that $\langle \varphi, z \rangle = f(z)\Theta(z)$.*

Proof. See [4] or [14].

Definition 2.5. In Proposition 2.4, the holomorphic function $f(z)$ is denoted by $\langle\langle \varphi, z\rangle\rangle$ and called the *regularized Jackson integral*. That is,

$$\langle \varphi, z\rangle = \langle\langle \varphi, z\rangle\rangle\, \Theta(z).$$

From (5) and (8), the regularized Jackson integral $\langle\langle \varphi, z\rangle\rangle$ is also W-symmetric.

In particular, if $s = n$, it is confirmed from (7) that the function $\Theta(z)$ is periodic for the shifts $z \to q^{\nu} z$ for $\nu \in \mathbb{Z}^n$. This implies that the holomorphic function $\langle\langle \varphi, z\rangle\rangle$ becomes a constant that does not depend on z if $s = n$. The following result was proven by Gustafson [11, p. 77, (2.6)]:

Lemma 2.6 (Gustafson). *If $s = n$ and $x = (x_1, x_2, \ldots, x_n) \in (\mathbb{C}^*)^n$. Then*

$$\langle\langle 1, x\rangle\rangle = (1-q)^n (q)_\infty^n \frac{\prod_{1\le i<j\le 2n+2}(q a_i^{-1} a_j^{-1})_\infty}{(q a_1^{-1} a_2^{-1} \cdots a_{2n+2}^{-1})_\infty}. \tag{9}$$

Proof. See [11] or [15].

We call (9) *Gustafson's bilateral C_n-type summation formula*, which is a multiple extension of Bailey's ${}_6\psi_6$ summation formula [9, p. 140, (5.3.1)]. As we will see in Lemma 4.2, (9) can be rewritten in the form of Vandermonde-type determinant.

3 Cauchy–Sylvester's Compound Determinant

We denote by $<$ the lexicographic ordering on $\mathbb{Z}^n$. That is, for $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{Z}^n$, we write $\mu < \nu$ if there exists an index k such that

$$\mu_1 = \nu_1,\ \ldots,\ \mu_{k-1} = \nu_{k-1} \quad \text{and} \quad \mu_k < \nu_k.$$

Let s and n be positive integers satisfying $s \ge n$. We put

$$Z = \{\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{Z}^n\,;\, 1 \le \mu_1 < \mu_2 < \cdots < \mu_n \le s\}. \tag{10}$$

Let $A = (a_{ij})_{1\le i,j\le s}$ be an arbitrary $s \times s$ matrix. For μ and $\nu \in Z$, we denote by $A_{\mu,\nu}$ the submatrix of A obtained by choosing rows $\mu_1, \ldots, \mu_n$ and columns $\nu_1, \ldots, \nu_n$:

$$A_{\mu,\nu} = \left(a_{\mu_i,\nu_j}\right)_{1\le i,j\le n}.$$

Then the nth *compound matrix* $A^{(n)}$ of A is defined to be

$$A^{(n)} = \left(\det A_{\mu,\nu}\right)_{\mu,\nu\in Z},$$

where the rows and columns are arranged in increasing ordering on Z.

The following formula was obtained by Cauchy [7] and Sylvester [21] (see [20, pp. 99–131 of vol. I, pp. 193–197 of vol. II] or [22, pp. 87–89]).

Proposition 3.1. *Let $A = (a_{ij})_{1 \le i,j \le s}$ be an arbitrary square matrix of order s. The determinant of the nth compound matrix $A^{(n)}$ is given by*

$$\det A^{(n)} = (\det A)^{\binom{s-1}{n-1}}. \tag{11}$$

By applying this formula, we can give a much simpler proof to the determinant identities for classical group characters in [18], some of which are the special cases (or limiting cases) of our main determinant identity in Theorem 1.1.

Let B be the set of partitions whose Young diagram is contained in an $n \times (s-n)$ rectangle:

$$B = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n\,;\, s-n \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0\}. \tag{12}$$

Then the correspondence

$$Z \ni \kappa = (\kappa_1, \kappa_2, \dots, \kappa_n) \longmapsto \lambda = (s-\kappa_1-n+1, s-\kappa_2-n+2, \dots, s-\kappa_n) \in B$$

gives a bijection between Z and B. And this bijection reverses the lexicographic ordering on Z and B induced from that on $\mathbb{Z}^n$.

By applying the above formula (11), we can derive the following determinant identities involving characters.

Proposition 3.2. (1) *For symplectic Schur functions χ_λ, we have*

$$\det\left(\chi_\lambda(x_{(\mu)})\right)_{\substack{\lambda \in B \\ \mu \in Z}} = \left[\prod_{1 \le j < k \le s} \frac{(1-x_j/x_k)(1-x_jx_k)}{x_j}\right]^{\binom{s-2}{n-1}}. \tag{13}$$

(2) *For usual Schur functions S_λ, we have*

$$\det\left(S_\lambda(x_{(\mu)})\right)_{\substack{\lambda \in B \\ \mu \in Z}} = \left[\prod_{1 \le i < j \le s} (x_i - x_j)\right]^{\binom{s-2}{n-1}}. \tag{14}$$

In [18], Proposition 3.2 and its variations have been proven by some elementary calculation. Note that, if $n = 1$, then (13) and (14) are exactly Weyl's denominator formulae (or the Vandermonde determinant).

Proof. Apply (11) to the matrices $A = (x_j^{s-i+1} - x_j^{-s+i-1})_{1 \le i,j \le s}$ and $(x_j^{s-i})_{1 \le i,j \le s}$.

Remark. In the determinant identity in Theorem 1.1, we take the limit $q \to 0$ after putting $a_i = x_i$ for $i = 1, 2, \dots, s$. Then, since $\langle\langle \chi_\lambda, x_{(\mu)} \rangle\rangle \to \chi_\lambda(x_{(\mu)})$, we can deduce the determinant identity (13) for symplectic Schur functions. Moreover, considering the principal term of asymptotic behavior of both sides of (13) at $x_i \to +\infty$ $(1 \le i \le s)$, we immediately obtain the determinant identity (14) for Schur functions.

4 Proof of Theorem 1.1

In this section, we use Cauchy–Sylvester's theorem (Proposition 3.1) to give a proof of Theorem 1.1.

For simplicity, set

$$D(u,v) := \frac{\theta(u/v)\theta(uv)}{u}.$$

To specify the number n of variables $z_1, z_2, \ldots, z_n$, we use the notations $\Phi^{(n)}(z)$, $\Delta^{(n)}(z)$, and $\Theta^{(n)}(z)$ instead of $\Phi(z)$, $\Delta(z)$, and $\Theta(z)$, which are defined by (3), (4), and (6) respectively. Then we have

$$\Phi^{(n)}(z) = \prod_{i=1}^{n} \Phi^{(1)}(z_i), \quad \Theta^{(n)}(z) = \prod_{i=1}^{n} \Theta^{(1)}(z_i) \prod_{1 \le j < k \le n} D(z_j, z_k). \tag{15}$$

By definition, the symplectic Schur functions $\chi_{(i)}(z)$ in one variable $z \in \mathbb{C}^*$ are written in the form

$$\chi_{(i)}(z) = \frac{z^{i+1} - z^{-i-1}}{z - z^{-1}} \quad (i = 0, 1, 2, \ldots).$$

Lemma 4.1. *If* $\lambda = (\lambda_1, \ldots, \lambda_n) \in B$ *and* $\mu = (\mu_1, \ldots, \mu_n) \in Z$*, then*

$$\det\left(\langle\langle \chi_{(\lambda_i+n-i)}, x_{\mu_j} \rangle\rangle\right)_{1 \le i,j \le n} = \langle\langle \chi_\lambda, x_{(\mu)} \rangle\rangle \prod_{1 \le j < k \le n} D(x_{\mu_j}, x_{\mu_k})$$

for $x = (x_1, x_2, \ldots, x_s) \in (\mathbb{C}^*)^s$.

Proof. From the definition of the BC_1-type Jackson integral $\langle\langle \chi_{(\lambda_i+n-i)}, x_{\mu_j} \rangle\rangle$, we have

$$\begin{aligned}
&\det\left(\langle\langle \chi_{(\lambda_i+n-i)}, x_{\mu_j} \rangle\rangle\right)_{1 \le i,j \le n} \prod_{j=1}^{n} \Theta^{(1)}(x_{\mu_j}) \\
&= \det\left(\langle \chi_{(\lambda_i+n-i)}, x_{\mu_j} \rangle\right)_{1 \le i,j \le n} \\
&= \det\left(\int_0^{x_{\mu_j}\infty} \chi_{(\lambda_i+n-i)}(z_j)\, \Phi^{(1)}(z_j)\, \Delta^{(1)}(z_j) \frac{d_q z_j}{z_j}\right)_{1 \le i,j \le n}
\end{aligned}$$

$$
\begin{aligned}
&= \iint \cdots \int_0^{x_{(\mu)}\infty} \det\Big(\chi_{(\lambda_i+n-i)}(z_j)\Delta^{(1)}(z_j)\Big)_{1\le i,j\le n} \Phi^{(n)}(z)\frac{d_q z_1}{z_1}\wedge\cdots\wedge\frac{d_q z_n}{z_n}\\
&= (-1)^n \iint \cdots \int_0^{x_{(\mu)}\infty} \det\Big(z_j^{\lambda_i+n-i+1} - z_j^{-\lambda_i-n+i-1}\Big)_{1\le i,j\le n} \Phi^{(n)}(z)\frac{d_q z_1}{z_1}\wedge\cdots\wedge\frac{d_q z_n}{z_n}\\
&= (-1)^n \iint \cdots \int_0^{x_{(\mu)}\infty} A_{\lambda+\rho}(z)\,\Phi^{(n)}(z)\frac{d_q z_1}{z_1}\wedge\cdots\wedge\frac{d_q z_n}{z_n}\\
&= \iint \cdots \int_0^{x_{(\mu)}\infty} \chi_\lambda(z)\,\Phi^{(n)}(z)\,\Delta^{(n)}(z)\frac{d_q z_1}{z_1}\wedge\cdots\wedge\frac{d_q z_n}{z_n}\\
&= \langle \chi_\lambda, x_{(\mu)}\rangle\\
&= \langle\langle \chi_\lambda, x_{(\mu)}\rangle\rangle\,\Theta^{(n)}(x_{(\mu)}).
\end{aligned}
$$

Using (15) for the above equation, we obtain Lemma 4.1.

Lemma 4.2. *The following holds for $x=(x_1,\ldots,x_s)\in(\mathbb{C}^*)^s$:*

$$
\det\big(\langle\langle\chi_{(s-i)},x_j\rangle\rangle\big)_{1\le i,j\le s} = (1-q)^s(q)_\infty^s \frac{\prod_{1\le i<j\le 2s+2}(qa_i^{-1}a_j^{-1})_\infty}{(qa_1^{-1}a_2^{-1}\cdots a_{2s+2}^{-1})_\infty}\prod_{1\le j<k\le s} D(x_j,x_k).
$$

Remark. As we will see in the proof below (see (16)), Lemma 4.2 is just a restatement of Gustafson's bilateral C_s-type summation formula. In this sense, Gustafson's formula (9) itself can be regarded as a generalization of the original Vandermonde determinant.

Proof. From the definition of the BC_1-type Jackson integral $\langle\langle\chi_{(s-i)},x_j\rangle\rangle$, we have

$$
\begin{aligned}
&\det\big(\langle\langle\chi_{(s-i)},x_j\rangle\rangle\big)_{1\le i,j\le s}\prod_{j=1}^s\Theta^{(1)}(x_j)\\
&\quad= \det\big(\langle\chi_{(s-i)},x_j\rangle\big)_{1\le i,j\le s}\\
&\quad= \det\left(\int_0^{x_j\infty}\chi_{(s-i)}(z_j)\,\Phi^{(1)}(z_j)\,\Delta^{(1)}(z_j)\frac{d_q z_j}{z_j}\right)_{1\le i,j\le s}\\
&\quad= \iint\cdots\int_0^{x\infty}\det\Big(\chi_{(s-i)}(z_j)\,\Delta^{(1)}(z_j)\Big)_{1\le i,j\le s}\Phi^{(s)}(z)\frac{d_q z_1}{z_1}\wedge\cdots\wedge\frac{d_q z_s}{z_s}\\
&\quad= (-1)^s\iint\cdots\int_0^{x\infty}\det\Big(z_j^{s-i+1}-z_j^{-s+i-1}\Big)_{1\le i,j\le s}\Phi^{(s)}(z)\frac{d_q z_1}{z_1}\wedge\cdots\wedge\frac{d_q z_s}{z_s}\\
&\quad= (-1)^s\iint\cdots\int_0^{x\infty}A_\rho(z)\,\Phi^{(s)}(z)\frac{d_q z_1}{z_1}\wedge\cdots\wedge\frac{d_q z_s}{z_s}
\end{aligned}
$$

$$= \iint \cdots \int_0^{x\infty} \Phi^{(s)}(z)\, \Delta^{(s)}(z) \frac{d_q z_1}{z_1} \wedge \cdots \wedge \frac{d_q z_s}{z_s}$$
$$= \langle 1, x \rangle$$
$$= \langle\langle 1, x \rangle\rangle\, \Theta^{(s)}(x),$$

where $\langle\langle 1,x\rangle\rangle$ is the BC_s-type Jackson integral with parameters $a_1, a_2, \ldots, a_{2s+2}$. Using (15) for the above equation, it follows that

$$\det\left(\langle\langle \chi_{(s-i)}, x_j \rangle\rangle\right)_{1\le i,j\le s} = \langle\langle 1,x\rangle\rangle \prod_{1\le j<k\le s} D(x_j, x_k). \tag{16}$$

From Lemma 2.6 and (16), we obtain Lemma 4.2.

Proof of Theorem 1.4. We apply Cauchy–Sylvester's theorem on compound determinants (11) to the matrix

$$A = \left(\langle\langle \chi_{(s-i)}, x_j \rangle\rangle\right)_{1\le i,j\le s}.$$

Let κ, $\mu \in Z$ and $\lambda \in B$ be the partition corresponding to κ, i.e., $\lambda_i = s - \kappa_i - (n-i)$ $(1 \le i \le n)$. Then it follows from Lemma 4.1 that the (κ,μ)-entry of the nth compound matrix $A^{(n)}$ is given by

$$\det A_{\kappa,\mu} = \det\left(\langle\langle \chi_{(\lambda_i+n-i)}, x_{\mu_j} \rangle\rangle\right)_{1\le i,j\le n} = \langle\langle \chi_\lambda, x_{(\mu)} \rangle\rangle \prod_{1\le j<k\le n} D(x_{\mu_j}, x_{\mu_k}).$$

Hence we have

$$\begin{aligned}
\det A^{(n)} &= \det\left(\det\left(\langle\langle \chi_{(\lambda_i+n-i)}, x_{\mu_j} \rangle\rangle\right)_{1\le i,j\le n}\right)_{\substack{\lambda\in B\\ \mu\in Z}} \\
&= \det\left(\langle\langle \chi_\lambda, x_{(\mu)} \rangle\rangle \prod_{1\le j<k\le n} D(x_{\mu_j}, x_{\mu_k})\right)_{\substack{\lambda\in B\\ \mu\in Z}} \\
&= \det\left(\langle\langle \chi_\lambda, x_{(\mu)} \rangle\rangle\right)_{\substack{\lambda\in B\\ \mu\in Z}} \prod_{\mu\in Z} \prod_{1\le i<j\le n} D(x_{\mu_i}, x_{\mu_j}) \\
&= \det\left(\langle\langle \chi_\lambda, x_{(\mu)} \rangle\rangle\right)_{\substack{\lambda\in B\\ \mu\in Z}} \prod_{1\le i<j\le s} D(x_i, x_j)^{\binom{s-2}{n-2}}.
\end{aligned}$$

On the other hand, Lemma 4.2 tells us that

$$\begin{aligned}
\det A &= \det\left(\langle\langle \chi_{(s-i)}, x_j \rangle\rangle\right)_{1\le i,j\le s} \\
&= (1-q)^s (q)_\infty^s \frac{\prod_{1\le i<j\le 2s+2} (q a_i^{-1} a_j^{-1})_\infty}{(q a_1^{-1} a_2^{-1} \ldots a_{2s+2}^{-1})_\infty} \prod_{1\le j<k\le s} D(x_j, x_k).
\end{aligned}$$

Therefore, by using $s\binom{s-1}{n-1} = n\binom{s}{n}$ and $\binom{s-1}{n-1} - \binom{s-2}{n-2} = \binom{s-2}{n-1}$, we have

$$\det\big(\langle\langle \chi_\lambda, x_{(\mu)}\rangle\rangle\big)_{\substack{\lambda\in B\\ \mu\in Z}}$$
$$= \{(1-q)(q)_\infty\}^{n\binom{s}{n}} \left[\frac{\prod_{1\le i<j\le 2s+2}(qa_i^{-1}a_j^{-1})_\infty}{(qa_1^{-1}a_2^{-1}\dots a_{2s+2}^{-1})_\infty}\right]^{\binom{s-1}{n-1}} \left[\prod_{1\le j<k\le s} D(x_j,x_k)\right]^{\binom{s-2}{n-1}},$$

which is the desired identity. We complete the proof. □

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Multiple Generalizations of *q*-Series Identities and Related Formulas

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Abstract A number of multiple generalization of "familiar" and fundamental q-series transformation formulas in Chap. 16 of Ramanujan's notebook are obtained from ${}_3\phi_2$ transformations for Milne's multivariate basic hypergeometric series in our previous work. A generalization of ${}_1\phi_1$ transformation related to the basic Lauricella function ϕ_D is also presented.

Keywords Multivariate basic hypergeometric series • q-series transformations • Ramanujan's notebook • Partition function identities

Mathematics Subject Classification: Primary: 33D67; Secondary: 05A19, 11B65, 33C67, 33D90

1 Introduction

The famous Indian mathematician Ramanujan has recorded a number of elegant formulas without proof in his notebooks which nowadays are called as Ramanujan's Note-books. Bruce Berndt and his collaborators have published detailed proofs of several of Ramanujan's formulae which are of interest to the expert and non-expert alike. Their work may be found in the series of books published under the title Ramanujan's Note-books (of relevance here will be Part III of series [5]).

Not only are the formulas beautiful, but also it is known that Ramanujan's formulas have arisen in many areas of mathematics and mathematical physics. Furthermore, some of his formulas are well recognized to play important roles in these fields.

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K. Alladi and F. Garvan (eds.), *Partitions, q-Series, and Modular Forms*, Developments in Mathematics 23, DOI 10.1007/978-1-4614-0028-8_11,

In this paper, we focus on Chap. 16 of Ramanujan's notebook [1]. In particular, we give a number of new multivariate generalizations of "familiar" (as in [1]) and fundamental q-series transformations and allied formulas in Chap. 16 of the Ramanujan's notebook. Most of the formulas have been obtained from ${}_3\phi_2$ transformation formulas for Milne's multivariate basic hypergeometric series of A type in our previous work by limiting procedure.

We expect that some q-series identities presented in this paper will be worthwhile for future investigation of generalizations of minimal models in conformal field theory and representation theory of Virasoro algebra and W-algebra where several remarkable q-series identities arise naturally (see [16] and references therein).

The organization of this paper is as follows: After summarizing notations and basic definitions of multivariate (basic) hypergeometric series, in the next section, we present some multivariate ${}_3\phi_2$ transformation formulas which are sources of the multivariate q-series transformations in this paper. In Sect. 3, we give certain types of multivariate Heine transformations from ${}_3\phi_2$ transformations. In Sect. 4, we give some (q-)Pfaff transformations. Sections 5 and 6 are devoted to the derivations of multivariate generalizations of transformation and summation formulas for some partition functions. In particular, we give multivariate generalization of Entry 9 in Chap. 16 of Ramanujan's notebook in Sect. 6. We close this paper to present a generalization of ${}_1\phi_1$ transformation formula and related q-series transformation for partition functions. Throughout of this paper we discuss some special and limiting cases of q-series transformations in detail.

2 Preliminaries

Let $\mathbb{N}$ be a set of nonnegative integers. Hypergeometric series ${}_rF_s$ is defined by

$$ {}_rF_s\left[\begin{matrix}\{a_i\}_r\\ \{c_i\}_s\end{matrix};z\right] = {}_rF_s\left[\begin{matrix}a_1,a_2,\dots,a_r\\ c_1,c_2,\dots,c_s\end{matrix};q;z\right] = \sum_{n\in\mathbb{N}} \frac{[a_1,\dots,a_r]_n}{[c_1,\dots,c_s]_n\, n!} z^n, \tag{2.1} $$

where

$$ [a]_n := a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{2.2} $$

is a shifted factorial, and we employ the notation

$$ [a_1,a_2,\dots,a_r]_n := [a_1]_n[a_2]_n\cdots[a_r]_n. \tag{2.3} $$

Similarly, we denote basic hypergeometric series ${}_r\phi_s$ by

$$ {}_r\phi_s\left[\begin{matrix}\{a_i\}_r\\ \{c_k\}_s\end{matrix};q;z\right] = \sum_{n\in\mathbb{N}} \frac{(a_1,\dots,a_r)_n}{(c_1,\dots,c_s,q)_n} z^n \left((-1)^n q^{\frac{1}{2}n(n-1)}\right)^{s-r+1}, \tag{2.4} $$

where

$$(a;q)_\infty := \prod_{k\in\mathbb{N}}(1-aq^k), \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) = \frac{(a;q)_\infty}{(aq^n;q)_\infty} \tag{2.5}$$

are q-shifted factorials. Throughout this paper, we assume that q is a complex number such that $0 < |q| < 1$. Unless otherwise stated, we omit the basis q in q-shifted factorials. Namely, we denote $(a;q)_k$ as $(a)_k$ for instance. We also use the notation

$$(a_1,a_2,\ldots,a_r)_n := (a_1)_n(a_2)_n\cdots(a_r)_n. \tag{2.6}$$

For a multi-index $\beta = (\beta_1,\ldots,\beta_n) \in \mathbb{N}^n$, $|\beta| = \sum_{i=1}^n \beta_i$ stands for the length of β. In this paper, a multiple series $\sum_{\beta\in\mathbb{N}^n} S(\beta)$ is called A_n basic hypergeometric series if (1) the series has a form

$$\sum_{\beta\in\mathbb{N}^n} \frac{\Delta(xq^\beta)}{\Delta(x)} u_1^{\beta_1}\cdots u_n^{\beta_n} \times (\text{ratios of } q\text{-shifted factorials}), \tag{2.7}$$

where

$$\Delta(x) = \prod_{1\le i<j\le n}(x_i - x_j), \qquad \Delta(xq^\beta) = \prod_{1\le i<j\le n}(x_iq^{\beta_i} - x_jq^{\beta_j}) \tag{2.8}$$

are the Vandermonde determinants of $x = (x_1,\ldots,x_n)$ and $xq^\beta = (x_1q^{\beta_1},\ldots,x_nq^{\beta_n})$ respectively. (2) The multiple series is symmetric with respect to the subscript $1 \le i \le n$. (3) In the case when $n = 1$, the multiple series reduces to basic hypergeometric series.

Now we present some multiple nonterminating ${}_3\phi_2$ transformation formulas from multiple ${}_4\phi_3$ transformations in our previous work [8].

2.1 Nonterminating ${}_3\phi_2$ Transformations with Different Dimensions

$$\begin{aligned}
&\sum_{\gamma\in\mathbb{N}^n}\left(\frac{d^me}{aBC}\right)^{|\gamma|}\frac{\Delta(xq^\gamma)}{\Delta(x)}\frac{(a)_{|\gamma|}}{(e)_{|\gamma|}}\prod_{1\le i,j\le n}\frac{(b_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}\prod_{1\le i\le n,1\le k\le m}\frac{(c_kx_iy_k)_{\gamma_i}}{(dx_iy_k)_{\gamma_i}}\\
&=\frac{(e/a,d^me/BC)_\infty}{(e,d^me/aBC)_\infty}\sum_{\delta\in\mathbb{N}^m}\left(\frac{e}{a}\right)^{|\delta|}\frac{\Delta(yq^\delta)}{\Delta(y)}\prod_{1\le k,l\le m}\frac{((d/c_l)y_k/y_l)_{\delta_k}}{(qy_k/y_l)_{\delta_k}}\\
&\quad\times\frac{(a)_{|\delta|}}{(d^me/BC)_{|\delta|}}\prod_{1\le i\le n,1\le k\le m}\frac{((d/b_i)x_iy_k)_{\delta_k}}{(dx_iy_k)_{\delta_k}},
\end{aligned} \tag{2.9}$$

where $B = b_1 b_2 \cdots b_n$ and $C = c_1 c_2 \cdots c_m$. We use such notation throughout. In the case when $m = 1$ and $y_1 = 1$, (2.9) reduces to

$$\sum_{\gamma\in\mathbb{N}^n} \left(\frac{de}{aBc}\right)^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \frac{(a)_{|\gamma|}}{(e)_{|\gamma|}} \prod_{1\le i,j\le n} \frac{(b_j x_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}} \prod_{1\le i\le n} \frac{(cx_i)_{\gamma_i}}{(dx_i)_{\gamma_i}}$$
$$= \frac{(e/a, de/Bc)_\infty}{(e, de/aBc)_\infty} {}_{n+2}\phi_{n+1}\left[\begin{matrix} a, d/c, \{(d/b_i)x_i\}_n \\ de/Bc, \{dx_i\}_n \end{matrix}; q, \frac{e}{a}\right]. \tag{2.10}$$

In the case when $m = n = 1$ and $x_1 = y_1 = 1$, (2.9) reduces to

$${}_3\phi_2\left[\begin{matrix} a,b,c \\ d,e \end{matrix}; q, \frac{de}{abc}\right] = \frac{(e/a, de/bc)_\infty}{(e, de/abc)_\infty} {}_3\phi_2\left[\begin{matrix} a, d/b, d/c \\ d, de/bc \end{matrix}; q, \frac{e}{a}\right]. \tag{2.11}$$

In [6], (2.9) was obtained from the following multiple Sears transformation:

$$\sum_{\gamma\in\mathbb{N}^n} q^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \frac{(q^{-N}, a)_{|\gamma|}}{(e, aBCq^{1-N}/d^m e)_{|\gamma|}} \prod_{1\le i,j\le n} \frac{(b_j x_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}} \prod_{1\le i\le n, 1\le k\le m} \frac{(c_k x_i y_k)_{\gamma_i}}{(dx_i y_k)_{\gamma_i}}$$
$$= \frac{(e/a, d^m e/BC)_N}{(e, d^m e/aBC)_N} \sum_{\delta\in\mathbb{N}^m} q^{|\delta|} \frac{\Delta(yq^\delta)}{\Delta(y)} \prod_{1\le k,l\le m} \frac{((d/c_l)y_k/y_l)_{\delta_k}}{(qy_k/y_l)_{\delta_k}}$$
$$\times \frac{(q^{-N}, a)_{|\delta|}}{(q^{1-N}a/e, d^m e/BC)_{|\delta|}} \prod_{1\le i\le n, 1\le k\le m} \frac{((d/b_i)x_i y_k)_{\delta_k}}{(dx_i y_k)_{\delta_k}}, \tag{2.12}$$

by taking the limit $N \to \infty$. In the case when $m = n = 1$ and $x_1 = y_1 = 1$, (2.12) reduces to the Sears transformation for terminating balanced ${}_4\phi_3$ series:

$${}_4\phi_3\left[\begin{matrix} q^{-N}, a, b, c \\ d, e, abcq^{1-N}/de \end{matrix}; q, q\right] = \frac{(e/a, de/bc)_N}{(e, de/abc)_N} {}_4\phi_3\left[\begin{matrix} q^{-N}, a, d/b, d/c \\ d, aq^{1-N}/e, de/bc \end{matrix}; q, q\right]. \tag{2.13}$$

2.2 Reversing Version

$$\sum_{\gamma\in\mathbb{N}^n} \left(\frac{Ef}{a^m Bc}\right)^{|\gamma|} x_1^{-\gamma_1} \cdots x_n^{-\gamma_n} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1\le i,j\le n} \frac{(b_j x_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}} \prod_{1\le i\le n} (cx_i)_{\gamma_i}$$
$$\times (f)_{|\gamma|}^{-1} \prod_{1\le k\le m} \frac{(ay_k)_{|\gamma|}}{(e_k y_k)_{|\gamma|}} = \frac{(Ef/a^m B)_\infty}{(f)_\infty} \prod_{1\le k\le m} \frac{(ay_k)_\infty}{(e_k y_k)_\infty} \prod_{1\le i\le n} \frac{((Ef/a^m c)z_i)_\infty}{((Ef/a^m Bc)z_i)_\infty}$$

$$\times \sum_{\delta\in\mathbb{N}^m} (a)^{|\delta|} w_1^{-\delta_1}\cdots w_m^{-\delta_m} \frac{\Delta(wq^\delta)}{\Delta(w)} \prod_{1\le k,l\le m} \frac{((e_l/a)w_k/w_l)_{\delta_k}}{(qw_k/w_l)_{\delta_k}} \prod_{1\le k\le m} ((f/a)w_k)_{\delta_k}$$

$$\times (Ef/a^mB)_{|\delta|}{}^{-1} \prod_{1\le i\le n} \frac{((Ef/a^mb_ic)z_i)_{|\delta|}}{((Ef/a^mc)z_i)_{|\delta|}}, \tag{2.14}$$

where $z_i = b_i/Bx_i$ $(1\le i\le n)$ and $w_k = y_k^{-1}$ $(1\le k\le m)$, respectively.

In the case when $m=1$ and $y_1=1$, (2.14) reduces to

$$\sum_{\gamma\in\mathbb{N}^n} \left(\frac{ef}{aBc}\right)^{|\gamma|} x_1^{-\gamma_1}\cdots x_n^{-\gamma_n} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1\le i,j\le n} \frac{(b_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}} \prod_{1\le i\le n} (cx_i)_{\gamma_i} \frac{(a)_{|\gamma|}}{(e,f)_{|\gamma|}}$$

$$= \frac{(ef/aB,a)_\infty}{(e,f)_\infty} \prod_{1\le i\le n} \frac{((ef/ac)z_i)_\infty}{((ef/aBc)z_i)_\infty} {}_{n+2}\phi_{n+1}\left[\begin{matrix}\{(ef/ab_ic)z_i\}_n, e/a, f/a\\ \{(ef/ac)z_i\}_n, ef/aB\end{matrix}; q; a\right], \tag{2.15}$$

where $z_i = b_i/Bx_i$ $(1\le i\le n)$.

In the case when $f=a$, (2.15) reduces to the following A_n q-Gauss summation formula for ${}_2\phi_1$ series:

$$\sum_{\gamma\in\mathbb{N}^n} \left(\frac{c}{aB}\right)^{|\gamma|} x_1^{-\gamma_1}\cdots x_n^{-\gamma_n} \frac{\Delta(xq^\gamma)}{\Delta(x)} (c)_{|\gamma|}{}^{-1} \prod_{1\le i,j\le n} \frac{(b_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}} \prod_{1\le i\le n} (ax_i)_{\gamma_i}$$

$$= \frac{(c/B)_\infty}{(c)_\infty} \prod_{1\le i\le n} \frac{((c/a)z_i)_\infty}{((c/ab_i)z_i)_\infty}, \qquad z_i = b_i/Bx_i, \quad (1\le i\le n). \tag{2.16}$$

In the case when $n=1$, (2.16) reduces to the q-Gauss summation

$${}_2\phi_1\left[\begin{matrix}a,b\\ c\end{matrix}; q, \frac{c}{ab}\right] = \frac{(c/a, c/bc)_\infty}{(c, c/ab)_\infty}. \tag{2.17}$$

Remark 2.1. Formula (2.16) can be obtained from Theorem 6.5 (A_n Sears transformation formula) of Milne-Lilly [12].

Equation (2.14) is obtained from the following multiple ${}_4\phi_3$ transformation formula ((7.6) in [8] with a different arrangement of parameters):

$$\sum_{\gamma\in\mathbb{N}^n} q^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1\le i,j\le n} \frac{(b_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}} \prod_{1\le i\le n} \frac{(cx_i)_{\gamma_i}}{((a^mBCq^{1-N}/Ef)x_i)_{\gamma_i}}$$

$$\times \frac{(q^{-N})_{|\gamma|}}{(f)_{|\gamma|}} \prod_{1\le k\le m} \frac{(ay_k)_{|\gamma|}}{(e_ky_k)_{|\gamma|}}$$

$$
\begin{aligned}
&= \frac{(Ef/a^mB)_N}{(f)_N} \prod_{1\le k\le m} \frac{(ay_k)_N}{(e_ky_k)_N} \prod_{1\le i\le n} \frac{((Ef/a^mc)z_i)_N}{((Ef/a^mBc)z_i)_N} \\
&\times \sum_{\delta\in\mathbb{N}^m} q^{|\delta|} \frac{\Delta(wq^\delta)}{\Delta(w)} \prod_{1\le k,l\le m} \frac{((e_l/a)w_k/w_l)_{\delta_k}}{(qw_k/w_l)_{\delta_k}} \prod_{1\le k\le m} \frac{((f/a)w_k)_{\delta_k}}{((q^{1-N}/a)w_k)_{\delta_k}} \\
&\times \frac{(q^{-N})_{|\delta|}}{(Ef/a^mB)_{|\delta|}} \prod_{1\le i\le n} \frac{((Ef/a^mb_ic)z_i)_{|\delta|}}{((Ef/a^mc)z_i)_{|\delta|}},
\end{aligned}
\tag{2.18}
$$

where $z_i = b_i/Bx_i$ $(1 \le i \le n)$ and $w_k = y_k^{-1}$ $(1 \le k \le m)$, respectively.

In the case when $m = n = 1$ and $x_1 = y_1 = 1$, (2.14) reduces to

$$
{}_3\phi_2\left[\begin{matrix} a,b,c \\ e,f \end{matrix};q;\frac{ef}{abc}\right] = \frac{(ef/ab,ef/ac,a)_\infty}{(e,f,ef/abc)_\infty}\,{}_3\phi_2\left[\begin{matrix} ef/abc,e/a,f/a \\ ef/ab,ef/ac \end{matrix};q;a\right]. \tag{2.19}
$$

We also mention that, in the case when $m = n = 1$ and $x_1 = y_1 = 1$, (2.18) reduces to

$$
{}_4\phi_3\left[\begin{matrix} q^{-N},a,b,c \\ abcq^{1-N}/ef,e,f \end{matrix};q;q\right] = \frac{(ef/ab,ef/ac,a)_N}{(e,f,ef/abc)_N}\,{}_4\phi_3\left[\begin{matrix} q^{-N},ef/abc,e/a,f/a \\ q^{1-N}/a,ef/ab,ef/ac \end{matrix};q;q\right]. \tag{2.20}
$$

Note that (2.20) is obtained by reversing the order of the summation of the Sears transformation (2.13) and is also verified by iterating Sears transformation twice in an appropriate manner.

3 Passage to Heine's Transformations

In this section, we give a number of multivariate generalizations of Heine transformations for ${}_2\phi_1$ series [9]. (The first one is equivalent to Entry 6 of Chap. 16 of Ramanujan's notebook):

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a,b \\ c \end{matrix};q,u\right] &= \frac{(a,bu)_\infty}{(c,u)_\infty}\,{}_2\phi_1\left[\begin{matrix} c/a,u \\ bu \end{matrix};q,a\right] && \text{(1st)} \\
&= \frac{(c/a,au)_\infty}{(c,u)_\infty}\,{}_2\phi_1\left[\begin{matrix} a,abu/c \\ au \end{matrix};q,c/a\right] && \text{(2nd)} \\
&= \frac{(abu/c)_\infty}{(u)_\infty}\,{}_2\phi_1\left[\begin{matrix} c/b,c/a \\ c \end{matrix};q,abu/c\right]. && \text{(3rd)}
\end{aligned}
\tag{3.1}
$$

3.1 First Heine Transformation Formula for Basic Hypergeometric Series of Type A

Here we present a generalization of the first Heine transformation formula ((1st) of (3.1)):

$$ {}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix};q,u\right]=\frac{(a,bu)_\infty}{(c,u)_\infty}{}_2\phi_1\left[\begin{matrix}c/a,u\\bu\end{matrix};q,a\right] \tag{3.2}$$

for multiple basic hypergeometric series of type A, namely, with different dimensions.

Replace $f=a^mBcu/E$ in (2.14). Then let c tend to 0. After rearranging the parameters, we have

$$\begin{aligned}&\sum_{\gamma\in\mathbb{N}^n}u^{|\gamma|}x_1^{-\gamma_1}\cdots x_n^{-\gamma_n}\frac{\Delta(xq^\gamma)}{\Delta(x)}\prod_{1\le i,j\le n}\frac{(b_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}\prod_{1\le k\le m}\frac{(ay_k)_{|\gamma|}}{(c_ky_k)_{|\gamma|}}\\&=\prod_{1\le k\le m}\frac{(ay_k)_\infty}{(c_ky_k)_\infty}\prod_{1\le i\le n}\frac{(Buz_i)_\infty}{(uz_i)_\infty}\sum_{\delta\in\mathbb{N}^m}a^{|\delta|}w_1^{-\delta_1}\cdots w_m^{-\delta_m}\frac{\Delta(wq^\delta)}{\Delta(w)}\\&\times\prod_{1\le k,l\le m}\frac{((c_l/a)w_k/w_l)_{\delta_k}}{(qw_k/w_l)_{\delta_k}}\prod_{1\le i\le n}\frac{((Bu/b_i)z_i)_{|\delta|}}{(Buz_i)_{|\delta|}},\\&\qquad z_i=b_i/Bx_i,\ (1\le i\le n)\qquad w_k=y_k^{-1},\ (1\le k\le m).\end{aligned}\tag{3.3}$$

Note that (3.3) also has an alternative expression:

$$\begin{aligned}&\sum_{\gamma\in\mathbb{N}^n}u^{|\gamma|}x_1^{-\gamma_1}\cdots x_n^{-\gamma_n}\frac{\Delta(xq^\gamma)}{\Delta(x)}\prod_{1\le i,j\le n}\frac{(b_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}\prod_{1\le k\le m}\frac{(ay_k)_{|\gamma|}}{(c_ky_k)_{|\gamma|}}\\&=\prod_{1\le k\le m}\frac{(ay_k)_\infty}{(c_ky_k)_\infty}\prod_{1\le i\le n}\frac{(b_iux_i^{-1})_\infty}{((b_iu/B)x_i^{-1})_\infty}\sum_{\delta\in\mathbb{N}^m}a^{|\delta|}y_1^{\delta_1}\cdots y_m^{\delta_m}\frac{\Delta(y^{-1}q^\delta)}{\Delta(y^{-1})}\\&\times\prod_{1\le k,l\le m}\frac{((c_l/a)y_l/y_k)_{\delta_k}}{(qy_l/y_k)_{\delta_k}}\prod_{1\le i\le n}\frac{(ux_i^{-1})_{|\delta|}}{(b_iux_i^{-1})_{|\delta|}}.\end{aligned}\tag{3.4}$$

In the case when $m=1$ and $y_1=1$, (3.4) reduces to

$$\begin{aligned}&\sum_{\gamma\in\mathbb{N}^n}u^{|\gamma|}x_1^{-\gamma_1}\cdots x_n^{-\gamma_n}\frac{\Delta(xq^\gamma)}{\Delta(x)}\prod_{1\le i,j\le n}\frac{(b_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}\frac{(a)_{|\gamma|}}{(c)_{|\gamma|}}\\&=\frac{(a)_\infty}{(c)_\infty}\prod_{1\le i\le n}\frac{(b_iux_i^{-1})_\infty}{((b_iu/B)x_i^{-1})_\infty}{}_{n+1}\phi_n\left[\begin{matrix}c/a,\{ux_i^{-1}\}_n\\\{b_iux_i^{-1}\}_n\end{matrix};q,a\right].\end{aligned}\tag{3.5}$$

In the case when $m=n=1$ and $x_1=y_1=1$, (3.3) (equivalently (3.4)) reduces to the first Heine transformation (3.2).

In the case when $c_k=a$ for $1\le k\le m$, (3.3) reduces to the following A_n generalization of q-binomial theorem:

$$\sum_{\gamma\in\mathbb{N}^n} u^{|\gamma|} x_1^{-\gamma_1}\cdots x_n^{-\gamma_n}\frac{\Delta(xq^\gamma)}{\Delta(x)}\prod_{1\le i,j\le n}\frac{(b_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}=\prod_{1\le i\le n}\frac{(b_iux_i^{-1})_\infty}{((b_iu/B)x_i^{-1})_\infty}. \tag{3.6}$$

In the case when $n=1$ and $x_1=1$, (3.6) reduces to q-binomial theorem:

$$\sum_{k\in\mathbb{N}} u^k\frac{(b)_k}{(q)_k}=\frac{(bu)_\infty}{(u)_\infty}. \tag{3.7}$$

3.2 A_n Second Heine Transformations

We give a multiple generalization of second Heine transformation formula for ${}_2\phi_1$ series ((2nd) of (3.1)):

$${}_2\phi_1\left[\begin{matrix}a,b\\ c\end{matrix};q,u\right]=\frac{(c/a,au)_\infty}{(c,u)_\infty}{}_2\phi_1\left[\begin{matrix}a,abu/c\\ au\end{matrix};q,c/a\right] \tag{3.8}$$

from $m=n$ case of multiple ${}_3\phi_2$ transformation formula (2.9). Namely,

$$\begin{aligned}&\sum_{\gamma\in\mathbb{N}^n}\left(\frac{d^ne}{aBC}\right)^{|\gamma|}\frac{\Delta(xq^\gamma)}{\Delta(x)}\frac{(a)_{|\gamma|}}{(e)_{|\gamma|}}\prod_{1\le i,j\le n}\frac{(b_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}\prod_{1\le i,k\le n}\frac{(c_kx_iy_k)_{\gamma_i}}{(dx_iy_k)_{\gamma_i}}\\
&=\frac{(e/a,d^ne/BC)_\infty}{(e,d^ne/aBC)_\infty}\sum_{\delta\in\mathbb{N}^n}\left(\frac{e}{a}\right)^{|\delta|}\frac{\Delta(yq^\delta)}{\Delta(y)}\prod_{1\le k,l\le n}\frac{((d/c_l)y_k/y_l)_{\delta_k}}{(qy_k/y_l)_{\delta_k}}\\
&\times\frac{(a)_{|\delta|}}{(d^ne/BC)_{|\delta|}}\prod_{1\le k,i\le n}\frac{((d/b_i)x_iy_k)_{\delta_k}}{(dx_iy_k)_{\delta_k}}.\end{aligned} \tag{3.9}$$

Set $b_i=\frac{d}{u_i}\sqrt[n]{ac/e}$ in (3.9). Then let d tend to 0. Thus we have another type of A_n second Heine transformation:

$$\begin{aligned}&\sum_{\gamma\in\mathbb{N}^n}U^{|\gamma|}\frac{\Delta(xq^\gamma)}{\Delta(x)}\frac{(a)_{|\gamma|}}{(e)_{|\gamma|}}\prod_{1\le i,j\le n}(qx_i/x_j)_{\gamma_i}{}^{-1}\prod_{1\le i,k\le n}(c_kx_iy_k)_{\gamma_i}\\
&=\frac{(e/a,aU)_\infty}{(e,U)_\infty}\sum_{\delta\in\mathbb{N}^n}\left(\frac{e}{a}\right)^{|\delta|}\frac{\Delta(yq^\delta)}{\Delta(y)}\prod_{1\le k,l\le n}(qy_k/y_l)_{\delta_k}{}^{-1}\\
&\times\frac{(a)_{|\delta|}}{(aU)_{|\delta|}}\prod_{1\le k,i\le n}((u_i\sqrt[n]{aC/e})x_iy_k)_{\delta_k}.\end{aligned} \tag{3.10}$$

In the case when $n = 1$ and $x_1 = y_1 = 1$, (3.10) reduces to the second Heine transformation (3.8).

3.3 A_n Third Heine Transformation Formula

From (2.9), one can recover multivariate generalization of q-Euler transformation formula [8] with different dimensions:

$$\sum_{\gamma\in\mathbb{N}^n} u^{|\gamma|}\frac{\Delta(xq^{\gamma})}{\Delta(x)}\prod_{1\le i,j\le n}\frac{(b_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}\prod_{1\le i\le n,1\le k\le m}\frac{(c_kx_iy_k)_{\gamma_i}}{(dx_iy_k)_{\gamma_i}}$$
$$=\frac{(BCu/d^m)_\infty}{(u)_\infty}\sum_{\delta\in\mathbb{N}^m}\left(\frac{BCu}{d^m}\right)^{|\delta|}\frac{\Delta(yq^{\delta})}{\Delta(y)}\prod_{1\le k,l\le m}\frac{((d/c_l)y_k/y_l)_{\delta_k}}{(qy_k/y_l)_{\delta_k}}$$
$$\times\prod_{1\le i\le n,1\le k\le m}\frac{((d/b_i)x_iy_k)_{\delta_k}}{(dx_iy_k)_{\delta_k}}, \tag{3.11}$$

which generalize the third Heine transformation formula

$${}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix};q,u\right]=\frac{(abu/c)_\infty}{(u)_\infty}{}_2\phi_1\left[\begin{matrix}c/b,c/a\\c\end{matrix};q,\frac{ab}{c}u\right]. \tag{3.12}$$

(In the case when $m = n = 1$ and $x_1 = y_1 = 1$, (3.11) reduces to (3.12).)

4 Entry 8 as q-Pfaff Transformation

The purpose of this note is to present some multiple generalizations of q-Pfaff transformation formula:

$$\sum_{k\in\mathbb{N}}\frac{(a,b)_k}{(q,d)_k}x^k=\frac{(ax)_\infty}{(x)_\infty}\sum_{k\in\mathbb{N}}\frac{(a,d/b)_k}{(q,d,ax)_k}(-bx)^k q^{\binom{k}{2}}. \tag{4.1}$$

The q-Pfaff transformation formula (4.1) is a basic analogue of the Pfaff transformation formula for ${}_2F_1$ series (see (4.9)). Equation (4.1) can be expressed in terms of basic hypergeometric series:

$${}_2\phi_1\left[\begin{matrix}a,b\\d\end{matrix};q;x\right]=\frac{(ax)_\infty}{(x)_\infty}{}_2\phi_2\left[\begin{matrix}a,d/b\\d,ax\end{matrix};q;bx\right]. \tag{4.2}$$

Equation (4.1) has appeared as Entry 8 of Chap. 16 of Ramanujan's second notebook [1]. Analogous to the proof of (4.1) by Andrews [3], we derived multiple q-Pfaff transformation formulas by taking limits in multiple ${}_3\phi_2$ transformations in Sect. 2.

4.1 Preliminary Lemmas

Before beginning the derivation of the multivariate (q-)Pfaff transformation formula, we present lemmas concerning the limiting procedure for q-shifted factorials, and we will be using these frequently in the sequel.

First of all, the following is fundamental and is easy to see.

Lemma 1.

$$\lim_{b\to\infty}(bu)_k b^{-k} = (-u)^k q^{\frac{1}{2}k(k-1)}. \tag{4.3}$$

Equivalently, $(b)_k \sim (-b)^k q^{\frac{1}{2}k(k-1)}$ *as* $b\to\infty$.

Successive use of the above lemma leads to the following.

Lemma 2.

$$\lim_{b_1,\cdots,b_n\to\infty}\left[\prod_{1\le i,j\le n}(b_j x_i/x_j)_{\gamma_i}\right]B^{-|\gamma|} = (-1)^{n|\gamma|}q^{n\sum_{i=1}^n\binom{\gamma_i}{2}}(x_1\cdots x_n)^{-|\gamma|}\times x_1^{n\gamma_1}\cdots x_n^{n\gamma_n}. \tag{4.4}$$

Proof.

$$\begin{aligned}
&\left[\prod_{1\le i,j\le n}(b_j x_i/x_j)_{\gamma_i}\right]B^{-|\gamma|}\\
&=\prod_{1\le i\le n}\left(\prod_{1\le j\le n}(b_j x_i/x_j)_{\gamma_i}b_j^{-\gamma_i}\right)\xrightarrow{b_1,\ldots,b_n\to\infty}\prod_{1\le i\le n}\prod_{1\le j\le n}\left[(-1)^{\gamma_i}q^{\binom{\gamma_i}{2}}(x_i/x_j)^{\gamma_i}\right]\\
&=(-1)^{n|\gamma|}q^{n\sum_{i=1}^n\binom{\gamma_i}{2}}(x_1\cdots x_n)^{-|\gamma|}\times x_1^{n\gamma_1}\cdots x_n^{n\gamma_n}.
\end{aligned} \tag{4.5}$$

□

4.2 q-Pfaff Transformations with Different Dimensions

Take the limit b_i to infinity for $i=1,\ldots,n$ in (2.9). Replacing the parameters as $u=e/a$ and $b_k=d/c_k$ for $k=1,\ldots,m$ in the resulting identity leads the following generalization of q-Pfaff transformation:

$$\sum_{\delta\in\mathbb{N}^m} u^{|\delta|}\frac{\Delta(yq^\delta)}{\Delta(y)}\prod_{1\le k,l\le m}\frac{(b_l y_k/y_l)_{\delta_k}}{(qy_k/y_l)_{\delta_k}}(a)_{|\delta|}\prod_{1\le i\le n,1\le k\le m}\left[(dx_iy_k)_{\delta_k}\right]^{-1}$$

$$=\frac{(au)_\infty}{(u)_\infty}\sum_{\gamma\in\mathbb{N}^n}((-1)^nBu)^{|\gamma|}q^{n\sum_{i=1}^n\binom{\gamma_i}{2}}(x_1\cdots x_n)^{-|\gamma|}x_1^{n\gamma_1}\cdots x_n^{n\gamma_n}$$

$$\times\frac{\Delta(xq^\gamma)}{\Delta(x)}\frac{(a)_{|\gamma|}}{(au)_{|\gamma|}}\prod_{1\le i\le n,1\le k\le m}\frac{((d/b_k)x_iy_k)_{\gamma_i}}{(dx_iy_k)_{\gamma_i}}\prod_{1\le i,j\le n}\left[(qx_i/x_j)_{\gamma_i}\right]^{-1}. \tag{4.6}$$

In the case when $n=1$ and $x_1=1$, (4.6) reduces to

$$\sum_{\delta\in\mathbb{N}^m} u^{|\delta|}\frac{\Delta(yq^\delta)}{\Delta(y)}\prod_{1\le k,l\le m}\frac{(b_l y_k/y_l)_{\delta_k}}{(qy_k/y_l)_{\delta_k}}(a)_{|\delta|}\prod_{1\le k\le m}\left[(dy_k)_{\delta_k}\right]^{-1}$$

$$=\frac{(au)_\infty}{(u)_\infty}\sum_{\gamma\in\mathbb{N}}(-Bu)^\gamma q^{\binom{\gamma}{2}}\frac{(a)_\gamma}{(au,q)_\gamma}\prod_{1\le k\le m}\frac{((d/b_k)y_k)_\gamma}{(dy_k)_\gamma}. \tag{4.7}$$

In the case when $m=1$ and $y_1=1$, (4.6) reduces to

$$\sum_{\delta\in\mathbb{N}} u^\delta\frac{(a,b)_\delta}{(q)_\delta}\prod_{1\le i\le n}[(dx_i)_\delta]^{-1}$$

$$=\frac{(au)_\infty}{(u)_\infty}\sum_{\gamma\in\mathbb{N}^n}((-1)^nBu)^{|\gamma|}q^{n\sum_{i=1}^n\binom{\gamma_i}{2}}(x_1\cdots x_n)^{-|\gamma|}x_1^{n\gamma_1}\cdots x_n^{n\gamma_n}$$

$$\times\frac{\Delta(xq^\gamma)}{\Delta(x)}\frac{(a)_{|\gamma|}}{(au)_{|\gamma|}}\prod_{1\le i\le n}\frac{((d/b_k)x_i)_{\gamma_i}}{(dx_i)_{\gamma_i}}\prod_{1\le i,j\le n}\left[(qx_i/x_j)_{\gamma_i}\right]^{-1}.$$

In (4.6), we replace the parameters $x_i\to q^{x_i}(1\le i\le n)$, $y_k\to q^{y_k}(1\le k\le m), a_i\to q^{a_i}(1\le i\le n), b_k\to q^{b_k}(1\le k\le m)$, and $d\to q^d$, and then take the limit $q\to 1$ to obtain the following multiple generalization of Pfaff transformation for Gauss hypergeometric function ${}_2F_1$:

$$\sum_{\delta\in\mathbb{N}^m} u^{|\delta|}\frac{\Delta(y+\delta)}{\Delta(y)}\prod_{1\le k,l\le m}\frac{[b_l+y_k-y_l]_{\delta_k}}{[1+y_k-y_l]_{\delta_k}}[a]_{|\delta|}\prod_{1\le i\le n,1\le k\le m}\left[[d+x_i+y_k]_{\delta_k}\right]^{-1}$$

$$=(1-u)^{-a}\sum_{\gamma\in\mathbb{N}^n}\left(\frac{u}{u-1}\right)^{|\gamma|}\frac{\Delta(x+\gamma)}{\Delta(x)}[a]_{|\gamma|}$$

$$\times\prod_{1\le i\le n,1\le k\le m}\frac{[d-b_k+x_i+y_k]_{\gamma_i}}{[d+x_i+y_k]_{\gamma_i}}\prod_{1\le i,j\le n}\left[[1+x_i-x_j]_{\gamma_i}\right]^{-1}. \tag{4.8}$$

Remark 4.2. In the case when $n=m=1$ and $x_1=y_1=0$, (4.8) reduces to the Pfaff transformation formula for Gauss hypergeometric series ${}_2F_1$:

$$ {}_2F_1\left[\begin{matrix}a,b\\ d\end{matrix};u\right] = (1-u)^{-a}{}_2F_1\left[\begin{matrix}a,d-b\\ d\end{matrix};\frac{u}{u-1}\right]. \tag{4.9}$$

In the case when $n=1$ and $x_1=0$, (4.8) reduces to

$$\sum_{\delta\in\mathbb{N}^m} u^{|\delta|}\frac{\Delta(y+\delta)}{\Delta(y)}\prod_{1\le k,l\le m}\frac{[b_l+y_k-y_l]_{\delta_k}}{[1+y_k-y_l]_{\delta_k}}\,[a]_{|\delta|}\prod_{1\le k\le m}\left[[d+y_k]_{\delta_k}\right]^{-1}$$

$$= (1-u)^{-a}{}_{m+1}F_m\left[\begin{matrix}a,\{d-b_k+y_k\}_m\\ \{d+y_k\}_m\end{matrix};\frac{u}{u-1}\right]. \tag{4.10}$$

In the case when $m=1$ and $y_1=0$, (4.8) reduces to

$$\sum_{\delta\in\mathbb{N}} u^{\delta}\frac{[a,b]_\delta}{\delta!}\prod_{1\le i\le n}[[d+x_i]_\delta]^{-1} = (1-u)^{-a}\sum_{\gamma\in\mathbb{N}^n}\left(\frac{u}{u-1}\right)^{|\gamma|}\frac{\Delta(x+\gamma)}{\Delta(x)}[a]_{|\gamma|}$$

$$\times\prod_{1\le i\le n}\frac{[d-b_k+x_i]_{\gamma_i}}{[d+x_i]_{\gamma_i}}\prod_{1\le i,j\le n}\left[[1+x_i-x_j]_{\gamma_i}\right]^{-1}. \tag{4.11}$$

Next, for $k=1,\dots,m$, let c_k tend to infinity in (2.9) and rearrange the parameters as $u=e/a$ and $c_i=d/b_i$ for $i=1,\dots,n$. Then we obtain the following generalization of q-Pfaff transformation:

$$\sum_{\delta\in\mathbb{N}^m} u^{|\delta|}\frac{\Delta(yq^\delta)}{\Delta(y)}(a)_{|\delta|}\prod_{1\le i\le n,1\le k\le m}\frac{(b_ix_iy_k)_{\delta_k}}{(dx_iy_k)_{\delta_k}}\prod_{1\le k,l\le m}\left[(qy_k/y_l)_{\delta_k}\right]^{-1}$$

$$=\frac{(au)_\infty}{(u)_\infty}\sum_{\gamma\in\mathbb{N}^n}((-1)^mBu)^{|\gamma|}x_1^{m\gamma_1}\cdots x_n^{m\gamma_n}(y_1\cdots y_m)^{n|\gamma|}q^{\frac12\sum_{1\le i\le n}\gamma_i(\gamma_i-1)}$$

$$\times\frac{\Delta(xq^\gamma)}{\Delta(x)}\frac{(a)_{|\gamma|}}{(au)_{|\gamma|}}\prod_{1\le i\le n,1\le k\le m}\left[(dx_iy_k)_{\gamma_i}\right]^{-1}\prod_{1\le i,j\le n}\frac{((d/b_j)x_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}. \tag{4.12}$$

In the case when $n=1$ and $x_1=1$, (4.12) reduces to

$$\sum_{\delta\in\mathbb{N}^m} u^{|\delta|}\frac{\Delta(yq^\delta)}{\Delta(y)}(a)_{|\delta|}\prod_{1\le k\le m}\frac{(by_k)_{\delta_k}}{(dy_k)_{\delta_k}}\prod_{1\le k,l\le m}\left[(qy_k/y_l)_{\delta_k}\right]^{-1}$$

$$=\frac{(au)_\infty}{(u)_\infty}\sum_{\gamma\in\mathbb{N}}((-1)^mbu)^\gamma(y_1\cdots y_m)^\gamma q^{\binom{\gamma}{2}}\frac{(a,d/b)_{|\gamma|}}{(au,q)_{|\gamma|}}\prod_{1\le k\le m}\left[(dy_k)_{\gamma_i}\right]^{-1}. \tag{4.13}$$

In the case when $m=1$ and $y_1=0$, (4.12) reduces to

$$ {}_{n+1}\phi_n\left[\begin{matrix}a,\{b_ix_i\}_n\\ \{dx_i\}_n\end{matrix};q;u\right]=\frac{(au)_\infty}{(u)_\infty}\sum_{\gamma\in\mathbb{N}^n}(-Bu)^{|\gamma|}x_1^{\gamma_1}\cdots x_n^{\gamma_n}q^{\frac{1}{2}\sum_{1\le i\le n}\gamma_i(\gamma_i-1)} $$

$$ \times\frac{\Delta(xq^{\gamma})}{\Delta(x)}\frac{(a)_{|\gamma|}}{(au)_{|\gamma|}}\prod_{1\le i\le n}\left[(dx_i)_{\gamma_i}\right]^{-1}\prod_{1\le i,j\le n}\frac{((d/b_j)x_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}. \tag{4.14} $$

Similarly to obtain (4.8), taking $q\to1$ limit leads to the following multiple Pfaff transformation:

$$ \sum_{\delta\in\mathbb{N}^m}u^{|\delta|}\frac{\Delta(y+\delta)}{\Delta(y)}[a]_{|\delta|}\prod_{1\le i\le n,1\le k\le m}\frac{[b_i+x_i+y_k]_{\delta_k}}{[d+x_i+y_k]_{\delta_k}}\prod_{1\le k,l\le m}\left[[1+y_k-y_l]_{\delta_k}\right]^{-1} $$

$$ =(1-u)^{-a}\sum_{\gamma\in\mathbb{N}^n}\left(\frac{u}{u-1}\right)^{|\gamma|}\frac{\Delta(x+\gamma)}{\Delta(x)}[a]_{|\gamma|} $$

$$ \times\prod_{1\le i\le n,1\le k\le m}\left[[d+x_i+y_k]_{\gamma_i}\right]^{-1}\prod_{1\le i,j\le n}\frac{[d-b_j+x_i-x_j]_{\gamma_i}}{[1+x_i-x_j]_{\gamma_i}}. \tag{4.15} $$

In the case when $n=1$ and $x_1=0$, (4.15) reduces to

$$ \sum_{\delta\in\mathbb{N}^m}u^{|\delta|}\frac{\Delta(y+\delta)}{\Delta(y)}[a]_{|\delta|}\prod_{1\le k\le m}\frac{[b+y_k]_{\delta_k}}{[d+y_k]_{\delta_k}}\prod_{1\le k,l\le m}\left[[1+y_k-y_l]_{\delta_k}\right]^{-1} $$

$$ =(1-u)^{-a}\sum_{\gamma\in\mathbb{N}}\left(\frac{u}{u-1}\right)^{\gamma}\frac{[a,d-b]_\gamma}{\gamma!}\prod_{1\le k\le m}\left[[d+y_k]_\gamma\right]^{-1}. \tag{4.16} $$

In the case when $m=1$ and $y_1=0$, (4.15) reduces to

$$ {}_{n+1}F_n\left[\begin{matrix}a,\{b_i+x_i\}_n\\ \{d+x_i\}_n\end{matrix};u\right]=(1-u)^{-a}\sum_{\gamma\in\mathbb{N}^n}\left(\frac{u}{u-1}\right)^{|\gamma|}\frac{\Delta(x+\gamma)}{\Delta(x)}[a]_{|\gamma|} $$

$$ \times\prod_{1\le i\le n}\left[[d+x_i]_{\gamma_i}\right]^{-1}\prod_{1\le i,j\le n}\frac{[d-b_j+x_i-x_j]_{\gamma_i}}{[1+x_i-x_j]_{\gamma_i}}. \tag{4.17} $$

In the case when $m=n=1$ and $x_1=y_1=0$, (4.15) reduces to the Pfaff transformation (4.9).

5 Entry 9 of Chap. 16 in Ramanujan's Notebook and Related Formulas

This section and the last are devoted to multivariate generalizations of the q-series transformations for some partition functions. Particularly, the purpose of this section is to present some multiple generalizations of Entry 9

$$(aq)_\infty \sum_{k\in\mathbb{N}} \frac{b^k q^{k^2}}{(q,aq)_k} = \sum_{k\in\mathbb{N}} \frac{(b/a)_k}{(q)_k} (-a)^k q^{\frac{1}{2}k(k+1)} \tag{5.1}$$

of Chap. 16 of Ramanujan's second notebook [1]. We derive them by taking certain limits to multiple nonterminating ${}_3\phi_2$ transformations. We also give some generalizations of several classical partition function identities as special cases of them.

In (2.9), let a and b_i for $i = 1,\ldots,n$ tend to ∞. Now we suppose that $c_k = c$ for all $k = 1,\ldots,m$ in the resulting identity. Then replace the parameters as $d = bc/a$ and $e = a^m q$. By letting c tend to 0, we arrive at the following multiple generalization of Entry 9 (5.1):

$$\begin{aligned}(a^m q)_\infty &\sum_{\gamma\in\mathbb{N}^n} \left((-1)^{n+1} b^m\right)^{|\gamma|} (x_1\cdots x_n)^{-|\gamma|} x_1^{n\gamma_1}\cdots x_n^{n\gamma_n} q^{\frac{1}{2}|\gamma|(|\gamma|+1)+\frac{n}{2}\sum_{i=1}^n \gamma_i(\gamma_i-1)} \\ &\times \frac{\Delta(xq^\gamma)}{\Delta(x)} \left[(a^m q)_{|\gamma|}\right]^{-1} \prod_{1\le i,j\le n} \left[(qx_i/x_j)_{\gamma_i}\right]^{-1} \\ = &\sum_{\delta\in\mathbb{N}^m} (-a^m)^{|\delta|} q^{\frac{1}{2}|\delta|(|\delta|+1)} \frac{\Delta(yq^\delta)}{\Delta(y)} \prod_{1\le k,l\le m} \frac{((b/a)y_k/y_l)_{\delta_k}}{(qy_k/y_l)_{\delta_k}}.\end{aligned} \tag{5.2}$$

It is interesting to note that, since

$$\sum_{\gamma\in\mathbb{N}^n, |\gamma|=N} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1\le i,j\le n} \frac{(a_j x_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}} = \frac{(a_1 a_2\cdots a_n)_N}{(q)_N}, \tag{5.3}$$

which is equivalent to the A_n generalization of the Rogers' q-Dougall summation formula for very well-poised ${}_6\phi_5$ series and is a direct consequence of A_n q-binomial theorem by Milne [10]:

$$\sum_{\gamma\in\mathbb{N}^n} u^{|\gamma|} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1\le i,j\le n} \frac{(a_j x_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}} = \frac{(a_1 a_2\cdots a_n u)_\infty}{(u)_\infty}, \tag{5.4}$$

holds, the right hand side of (5.2) essentially reduces to $m = 1$ case:

$$(aq)_\infty \sum_{\gamma\in\mathbb{N}^n} \left((-1)^{n+1}b\right)^{|\gamma|} (x_1\cdots x_n)^{-|\gamma|} x_1^{n\gamma_1}\cdots x_n^{n\gamma_n} q^{\frac{1}{2}|\gamma|(|\gamma|+1)+\frac{n}{2}\sum_{i=1}^n \gamma_i(\gamma_i-1)}$$
$$\times \frac{\Delta(xq^\gamma)}{\Delta(x)} \left[(aq)_{|\gamma|}\right]^{-1} \prod_{1\le i,j\le n} \left[(qx_i/x_j)_{\gamma_i}\right]^{-1} = \sum_{k\in\mathbb{N}} (-a)^k q^{\frac{1}{2}k(k-1)} \frac{(b/a)_k}{(q)_k}. \tag{5.5}$$

In the case when $n = 1$ and $x_1 = 1$, (5.5) reduces to Entry 9 (5.1) of Chap. 16 of Ramanujan's notebook.

We are going to present special cases of (5.5). In the case when $a = b$, (5.5) reduces to

$$\sum_{\gamma\in\mathbb{N}^n} \left((-1)^{n+1}a\right)^{|\gamma|} (x_1\cdots x_n)^{-|\gamma|} x_1^{n\gamma_1}\cdots x_n^{n\gamma_n} q^{\frac{1}{2}|\gamma|(|\gamma|+1)+\frac{n}{2}\sum_{i=1}^n \gamma_i(\gamma_i-1)}$$
$$\times \frac{\Delta(xq^\gamma)}{\Delta(x)} \left[(aq)_{|\gamma|}\right]^{-1} \prod_{1\le i,j\le n} \left[(qx_i/x_j)_{\gamma_i}\right]^{-1} = \frac{1}{(aq)_\infty}. \tag{5.6}$$

Note that, in the case when $n = 1$ and $x_1 = 1$, (5.6) reduces to Entry 3 of Chap. 16 of Ramanujan's notebook [1]:

$$\frac{1}{(aq)_\infty} = \sum_{k\in\mathbb{N}} \frac{a^k q^{k^2}}{(q,aq)_k}. \tag{5.7}$$

In the case when $a = 1,\ b = q$, (5.5) reduces to

$$(q)_\infty \sum_{\gamma\in\mathbb{N}^n} (-1)^{(n+1)|\gamma|} (x_1\cdots x_n)^{-|\gamma|} x_1^{n\gamma_1}\cdots x_n^{n\gamma_n} q^{\frac{1}{2}|\gamma|(|\gamma|+3)+\frac{n}{2}\sum_{i=1}^n \gamma_i(\gamma_i-1)}$$
$$\times \frac{\Delta(xq^\gamma)}{\Delta(x)} \left[(q)_{|\gamma|}\right]^{-1} \prod_{1\le i,j\le n} \left[(qx_i/x_j)_{\gamma_i}\right]^{-1} = \sum_{k\in\mathbb{N}} (-1)^k q^{\frac{1}{2}k(k-1)}. \tag{5.8}$$

Note that, in the case when $n = 1$ and $x_1 = 1$, (5.8) reduces to the following q-series transformation formula:

$$(q)_\infty \sum_{k\in\mathbb{N}} \frac{q^{k^2+k}}{(q)_k^2} = \sum_{k\in\mathbb{N}} (-1)^k q^{\frac{1}{2}k(k+1)}, \tag{5.9}$$

which is corollary (i) of Entry 9 in [1] and which is originally due to Gauss.

Replace q by q^2 and let $a=q^{-1}, b=q$ in (5.5) to obtain the following formula:

$$(q;q^2)_\infty \sum_{\gamma\in\mathbb{N}^n} \left((-1)^{n+1}\right)^{|\gamma|} (x_1\cdots x_n)^{-|\gamma|} x_1^{n\gamma_1}\cdots x_n^{n\gamma_n} q^{|\gamma|(|\gamma|+2)+n\sum_{i=1}^n \gamma_i(\gamma_i-1)}$$

$$\times \frac{\Delta(xq^{2\gamma})}{\Delta(x)} \left[(q;q^2)_{|\gamma|}\right]^{-1} \prod_{1\le i,j\le n} \left[(q^2x_i/x_j;q^2)_{\gamma_i}\right]^{-1} = \sum_{k\in\mathbb{N}} (-1)^k q^{k^2}. \tag{5.10}$$

Note that, in the case when $n=1$ and $x_1=1$, (5.10) reduces to the following q-series transformation:

$$(q;q^2)_\infty \sum_{k\in\mathbb{N}} \frac{q^{2k^2+k}}{(q)_{2k}} = \sum_{k\in\mathbb{N}} (-1)^k q^{k^2}, \tag{5.11}$$

which is corollary (ii) of Entry 9 in [1].

6 ${}_1\phi_1$ Transformation Formula Related to q-Lauricella Function

In this section we give a multiple generalization of the following q-series transformation formula:

$$(b)_\infty \sum_{k\in\mathbb{N}} \frac{(\lambda/a)_k}{(b,q)_k} (-a)^k q^{\binom{k}{2}} = (a)_\infty \sum_{k\in\mathbb{N}} \frac{(\lambda/b)_k}{(a,q)_k} (-b)^k q^{\binom{k}{2}}. \tag{6.1}$$

Equation (6.1) is the transformation formula for ${}_1\phi_1$ series:

$${}_1\phi_1 \left[\begin{matrix}\lambda/a \\ b\end{matrix}; q; a\right] = \frac{(a)_\infty}{(b)_\infty} {}_1\phi_1 \left[\begin{matrix}\lambda/b \\ a\end{matrix}; q; b\right]. \tag{6.2}$$

Note that (6.1) is symmetric with respect to the parameters a and b. In [15], Srivastava found that Entry 9 of Chap. 16 of Ramanujan's second notebook [1] can be obtained from (6.1) and verified other q-series transformation and summation formulas.

We give a result by using a multiple generalization of the second Heine's transformation between A_n basic hypergeometric series and basic Lauricella function $\phi_D^{(n)}$ (6.3) appeared in our previous work [8].

6.1 Second Heine Transformation Formula Between A_n ${}_2\phi_1$ Series and Basic Lauricella Series $\phi_D^{(n)}$ [8]

$$\sum_{\gamma\in\mathbb{N}^n} u^{|\gamma|}\frac{\Delta(xq^\gamma)}{\Delta(x)}\prod_{1\le i,j\le n}\frac{(a_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}\prod_{1\le i\le n}\frac{(bx_i)_{\gamma_i}}{(cx_i)_{\gamma_i}} = \frac{(Au)_\infty}{(u)_\infty}\prod_{1\le i\le n}\frac{((c/a_i)x_i)_\infty}{(cx_i)_\infty}$$
$$\times\phi_D^{(n)}\left[\begin{matrix}Abu/c;a_1,\ldots,a_n\\ Au\end{matrix};q;(c/a_1)x_1,\ldots,(c/a_{n-1})x_{n-1},(c/a_n)x_n\right]$$
$$=\frac{(Au)_\infty}{(u)_\infty}\prod_{1\le i\le n}\frac{((c/a_i)x_i)_\infty}{(cx_i)_\infty}\sum_{\alpha\in\mathbb{N}^n}\frac{(Abu/c)_{|\alpha|}(a_1)_{\alpha_1}(a_2)_{\alpha_2}\ldots(a_n)_{\alpha_n}}{(Au)_{|\alpha|}(q)_{\alpha_1}(q)_{\alpha_2}\ldots(q)_{\alpha_n}}$$
$$\times\prod_{1\le i\le n}[(c/a_i)x_i]^{\alpha_i}. \tag{6.3}$$

Formula (6.3) has been obtained by combining the multivariate Euler transformation in our previous work [8] and Andrews' transformation (see [2] (the case of $l=2$) and [4])

$$\phi_D^{(l)}\left[\begin{matrix}a;b_1,b_2,\ldots,b_l\\ c\end{matrix};q;x_1,x_2,\ldots,x_l\right]=\frac{(a)_\infty}{(c)_\infty}\prod_{k=1}^{l}\frac{(b_kx_k)_\infty}{(x_k)_\infty}{}_{l+1}\phi_l\left[\begin{matrix}c/a,x_1,\ldots,x_l\\ b_1x_1,\ldots,b_lx_l\end{matrix};q;a\right] \tag{6.4}$$

between the basic hypergeometric series ${}_{n+1}\phi_n$ and the basic Lauricella series $\phi_D^{(n)}$, which is defined by

$$\phi_D^{(l)}\left[\begin{matrix}a;b_1,b_2,\ldots,b_l\\ c\end{matrix};q;x_1,x_2,\ldots,x_l\right]=\sum_{\alpha\in\mathbb{N}^l}\frac{(a)_{|\alpha|}(b_1)_{\alpha_1}(b_2)_{\alpha_2}\ldots(b_l)_{\alpha_l}}{(c)_{|\alpha|}(q)_{\alpha_1}(q)_{\alpha_2}\ldots(q)_{\alpha_l}}x^\alpha. \tag{6.5}$$

Remark 6.3. In the case when $n=1$, (6.3) reduces to the second Heine's transformation formula (3.8).

6.2 Multivariate ${}_1\phi_1$ Transformation

In (6.3), replace u to u/A. Then let a_i tend to infinity for $i=1,\ldots,n$. By setting $a=u, b=\lambda/a$, and $c=b$, respectively, we arrive at a multiple generalization of ${}_1\phi_1$ transformation:

$$\sum_{\gamma\in\mathbb{N}^n} ((-1)^n a)^{|\gamma|} (x_1\cdots x_n)^{-|\gamma|} x_1^{n\gamma_1}\cdots x_n^{n\gamma_n} q^{\frac{n}{2}\sum_i \gamma_i(\gamma_i-1)}$$
$$\times \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1\le i,j\le n} (qx_i/x_j)_{\gamma_i}^{-1} \prod_{1\le i\le n} \frac{((\lambda/a)x_i)_{\gamma_i}}{(bx_i)_{\gamma_i}}$$
$$= (a)_\infty \prod_{1\le i\le n} (bx_i)_\infty^{-1} \sum_{\alpha\in\mathbb{N}^n} \frac{(\lambda/b)_{|\alpha|}}{(a)_{|\alpha|}(q)_{\alpha_1}(q)_{\alpha_2}\cdots(q)_{\alpha_n}} (-b)^{|\alpha|} x^\alpha q^{\frac{1}{2}\sum_i \alpha_i(\alpha_i-1)}. \tag{6.6}$$

In the case when $n=1$, (6.6) reduces to ${}_1\phi_1$ transformation (6.1). Note that (6.6) is *not* symmetric with respect to the parameters a and b, but also the forms of the multiple series in both sides are different.

In (6.6), replace $a\to aq, b\to bq$, and $\lambda\to\lambda q$. Next let a tend to 0. Then we have a multiple generalization of Entry 9 of Chap. 16 of Ramanujan's notebook:

$$\prod_{1\le i\le n} (bqx_i)_\infty \sum_{\gamma\in\mathbb{N}^n} \left((-1)^{n+1}\lambda\right)^{|\gamma|} (x_1\cdots x_n)^{-|\gamma|} x_1^{(n+1)\gamma_1}\cdots x_n^{(n+1)\gamma_n}$$
$$\times q^{\frac{n+1}{2}\sum_i \gamma_i(\gamma_i-1)} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1\le i,j\le n} (qx_i/x_j)_{\gamma_i}^{-1} \prod_{1\le i\le n} (bqx_i)_{\gamma_i}^{-1}$$
$$= \sum_{\alpha\in\mathbb{N}^n} \frac{(\lambda/b)_{|\alpha|}}{(q)_{\alpha_1}(q)_{\alpha_2}\cdots(q)_{\alpha_n}} (-b)^{|\alpha|} x^\alpha q^{\frac{1}{2}\sum_i \alpha_i(\alpha_i-1)}. \tag{6.7}$$

In the case when $n=1$, (6.7) reduces to Entry 9 (5.1).

In (6.6), set $\lambda=b$. Then we obtain

$$\sum_{\gamma\in\mathbb{N}^n} ((-1)^n a)^{|\gamma|} (x_1\cdots x_n)^{-|\gamma|} x_1^{n\gamma_1}\cdots x_n^{n\gamma_n} q^{\frac{n}{2}\sum_i \gamma_i(\gamma_i-1)}$$
$$\times \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1\le i,j\le n} (qx_i/x_j)_{\gamma_i}^{-1} \prod_{1\le i\le n} \frac{((b/a)x_i)_{\gamma_i}}{(bx_i)_{\gamma_i}} = (a)_\infty \prod_{1\le i\le n} (bx_i)_\infty^{-1}. \tag{6.8}$$

In the case when $n=1$, (6.8) reduces to

$$\sum_{k\in\mathbb{N}} \frac{(y/x)_k}{(y,q)_k} (-x)^k q^{\binom{k}{2}} = \frac{(x)_\infty}{(y)_\infty}. \tag{6.9}$$

By replacing $b\to bq$ and taking the limit $a\to 0$ in (6.8), we have an another A_n generalization of Entry 3:

$$\sum_{\gamma\in\mathbb{N}^n} \left((-1)^{n+1} b\right)^{|\gamma|} (x_1\cdots x_n)^{-|\gamma|} x_1^{(n+1)\gamma_1}\cdots x_n^{(n+1)\gamma_n} q^{|\gamma|+\frac{n+1}{2}\sum_i \gamma_i(\gamma_i-1)}$$
$$\times \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1\le i,j\le n} (qx_i/x_j)_{\gamma_i}^{-1} \prod_{1\le i\le n} (bqx_i)_{\gamma_i}^{-1} = \prod_{1\le i\le n} (bqx_i)_\infty^{-1}. \tag{6.10}$$

In the case when $n=1$, (6.10) reduces to Entry 3 (5.7).

In (6.6), set $b = q, \lambda = q^2$, and $a \to aq$. Then we obtain

$$\sum_{\gamma \in \mathbb{N}^n} ((-1)^n a)^{|\gamma|} (x_1 \cdots x_n)^{-|\gamma|} x_1^{n\gamma_1} \cdots x_n^{n\gamma_n} q^{|\gamma| + \frac{n}{2} \sum_i \gamma_i(\gamma_i - 1)}$$
$$\times \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \le i,j \le n} (qx_i/x_j)_{\gamma_i}^{-1} \prod_{1 \le i \le n} \frac{((q/a)x_i)_{\gamma_i}}{(qx_i)_{\gamma_i}}$$
$$= (aq)_\infty \prod_{1 \le i \le n} (qx_i)_\infty^{-1} \sum_{\alpha \in \mathbb{N}^n} \frac{(q)_{|\alpha|}}{(aq)_{|\alpha|} (q)_{\alpha_1} (q)_{\alpha_2} \cdots (q)_{\alpha_n}} (-1)^{|\alpha|} x^\alpha q^{\frac{1}{2} \sum_i \alpha_i(\alpha_i + 1)}. \tag{6.11}$$

In the case when $n = 1$, (6.11) reduces to

$$\sum_{k \in \mathbb{N}} \frac{(q/x)_k}{(q)_k^2} (-x)^k q^{\frac{1}{2}k(k+1)} = \frac{(xq)_\infty}{(q)_\infty} \sum_{k \in \mathbb{N}} \frac{1}{(xq)_k} (-1)^k q^{\frac{1}{2}k(k+1)}. \tag{6.12}$$

By letting a tend to 0 in (6.11), we have

$$\sum_{\gamma \in \mathbb{N}^n} (-1)^{(n+1)|\gamma|} (x_1 \cdots x_n)^{-|\gamma|} x_1^{(n+1)\gamma_1} \cdots x_n^{(n+1)\gamma_n} q^{|\gamma| + \frac{n+1}{2} \sum_i \gamma_i^2}$$
$$\times \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1 \le i,j \le n} (qx_i/x_j)_{\gamma_i}^{-1} \prod_{1 \le i \le n} (qx_i)_{\gamma_i}^{-1}$$
$$= \prod_{1 \le i \le n} (qx_i)_\infty^{-1} \sum_{\alpha \in \mathbb{N}^n} \frac{(q)_{|\alpha|}}{(q)_{\alpha_1} (q)_{\alpha_2} \cdots (q)_{\alpha_n}} (-1)^{|\alpha|} x^\alpha q^{\frac{1}{2} \sum_i \alpha_i(\alpha_i + 1)}. \tag{6.13}$$

In the case when $n = 1$, (6.13) reduces to the Gauss' formula (5.9).
In (6.6), replace q by q^2. By setting $b = q, \lambda = q^3$, and $a \to aq^2$, we obtain

$$\sum_{\gamma \in \mathbb{N}^n} ((-1)^n a)^{|\gamma|} (x_1 \cdots x_n)^{-|\gamma|} x_1^{n\gamma_1} \cdots x_n^{n\gamma_n} q^{2|\gamma| + n \sum_i \gamma_i(\gamma_i - 1)}$$
$$\times \frac{\Delta(xq^{2\gamma})}{\Delta(x)} \prod_{1 \le i,j \le n} (q^2 x_i/x_j; q^2)_{\gamma_i}^{-1} \prod_{1 \le i \le n} \frac{((q/a)x_i; q^2)_{\gamma_i}}{(qx_i; q^2)_{\gamma_i}}$$
$$= (aq^2; q^2)_\infty \prod_{1 \le i \le n} (qx_i; q^2)_\infty^{-1}$$
$$\times \sum_{\alpha \in \mathbb{N}^n} \frac{(q^2; q^2)_{|\alpha|}}{(aq^2; q^2)_{|\alpha|} (q^2; q^2)_{\alpha_1} (q^2; q^2)_{\alpha_2} \cdots (q^2; q^2)_{\alpha_n}} (-1)^{|\alpha|} x^\alpha q^{\sum_i \alpha_i^2}. \tag{6.14}$$

In the case when $n=1$, (6.14) reduces to

$$\sum_{k\in\mathbb{N}} \frac{(q/x;q^2)_k}{(q,q^2;q^2)_k} (-x)^k q^{k^2+k} = \sum_{k\in\mathbb{N}} \frac{(q/x;q^2)_k}{(q)_{2k}} (-x)^k q^{k^2+k}$$
$$= \frac{(xq^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{k\in\mathbb{N}} \frac{1}{(xq^2;q^2)_k} (-1)^k q^{k^2}.$$

In (6.14), let a tend to 0. Then we obtain

$$\sum_{\gamma\in\mathbb{N}^n} (-1)^{(n+1)|\gamma|} (x_1\cdots x_n)^{-|\gamma|} x_1^{n\gamma_1}\cdots x_n^{n\gamma_n} q^{3|\gamma|+(n+1)\sum_i \gamma_i(\gamma_i-1)}$$
$$\times \frac{\Delta(xq^{2\gamma})}{\Delta(x)} \prod_{1\le i,j\le n} (q^2 x_i/x_j;q^2)_{\gamma_i}^{-1} \prod_{1\le i\le n} (qx_i;q^2)_{\gamma_i}^{-1}$$
$$= \prod_{1\le i\le n} (qx_i;q^2)_\infty^{-1} \sum_{\alpha\in\mathbb{N}^n} \frac{(q^2;q^2)_{|\alpha|}}{(q^2;q^2)_{\alpha_1}(q^2;q^2)_{\alpha_2}\cdots(q^2;q^2)_{\alpha_n}} (-1)^{|\alpha|} x^\alpha q^{\sum_i \alpha_i^2}. \tag{6.15}$$

In the case when $n=1$, (6.11) reduces to (5.11).

It is an interesting feature of our multivariate ${}_1\phi_1$ transformation (6.6) that the series in both sides have a different structure from each other. So one obtains other q-series transformation and summation formulas by reversing both sides in (6.6) and similar specializations as above.

In (6.6), replace $a\to aq, b\to bq$, and $\lambda\to\lambda q$. Next let b tend to infinity. Then we obtain another type of multivariate generalization of Entry 9 (5.1):

$$\sum_{\alpha\in\mathbb{N}^n} \frac{(-\lambda)^{|\alpha|} x^\alpha q^{\frac{1}{2}|\alpha|(|\alpha|+1)+\frac{1}{2}\sum_i \alpha_i(\alpha_i-1)}}{(aq)_{|\alpha|}(q)_{\alpha_1}(q)_{\alpha_2}\cdots(q)_{\alpha_n}}$$
$$= (aq)_\infty^{-1} \sum_{\gamma\in\mathbb{N}^n} ((-1)^n a)^{|\gamma|} (x_1\cdots x_n)^{-|\gamma|} x_1^{n\gamma_1}\cdots x_n^{n\gamma_n} q^{\frac{n}{2}\sum_i \gamma_i(\gamma_i-1)}$$
$$\times \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1\le i,j\le n} (qx_i/x_j)_{\gamma_i}^{-1} \prod_{1\le i\le n} ((\lambda/a)x_i)_{\gamma_i}. \tag{6.16}$$

Indeed, in the case when $n=1$, (6.16) reduces to (5.1).

By setting $\lambda=b$ in (6.6), we have

$$\sum_{\alpha\in\mathbb{N}^n} \frac{(a/b)_{|\alpha|}}{(a)_{|\alpha|}(q)_{\alpha_1}(q)_{\alpha_2}\cdots(q)_{\alpha_n}} (-b)^{|\alpha|} x^\alpha q^{\frac{1}{2}\sum_i \alpha_i(\alpha_i-1)} = (a)_\infty^{-1} \prod_{1\le i\le n} (bx_i)_\infty. \tag{6.17}$$

In the case when $n=1$, (6.17) reduces to (6.9).

Replace $a \to aq$ in (6.17) and let b tend to 0. Then we obtain

$$\sum_{\alpha\in\mathbb{N}^n} \frac{(-1)^{|\alpha|} x^{\alpha} q^{\frac{1}{2}|\alpha|(|\alpha|+1)+\frac{1}{2}\sum_i \alpha_i(\alpha_i-1)}}{(a)_{|\alpha|}(q)_{\alpha_1}(q)_{\alpha_2}\cdots(q)_{\alpha_n}} = (a)_\infty^{-1}. \tag{6.18}$$

In the case when $n=1$, (6.18) reduces to Entry 3 (5.7).
Put $a=q, \lambda=q^2$, and $b\to bq$ in (6.6), we have

$$\sum_{\alpha\in\mathbb{N}^n} \frac{(q/b)_{|\alpha|}}{(q)_{|\alpha|}(q)_{\alpha_1}(q)_{\alpha_2}\cdots(q)_{\alpha_n}} (-b)^{|\alpha|} x^{\alpha} q^{\frac{1}{2}\sum_i \alpha_i(\alpha_i+1)}$$
$$= (q)_\infty^{-1} \prod_{1\le i\le n} (bqx_i)_\infty \sum_{\gamma\in\mathbb{N}^n} ((-1)^n a)^{|\gamma|} (x_1\cdots x_n)^{-|\gamma|} x_1^{n\gamma_1}\cdots x_n^{n\gamma_n} q^{\frac{n}{2}\sum_i \gamma_i(\gamma_i+1)}$$
$$\times \frac{\Delta(xq^{\gamma})}{\Delta(x)} \prod_{1\le i,j\le n} (qx_i/x_j)_{\gamma_i}^{-1} \prod_{1\le i\le n} \frac{(qx_i)_{\gamma_i}}{(bqx_i)_{\gamma_i}}. \tag{6.19}$$

Let b tend to 0 in (6.19). We obtain

$$\sum_{\alpha\in\mathbb{N}^n} \frac{x^{\alpha} q^{\frac{1}{2}|\alpha|(|\alpha|+1)+\frac{1}{2}\sum_i \alpha_i(\alpha_i+1)}}{(q)_{|\alpha|}(q)_{\alpha_1}(q)_{\alpha_2}\cdots(q)_{\alpha_n}}$$
$$= (q)_\infty^{-1} \sum_{\gamma\in\mathbb{N}^n} ((-1)^n a)^{|\gamma|} (x_1\cdots x_n)^{-|\gamma|} x_1^{n\gamma_1}\cdots x_n^{n\gamma_n} q^{\frac{n}{2}\sum_i \gamma_i(\gamma_i+1)}$$
$$\times \frac{\Delta(xq^{\gamma})}{\Delta(x)} \prod_{1\le i,j\le n} (qx_i/x_j)_{\gamma_i}^{-1} \prod_{1\le i\le n} (qx_i)_{\gamma_i}. \tag{6.20}$$

In the case when $n=1$, (6.20) reduces to (5.9).
Replace $q\to q^2$ in (6.6). Then, by setting $a=q, \lambda=q^3$, and $b\to bq^2$ we get

$$\sum_{\alpha\in\mathbb{N}^n} \frac{(q/b;q^2)_{|\alpha|}}{(q;q^2)_{|\alpha|}(q^2;q^2)_{\alpha_1}(q^2;q^2)_{\alpha_2}\cdots(q^2;q^2)_{\alpha_n}} (-b)^{|\alpha|} x^{\alpha} q^{\sum_i \alpha_i(\alpha_i+1)}$$
$$= (q;q^2)_\infty^{-1} \prod_{1\le i\le n} (bq^2x_i;q^2)_\infty \sum_{\gamma\in\mathbb{N}^n} (-1)^{n|\gamma|} (x_1\cdots x_n)^{-|\gamma|} x_1^{n\gamma_1}\cdots x_n^{n\gamma_n} q^{|\gamma|+n\sum_i \gamma_i(\gamma_i-1)}$$
$$\times \frac{\Delta(xq^{2\gamma})}{\Delta(x)} \prod_{1\le i,j\le n} (q^2x_i/x_j;q^2)_{\gamma_i}^{-1} \prod_{1\le i\le n} \frac{(q^2x_i;q^2)_{\gamma_i}}{(bq^2x_i;q^2)_{\gamma_i}}. \tag{6.21}$$

In the case when $n=1$, (6.21) reduces to (6.15).

Let b tend to 0 in (6.21). We have

$$\sum_{\alpha\in\mathbb{N}^n}\frac{x^{\alpha}q^{|\alpha|^2+\sum_i\alpha_i(\alpha_i+1)}}{(q;q^2)_{|\alpha|}(q^2;q^2)_{\alpha_1}(q^2;q^2)_{\alpha_2}\dots(q^2;q^2)_{\alpha_n}}$$
$$=(q;q^2)_{\infty}^{-1}\sum_{\gamma\in\mathbb{N}^n}(-1)^{n|\gamma|}(x_1\cdots x_n)^{-|\gamma|}x_1^{n\gamma_1}\cdots x_n^{n\gamma_n}q^{|\gamma|+n\sum_i\gamma_i(\gamma_i-1)}$$
$$\times\frac{\Delta(xq^{2\gamma})}{\Delta(x)}\prod_{1\le i,j\le n}(q^2x_i/x_j;q^2)_{\gamma_i}^{-1}\prod_{1\le i\le n}(q^2x_i;q^2)_{\gamma_i}. \tag{6.22}$$

In the case when $n=1$, (6.22) reduces to (5.11).

Acknowledgments I express my sincere thanks to the referee for his valuable comments and suggestions. I am grateful to the organizers of the conference for providing a warm and stimulating atmosphere.

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Nonterminating q-Whipple Transformations for Basic Hypergeometric Series in U(n)

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Abstract In this paper we derive multivariable generalizations of Bailey's classical nonterminating q-Whipple and q-Saalschütz transformations. We work in the setting of multiple basic hypergeometric series very-well-poised on unitary groups $\mathrm{U}(n+1)$, multiple series that are associated to the root system A_n. We extend Bailey's proofs of these transformations by first taking suitable limits of our $\mathrm{U}(n+1)$ ${}_{10}\phi_9$ transformation formula, in which the multiple sums are taken over an n-dimensional tetrahedron (n-simplex). A natural partition of the (finite) n-simplex combines with our analysis of the convergence of the multiple series to yield our transformations. We expect that all of these results will directly extend to the analogous case of multiple basic hypergeometric series associated to the root system D_n.

Keywords Multiple basic hypergeometric series associated to root systems • $\mathrm{U}(n+1)$ ${}_{10}\phi_9$ transformation formulas • $\mathrm{U}(n+1)$ nonterminating q-Whipple transformations • $\mathrm{U}(n+1)$ nonterminating q-Saalschütz transformations

Mathematics Subject Classification: Primary: 33D70, 05A19; Secondary: 05A30

[1]The first author was partially supported by NSF grants DMS 86-04232, DMS 89-04455, DMS 90-96254, NSA supplements to these NSF grants, by NSF grant DMS 0100288, and by NSA grants MDA 904-88-H-2010, MDA 904-91-H-0055, MDA 904-93-H-3032, MDA 904-97-1-0019, MDA 904-99-1-0003, H98230-06-1-0064, and H98230-08-1-0093. Both authors were partially supported by NSA grant MDA 904-91-H-0055.

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K. Alladi and F. Garvan (eds.), *Partitions, q-Series, and Modular Forms*, Developments in Mathematics 23, DOI 10.1007/978-1-4614-0028-8_12,

1 Introduction

For much of the last 30 years, a natural multivariable extension of hypergeometric series known as hypergeometric series associated to root systems has been intensively studied and developed by many authors. The classical, or $q = 1$ case, for the root system A_n, first appeared in the 1976 work of Biedenharn, Holman, and Louck [29], where they were employed to find useful formulas for the multiplicity-free $6j$-symbols of the group $\mathrm{SU}(n)$. The basic, or q-case, for the root system A_n was initiated by Milne in the 1985 papers [44–46], while the very-well-poised $\mathrm{U}(n+1)$, or A_n series, were first introduced in [48, Definition 1.10]. Gustafson followed up with very-well-poised series for the other root systems in [26]. These, and more general, multiple series are classified according to the type of specific factors (such as a Vandermonde determinant) appearing in the summand. For a precise definition of the A_n, C_n, and D_n multiple q-series, see the papers of Bhatnagar [3] or Milne [58, Sect. 5].

A second deep impact from mathematical physics appeared in the 1997 paper [22] of Frenkel and Turaev. Motivated by their study of elliptic $6j$-symbols in statistical mechanics, Frenkel and Turaev introduced 1-variable elliptic or modular hypergeometric series. Elliptic hypergeometric series sit at the top level of the hierarchy "rational – trigonometric – elliptic" of their term ratios. That is, a series $\sum_k a_k$ is called hypergeometric if the term ratio $f(k) = a_{k+1}/a_k$ is a rational function of k, a q-series if f is a rational function of q^k for a fixed q, and an elliptic series if f is an elliptic function. For technical reasons involving addition formulas for suitably defined theta functions, the development of the theory of elliptic hypergeometric series was forced to begin by extending the deepest classical hypergeometric results such as Jackson's ${}_8W_7$ summation and Bailey's ${}_{10}W_9$ transformation (or their multivariable analogues). One could not start with or recover the simplest classical results such as the binomial theorem and Gauss's ${}_2F_1$ summation theorem.

In recent years, the above two fields have been unified to yield the powerful, deep subject of *elliptic hypergeometric series on root systems*. Warnaar, in [82], was one of the first to enter this new unified subject by proving one multivariable elliptic C_n analogue of Jackson's ${}_8W_7$ summation formula and conjecturing another. Van Diejen and Spiridonov [17] showed that Warnaar's conjectured summation could be derived from a certain conjectured elliptic Selberg integral. Rosengren, in [69], then gave an inductive proof of Warnaar's second summation, using a very special case of his first summation in the inductive step. This unified subject has since expanded rapidly.

In the sequel (in alphabetical order), since 1976, Bhatnagar [2, 3], Bhatnagar and Milne [4], Bhatnagar and Schlosser [5], van de Bult and Rains [8], Coskun [9, 11–13], Coskun and Gustafson [14], Degenhardt and Milne [15], van Diejen [16], van Diejen and Spiridonov [17–19], Frenkel and Turaev [22], Gessel and Krattenthaler [24], Gustafson [25–27], Gustafson and Rakha [28], Ito [31, 32], Kajihara [33–35], Kajihara and Noumi [36, 37], Krattenthaler [38], Krattenthaler and Schlosser [39], Lascoux et al. [40], Lassalle and Schlosser [41], Lilly and Milne [42], Milne [44–59], Milne and Lilly [60], Milne and Newcomb [61], Milne and Schlosser [62], Rains [63–65], Rakha [66, 67], Rakha and Siddiqi [68], Rosengren

[69–71], Rosengren and Schlosser [72, 73], Schlosser [74–78], Spiridonov [80, 81], Warnaar [82], Warnaar and Zudilin [83], and many others have developed this theory and uncovered many applications to number theory, combinatorics, and mathematical physics, to name a few. For more complete references to this rapidly growing, vast subject, see Chap. 11 of [23], the survey paper [58], the papers [56, 70, 71, 80, 81], and the references therein. Perhaps the deepest application so far is the new infinite families of sums of squares formulas in [55, 57, 59].

The purpose of this paper is to derive multivariable generalizations of Bailey's classical nonterminating q-Whipple and q-Saalschütz transformations, in [23, (III.36) on pp. 364] and [23, (II.24) on pp. 356], respectively. In order to achieve the deepest possible results, we extend Bailey's proofs from [23, pp. 50–51] of these transformations by first taking suitable limits of our $\mathrm{U}(n+1)$ ${}_{10}\phi_9$ transformation formula in [61, Theorem 3.3], in which the multiple sums are taken over an n-dimensional tetrahedron (n-simplex). A natural partition of the (finite) n-simplex combines with our analysis of the convergence of the multiple series to yield our transformations. The $n+1$ nonterminating multiple sums on the right-hand sides of our two $\mathrm{U}(n+1)$ nonterminating q-Whipple transformations correspond to the $n+1$ vertices of the n-dimensional tetrahedron (n-simplex) that contained the original (before taking limits) indices of summation $y_1,\ldots,y_n \geq 0$ and $0 \leq y_1+\cdots+y_n \leq N$, where N is a nonnegative integer. As a result, our two $\mathrm{U}(n+1)$ nonterminating q-Saalschütz transformations expand an infinite product into the sum of $n+1$ nonterminating multiple sums. In [15], our $\mathrm{U}(n+1)$ nonterminating q-Whipple and q-Saalschütz transformations are written (and applied) in a more compact form using multiple q-integrals.

Our transformations here are not direct consequences of the fundamental theorem for $\mathrm{U}(n+1)$ series in [45, Theorem 1.49]. That is, they lie deeper than the Macdonald identities for the root system A_n. As discussed in Sect. 3 of [58] and Sect. 11.7 of the book [23], Theorem 1.49 of [45] is the foundation on which the theory and application of $\mathrm{U}(n+1)$ or A_n q-series (and the more general series for other root systems) is built. This "fundamental theorem" is stated in modern classical notation in [56, Lemma 7.3, pp. 163], [58, Theorem 3.1, pp. 207], [62, Corollary 4.4, pp. 768–769], and [23, (11.7.1), pp. 331].

Coskun and Kajihara have studied nonterminating q-Whipple transformations for multiple hypergeometric series that are quite different than those in this paper. In the bottom of page 9 and top of page 10 of [10], Coskun claims (without any details) to have a "multiple nonterminating [q-] Watson transformation" with the "multiple series running over partitions." Such BC_n series, evaluated at arguments that are generalizations of Macdonald (symmetric) polynomials, are different from the multiple q-series associated to root systems that we study in this paper. The 3-term $\mathrm{U}(n+1)$ nonterminating q-Whipple transformations in Sect. 6 of Kajihara's preprint [35] are easy, direct consequences of Bailey's classical 3-term one-variable case, are not as deep as the $n+2$-term transformations in this paper, and do not extend Bailey's original proof to the level of $\mathrm{U}(n+1)$ multiple q-series. In [77, Corollary 5.1], Schlosser used multiple q-integrals and a determinant evaluation to establish a nonterminating ${}_8\phi_7$ summation for the root system C_n, in which the "sum side" has 2^n multiple sums. Specializing this result, he also obtained, in Corollary 6.2, an A_n nonterminating ${}_3\phi_2$ summation, in which the sum side also has 2^n multiple sums. Corollaries 5.1 and 6.2 of [77] are not as deep as our $\mathrm{U}(n+1)$ nonterminating q-Dougall summation theorem in [15], or $\mathrm{U}(n+1)$ nonterminating q-Saalschütz transformations in this paper, in which the sum sides have $n+1$ multiple sums. Nonetheless, Corollaries 5.1 and 6.2 of [77] are of significant interest as they are strikingly similar to the identities supporting the combinatorial applications in the papers of Krattenthaler [38] and Gessel and Krattenthaler [24]. Building on the methods in [77], Rosengren and Schlosser [72, Corollaries 4.1 and 4.2] establish C_n nonterminating ${}_{10}\phi_9$ transformations, in which each side has 2^n multiple sums. These are the first multivariable generalizations of Bailey's nonterminating ${}_{10}\phi_9$ transformation (see (III.39) on page 365 of [23]) that have appeared in the literature.

Our main motivation for deriving our $\mathrm{U}(n+1)$ nonterminating q-Whipple and q-Saalschütz transformations was to obtain, in [15], our $\mathrm{U}(n+1)$ nonterminating q-Dougall summation theorem, the related (classical and q-) multiple beta integrals, and a $\mathrm{U}(n+1)$ extension of Jacobi's classical identity for the 8th powers of theta functions in [21, (21), pp. xxviii] and [84, pp. 470]. Another strong motivation is to extend the results in this paper, and [15], to the C_n and D_n cases. Keeping in mind that the sums of squares applications involved C_n nonterminating multiple basic hypergeometric series, such extensions could lead to additional deep number theoretic applications. We expect that all of the results in this paper will directly extend to the analogous case of multiple basic hypergeometric series associated to the root system D_n.

In a subsequent paper, we will pursue this D_n case. A very strong foundation for this work is provided by the following situation: In the paper [5], at the end of Sect. 5, Bhatnagar and Schlosser propose using their results to derive additional C_n and D_n ${}_{10}\phi_9$ transformations, some of which had multiple sums over an n-simplex. Now, the inductive constructions and related algebraic manipulations in this paper and [15] are consistent with the $p=0$ case of the more general elliptic hypergeometric series associated to the root systems A_n and D_n. In [70], Rosengren carried out the calculations proposed by Bhatnagar and Schlosser at the elliptic level, and obtained in Sect. 8, on pages 439–444, ${}_{10}\phi_9$ transformations of A_n and D_n elliptic hypergeometric series, with the sums over an n-simplex. (See Corollary 8.1 on page 439 for his first A_n case, and Corollary 8.2 on page 441 for a second. Note that the n-simplex version of Corollary 8.4 on page 443 is a D_n case, and the n-simplex version of Corollary 8.5 on page 444 is another.) More evidence of the parallel developments is at the bottom of page 441 of [70]. Taking the $p=0$ (or trigonometric case) of Rosengren's A_n and D_n ${}_{10}\phi_9$ transformations brings us to the starting point of this paper in Theorem A.1, and also gives us the ${}_{10}\phi_9$ transformations for multiple basic hypergeometric series associated to the root system D_n that we need to get started.

We also expect to be able to recover known, or derive new, A_n and D_n Ramanujan ${}_1\Psi_1$ Summation Theorems, by using our corresponding multivariable nonterminating q-Saalschütz transformations and other current Heine transformations. This program would extend the analysis on page 52 of [23]. In addition, starting with our multiple q-integral representation, in [15], of our $\mathrm{U}(n+1)$ nonterminating q-Dougall summation theorem, and a D_n analog, we should be able to derive deep new A_n and

D_n nonterminating ${}_{10}\phi_9$ transformations, in which each side has $n+1$ multiple sums. Such analysis would extend Sect. 2.12 on pages 55 to 58 of [23].

Our present paper is organized as follows. In Sect. 2 we give some background information and notation for classical basic hypergeometric series of one variable. Section 3 provides a self-contained account of Bailey's proof of the classical nonterminating q-Whipple transformation, and his classical nonterminating q-Saalschütz transformation. We derive our $\mathrm{U}(n+1)$ nonterminating q-Whipple transformations in Sect. 4 by letting $N \to \infty$, in an appropriate manner, on both sides of the $\mathrm{U}(n+1)$ ${}_{10}\phi_9$ transformation formula given by Theorem A.1. Our $\mathrm{U}(n+1)$ nonterminating q-Saalschütz transformations are obtained in Sect. 5. Appendix A contains the $\mathrm{U}(n+1)$ basic hypergeometric series summation theorems and transformations that we need in the derivations of our $\mathrm{U}(n+1)$ nonterminating q-Whipple and q-Saalschütz transformations. In Appendix B we discuss and prove the absolute convergence of the nonterminating multiple series in this paper.

2 Background and Notation

In this section we give some background information and notation for classical basic hypergeometric series of one variable. We include some simple identities that are commonly used to transform or simplify q-factorials.

Throughout this paper we will let q be a complex number such that $|q| < 1$.

Define

$$(\alpha)_\infty = (\alpha;q)_\infty := \prod_{k\geq 0}(1-\alpha q^k) \tag{2.1}$$

and thus

$$(\alpha)_n = (\alpha;q)_n := \frac{(\alpha)_\infty}{(\alpha q^n)_\infty} = \prod_{i=1}^{n}(1-\alpha q^{i-1}). \tag{2.2}$$

Definition 2.1 ($_r\phi_s$ basic hypergeometric series). Let $a_1,\ldots,a_r$, $b_1,\ldots,b_s$, and z be complex numbers. Define

$${}_r\phi_s\left[\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,z\right] := \sum_{k=0}^{\infty}\frac{(a_1,\cdots,a_r)_k}{(b_1,\cdots,b_s,q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k, \tag{2.3}$$

where $(a_1,\ldots,a_n)_m := (a_1)_m\cdots(a_n)_m$, $\binom{k}{2} = \frac{k(k-1)}{2}$, and $q \neq 0$ when $r > s+1$.

This definition is given in [23]. The parameters $b_1,\ldots,b_s$ are such that the denominator factors in the terms of the series (2.3) are never 0. Since $(q^{-m};q)_k = 0$, if $k = m+1, m+2, \ldots$, an ${}_r\phi_s$ series terminates if one of its numerator parameters is of the form q^{-m} with $m = 0,1,2,\ldots$, and $q \neq 0$.

Unless stated otherwise, when dealing with nonterminating (multiple) basic hypergeometric series we shall assume that $|q| < 1$ and that the parameters and variables are such that the series converges absolutely. Note that if $r = s+1$, the $[(-1)^k q^{\binom{k}{2}}]^{1+s-r}$ factor in (2.3) drops out. This case gives the earlier definition for ${}_r\phi_s$ from [79], and others. The additional factor is present in [23] so that inverting the base q and simplifying yields a similar series when $|q| > 1$. Also observe that after replacing each parameter a by q^a in (2.3) and then letting $q \to 1^-$, the ${}_r\phi_s$ series becomes the classical ${}_rF_s$ hypergeometric series. Such $q \to 1^-$ limit computations can also be carried out for the $\mathrm{U}(n+1)$ nonterminating q-Whipple and q-Saalschütz transformations in this paper.

The series ${}_{r+1}\phi_r$ is said to be *k-balanced* if $b_1 b_2 \cdots b_r = q^k a_1 a_2 \cdots a_{r+1}$ and $z = q$. A 1-balanced series is called *balanced* (or *Saalschützian*). The ${}_{r+1}\phi_r$ series is said to be *well-poised* if $qa_1 = a_2 b_1 = a_3 b_2 = \cdots = a_{r+1} b_r$, and *very-well-poised* if it is well-poised and $a_2 = qa_1^{1/2}$, $a_3 = -qa_1^{1/2}$.

The following identities come up frequently in the analysis of multiple basic hypergeometric series when powers of q^{-N} are factored out of q-factorials, by reversing the order of their factors:

$$(a;q)_n = (q^{1-n}/a;q)_n (-a)^n q^{\binom{n}{2}}, \tag{2.4}$$

$$(aq^k;q)_{n-k} = \frac{(a;q)_n}{(a;q)_k}, \tag{2.5}$$

$$(aq^{-n};q)_n = (q/a;q)_n \left(-\frac{a}{q}\right)^n q^{-\binom{n}{2}}, \tag{2.6}$$

and

$$\binom{y_1+\cdots+y_n}{2} = e_2(y_1,\ldots,y_n) + \left[\binom{y_1}{2} + \cdots + \binom{y_n}{2}\right], \tag{2.7}$$

where $e_2(y_1,\ldots,y_n)$ is the second elementary symmetric function of $\{y_1,\ldots,y_n\}$.

Remark. Equations (2.4), (2.5), and (2.6) are given by (I7), (I20), and (I8) on pages 351, 352, and 351, respectively, of [23]. The elementary identity in (2.7) is useful in simplifying powers of q. We have previously applied (2.7) in [53, pp. 42] and [61, pp. 280] and (2.4), (2.5), and (2.6) in much of our earlier work in this area.

3 Classical Nonterminating *q*-Whipple Transformation

In this section we give a self-contained account of Bailey's proof of the classical nonterminating q-Whipple transformation, as well as of his classical nonterminating q-Saalschütz transformation. We follow the outline on pages 50–51 of [23]. One of the main features and motivations of this paper is our extension of these classical proofs of these two results to the $\mathrm{U}(n+1)$ case.

The proofs begin with Bailey's ${}_{10}\phi_9$ transformation formula in (III.28) of page 363 of [23].

Theorem 3.1 (Bailey's ${}_{10}\phi_9$ transformation formula). *Let a, b, c, d, e, f, and λ be indeterminate, n be a nonnegative integer, and suppose that none of the denominators in* (3.1) *vanish. Then*

$$\text{(3.1a)}\quad {}_{10}\phi_9\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f, \frac{\lambda a q^{n+1}}{ef}, q^{-n} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f}, \frac{efq^{-n}}{\lambda}, aq^{n+1} \end{matrix}; q, q\right]$$

$$\text{(3.1b)}\quad = \frac{\left(aq, \frac{aq}{ef}, \frac{\lambda q}{e}, \frac{\lambda q}{f}; q\right)_n}{\left(\frac{aq}{e}, \frac{aq}{f}, \frac{\lambda q}{ef}, \lambda q; q\right)_n}$$

$$\text{(3.1c)}\quad \times {}_{10}\phi_9\left[\begin{matrix} \lambda, q\sqrt{\lambda}, -q\sqrt{\lambda}, \frac{\lambda b}{a}, \frac{\lambda c}{a}, \frac{\lambda d}{a}, e, f, \frac{\lambda a q^{n+1}}{ef}, q^{-n} \\ \sqrt{\lambda}, -\sqrt{\lambda}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{\lambda q}{e}, \frac{\lambda q}{f}, \frac{efq^{-n}}{a}, \lambda q^{n+1} \end{matrix}; q, q\right],$$

where $\lambda = qa^2/bcd$.

Equation (3.1) is (III.28) on page 363 of [23].

Letting $d \to a^3q^{2+n}/bcdef$, followed by $n \to \infty$ in (3.1) yields Theorem 3.2.

Theorem 3.2 (Classical nonterminating q-Whipple transformation). *Let a, b, c, d, e, and f be indeterminate. Suppose that none of the denominators in* (3.2) *vanish, and that $0 < |q| < 1$ and $|a^2q^2/bcdef| < 1$. Then*

$$\text{(3.2a)}\quad {}_8\phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f} \end{matrix}; q, \frac{a^2q^2}{bcdef}\right]$$

$$\text{(3.2b)}\quad = \frac{\left(aq, \frac{aq}{de}, \frac{aq}{df}, \frac{aq}{ef}; q\right)_\infty}{\left(\frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f}, \frac{aq}{def}; q\right)_\infty} {}_4\phi_3\left[\begin{matrix} \frac{aq}{bc}, d, e, f \\ \frac{aq}{b}, \frac{aq}{c}, \frac{def}{a} \end{matrix}; q, q\right]$$

$$\text{(3.2c)}\quad + \frac{\left(aq, \frac{aq}{bc}, d, e, f, \frac{a^2q^2}{bdef}, \frac{a^2q^2}{cdef}; q\right)_\infty}{\left(\frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f}, \frac{a^2a^2}{bcdef}, \frac{def}{aq}; q\right)_\infty} {}_4\phi_3\left[\begin{matrix} \frac{aq}{de}, \frac{aq}{df}, \frac{aq}{ef}, \frac{a^2q^2}{bcdef} \\ \frac{a^2q^2}{bdef}, \frac{a^2q^2}{cdef}, \frac{aq^2}{def} \end{matrix}; q, q\right].$$

Equation (3.2) is (III.36) on page 364 of [23].

Proof. We give a brief summary of Bailey's proof, outlined on page 50 of [23]. In (3.1), let $d \mapsto \frac{a^3q^{2+n}}{bcdef}$. Then (3.1a) does not change, and as $n \to \infty$, (3.1a) converges to (3.2a). The products in (3.1b) converge to

$$\text{(3.3)}\quad \frac{\left(aq, \frac{aq}{de}, \frac{aq}{df}, \frac{aq}{ef}; q\right)_\infty}{\left(\frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f}, \frac{aq}{def}; q\right)_\infty}.$$

However, for large n, (3.1c) becomes relatively large at both ends and small in the middle of the series, preventing us from taking the term-by-term limit directly. To overcome this difficulty, Bailey used the following scheme for an arbitrary sequence $\{\lambda_k\}$: Set $n = 2m+1$, and break up (3.1c) according to (3.4a) in

$$\sum_{k=0}^{2m+1} \lambda_k = \sum_{k=0}^{m} \lambda_k + \sum_{k=m+1}^{2m+1} \lambda_k \tag{3.4a}$$

$$= \sum_{k=0}^{m} \lambda_k + \sum_{k=0}^{m} \lambda_{2m+1-k}, \tag{3.4b}$$

and then reverse the order of the second series as in (3.4b).

Use the summand in (3.1c) for λ_k in (3.4). Then let $m \to \infty$ (and hence $n \to \infty$) and multiply by (3.3) to obtain (3.2b) and (3.2c). $\square$

If, in (3.2), $aq/cd = 1$, the ${}_4\phi_3$s reduce to ${}_3\phi_2$s, and the ${}_8\phi_7$ series reduces to a ${}_6\phi_5$. The resulting ${}_6\phi_5$ is summable by means of the following.

Theorem 3.3 (Classical nonterminating ${}_6\phi_5$ summation theorem). *Let a, b, c, and d be indeterminate. Suppose that none of the denominators in* (3.5) *vanish, and that* $0 < |q| < 1$ *and* $|aq/bcd| < 1$. *Then*

$${}_6\phi_5 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d} \end{matrix} ; q, \frac{aq}{bcd} \right] = \frac{\left(aq, \frac{aq}{bc}, \frac{aq}{bd}, \frac{aq}{cd}; q\right)_\infty}{\left(\frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{bcd}; q\right)_\infty}. \tag{3.5}$$

Equation (3.5) is (II.20) on page 356 of [23].

Solving for the first ${}_3\phi_2$ on the resulting right-hand side and relabelling we obtain Theorem 3.4.

Theorem 3.4 (Classical nonterminating q-Saalschütz transformation). *Let a, b, c, e, and f be indeterminate. Suppose that none of the denominators in* (3.6) *vanish, and that* $0 < |q| < 1$. *Then*

$$\begin{aligned} {}_3\phi_2 \left[\begin{matrix} a, b, c \\ e, f \end{matrix} ; q, q \right] &= \frac{\left(\frac{q}{e}, \frac{f}{a}, \frac{f}{b}, \frac{f}{c}; q\right)_\infty}{\left(\frac{aq}{e}, \frac{bq}{e}, \frac{cq}{e}, f; q\right)_\infty} \\ &\quad - \frac{\left(\frac{q}{e}, a, b, c, \frac{qf}{e}; q\right)_\infty}{\left(\frac{e}{q}, \frac{aq}{e}, \frac{bq}{e}, \frac{cq}{e}, f; q\right)_\infty} {}_3\phi_2 \left[\begin{matrix} \frac{aq}{e}, \frac{bq}{e}, \frac{cq}{e} \\ \frac{q^2}{e}, \frac{qf}{e} \end{matrix} ; q, q \right], \end{aligned} \tag{3.6}$$

where $ef = abcq$.

Equation (3.6) is (II.24) on page 356 of [23].

4 U$(n+1)$ Nonterminating q-Whipple Transformations

In this section we derive our U$(n+1)$ nonterminating q-Whipple transformations by letting $N \to \infty$, in an appropriate manner, on both sides of the U$(n+1)$ ${}_{10}\phi_9$ transformation formula given by Theorem A.1. (This multivariable extension of Bailey's classical ${}_{10}\phi_9$ transformation was first given by Theorem 3.3 of [61].) The key to our extension of Bailey's classical proof from Sect. 3 is the partition of the n-dimensional tetrahedron (n-simplex) determined by (4.13), the analysis in Appendix B involving the multiple power series ratio test of Lemma B.1, and the Dominated Convergence Theorem.

We give the formal aspects of our proofs in this section, while the absolute convergence of the nonterminating multiple series is established in Appendix B.

Theorem 4.1 (First U$(\mathbf{n+1})$ nonterminating q-Whipple transformation). *Let a, b, c, d, $e_1, \ldots, e_n, f$, and $x_1, \ldots, x_n$ be indeterminate, with $n \geq 1$. Take $0 < |q| < 1$ and*

$$\left| \frac{a^2q^2}{bcde_1 \cdots e_n f} \right| < 1. \tag{4.1}$$

Suppose that none of the denominators in (4.2) vanish. Then

$$\sum_{y_1,\ldots,y_n \geq 0} \Bigg\{ \prod_{i=1}^{n} \left[\frac{1 - \frac{x_i}{x_n} a q^{y_i + (y_1+\cdots+y_n)}}{1 - \frac{x_i}{x_n} a} \right]$$

$$\times \frac{(b)_{y_1+\cdots+y_n} (c)_{y_1+\cdots+y_n}}{\left(\frac{aq}{d}\right)_{y_1+\cdots+y_n} \left(\frac{aq}{f}\right)_{y_1+\cdots+y_n}} \prod_{i=1}^{n} \left[\frac{\left(\frac{x_i}{x_n} d\right)_{y_i} \left(\frac{x_i}{x_n} f\right)_{y_i}}{\left(\frac{x_i}{x_n} \frac{aq}{b}\right)_{y_i} \left(\frac{x_i}{x_n} \frac{aq}{c}\right)_{y_i}} \right]$$

$$\times \prod_{i=1}^{n} \left[\frac{\left(\frac{x_i}{x_n} a\right)_{y_1+\cdots+y_n}}{\left(\frac{x_i}{x_n} \frac{aq}{e_i}\right)_{y_1+\cdots+y_n}} \right] \prod_{r,s=1}^{n} \left[\frac{\left(\frac{x_r}{x_s} e_s\right)_{y_r}}{\left(q \frac{x_r}{x_s}\right)_{y_r}} \right]$$

$$\times \prod_{1 \leq r < s \leq n} \left[\frac{1 - \frac{x_r}{x_s} q^{y_r - y_s}}{1 - \frac{x_r}{x_s}} \right]$$

$$\times \left(\frac{a^2 q}{bcde_1 \cdots e_n f} \right)^{y_1+\cdots+y_n} q^{y_1+2y_2+\cdots+ny_n} \Bigg\} \tag{4.2a}$$

$$= \frac{\left(\frac{aq}{de_1\cdots e_n}\right)_\infty \left(\frac{aq}{e_1\cdots e_n f}\right)_\infty}{\left(\frac{aq}{d}\right)_\infty \left(\frac{aq}{f}\right)_\infty} \prod_{i=1}^{n} \left[\frac{\left(\frac{x_i}{x_n} aq\right)_\infty \left(\frac{x_n}{x_i} \frac{ae_i q}{de_1\cdots e_n f}\right)_\infty}{\left(\frac{x_n}{x_i} \frac{aq}{de_1\cdots e_n f}\right)_\infty \left(\frac{x_i}{x_n} \frac{aq}{e_i}\right)_\infty} \right] \tag{4.2b}$$

$$\sum_{y_1,\ldots,y_n\geq 0}\left\{\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}\frac{aq}{bc}\right)_{y_i}\left(\frac{x_i}{x_n}d\right)_{y_i}\left(\frac{x_i}{x_n}f\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{aq}{c}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{de_1\cdots e_nf}{a}\right)_{y_i}}\right]\right.$$

$$\left.\times\prod_{r,s=1}^{n}\left[\frac{\left(\frac{x_r}{x_s}e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\right]\prod_{1\leq r<s\leq n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]q^{y_1+2y_2+\cdots ny_n}\right\}\tag{4.2c}$$

$$+\frac{\left(\frac{a^2q^2}{bde_1\cdots e_nf}\right)_\infty\left(\frac{a^2q^2}{cde_1\cdots e_nf}\right)_\infty(q)_\infty}{\left(\frac{a^2q^2}{bcde_1\cdots e_nf}\right)_\infty\left(\frac{aq}{d}\right)_\infty\left(\frac{aq}{f}\right)_\infty}\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}aq\right)_\infty}{\left(\frac{x_i}{x_n}\frac{aq}{e_i}\right)_\infty\left(1-\frac{x_n}{x_i}\frac{aq}{de_1\cdots e_nf}\right)}\right]\tag{4.2d}$$

$$\sum_{j=1}^{n}\left\{\frac{\left(\frac{x_j}{x_n}\frac{aq}{bc}\right)_\infty\left(\frac{x_j}{x_n}d\right)_\infty\left(\frac{x_j}{x_n}f\right)_\infty}{\left(\frac{x_j}{x_n}\frac{aq}{b}\right)_\infty\left(\frac{x_j}{x_n}\frac{aq}{c}\right)_\infty\left(\frac{x_j}{x_n}\frac{de_1\cdots e_nf}{a}\right)_\infty}\prod_{s=1}^{n}\left[\frac{\left(\frac{x_j}{x_s}e_s\right)_\infty}{\left(q\frac{x_j}{x_s}\right)_\infty}\right]\right.$$

$$\times\sum_{y_1,\ldots,y_n\geq 0}\prod_{\substack{1\leq i\leq n\\ i\neq j}}\left[1-\frac{x_n}{x_i}\frac{aq}{de_1\cdots e_nf}q^{y_j-y_i}\right]$$

$$\times\prod_{\substack{1\leq i\leq n\\ i\neq j}}\left[\frac{\left(\frac{x_i}{x_n}\frac{aq}{bc}\right)_{y_i}\left(\frac{x_i}{x_n}d\right)_{y_i}\left(\frac{x_i}{x_n}f\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{aq}{c}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{de_1\cdots e_nf}{a}\right)_{y_i}}\right]$$

$$\times\left[\frac{\left(\frac{a^2q^2}{bcde_1\cdots e_nf}\right)_{y_j}\left(\frac{aq}{de_1\cdots e_n}\right)_{y_j}\left(\frac{aq}{e_1\cdots e_nf}\right)_{y_j}}{\left(\frac{a^2q^2}{bde_1\cdots e_nf}\right)_{y_j}\left(\frac{a^2q^2}{cde_1\cdots e_nf}\right)_{y_j}(q)_{y_j}}\right]$$

$$\times\prod_{i=1}^{n}\left[\frac{\left(\frac{x_n}{x_i}\frac{ae_iq}{de_1\cdots e_nf}\right)_{y_j}}{\left(\frac{x_n}{x_i}\frac{aq^2}{de_1\cdots e_nf}\right)_{y_j}}\right]\prod_{\substack{1\leq r,s\leq n\\ r\neq j}}\left[\frac{\left(\frac{x_r}{x_s}e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\right]$$

$$\times\prod_{r=1}^{j-1}\left[\frac{1}{1-\frac{x_r}{x_j}}\right]\prod_{s=j+1}^{n}\left[\frac{1}{1-\frac{x_j}{x_s}}\right]$$

$$\times(-1)^{\,j}\left(\frac{aq}{de_1\cdots e_nf}\right)\left(\frac{x_n}{x_j}\right)\prod_{\substack{1\leq r<s\leq n\\ r,s\neq j}}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]$$

$$\left.\times q^{y_1+\cdots+y_j}q^{y_1+2y_2+\cdots+(j-1)y_{j-1}+(j+1)y_{j+1}+\cdots+ny_n}\prod_{r=1}^{j-1}\left(\frac{x_r}{x_j}\right)\right\}.\tag{4.2e}$$

Remark. The $n=1$ and $e_1 \mapsto e$ case of (4.2) is (III.36) on page 364 of [23].

Proof. Begin with Theorem A.1, and replace λ with qa^2/bcd. The sum (A.1a) becomes

$$\sum_{\substack{y_1,\dots,y_n\geq 0\\ 0\leq y_1+\cdots+y_n\leq N}} \Bigg\{ \prod_{i=1}^{n}\left[\frac{1-\frac{x_i}{x_n}aq^{y_i+(y_1+\cdots+y_n)}}{1-\frac{x_i}{x_n}a}\right] \tag{4.3a}$$

$$\times \prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}d\right)_{y_i}\left(\frac{x_i}{x_n}f\right)_{y_i}\left(\frac{x_i}{x_n}\frac{a^3q^2}{bcde_1\cdots e_nf}q^N\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{aq}{c}\right)_{y_i}\left(\frac{x_i}{x_n}aq^{1+N}\right)_{y_i}}\right] \tag{4.3b}$$

$$\times\left[\frac{(b)_{y_1+\cdots+y_n}(c)_{y_1+\cdots+y_n}\left(q^{-N}\right)_{y_1+\cdots+y_n}}{\left(\frac{aq}{d}\right)_{y_1+\cdots+y_n}\left(\frac{aq}{f}\right)_{y_1+\cdots+y_n}\left(\frac{bcde_1\cdots e_nf}{qa^2}q^{-N}\right)_{y_1+\cdots+y_n}}\right] \tag{4.3c}$$

$$\times\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}a\right)_{y_1+\cdots+y_n}}{\left(\frac{x_i}{x_n}\frac{aq}{e_i}\right)_{y_1+\cdots+y_n}}\right]\prod_{r,s=1}^{n}\left[\frac{\left(\frac{x_r}{x_s}e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\right] \tag{4.3d}$$

$$\times\prod_{1\leq r<s\leq n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]q^{y_1+2y_2+\cdots+ny_n}\Bigg\} \tag{4.3e}$$

By interchanging pairs of factors in (4.3b) and (4.3c), observe that the substitution

$$d\mapsto \frac{a^3q^{2+N}}{bcde_1\cdots e_nf} \tag{4.4}$$

leaves the sum (4.3) (termwise) unchanged. In Appendix B we show that (4.3) becomes (4.2a) as $N\to\infty$.

Since, originally, $\lambda=\frac{qa^2}{bcd}$, we have from (4.4) that

$$\lambda\mapsto\frac{de_1\cdots e_nf}{a}q^{-1-N}. \tag{4.5}$$

Use (4.5) and then apply (2.6) to rewrite the original products in (A.1b). We obtain

$$\frac{\left(\frac{de_1\cdots e_n}{a}q^{-N}\right)_N\left(\frac{aq}{e_1\cdots e_nf}\right)_N}{\left(\frac{d}{a}q^{-N}\right)_N\left(\frac{aq}{f}\right)_N}\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}aq\right)_N\left(\frac{x_i}{x_n}\frac{de_1\cdots e_nf}{ae_i}q^{-N}\right)_N}{\left(\frac{x_i}{x_n}\frac{de_1\cdots e_nf}{a}q^{-N}\right)_N\left(\frac{x_i}{x_n}\frac{aq}{e_i}\right)_N}\right] \tag{4.6a}$$

$$(4.6b)\qquad = \frac{\left(\frac{aq}{de_1\cdots e_n}\right)_N \left(\frac{aq}{e_1\cdots e_n f}\right)_N}{\left(\frac{aq}{d}\right)_N \left(\frac{aq}{f}\right)_N} \prod_{i=1}^{n} \left[\frac{\left(\frac{x_i}{x_n} aq\right)_N \left(\frac{x_n}{x_i} \frac{ae_i q}{de_1\cdots e_n f}\right)_N}{\left(\frac{x_n}{x_i} \frac{aq}{de_1\cdots e_n f}\right)_N \left(\frac{x_i}{x_n} \frac{aq}{e_i}\right)_N} \right].$$

As $N \to \infty$, the (rewritten) products in (4.6b) clearly become the products in (4.2b) given by

$$(4.7)\qquad \frac{\left(\frac{aq}{de_1\cdots e_n}\right)_\infty \left(\frac{aq}{e_1\cdots e_n f}\right)_\infty}{\left(\frac{aq}{d}\right)_\infty \left(\frac{aq}{f}\right)_\infty} \prod_{i=1}^{n} \left[\frac{\left(\frac{x_i}{x_n} aq\right)_\infty \left(\frac{x_n}{x_i} \frac{ae_i q}{de_1\cdots e_n f}\right)_\infty}{\left(\frac{x_n}{x_i} \frac{aq}{de_1\cdots e_n f}\right)_\infty \left(\frac{x_i}{x_n} \frac{aq}{e_i}\right)_\infty} \right].$$

For convenience, in (A.1c), first apply the substitution (4.4), and then (4.5). With this, (A.1c) becomes

$$(4.8a)\qquad \sum_{\substack{y_1,\ldots,y_n \ge 0 \\ 0 \le y_1+\cdots+y_n \le N}} \left\{ \prod_{i=1}^{n} \left[\frac{1 - \frac{x_i}{x_n} \frac{de_1\cdots e_n f}{a} q^{-1-N} q^{y_i + (y_1+\cdots+y_n)}}{1 - \frac{x_i}{x_n} \frac{de_1\cdots e_n f}{a} q^{-1-N}} \right] \right.$$

$$\times \prod_{i=1}^{n} \left[\frac{\left(\frac{x_i}{x_n} \frac{aq}{bc}\right)_{y_i} \left(\frac{x_i}{x_n} d\right)_{y_i} \left(\frac{x_i}{x_n} f\right)_{y_i}}{\left(\frac{x_i}{x_n} \frac{aq}{b}\right)_{y_i} \left(\frac{x_i}{x_n} \frac{aq}{c}\right)_{y_i} \left(\frac{x_i}{x_n} \frac{de_1\cdots e_n f}{a}\right)_{y_i}} \right]$$

$$(4.8b)\qquad \times \left[\frac{\left(\frac{bde_1\cdots e_n f}{a^2} q^{-1-N}\right)_{y_1+\cdots+y_n} \left(\frac{cde_1\cdots e_n f}{a^2} q^{-1-N}\right)_{y_1+\cdots+y_n}}{\left(\frac{bcde_1\cdots e_n f}{a^2} q^{-1-N}\right)_{y_1+\cdots+y_n} \left(\frac{de_1\cdots e_n}{a} q^{-N}\right)_{y_1+\cdots+y_n}} \right]$$

$$(4.8c)\qquad \times \frac{\left(q^{-N}\right)_{y_1+\cdots+y_n}}{\left(\frac{e_1\cdots e_n f}{a} q^{-N}\right)_{y_1+\cdots+y_n}} \prod_{r,s=1}^{n} \left[\frac{\left(\frac{x_r}{x_s} e_s\right)_{y_r}}{\left(q \frac{x_r}{x_s}\right)_{y_r}} \right]$$

$$(4.8d)\qquad \times \prod_{i=1}^{n} \left[\frac{\left(\frac{x_i}{x_n} \frac{de_1\cdots e_n f}{a} q^{-1-N}\right)_{y_1+\cdots+y_n}}{\left(\frac{x_i}{x_n} \frac{de_1\cdots e_n f}{ae_i} q^{-N}\right)_{y_1+\cdots+y_n}} \right]$$

$$\left. \times \prod_{1 \le r < s \le n} \left[\frac{1 - \frac{x_r}{x_s} q^{y_r - y_s}}{1 - \frac{x_r}{x_s}} \right] q^{y_1 + 2y_2 + \cdots + n y_n} \right\}.$$

We next rewrite the sum (4.8) by using (2.4) to reverse the order of all the products that contain a q^{-N}. Factoring q^{-N} out of (4.8a), and simplifying, gives the following sum, in which each term only involves q^N:

$$\text{(4.9a)}\qquad \sum_{\substack{y_1,\dots,y_n\geq 0\\ 0\leq y_1+\cdots+y_n\leq N}} \left\{ \prod_{i=1}^{n}\left[\frac{1-\frac{x_n}{x_i}\frac{a}{de_1\cdots e_nf}q^{1+N}q^{-(y_i+(y_1+\cdots+y_n))}}{1-\frac{x_n}{x_i}\frac{a}{de_1\cdots e_nf}q^{1+N}}\right]\right.$$

$$\text{(4.9b)}\qquad \times\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}\frac{aq}{bc}\right)_{y_i}\left(\frac{x_i}{x_n}d\right)_{y_i}\left(\frac{x_i}{x_n}f\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{aq}{c}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{de_1\cdots e_nf}{a}\right)_{y_i}}\right]$$

$$\text{(4.9c)}\qquad \times\left[\frac{\left(\frac{a^2}{bde_1\cdots e_nf}q^{2+N-(y_1+\cdots+y_n)}\right)_{y_1+\cdots+y_n}}{\left(\frac{a^2}{bcde_1\cdots e_nf}q^{2+N-(y_1+\cdots+y_n)}\right)_{y_1+\cdots+y_n}}\right]$$

$$\text{(4.9d)}\qquad \times\left[\frac{\left(\frac{a^2}{cde_1\cdots e_nf}q^{2+N-(y_1+\cdots+y_n)}\right)_{y_1+\cdots+y_n}}{\left(\frac{a}{de_1\cdots e_n}q^{1+N-(y_1+\cdots+y_n)}\right)_{y_1+\cdots+y_n}}\right]$$

$$\text{(4.9e)}\qquad \times\left[\frac{\left(q^{1+N-(y_1+\cdots+y_n)}\right)_{y_1+\cdots+y_n}}{\left(\frac{a}{e_1\cdots e_nf}q^{1+N-(y_1+\cdots+y_n)}\right)_{y_1+\cdots+y_n}}\right]$$

$$\text{(4.9f)}\qquad \times\prod_{i=1}^{n}\left[\frac{\left(\frac{x_n}{x_i}\frac{a}{de_1\cdots e_nf}q^{2+N-(y_1+\cdots+y_n)}\right)_{y_1+\cdots+y_n}}{\left(\frac{x_n}{x_i}\frac{ae_i}{de_1\cdots e_nf}q^{1+N-(y_1+\cdots+y_n)}\right)_{y_1+\cdots+y_n}}\right]$$

$$\text{(4.9g)}\qquad \times\prod_{r,s=1}^{n}\left[\frac{\left(\frac{x_r}{x_s}e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\right]\prod_{1\leq r<s\leq n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]$$

$$\text{(4.9h)}\qquad \left.\times q^{y_1+2y_2+\cdots+ny_n}\right\}.$$

In (4.9), all the powers of q, involving N, that appear in each factor, are nonnegative, except for those in the numerator of (4.9a). The only other negative powers of q appear in the Vandermonde product in (4.9g). Now, termwise, as $N\to\infty$, it is not hard to see that the other factors, involving N, in (4.9) approach a nonzero constant. The above two factors, as well as the power of q in (4.9h), require further analysis. They are

$$\prod_{i=1}^{n}\left(1-\frac{x_n}{x_i}\frac{a}{de_1\cdots e_n f}q^{1+N}q^{-(y_i+(y_1+\cdots+y_n))}\right)$$

$$\times\prod_{1\le r<s\le n}\left[1-\frac{x_r}{x_s}q^{y_r-y_s}\right]q^{y_1+2y_2+\cdots+ny_n}. \tag{4.10}$$

Rewrite (4.10) as

$$\prod_{i=1}^{n}\left[q^{y_i}-\frac{x_n}{x_i}\frac{a}{de_1\cdots e_n f}q^{1+N-(y_1+\cdots+y_n)}\right]\prod_{1\le r<s\le n}\left[q^{y_s}-\frac{x_r}{x_s}q^{y_r}\right]. \tag{4.11}$$

By putting $y_i=\alpha_i N$, with the α_i nonnegative, where $0\le\alpha_1+\cdots+\alpha_n\le 1$, (4.11) it becomes

$$\prod_{i=1}^{n}\left[q^{\alpha_i N}-\frac{x_n}{x_i}\frac{a}{de_1\cdots e_n f}q^{1+N(1-(\alpha_1+\cdots\alpha_n))}\right] \tag{4.12a}$$

$$\times\prod_{1\le r<s\le n}\left[q^{\alpha_s N}-\frac{x_r}{x_s}q^{\alpha_r N}\right]. \tag{4.12b}$$

In order to investigate the behavior of (4.12) as $N\to\infty$, there are four main cases to consider, each involving the n-dimensional tetrahedron (n-simplex) that contains the indices of summation:

Case 1. $\alpha_1+\cdots+\alpha_n=0$. This case at the origin trivially gives a constant, as each α_i is 0.

Case 2. One $\alpha_i=1$, and all the others are 0. Here, (4.12) becomes constant as $N\to\infty$. This case corresponds to the n vertices of the n-dimensional tetrahedron that contains the indices of summation.

Case 3. $0<\alpha_1+\cdots+\alpha_n<1$, with at least one $\alpha_i>0$. In this case (4.12a) will approach 0 and (4.12b) will approach a constant. That is, (4.12) goes to 0.

Case 4. $\alpha_1+\cdots+\alpha_n=1$, with no $\alpha_i=1$. In this case, at least two α_is are nonzero, so (4.12b) will approach 0 and (4.12a) will approach a constant. That is, (4.12) goes to 0.

Remark. In the above cases 1–4, as $N\to\infty$, the factors in (4.12) go to constants on the $n+1$ vertices of the n-dimensional tetrahedron, while going to 0 in the region near the center. This implies that the behavior of (4.9) is analogous to the classical 1-variable case on page 50 of [23], which we outlined above in Bailey's proof of Theorem 3.2. Note that this classical (limit) analysis does not extend to the U(n) ${}_{10}\phi_9$ transformation formulas, involving sums over a "square" or "hyperrectangle," given by Theorems 3.1 and 3.2 of [61]. (Here, our limits tend to constants on the vertices of the "squares," while diverging on the interiors.) That is why, in this paper, we started with the $\mathrm{U}(n)$ ${}_{10}\phi_9$ transformation formula of Theorem A.1, involving sums over a "triangle" or "n-simplex," first given by Theorem 3.3 of [61].

Given the above motivation, we utilize the following scheme for dealing with the limit of (4.9):

Split (4.9) into $n+1$ sums according to

$$\sum_{\substack{y_1,\ldots,y_n\geq 0\\ 0\leq y_1+\cdots+y_n\leq N}} \lambda_{y_1,\ldots,y_n} \tag{4.13a}$$

$$= \sum_{\substack{0\leq y_1,\ldots,y_n\leq N/2\\ 0\leq y_1+\cdots+y_n\leq N}} \lambda_{y_1,\ldots,y_n} \tag{4.13b}$$

$$+\sum_{j=1}^{n} \sum_{\substack{y_1,\ldots,y_n\geq 0\\ N/2<y_1+\cdots+y_n\leq N\\ N/2<y_j\leq N}} \lambda_{y_1,\ldots,y_n} \tag{4.13c}$$

$$= \sum_{\substack{0\leq y_1,\ldots,y_n\leq N/2\\ 0\leq y_1+\cdots+y_n\leq N}} \lambda_{y_1,\ldots,y_n} \tag{4.13d}$$

$$+\sum_{j=1}^{n} \sum_{\substack{y_1,\ldots,y_n\geq 0\\ 0\leq y_1+\cdots+y_n<N/2\\ 0\leq y_j<N/2}} \lambda_{y_1,\ldots,y_{j-1},N-(y_1+\cdots+y_n),y_{j+1},\ldots,y_n}. \tag{4.13e}$$

The multiple sum in (4.13a) is transformed into the $n+1$ multiple sums in (4.13b) and (4.13c) by splitting up the index of summation set determined by $y_1,\ldots,y_n\geq 0$ and $0\leq y_1+\cdots+y_n\leq N$, into the disjoint union of subsets, in which none, or exactly one, respectively, of the conditions $y_j>N/2$ hold. In (4.13c), the conditions $y_1,\ldots,y_n\geq 0$ and $y_j>N/2$ immediately imply $N/2<y_1+\cdots+y_n$. Furthermore, the three index of summation conditions for the inner multiple sum in (4.13c) are equivalent to the two conditions $y_1,\ldots,y_n\geq 0$ and $N/2<y_j\leq N-(y_1+\cdots+y_{j-1}+y_{j+1}+\cdots+y_n)$. In order to obtain (4.13e) from (4.13c), the inner multiple sum in (4.13c) is shifted and then reversed *only in the j-th coordinate*. Shifting y_j down by $N/2$ gives $y_1,\ldots,y_n\geq 0$ and $0<y_j\leq N/2-(y_1+\cdots+y_{j-1}+y_{j+1}+\cdots+y_n)$, while changing $\lambda_{y_1,\ldots,y_n}$ to $\lambda_{y_1,\ldots,y_{j-1},y_j+N/2,y_{j+1},\ldots,y_n}$. Reversing the order of summation in y_j now gives $y_1,\ldots,y_n\geq 0$ and $0\leq y_j<N/2-(y_1+\cdots+y_{j-1}+y_{j+1}+\cdots+y_n)$, while changing $\lambda_{y_1,\ldots,y_{j-1},y_j+N/2,y_{j+1},\ldots,y_n}$ to $\lambda_{y_1,\ldots,y_{j-1},N-(y_1+\cdots+y_n),y_{j+1},\ldots,y_n}$, as in the summand, we replaced y_j by $N/2-(y_1+\cdots+y_{j-1}+y_{j+1}+\cdots+y_n)-y_j$, which equals $N/2-(y_1+\cdots+y_n)$. Finally, note that the two conditions $y_1,\ldots,y_n\geq 0$ and $0\leq y_j<N/2-(y_1+\cdots+y_{j-1}+y_{j+1}+\cdots+y_n)$ are equivalent to the three index of summation conditions for the inner multiple sum in (4.13e).

We now replace $\lambda_{y_1,\ldots,y_n}$ in (4.13a) with the summand in (4.9).
The sum in (4.13d) then becomes

$$\sum_{\substack{0\le y_1,\ldots,y_n\le N/2\\ 0\le y_1+\cdots+y_n\le N}} \Bigg\{ \prod_{i=1}^{n}\left[\frac{1-\frac{x_n}{x_i}\frac{aq}{de_1\cdots e_nf}q^{N-(y_i+(y_1+\cdots+y_n))}}{1-\frac{x_n}{x_i}\frac{aq}{de_1\cdots e_nf}q^{N}}\right] \tag{4.14a}$$

$$\times\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}\frac{aq}{bc}\right)_{y_i}\left(\frac{x_i}{x_n}d\right)_{y_i}\left(\frac{x_i}{x_n}f\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{aq}{c}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{de_1\cdots e_nf}{a}\right)_{y_i}}\right] \tag{4.14b}$$

$$\times\left[\frac{\left(\frac{a^2}{bde_1\cdots e_nf}q^{2+N-(y_1+\cdots+y_n)}\right)_{y_1+\cdots+y_n}}{\left(\frac{a^2}{bcde_1\cdots e_nf}q^{2+N-(y_1+\cdots+y_n)}\right)_{y_1+\cdots+y_n}}\right] \tag{4.14c}$$

$$\times\left[\frac{\left(\frac{a^2}{cde_1\cdots e_nf}q^{2+N-(y_1+\cdots+y_n)}\right)_{y_1+\cdots+y_n}}{\left(\frac{a}{de_1\cdots e_n}q^{1+N-(y_1+\cdots+y_n)}\right)_{y_1+\cdots+y_n}}\right] \tag{4.14d}$$

$$\times\left[\frac{\left(q^{1+N-(y_1+\cdots+y_n)}\right)_{y_1+\cdots+y_n}}{\left(\frac{a}{e_1\cdots e_nf}q^{1+N-(y_1+\cdots+y_n)}\right)_{y_1+\cdots+y_n}}\right] \tag{4.14e}$$

$$\times\prod_{i=1}^{n}\left[\frac{\left(\frac{x_n}{x_i}\frac{a}{de_1\cdots e_nf}q^{2+N-(y_1+\cdots+y_n)}\right)_{y_1+\cdots+y_n}}{\left(\frac{x_n}{x_i}\frac{ae_i}{de_1\cdots e_nf}q^{1+N-(y_1+\cdots+y_n)}\right)_{y_1+\cdots+y_n}}\right] \tag{4.14f}$$

$$\times\prod_{r,s=1}^{n}\left[\frac{\left(\frac{x_r}{x_s}e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\right]\prod_{1\le r<s\le n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \tag{4.14g}$$

$$\times q^{y_1+2y_2+\cdots+ny_n}\Bigg\}. \tag{4.14h}$$

As $N\to\infty$, we show in Appendix B that (4.14) becomes the sum in (4.2c). This represents some of the deepest analysis of this paper.

We next study the limit, as $N \to \infty$, of each inner sum in (4.13e). As these inner sums are similar, it is only necessary to examine the jth inner sum given by

$$\sum_{\substack{y_1,\dots,y_n \geq 0 \\ 0 \leq y_j < N/2 \\ 0 \leq y_1+\cdots+y_n < N/2}} \Bigg\{ \prod_{\substack{1\leq i\leq n \\ i\neq j}} \left[\frac{1-\frac{x_n}{x_i}\frac{a}{de_1\cdots e_nf}q^{1+y_j-y_i}}{1-\frac{x_n}{x_i}\frac{a}{de_1\cdots e_nf}q^{1+N}} \right] \tag{4.15a}$$

$$\times \left[\frac{1-\frac{x_n}{x_j}\frac{a}{de_1\cdots e_nf}q^{1-N+y_j+(y_1+\cdots+y_n)}}{1-\frac{x_n}{x_j}\frac{a}{de_1\cdots e_nf}q^{1+N}} \right] \tag{4.15b}$$

$$\times \prod_{\substack{1\leq i\leq n \\ i\neq j}} \left[\frac{\left(\frac{x_i}{x_n}\frac{aq}{bc}\right)_{y_i}\left(\frac{x_i}{x_n}d\right)_{y_i}\left(\frac{x_i}{x_n}f\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{aq}{c}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{de_1\cdots e_nf}{a}\right)_{y_i}} \right] \tag{4.15c}$$

$$\times \left[\frac{\left(\frac{x_j}{x_n}\frac{aq}{bc}\right)_{N-(y_1+\cdots+y_n)}\left(\frac{x_j}{x_n}f\right)_{N-(y_1+\cdots+y_n)}}{\left(\frac{x_j}{x_n}\frac{aq}{b}\right)_{N-(y_1+\cdots+y_n)}\left(\frac{x_j}{x_n}\frac{aq}{c}\right)_{N-(y_1+\cdots+y_n)}} \right] \tag{4.15d}$$

$$\times \left[\frac{\left(\frac{x_j}{x_n}d\right)_{N-(y_1+\cdots+y_n)}\left(q^{1+y_j}\right)_{N-y_j}}{\left(\frac{x_j}{x_n}\frac{de_1\cdots e_nf}{a}\right)_{N-(y_1+\cdots+y_n)}\left(\frac{a}{e_1\cdots e_nf}q^{1+y_j}\right)_{N-y_j}} \right] \tag{4.15e}$$

$$\times \left[\frac{\left(\frac{a^2}{bde_1\cdots e_nf}q^{2+y_j}\right)_{N-y_j}\left(\frac{a^2}{cde_1\cdots e_nf}q^{2+y_j}\right)_{N-y_j}}{\left(\frac{a^2}{bcde_1\cdots e_nf}q^{2+y_j}\right)_{N-y_j}\left(\frac{a}{de_1\cdots e_n}q^{1+y_j}\right)_{N-y_j}} \right] \tag{4.15f}$$

$$\times \prod_{i=1}^{n} \left[\frac{\left(\frac{x_n}{x_i}\frac{a}{de_1\cdots e_nf}q^{2+y_j}\right)_{N-y_j}}{\left(\frac{x_n}{x_i}\frac{ae_i}{de_1\cdots e_nf}q^{1+y_j}\right)_{N-y_j}} \right] \prod_{\substack{1\leq r,s\leq n \\ r\neq j}} \left[\frac{\left(\frac{x_r}{x_s}e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}} \right] \tag{4.15g}$$

$$\times \prod_{s=1}^{n} \left[\frac{\left(\frac{x_j}{x_s}e_s\right)_{N-(y_1+\cdots+y_n)}}{\left(q\frac{x_j}{x_s}\right)_{N-(y_1+\cdots+y_n)}} \right] \prod_{r=1}^{j-1} \left[\frac{1-\frac{x_r}{x_j}q^{-N+y_r+(y_1+\cdots+y_n)}}{1-\frac{x_r}{x_j}} \right] \tag{4.15h}$$

$$\times \prod_{s=j+1}^{n}\left[\frac{1-\frac{x_j}{x_s}q^{N-(y_s+(y_1+\cdots+y_n))}}{1-\frac{x_j}{x_s}}\right]\prod_{\substack{1\le r<s\le n\\ r,s\neq j}}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \tag{4.15i}$$

$$\left.\times\, q^{y_1+2y_2+\cdots(j-1)y_{j-1}+j(N-(y_1+\cdots+y_n))+(j+1)y_{j+1}+\cdots+ny_n}\right\}. \tag{4.15j}$$

We next rewrite the sum (4.15) by using (2.5) to express each product of the form $(Aq^{y_j};q)_{N-y_j}$ in (4.15e), (4.15f), and (4.15g) as a ratio of products. Factoring q^{-N} out of the numerators of (4.15b) and the second factor in (4.15h), and simplifying, gives the following sum, in which each term only involves q^N:

$$\sum_{\substack{y_1,\ldots,y_n\ge 0\\ 0\le y_j<N/2\\ 0\le y_1+\cdots+y_n<N/2}}\left\{\prod_{\substack{1\le i\le n\\ i\neq j}}\left[\frac{1-\frac{x_n}{x_i}\frac{aq}{de_1\cdots e_nf}q^{y_j-y_i}}{1-\frac{x_n}{x_i}\frac{aq}{de_1\cdots e_nf}q^{N}}\right]\right. \tag{4.16a}$$

$$\times\left[\frac{1-\frac{x_j}{x_n}\frac{de_1\cdots e_nf}{aq}q^{N-(y_j+(y_1+\cdots+y_n))}}{1-\frac{x_n}{x_j}\frac{aq}{de_1\cdots e_nf}q^{N}}\right] \tag{4.16b}$$

$$\times\prod_{\substack{1\le i\le n\\ i\neq j}}\left[\frac{\left(\frac{x_i}{x_n}\frac{aq}{bc}\right)_{y_i}\left(\frac{x_i}{x_n}d\right)_{y_i}\left(\frac{x_i}{x_n}f\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{aq}{c}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{de_1\cdots e_nf}{a}\right)_{y_i}}\right] \tag{4.16c}$$

$$\times\left[\frac{\left(\frac{x_j}{x_n}\frac{aq}{bc}\right)_{N-(y_1+\cdots+y_n)}\left(\frac{x_j}{x_n}d\right)_{N-(y_1+\cdots+y_n)}\left(\frac{x_j}{x_n}f\right)_{N-(y_1+\cdots+y_n)}}{\left(\frac{x_j}{x_n}\frac{aq}{b}\right)_{N-(y_1+\cdots+y_n)}\left(\frac{x_j}{x_n}\frac{aq}{c}\right)_{N-(y_1+\cdots+y_n)}\left(\frac{x_j}{x_n}\frac{de_1\cdots e_nf}{a}\right)_{N-(y_1+\cdots+y_n)}}\right] \tag{4.16d}$$

$$\times\left[\frac{\left(\frac{a^2q^2}{bde_1\cdots e_nf}\right)_N\left(\frac{a^2q^2}{cde_1\cdots e_nf}\right)_N(q)_N}{\left(\frac{a^2q^2}{bcde_1\cdots e_nf}\right)_N\left(\frac{aq}{de_1\cdots e_n}\right)_N\left(\frac{aq}{e_1\cdots e_nf}\right)_N}\right] \tag{4.16e}$$

$$\times\left[\frac{\left(\frac{a^2q^2}{bcde_1\cdots e_nf}\right)_{y_j}\left(\frac{aq}{de_1\cdots e_n}\right)_{y_j}\left(\frac{aq}{e_1\cdots e_nf}\right)_{y_j}}{\left(\frac{a^2q^2}{bde_1\cdots e_nf}\right)_{y_j}\left(\frac{a^2q^2}{cde_1\cdots e_nf}\right)_{y_j}(q)_{y_j}}\right] \tag{4.16f}$$

(4.16g) $$\times \prod_{i=1}^{n} \left[\frac{\left(\frac{x_n}{x_i}\frac{aq^2}{de_1\cdots e_nf}\right)_N \left(\frac{x_n}{x_i}\frac{ae_iq}{de_1\cdots e_nf}\right)_{y_j}}{\left(\frac{x_n}{x_i}\frac{ae_iq}{de_1\cdots e_nf}\right)_N \left(\frac{x_n}{x_i}\frac{aq^2}{de_1\cdots e_nf}\right)_{y_j}} \right]$$

(4.16h) $$\times \prod_{\substack{1\le r,s\le n\\ r\neq j}} \left[\frac{\left(\frac{x_r}{x_s}e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}} \right] \prod_{s=1}^{n} \left[\frac{\left(\frac{x_j}{x_s}e_s\right)_{N-(y_1+\cdots+y_n)}}{\left(q\frac{x_j}{x_s}\right)_{N-(y_1+\cdots+y_n)}} \right]$$

(4.16i) $$\times \prod_{r=1}^{j-1} \left[\frac{1-\frac{x_j}{x_r}q^{N-(y_r+(y_1+\cdots+y_n))}}{1-\frac{x_r}{x_j}} \right] \prod_{s=j+1}^{n} \left[\frac{1-\frac{x_j}{x_s}q^{N-(y_s+(y_1+\cdots+y_n))}}{1-\frac{x_j}{x_s}} \right]$$

(4.16j) $$\times \prod_{\substack{1\le r<s\le n\\ r,s\neq j}} \left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \right] \prod_{r=1}^{j-1} \left(\frac{x_r}{x_j}\right)$$

(4.16k) $$\times(-1)^j \left(\frac{aq}{de_1\cdots e_nf}\right)\left(\frac{x_n}{x_j}\right)$$

(4.16l) $$\left.\times q^{y_1+\cdots+y_j} q^{y_1+2y_2+\cdots+(j-1)y_{j-1}+(j+1)y_{j+1}+\cdots+ny_n} \right\}.$$

Letting $N\to\infty$, we show in Appendix B that (4.16) becomes

(4.17a) $$\left[\frac{\left(\frac{x_j}{x_n}\frac{aq}{bc}\right)_\infty \left(\frac{x_j}{x_n}d\right)_\infty \left(\frac{x_j}{x_n}f\right)_\infty}{\left(\frac{x_j}{x_n}\frac{aq}{b}\right)_\infty \left(\frac{x_j}{x_n}\frac{aq}{c}\right)_\infty \left(\frac{x_j}{x_n}\frac{de_1\cdots e_nf}{a}\right)_\infty} \right] \prod_{i=1}^{n} \left[\frac{\left(\frac{x_n}{x_i}\frac{aq^2}{de_1\cdots e_nf}\right)_\infty}{\left(\frac{x_n}{x_i}\frac{ae_iq}{de_1\cdots e_nf}\right)_\infty} \right]$$

(4.17b) $$\times \left[\frac{\left(\frac{a^2q^2}{bde_1\cdots e_nf}\right)_\infty \left(\frac{a^2q^2}{cde_1\cdots e_nf}\right)_\infty (q)_\infty}{\left(\frac{a^2q^2}{bcde_1\cdots e_nf}\right)_\infty \left(\frac{aq}{de_1\cdots e_n}\right)_\infty \left(\frac{aq}{e_1\cdots e_nf}\right)_\infty} \right] \prod_{s=1}^{n} \left[\frac{\left(\frac{x_j}{x_s}e_s\right)_\infty}{\left(\frac{x_j}{x_s}q\right)_\infty} \right]$$

(4.17c) $$\sum_{y_1,\ldots,y_n\ge 0} \left\{ \prod_{\substack{1\le i\le n\\ i\neq j}} \left[1-\frac{x_n}{x_i}\frac{aq}{de_1\cdots e_nf}q^{y_j-y_i} \right] \right.$$

(4.17d) $$\times \prod_{\substack{1\le i\le n\\ i\neq j}} \left[\frac{\left(\frac{x_i}{x_n}\frac{aq}{bc}\right)_{y_i} \left(\frac{x_i}{x_n}d\right)_{y_i} \left(\frac{x_i}{x_n}f\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i} \left(\frac{x_i}{x_n}\frac{aq}{c}\right)_{y_i} \left(\frac{x_i}{x_n}\frac{de_1\cdots e_nf}{a}\right)_{y_i}} \right]$$

$$\times \left[\frac{\left(\frac{a^2q^2}{bcde_1\cdots e_nf}\right)_{y_j} \left(\frac{aq}{de_1\cdots e_n}\right)_{y_j} \left(\frac{aq}{e_1\cdots e_nf}\right)_{y_j}}{\left(\frac{a^2q^2}{bde_1\cdots e_nf}\right)_{y_j} \left(\frac{a^2q^2}{cde_1\cdots e_nf}\right)_{y_j} (q)_{y_j}} \right] \tag{4.17e}$$

$$\times \prod_{i=1}^{n} \left[\frac{\left(\frac{x_n}{x_i}\frac{ae_iq}{de_1\cdots e_nf}\right)_{y_j}}{\left(\frac{x_n}{x_i}\frac{aq^2}{de_1\cdots e_nf}\right)_{y_j}} \right] \prod_{\substack{1\le r,s\le n\\ r\neq j}} \left[\frac{\left(\frac{x_r}{x_s}e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}} \right] \tag{4.17f}$$

$$\times \prod_{r=1}^{j-1} \left[\frac{1}{1-\frac{x_r}{x_j}} \right] \prod_{s=j+1}^{n} \left[\frac{1}{1-\frac{x_j}{x_s}} \right] \tag{4.17g}$$

$$\times \prod_{\substack{1\le r<s\le n\\ r,s\neq j}} \left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \right] \prod_{r=1}^{j-1} \left(\frac{x_r}{x_j}\right) \tag{4.17h}$$

$$\times (-1)^j \left(\frac{aq}{de_1\cdots e_nf}\right) \left(\frac{x_n}{x_j}\right) \tag{4.17i}$$

$$\left. \times q^{y_1+\cdots+y_j} q^{y_1+2y_2+\cdots+(j-1)y_{j-1}+(j+1)y_{j+1}+\cdots+ny_n} \right\}. \tag{4.17j}$$

Taking (4.17) for $j = 1,\ldots,n$, adding these to (4.2c), and then multiplying everything by (4.7) gives (4.2b)–(4.2e). □

Theorem 4.2 (Second $\mathbf{U}(n+1)$ nonterminating q-Whipple transformation). *Let $a,b,c,d,\ e_1,\ldots,e_n,f,$ and $x_1,\ldots,x_n$ be indeterminate, with $n \geq 1$. Take $0 < |q| < 1$ and*

$$\left| \frac{x_n}{x_i} \frac{a^2q^2}{bcde_1\cdots e_nf} \right| < 1, \quad \textit{for } i = 1,2,\ldots,n. \tag{4.18}$$

Suppose that none of the denominators in (4.19) vanish. Then

$$\sum_{y_1,\ldots,y_n\geq 0} \left\{ \prod_{i=1}^{n} \left[\frac{1-\frac{x_i}{x_n}aq^{y_i+(y_1+\cdots+y_n)}}{1-\frac{x_i}{x_n}a} \right] \right.$$

$$\times \prod_{i=1}^{n} \left[\frac{\left(\frac{x_i}{x_n}c\right)_{y_i} \left(\frac{x_i}{x_n}d\right)_{y_i} \left(\frac{x_i}{x_n}f\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i}} \right]$$

$$\times \left[\frac{(b)_{y_1+\cdots+y_n}}{\left(\frac{aq}{c}\right)_{y_1+\cdots+y_n} \left(\frac{aq}{d}\right)_{y_1+\cdots+y_n} \left(\frac{aq}{f}\right)_{y_1+\cdots+y_n}} \right]$$

$$\times \prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}a\right)_{y_1+\cdots+y_n}}{\left(\frac{x_i}{x_n}\frac{aq}{e_i}\right)_{y_1+\cdots+y_n}}\right]\prod_{r,s=1}^{n}\left[\frac{\left(\frac{x_r}{x_s}e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\right]$$

$$\times \prod_{1\le r<s\le n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\left(\frac{a^2q}{bcde_1\cdots e_nf}\right)^{y_1+\cdots+y_n}$$

$$\times q^{y_1+2y_2+\cdots+ny_n}q^{e_2(y_1,\ldots,y_n)}\prod_{i=1}^{n}\left(\frac{x_n}{x_i}\right)^{y_i}\Bigg\} \tag{4.19a}$$

$$= \frac{\left(\frac{aq}{de_1\cdots e_n}\right)_\infty\left(\frac{aq}{e_1\cdots e_nf}\right)_\infty}{\left(\frac{aq}{d}\right)_\infty\left(\frac{aq}{f}\right)_\infty}\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}aq\right)_\infty\left(\frac{x_n}{x_i}\frac{ae_iq}{de_1\cdots e_nf}\right)_\infty}{\left(\frac{x_n}{x_i}\frac{aq}{de_1\cdots e_nf}\right)_\infty\left(\frac{x_i}{x_n}\frac{aq}{e_i}\right)_\infty}\right] \tag{4.19b}$$

$$\sum_{y_1,\ldots,y_n\ge 0}\left\{\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}d\right)_{y_i}\left(\frac{x_i}{x_n}f\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{de_1\cdots e_nf}{a}\right)_{y_i}}\right]\left[\frac{\left(\frac{aq}{bc}\right)_{y_1+\cdots+y_n}}{\left(\frac{aq}{c}\right)_{y_1+\cdots+y_n}}\right]\right.$$

$$\left.\times\prod_{r,s=1}^{n}\left[\frac{\left(\frac{x_r}{x_s}e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\right]\prod_{1\le r<s\le n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]q^{y_1+2y_2+\cdots ny_n}\right\} \tag{4.19c}$$

$$+\frac{\left(\frac{a^2q^2}{bde_1\cdots e_nf}\right)_\infty\left(\frac{aq}{bc}\right)_\infty(q)_\infty}{\left(\frac{aq}{c}\right)_\infty\left(\frac{aq}{d}\right)_\infty\left(\frac{aq}{f}\right)_\infty}\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}aq\right)_\infty}{\left(\frac{x_i}{x_n}\frac{aq}{e_i}\right)_\infty\left(1-\frac{x_n}{x_i}\frac{aq}{de_1\cdots e_nf}\right)}\right] \tag{4.19d}$$

$$\times\sum_{j=1}^{n}\left\{\frac{\left(\frac{x_n}{x_j}\frac{a^2q^2}{cde_1\cdots e_nf}\right)_\infty\left(\frac{x_j}{x_n}d\right)_\infty\left(\frac{x_j}{x_n}f\right)_\infty}{\left(\frac{x_j}{x_n}\frac{aq}{b}\right)_\infty\left(\frac{x_n}{x_j}\frac{a^2q^2}{bcde_1\cdots e_nf}\right)_\infty\left(\frac{x_j}{x_n}\frac{de_1\cdots e_nf}{a}\right)_\infty}\prod_{s=1}^{n}\left[\frac{\left(\frac{x_j}{x_s}e_s\right)_\infty}{\left(\frac{x_j}{x_s}q\right)_\infty}\right]\right.$$

$$\times\sum_{y_1,\ldots,y_n\ge 0}\prod_{\substack{1\le i\le n\\ i\ne j}}\left[1-\frac{x_n}{x_i}\frac{aq}{de_1\cdots e_nf}q^{y_j-y_i}\right]$$

$$\times\prod_{\substack{1\le i\le n\\ i\ne j}}\left[\frac{\left(\frac{x_i}{x_n}d\right)_{y_i}\left(\frac{x_i}{x_n}f\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{de_1\cdots e_nf}{a}\right)_{y_i}}\right]$$

$$\times\left[\frac{\left(\frac{x_n}{x_j}\frac{a^2q^2}{bcde_1\cdots e_nf}\right)_{y_1+\cdots+y_n}\left(\frac{aq}{de_1\cdots e_n}\right)_{y_j}\left(\frac{aq}{e_1\cdots e_nf}\right)_{y_j}}{\left(\frac{a^2q^2}{bde_1\cdots e_nf}\right)_{y_j}\left(\frac{x_n}{x_j}\frac{a^2q^2}{cde_1\cdots e_nf}\right)_{y_1+\cdots+y_n}(q)_{y_j}}\right]$$

$$\times \prod_{i=1}^{n} \left[\frac{\left(\frac{x_n}{x_i} \frac{ae_i q}{de_1 \cdots e_n f} \right)_{y_j}}{\left(\frac{x_n}{x_i} \frac{aq^2}{de_1 \cdots e_n f} \right)_{y_j}} \right] \prod_{\substack{1 \le r,s \le n \\ r \neq j}} \left[\frac{\left(\frac{x_r}{x_s} e_s \right)_{y_r}}{\left(q \frac{x_r}{x_s} \right)_{y_r}} \right]$$

$$\times \prod_{r=1}^{j-1} \left[\frac{1}{1 - \frac{x_r}{x_j}} \right] \prod_{s=j+1}^{n} \left[\frac{1}{1 - \frac{x_j}{x_s}} \right]$$

$$\times (-1)^{j} \left(\frac{aq}{de_1 \cdots e_n f} \right) \left(\frac{x_n}{x_j} \right) \prod_{\substack{1 \le r < s \le n \\ r,s \neq j}} \left[\frac{1 - \frac{x_r}{x_s} q^{y_r - y_s}}{1 - \frac{x_r}{x_s}} \right]$$

$$\times q^{y_1 + \cdots + y_j} q^{y_1 + 2y_2 + \cdots + (j-1)y_{j-1} + (j+1)y_{j+1} + \cdots + ny_n} \prod_{r=1}^{j-1} \left(\frac{x_r}{x_j} \right) \Bigg\}, \tag{4.19e}$$

where $e_2(y_1, \ldots, y_n)$ is the second elementary symmetric function of $\{y_1, \ldots, y_n\}$.

Proof. To obtain this second transformation, first exchange c and d in (A.1a)–(A.1c). The rest of the proof, including the use of Appendix B, is essentially the same as that of Theorem 4.1. □

Remark. Note that $e_2(y_1, \ldots, y_n)$ only appears in (4.19a), but not the right-hand side of (4.19). Moreover, $e_2(y_1, \ldots, y_n)$ does not appear in Theorem 4.1. In the proof of Theorem 4.1, before taking $N \to \infty$ in the sum (4.3), we only had to apply (2.4) to the ratio of the two products in (4.3c), involving q^{-N}. The factors $q^{\binom{y_1 + \cdots + y_n}{2}}$ that came out cancelled. However, in the analogous computation in the proof of Theorem 4.2, such cancellation does not occur. Further simplification requires (2.7), introducing the additional power of q given by

$$q^{e_2(y_1, \ldots, y_n)}, \tag{4.20}$$

where $e_2(y_1, \ldots, y_n)$ is the second elementary symmetric function of $\{y_1, \ldots, y_n\}$.

Remark. The $n = 1$ and $e_1 \mapsto e$ case of (4.19) is (III.36) on page 364 of [23].

5 U$(n+1)$ Nonterminating q-Saalschütz Transformations

In this section we derive our U$(n+1)$ nonterminating q-Saalschütz transformations by applying Theorems A.3 and A.4, respectively, to suitable special cases of Theorems 4.1 and 4.2. This analysis extends Bailey's proof, summarized on page 51 of [23], of his classical nonterminating q-Saalschütz transformation.

We give the formal aspects of our proofs in this section, while the absolute convergence of the nonterminating multiple series is established in Appendix B.

Theorem 5.1 (First $\mathrm{U}(n+1)$ nonterminating q-Saalschütz transformation). *Let a,b, $c_1,\ldots,c_n,e,f$, and $x_1,\ldots,x_n$ be indeterminate, with $n\geq 1$. Take $0<|q|<1$. Suppose that none of the denominators in* (5.1) *vanish. Then*

$$\sum_{y_1,\ldots,y_n\geq 0}\left\{\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}a\right)_{y_i}\left(\frac{x_i}{x_n}b\right)_{y_i}}{\left(\frac{x_i}{x_n}e\right)_{y_i}\left(\frac{x_i}{x_n}f\right)_{y_i}}\right]\prod_{r,s=1}^{n}\left[\frac{\left(\frac{x_r}{x_s}c_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\right]\right.$$

$$\left.\times\prod_{1\leq r<s\leq n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]q^{y_1+2y_2+\cdots ny_n}\right\}\tag{5.1a}$$

$$=\frac{\left(\frac{f}{a}\right)_\infty\left(\frac{f}{b}\right)_\infty}{\left(\frac{aq}{e}\right)_\infty\left(\frac{bq}{e}\right)_\infty}\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}\frac{f}{c_i}\right)_\infty\left(\frac{x_n}{x_i}\frac{q}{e}\right)_\infty}{\left(\frac{x_i}{x_n}f\right)_\infty\left(\frac{x_n}{x_i}\frac{qc_i}{e}\right)_\infty}\right]\tag{5.1b}$$

$$-\frac{\left(\frac{fq}{e}\right)_\infty(q)_\infty}{\left(\frac{aq}{e}\right)_\infty\left(\frac{bq}{e}\right)_\infty}\prod_{i=1}^{n}\left[\frac{\left(\frac{x_n}{x_i}\frac{q}{e}\right)_\infty}{\left(\frac{x_n}{x_i}\frac{qc_i}{e}\right)_\infty\left(1-\frac{x_n}{x_i}\frac{q}{e}\right)}\right]$$

$$\sum_{j=1}^{n}\left\{\frac{\left(\frac{x_j}{x_n}a\right)_\infty\left(\frac{x_j}{x_n}b\right)_\infty}{\left(\frac{x_j}{x_n}e\right)_\infty\left(\frac{x_j}{x_n}f\right)_\infty}\prod_{s=1}^{n}\left[\frac{\left(\frac{x_j}{x_s}c_s\right)_\infty}{\left(\frac{x_j}{x_s}q\right)_\infty}\right]\right.$$

$$\times\sum_{y_1,\ldots,y_n\geq 0}\prod_{\substack{1\leq i\leq n\\ i\neq j}}\left[1-\frac{x_n}{x_i}\frac{1}{e}q^{1+y_j-y_i}\right]$$

$$\times\prod_{\substack{1\leq i\leq n\\ i\neq j}}\left[\frac{\left(\frac{x_i}{x_n}a\right)_{y_i}\left(\frac{x_i}{x_n}b\right)_{y_i}}{\left(\frac{x_i}{x_n}e\right)_{y_i}\left(\frac{x_i}{x_n}f\right)_{y_i}}\right]\cdot\left[\frac{\left(\frac{aq}{e}\right)_{y_j}\left(\frac{bq}{e}\right)_{y_j}}{\left(\frac{fq}{e}\right)_{y_j}(q)_{y_j}}\right]$$

$$\times\prod_{i=1}^{n}\left[\frac{\left(\frac{x_n}{x_i}\frac{qc_i}{e}\right)_{y_j}}{\left(\frac{x_n}{x_i}\frac{q^2}{e}\right)_{y_j}}\right]\prod_{\substack{1\leq r,s\leq n\\ r\neq j}}\left[\frac{\left(\frac{x_r}{x_s}c_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\right]$$

$$\times\prod_{r=1}^{j-1}\left[\frac{1}{1-\frac{x_r}{x_j}}\right]\prod_{s=j+1}^{n}\left[\frac{1}{1-\frac{x_j}{x_s}}\right]$$

$$\times \prod_{\substack{1\le r<s\le n\\ r,s\neq j}} \left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] (-1)^j \left(\frac{q}{e}\right)\left(\frac{x_n}{x_j}\right)\prod_{r=1}^{j-1}\left(\frac{x_r}{x_j}\right)$$

$$\left.\times q^{y_1+\cdots+y_j} q^{y_1+2y_2+\cdots+(j-1)y_{j-1}+(j+1)y_{j+1}+\cdots+ny_n}\right\}, \tag{5.1c}$$

where $a=\frac{ef}{bc_1\cdots c_n q}$.

Proof. In (4.2), we require that $aq/c=d$. Equation (4.2a) then reduces to (A.4a), and (4.2b)-(4.2c) becomes

$$\frac{\left(\frac{c}{e_1\cdots e_n}\right)_\infty \left(\frac{aq}{e_1\cdots e_n f}\right)_\infty}{\left(\frac{aq}{f}\right)_\infty (c)_\infty} \prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}aq\right)_\infty \left(\frac{x_n}{x_i}\frac{ce_i}{e_1\cdots e_n f}\right)_\infty}{\left(\frac{x_n}{x_i}\frac{c}{e_1\cdots e_n f}\right)_\infty \left(\frac{x_i}{x_n}\frac{aq}{e_i}\right)_\infty}\right]$$

$$\sum_{y_1,\ldots,y_n\ge 0}\left\{\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}\frac{aq}{bc}\right)_{y_i}\left(\frac{x_i}{x_n}f\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{qe_1\cdots e_n f}{c}\right)_{y_i}}\right]\right.$$

$$\left.\times\prod_{r,s=1}^{n}\left[\frac{\left(\frac{x_r}{x_s}e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\right]\prod_{1\le r<s\le n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]q^{y_1+2y_2+\cdots ny_n}\right\}. \tag{5.2}$$

With the same restriction, (4.2d)-(4.2e) becomes

$$\frac{\left(\frac{acq}{be_1\cdots e_n f}\right)_\infty \left(\frac{aq}{e_1\cdots e_n f}\right)_\infty (q)_\infty}{\left(\frac{aq}{be_1\cdots e_n f}\right)_\infty \left(\frac{aq}{f}\right)_\infty (c)_\infty}\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}aq\right)_\infty}{\left(\frac{x_i}{x_n}\frac{aq}{e_i}\right)_\infty\left(1-\frac{x_n}{x_i}\frac{c}{e_1\cdots e_n f}\right)}\right]$$

$$\sum_{j=1}^{n}\left\{\frac{\left(\frac{x_j}{x_n}\frac{aq}{bc}\right)_\infty\left(\frac{x_j}{x_n}f\right)_\infty}{\left(\frac{x_j}{x_n}\frac{aq}{b}\right)_\infty\left(\frac{x_j}{x_n}\frac{e_1\cdots e_n fq}{c}\right)_\infty}\prod_{s=1}^{n}\left[\frac{\left(\frac{x_j}{x_s}e_s\right)_\infty}{\left(\frac{x_j}{x_s}q\right)_\infty}\right]\right.$$

$$\times\sum_{y_1,\ldots,y_n\ge 0}\prod_{\substack{1\le i\le n\\ i\neq j}}\left[1-\frac{x_n}{x_i}\frac{c}{e_1\cdots e_n fq}q^{1+y_j-y_i}\right]$$

$$\times \prod_{\substack{1\leq i\leq n\\ i\neq j}} \left[\frac{\left(\frac{x_i}{x_n}\frac{aq}{bc}\right)_{y_i} \left(\frac{x_i}{x_n} f\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i} \left(\frac{x_i}{x_n}\frac{e_1\cdots e_n f q}{c}\right)_{y_i}} \right] \frac{\left(\frac{aq}{be_1\cdots e_n f}\right)_{y_j} \left(\frac{c}{e_1\cdots e_n}\right)_{y_j}}{\left(\frac{aqc}{be_1\cdots e_n f}\right)_{y_j} (q)_{y_j}}$$

$$\times \prod_{i=1}^{n} \left[\frac{\left(\frac{x_n}{x_i}\frac{ce_i}{e_1\cdots e_n f}\right)_{y_j}}{\left(\frac{x_n}{x_i}\frac{cq}{e_1\cdots e_n f}\right)_{y_j}} \right] \prod_{\substack{1\leq r,s\leq n\\ r\neq j}} \left[\frac{\left(\frac{x_r}{x_s} e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}} \right]$$

$$\times \prod_{r=1}^{j-1} \left[\frac{1}{1-\frac{x_r}{x_j}} \right] \prod_{s=j+1}^{n} \left[\frac{1}{1-\frac{x_j}{x_s}} \right]$$

$$\times \prod_{\substack{1\leq r<s\leq n\\ r,s\neq j}} \left[\frac{1-\frac{x_r}{x_s} q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \right] (-1)^j \left(\frac{c}{e_1\cdots e_n f}\right) \left(\frac{x_n}{x_j}\right)$$

$$\left. \times \prod_{r=1}^{j-1} \left(\frac{x_r}{x_j}\right) q^{y_1+\cdots+y_j} q^{y_1+2y_2+\cdots+(j-1)y_{j-1}+(j+1)y_{j+1}+\cdots+ny_n} \right\} . \tag{5.3}$$

Now, (A.4b) is the sum of (5.2) and (5.3). Solving for the sum in (5.2) and relabelling according to

$$e_s \mapsto c_s,\ f \mapsto b,\ c \mapsto \frac{bc_1\cdots c_n q}{e},\ a \mapsto \frac{1}{f},\ b \mapsto \frac{q}{f^2} \tag{5.4}$$

yields (5.1). □

Remark. After relabelling according to (5.4), we had no more a's, but instead, a large number of $ef/bc_1\cdots c_n q$'s. Just as in the one-variable classical case, we then set these expressions equal to a, giving the balanced condition $a = ef/bc_1\cdots c_n q$.

Remark. When we applied the nonterminating $\mathrm{U}(n+1)$ ${}_6\phi_5$ summation of Theorem A.3 to the $aq/c = d$ case of Theorem 4.1, we needed the convergence condition $0 < |q| < 1$, as well as those in equations (A.3) and (4.1). However, in Theorem 5.1, we only require the condition $0 < |q| < 1$. Just as in the one-variable classical case, the initial extra convergence conditions are removed by analytic continuation.

Remark. The $n = 1$ and $c_1 \mapsto c$ case of (5.1) is the classical nonterminating q-Saalschütz transformation in (II.24) on page 356 of [23].

Theorem 5.2 (Second $\mathrm{U}(n+1)$ nonterminating q-Saalschütz transformation). *Let $a, b, c_1, \ldots, c_n, e, f,$ and $x_1, \ldots, x_n$ be indeterminate, with $n \geq 1$. Take $0 < |q| < 1$. Suppose that none of the denominators in (5.5) vanish. Then*

$$\sum_{y_1,\dots,y_n\geq 0}\left\{\frac{(a)_{y_1+\cdots+y_n}}{(f)_{y_1+\cdots+y_n}}\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}b\right)_{y_i}}{\left(\frac{x_i}{x_n}e\right)_{y_i}}\right]\prod_{r,s=1}^{n}\left[\frac{\left(\frac{x_r}{x_s}c_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\right]\right.$$

(5.5a)
$$\left.\times\prod_{1\leq r<s\leq n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]q^{y_1+2y_2+\cdots+ny_n}\right\}$$

(5.5b)
$$=\frac{\left(\frac{f}{a}\right)_\infty\left(\frac{f}{c_1\cdots c_n}\right)_\infty}{(f)_\infty\left(\frac{bq}{e}\right)_\infty}\prod_{i=1}^{n}\left[\frac{\left(\frac{x_n}{x_i}\frac{fc_i}{bc_1\cdots c_n}\right)_\infty\left(\frac{x_n}{x_i}\frac{q}{e}\right)_\infty}{\left(\frac{x_n}{x_i}\frac{aq}{e}\right)_\infty\left(\frac{x_n}{x_i}\frac{qc_i}{e}\right)_\infty}\right]$$

$$-\frac{(a)_\infty(q)_\infty}{(f)_\infty\left(\frac{bq}{e}\right)_\infty}\prod_{i=1}^{n}\left[\frac{\left(\frac{x_n}{x_i}\frac{q}{e}\right)_\infty}{\left(\frac{x_n}{x_i}\frac{qc_i}{e}\right)_\infty\left(1-\frac{x_n}{x_i}\frac{q}{e}\right)}\right]$$

$$\sum_{j=1}^{n}\left\{\frac{\left(\frac{x_n}{x_j}\frac{fq}{e}\right)_\infty\left(\frac{x_j}{x_n}b\right)_\infty}{\left(\frac{x_j}{x_n}e\right)_\infty\left(\frac{x_n}{x_j}\frac{aq}{e}\right)_\infty}\prod_{s=1}^{n}\left[\frac{\left(\frac{x_j}{x_s}c_s\right)_\infty}{\left(\frac{x_j}{x_s}q\right)_\infty}\right]\right.$$

$$\times\sum_{y_1,\dots,y_n\geq 0}\prod_{\substack{1\leq i\leq n\\ i\neq j}}\left[1-\frac{x_n}{x_i}\frac{1}{e}q^{1+y_j-y_i}\right]$$

$$\times\prod_{\substack{1\leq i\leq n\\ i\neq j}}\left[\frac{\left(\frac{x_i}{x_n}b\right)_{y_i}}{\left(\frac{x_i}{x_n}e\right)_{y_i}}\right]\frac{\left(\frac{x_n}{x_j}\frac{aq}{e}\right)_{y_1+\cdots+y_n}\left(\frac{bq}{e}\right)_{y_j}}{\left(\frac{x_n}{x_j}\frac{fq}{e}\right)_{y_1+\cdots+y_n}(q)_{y_j}}$$

$$\times\prod_{i=1}^{n}\left[\frac{\left(\frac{x_n}{x_i}\frac{qc_i}{e}\right)_{y_j}}{\left(\frac{x_n}{x_i}\frac{q^2}{e}\right)_{y_j}}\right]\prod_{\substack{1\leq r,s\leq n\\ r\neq j}}\left[\frac{\left(\frac{x_r}{x_s}c_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\right]$$

$$\times\prod_{r=1}^{j-1}\left[\frac{1}{1-\frac{x_r}{x_j}}\right]\prod_{s=j+1}^{n}\left[\frac{1}{1-\frac{x_j}{x_s}}\right]$$

$$\times\prod_{\substack{1\leq r<s\leq n\\ r,s\neq j}}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right](-1)^j\left(\frac{q}{e}\right)\left(\frac{x_n}{x_j}\right)\prod_{r=1}^{j-1}\left(\frac{x_r}{x_j}\right)$$

(5.5c)
$$\left.\times q^{y_1+\cdots+y_j}q^{y_1+2y_2+\cdots+(j-1)y_{j-1}+(j+1)y_{j+1}+\cdots+ny_n}\right\},$$

where $a=\frac{ef}{bc_1\cdots c_nq}$.

Proof. In (4.19), first exchange b and c, then require that $aq/c = d$. Equation (4.19a) reduces to (A.6a). The remainder of the proof is then similar to that of Theorem 5.1. ☐

Remark. The $n = 1$ and $c_1 \mapsto c$ case of (5.5) is the classical nonterminating q-Saalschütz transformation in (II.24) on page 356 of [23].

Appendix A. Background Information: $\mathrm{U}(n+1)$ (or A_n) Basic Hypergeometric Summation Theorems

In this Appendix A we state the $\mathrm{U}(n+1)$ basic hypergeometric series summation theorems and transformations, from [48, 50, 51, 53, 61], that we need in the derivations of our $\mathrm{U}(n+1)$ nonterminating q-Whipple and q-Saalschütz transformations.

All of the finite (or terminating) multiple sums here are taken over a "triangle" or "n-simplex" determined by $y_1, \ldots, y_n \geq 0$ and $0 \leq y_1 + \cdots + y_n \leq N$, where N is a nonnegative integer. This choice of region of summation is needed in order to make our convergence arguments work.

Our two $\mathrm{U}(n+1)$ nonterminating q-Whipple transformations in Sect. 4 are a consequence of the following multivariable extension of Bailey's classical ${}_{10}\phi_9$ transformation given by Theorem 3.3 of [61].

Theorem A.1 (First $\mathbf{U}(n+1)$ ${}_{10}\phi_9$ transformation formula). *Let a, b, c, d, e_1, $\ldots$, e_n, f, and $x_1, \ldots, x_n$ be indeterminate, N be a nonnegative integer, $n \geq 1$, and suppose that none of the denominators in (A.1) vanish. Then*

$$\sum_{\substack{y_1,\ldots,y_n \geq 0 \\ 0 \leq y_1+\cdots+y_n \leq N}} \Bigg\{ \prod_{i=1}^{n} \left[\frac{1 - \frac{x_i}{x_n} a q^{y_i + (y_1+\cdots+y_n)}}{1 - \frac{x_i}{x_n} a} \right]$$

$$\times \prod_{i=1}^{n} \left[\frac{\left(\frac{x_i}{x_n} d\right)_{y_i} \left(\frac{x_i}{x_n} f\right)_{y_i} \left(\frac{x_i}{x_n} \frac{\lambda a}{e_1 \cdots e_n f} q^{1+N}\right)_{y_i}}{\left(\frac{x_i}{x_n} \frac{aq}{b}\right)_{y_i} \left(\frac{x_i}{x_n} \frac{aq}{c}\right)_{y_i} \left(\frac{x_i}{x_n} a q^{1+N}\right)_{y_i}} \right]$$

$$\times \left[\frac{(b)_{y_1+\cdots+y_n} (c)_{y_1+\cdots+y_n} \left(q^{-N}\right)_{y_1+\cdots+y_n}}{\left(\frac{aq}{d}\right)_{y_1+\cdots+y_n} \left(\frac{aq}{f}\right)_{y_1+\cdots+y_n} \left(\frac{e_1 \cdots e_n f}{\lambda} q^{-N}\right)_{y_1+\cdots+y_n}} \right]$$

$$\times \prod_{i=1}^{n} \left[\frac{\left(\frac{x_i}{x_n} a\right)_{y_1+\cdots+y_n}}{\left(\frac{x_i}{x_n} \frac{aq}{e_i}\right)_{y_1+\cdots+y_n}} \right] \prod_{r,s=1}^{n} \left[\frac{\left(\frac{x_r}{x_s} e_s\right)_{y_r}}{\left(q \frac{x_r}{x_s}\right)_{y_r}} \right]$$

$$\times \prod_{1\le r<s\le n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]q^{y_1+2y_2+\cdots+ny_n}\Bigg\} \tag{A.1a}$$

$$= \frac{\left(\frac{aq}{e_1\cdots e_n f}\right)_N\left(\frac{\lambda q}{f}\right)_N}{\left(\frac{\lambda q}{e_1\cdots e_n f}\right)_N\left(\frac{aq}{f}\right)_N}\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}aq\right)_N\left(\frac{x_i}{x_n}\frac{\lambda q}{e_i}\right)_N}{\left(\frac{x_i}{x_n}\lambda q\right)_N\left(\frac{x_i}{x_n}\frac{aq}{e_i}\right)_N}\right] \tag{A.1b}$$

$$\times \sum_{\substack{y_1,\dots,y_n\ge 0\\ 0\le y_1+\cdots+y_n\le N}}\Bigg\{\prod_{i=1}^{n}\left[\frac{1-\frac{x_i}{x_n}\lambda q^{y_i+(y_1+\cdots+y_n)}}{1-\frac{x_i}{x_n}\lambda}\right]$$

$$\times\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}\frac{\lambda d}{a}\right)_{y_i}\left(\frac{x_i}{x_n}f\right)_{y_i}\left(\frac{x_i}{x_n}\frac{\lambda a}{e_1\cdots e_n f}q^{1+N}\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{aq}{c}\right)_{y_i}\left(\frac{x_i}{x_n}\lambda q^{1+N}\right)_{y_i}}\right]$$

$$\times\left[\frac{\left(\frac{\lambda b}{a}\right)_{y_1+\cdots+y_n}\left(\frac{\lambda c}{a}\right)_{y_1+\cdots+y_n}\left(q^{-N}\right)_{y_1+\cdots+y_n}}{\left(\frac{aq}{d}\right)_{y_1+\cdots+y_n}\left(\frac{\lambda q}{f}\right)_{y_1+\cdots+y_n}\left(\frac{e_1\cdots e_n f}{a}q^{-N}\right)_{y_1+\cdots+y_n}}\right]$$

$$\times\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}\lambda\right)_{y_1+\cdots+y_n}}{\left(\frac{x_i}{x_n}\frac{\lambda q}{e_i}\right)_{y_1+\cdots+y_n}}\right]\prod_{r,s=1}^{n}\left[\frac{\left(\frac{x_r}{x_s}e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\right]$$

$$\times \prod_{1\le r<s\le n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]q^{y_1+2y_2+\cdots+ny_n}\Bigg\}, \tag{A.1c}$$

where $\lambda = qa^2/bcd$.

Remark. The $n=1$, $N=n$, $e_1\mapsto e$ case of (A.1) is (III.28) on page 363 of [23].

Our two $\mathrm{U}(n+1)$ nonterminating q-Saalschütz transformations in Sect. 5 arise by applying two $\mathrm{U}(n+1)$ nonterminating ${}_6\phi_5$ summation theorems from [48] and [53] to special cases of our two $\mathrm{U}(n+1)$ nonterminating q-Whipple transformations in Sect. 4.

We illustrate a simple case of letting $N\to\infty$, while using the Dominated Convergence Theorem, in a terminating multiple q-series, summed over a "triangle" or "n-simplex," by deriving the above two $\mathrm{U}(n+1)$ nonterminating ${}_6\phi_5$ summation theorems of [48] and [53] from the following $\mathrm{U}(n+1)$ q-Dougall summation formula from [50, 51].

Theorem A.2 (First $\mathrm{U}(n+1)$ q-Dougall summation formula). *Let a, b, c, e, $d_1,\dots,d_n$, and $x_1,\dots,x_n$ be indeterminate, N be a nonnegative integer, $n\ge 1$, and suppose that none of the denominators in* (A.2) *vanish. Then*

$$\sum_{\substack{y_1,\dots,y_n\geq 0\\ 0\leq y_1+\cdots+y_n\leq N}} \Bigg\{ \prod_{i=1}^{n}\left[\frac{1-\frac{x_i}{x_n}aq^{y_i+(y_1+\cdots+y_n)}}{1-\frac{x_i}{x_n}a}\right]$$

$$\times\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}b\right)_{y_i}\left(\frac{x_i}{x_n}e\right)_{y_i}}{\left(\frac{x_i}{x_n}aq^{1+N}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{aq}{c}\right)_{y_i}}\right]$$

$$\times\left[\frac{\left(q^{-N}\right)_{y_1+\cdots+y_n}(c)_{y_1+\cdots+y_n}}{\left(\frac{aq}{b}\right)_{y_1+\cdots+y_n}\left(\frac{aq}{e}\right)_{y_1+\cdots+y_n}}\right]$$

$$\times\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}a\right)_{y_1+\cdots+y_n}}{\left(\frac{x_i}{x_n}\frac{aq}{d_i}\right)_{y_1+\cdots+y_n}}\right]\prod_{r,s=1}^{n}\left[\frac{\left(\frac{x_r}{x_s}d_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\right]$$

$$\times\prod_{1\leq r<s\leq n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]q^{y_1+2y_2+\cdots+ny_n}\Bigg\} \tag{A.2a}$$

$$=\frac{\left(\frac{aq}{ce}\right)_N\left(\frac{aq}{d_1\cdots d_ne}\right)_N}{\left(\frac{aq}{e}\right)_N\left(\frac{aq}{cd_1\cdots d_ne}\right)_N}\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}aq\right)_N\left(\frac{x_i}{x_n}\frac{aq}{cd_i}\right)_N}{\left(\frac{x_i}{x_n}\frac{aq}{c}\right)_N\left(\frac{x_i}{x_n}\frac{aq}{d_i}\right)_N}\right], \tag{A.2b}$$

where $\frac{aq}{bc}=\frac{d_1\cdots d_ne}{a}q^{-N}$.

Remark. The $n=1$, $N=n$, $d_1\mapsto b$, $e\mapsto d$, $b\mapsto e$ case of (A.2) is (II.22) on page 356 of [23].

Remark. Theorem A.2, with different notation, has already appeared in [50, 51]. In particular, rewriting Theorem 6.17 of [50] and [51] by replacing n by $n+1$, m by N, and z_n/z_{n+1} by a, making the substitutions $z_i=x_i$ and q^{-N_i} by d_i, for $i=1,2,\dots,n$, and then taking $b_{n+1,n+2}=q/b$, $a_{n+1,n+1}=c/a$, and $a_{n+1,n+2}=e/a$, gives Theorem A.2. Moreover, Theorem A.2, written with the same notation as (A.2), has already been applied as Theorems A5 and A12 of [61] and [4], respectively.

We now obtain the first $\mathrm{U}(n+1)$ nonterminating ${}_6\phi_5$ summation theorem. First, set $b=\frac{a^2q^{1+N}}{cd_1\cdots d_ne}$ in (A.2), and then make the substitutions $c\mapsto b$, $e\mapsto f$, and $d_i\mapsto e_i$, for $i=1,2,\dots,n$. Next, as described in Appendix B, letting $N\to\infty$, while using the Dominated Convergence Theorem, yields the following theorem.

Theorem A.3 (First $\mathrm{U}(n+1)$ nonterminating ${}_6\phi_5$ summation theorem). *Let* a,b,f, $e_1,\dots,e_n$, *and* $x_1,\dots,x_n$ *be indeterminate, with* $n\geq 1$. *Take* $0<|q|<1$ *and*

$$\left|\frac{aq}{be_1\cdots e_nf}\right|<1. \tag{A.3}$$

Suppose that none of the denominators in (A.4) *vanish. Then*

$$\sum_{y_1,\ldots,y_n\geq 0}\left\{\prod_{i=1}^{n}\left[\frac{1-\frac{x_i}{x_n}aq^{y_i+(y_1+\cdots+y_n)}}{1-\frac{x_i}{x_n}a}\right]\right.$$

$$\times\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}f\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i}}\right]\left[\frac{(b)_{y_1+\cdots+y_n}}{\left(\frac{aq}{f}\right)_{y_1+\cdots+y_n}}\right]$$

$$\times\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}a\right)_{y_1+\cdots+y_n}}{\left(\frac{x_i}{x_n}\frac{aq}{e_i}\right)_{y_1+\cdots+y_n}}\right]\prod_{r,s=1}^{n}\left[\frac{\left(\frac{x_r}{x_s}e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\right]$$

$$\times\prod_{1\leq r<s\leq n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]$$

$$\text{(A.4a)}\qquad \left.\times\left(\frac{a}{be_1\cdots e_nf}\right)^{y_1+\cdots+y_n} q^{y_1+2y_2+\cdots+ny_n}\right\}$$

$$\text{(A.4b)}\qquad =\frac{\left(\frac{aq}{bf}\right)_\infty\left(\frac{aq}{e_1\cdots e_nf}\right)_\infty}{\left(\frac{aq}{be_1\cdots e_nf}\right)_\infty\left(\frac{aq}{f}\right)_\infty}\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}\frac{aq}{be_i}\right)_\infty\left(\frac{x_i}{x_n}aq\right)_\infty}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_\infty\left(\frac{x_i}{x_n}\frac{aq}{e_i}\right)_\infty}\right].$$

Remark. The $n=1$ and $e_1\mapsto b$, $b\mapsto c$, and $f\mapsto d$ case of (A.4) is the classical nonterminating ${}_6\phi_5$ summation in (II.20) on page 356 of [23].

Remark. Theorem A.3, with different notation, has already appeared in [48]. In particular, rewriting Theorem 1.44 of [48] by replacing n by $n+1$ and z_n/z_{n+1} by a, making the substitutions $z_i=x_i$ and $a_{ii}=e_i$, for $i=1,2,\ldots,n$, and then taking $a_{n+1,n+1}=f/a$, gives Theorem A.3. Moreover, Theorem A.3, written with the same notation as (A.4), with $n\mapsto\ell$, $b\mapsto c$, $f\mapsto d$, and $e_i\mapsto b_i$, for $i=1,2,\ldots,n$, has already been applied as Theorem 6.3 of [60].

We next obtain the second $\mathrm{U}(n+1)$ nonterminating ${}_6\phi_5$ summation theorem. First, set $c=\frac{a^2q^{1+N}}{bd_1\cdots d_ne}$ in (A.2) and then make the substitutions $e\mapsto f$, and $d_i\mapsto e_i$, for $i=1,2,\ldots,n$. Next, as described in Appendix B, letting $N\to\infty$, while using the Dominated Convergence Theorem, yields the following theorem.

Theorem A.4 (Second $\mathrm{U}(n+1)$ nonterminating ${}_6\phi_5$ summation theorem). *Let a,b,f, $e_1,\ldots,e_n$, and $x_1,\ldots,x_n$ be indeterminate, with $n\geq 1$. Take $0<|q|<1$ and*

$$\text{(A.5)}\qquad \left|\frac{x_n}{x_i}\frac{aq}{be_1\cdots e_nf}\right|<1,\quad \mathit{for}\ \ i=1,2,\ldots,n.$$

Suppose that none of the denominators in (A.6) *vanish. Then*

$$\sum_{y_1,\dots,y_n\geq 0} \left\{ \prod_{i=1}^{n} \left[\frac{1-\frac{x_i}{x_n} a q^{y_i+(y_1+\cdots+y_n)}}{1-\frac{x_i}{x_n} a} \right] \right.$$

$$\times \left[\left(\frac{aq}{b}\right)_{y_1+\cdots+y_n} \left(\frac{aq}{f}\right)_{y_1+\cdots+y_n} \right]^{-1} \prod_{i=1}^{n} \left[\left(\frac{x_i}{x_n} b\right)_{y_i} \left(\frac{x_i}{x_n} f\right)_{y_i} \right]$$

$$\times \prod_{i=1}^{n} \left[\frac{\left(\frac{x_i}{x_n} a\right)_{y_1+\cdots+y_n}}{\left(\frac{x_i}{x_n}\frac{aq}{e_i}\right)_{y_1+\cdots+y_n}} \right] \prod_{r,s=1}^{n} \left[\frac{\left(\frac{x_r}{x_s} e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}} \right]$$

$$\times \prod_{1\leq r<s\leq n} \left[\frac{1-\frac{x_r}{x_s} q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \right] \left(\frac{a}{b e_1 \cdots e_n f} \right)^{y_1+\cdots+y_n} q^{e_2(y_1,\dots,y_n)}$$

$$\left. \times q^{y_1+2y_2+\cdots+ny_n} \prod_{i=1}^{n} \left(\frac{x_n}{x_i}\right)^{y_i} \right\} \tag{A.6a}$$

$$= \frac{\left(\frac{aq}{be_1\cdots e_n}\right)_\infty \left(\frac{aq}{e_1\cdots e_n f}\right)_\infty}{\left(\frac{aq}{b}\right)_\infty \left(\frac{aq}{f}\right)_\infty} \prod_{i=1}^{n} \left[\frac{\left(\frac{x_n}{x_i}\frac{aqe_i}{be_1\cdots e_n f}\right)_\infty \left(\frac{x_i}{x_n} aq\right)_\infty}{\left(\frac{x_n}{x_i}\frac{aq}{be_1\cdots e_n f}\right)_\infty \left(\frac{x_i}{x_n}\frac{aq}{e_i}\right)_\infty} \right], \tag{A.6b}$$

where $e_2(y_1,\dots,y_n)$ is the second elementary symmetric function of $\{y_1,\dots,y_n\}$.

Remark. The $n=1$ and $e_1 \mapsto b$, $b \mapsto c$, and $f \mapsto d$ case of (A.6) is the classical nonterminating ${}_6\phi_5$ summation in (II.20) on page 356 of [23].

Remark. Theorem A.4, with different notation, has already appeared in [53]. In particular, rewriting Theorem 4.27 of [53] by replacing n by $n+1$, and z_n/z_{n+1} by a, making the substitutions $z_i = x_i$ and $a_{ii} = e_i$, for $i = 1, 2, \dots, n$, and then taking $a_{n+1,n+1} = f/a$ and $a_{n+1,n+2} = b/a$, gives Theorem A.4.

Remark. The plane partition enumeration work of Krattenthaler [38] and Gessel and Krattenthaler [24] suggests that the following direct special cases (which we do not write down here) of Theorems 4.1, 4.2, A.3, A.4, 5.1, and 5.2 may be of interest. They are obtained by setting $e_s = q$ in the first four theorems and $c_s = q$ in the last two.

Appendix B. Convergence of the (Multiple) (q-)Series

In this appendix we discuss and prove the absolute convergence of the nonterminating summations in this paper. Similar convergence proofs for multiple (basic) hypergeometric series have been given (in alphabetical order) by van Diejen

[16, Appendix], Milne [46, pp. 54–55], [47, pp. 269–270], [48, pp. 234–237, 242], [49, pp. 83–85], [56, pp. 132–134], Milne and Newcomb [61, pp. 274–276], and Schlosser [75, Appendix C, pp. 455–458].

Tannery's Theorem for Sums [7, pp. 136–138] is sufficient for the simpler convergence arguments. However, the Dominated Convergence Theorem (see [6, Theorem 6.19 on pp. 183] and [20, Theorem 2.24 on pp. 54, and Exercise 22 on pp. 59]) is more suitable and powerful for our analysis here.

We first complete the proof of Theorem 4.1 by showing that the sums (4.3), (4.14), and (4.16) become (4.2a), (4.2c), and (4.17), respectively, as $N \to \infty$. The following analysis is necessary for $n \geq 2$, while the $n = 1$ case is much simpler.

Our main analysis consists of four steps. First, observe that as $N \to \infty$, the termwise limits of the first three sums equals the terms, respectively, in the second three sums. Second, establish upper bounds (that are independent of N) of the absolute values of the terms in the first three sums. (This is done *before* taking $N \to \infty$, and utilizes the conditions on N that determine the region of (finite) summation of these sums.) Third, apply the "multiple power series ratio test" in Lemma B.1 to show that the "upper bound" sums are absolutely convergent. Finally, appeal to the Dominated Convergence Theorem. The convergence conditions in Theorem 4.1 are the same as for the upper bound sums.

Establishing that (4.14) becomes (4.2c), as $N \to \infty$, is much more subtle. This requires the additional step of multiplying out the Vandermonde determinate as in (B.4), interchanging summation, and carrying out a much more delicate four-step Dominated Convergence Theorem analysis of each of the $n!$ inner multiple sums corresponding to the permutations σ of $\mathcal{S}_n$. We then apply (B.4) in reverse to get the Vandermonde back and finally obtain the sum in (4.2c). (Our analysis also yields a simpler (but less sharp) upper bound for the sum (4.14), while avoiding applying (B.4) in reverse.) This situation here is much deeper than the previous convergence proofs for multiple (basic) hypergeometric series.

To carry out the above analysis we first need the multiple power series ratio test.

Lemma B.1 (The multiple power series ratio test). *Given*

$$\sum_{y_1,\ldots,y_n \geq 0} f(y_1,\ldots,y_n), \tag{B.1}$$

set

$$g_m(y_1,\ldots,y_n) = \left| \frac{f(y_1,\ldots,y_{m-1},y_m+1,y_{m+1},\ldots,y_n)}{f(y_1,\ldots,y_n)} \right|, \tag{B.2}$$

for $m = 1,\ldots,n$. Then, if

$$\lim_{\varepsilon \to \infty} g_m(\varepsilon y_1,\ldots,\varepsilon y_n) < 1, \tag{B.3}$$

for $m = 1,\ldots,n$, the multiple sum in (B.1) converges absolutely.

Remark. The multiple power series ratio test may be found in [1, 30, 43].

A useful technique appears in the proof of Theorem 5.6 in [56].

Obtaining the sum (4.2c) requires the Vandermonde determinant expansion

$$q^{y_1+2y_2+\cdots+ny_n} \prod_{1\le r<s\le n}\left[1-\frac{x_r}{x_s}q^{y_r-y_s}\right] \tag{B.4a}$$

$$= \prod_{i=1}^{n} x_i^{1-i} \sum_{\sigma\in\mathcal{S}_n} \epsilon(\sigma) \prod_{i=1}^{n} x_{\sigma(i)}^{i-1} \prod_{i=1}^{n} q^{\sigma^{-1}(i)y_i}, \tag{B.4b}$$

where $\epsilon(\sigma)$ is the sign of the permutation σ.

Unless otherwise noted, we take $0<|q|<1$.

In obtaining our various (term-wise) upper bounds we make frequent use of (B.5)–(B.9) below:

$$\lim_{N\to\infty} \left(Aq^N\right)_m = 1, \tag{B.5}$$

where A and m are independent of N.

Equation (B.5) also implies that

$$C_1 \le \left|\left(Aq^N\right)_m\right| \le C_2, \tag{B.6}$$

where A and m are independent of N, A is independent of m, and C_1 and C_2 are positive constants independent of N and m.

We also have

$$C_3 \le \left|\left(Aq^{N-m}\right)_m\right| \le C_4, \tag{B.7}$$

where $N-m\ge 0$, and the same conditions as in (B.6) apply.

The triangle inequality immediately gives

$$\prod_{1\le r<s\le n}\left|\left[q^{y_s}-\frac{x_r}{x_s}q^{y_r}\right]\right| \le \prod_{1\le r<s\le n}\left[1+\left|\frac{x_r}{x_s}\right|\right] \tag{B.8}$$

and

$$\prod_{i=1}^{n}\left|\left[1-\frac{x_i}{x_n}A_i q^{y_i+(y_1+\cdots+y_n)}\right]\right| \le \prod_{i=1}^{n}\left[1+\left|\frac{x_i}{x_n}A_i\right|\right], \tag{B.9}$$

where the $\{y_1,\ldots,y_n\}$ are nonnegative integers. Note that the right-hand sides of (B.8) and (B.9) are independent of $\{y_1,\ldots,y_n\}$. In particular, we also take A_i independent of $\{y_1,\ldots,y_n\}$.

As we are using the Dominated Convergence Theorem, we only need to apply Lemma B.1 to our various upper bound sums, whose terms and (infinite) region of summation are independent of N.

It is easy to see that the following factors (B.10) of f in (B.1) contribute 1 to the limit in (B.3).

$$\prod_{i=1}^{n}\left[\frac{1-\frac{x_i}{x_n}aq^{y_i+(y_1+\cdots+y_n)}}{1-\frac{x_i}{x_n}a}\right],\ \prod_{i=1}^{n}(a)_{y_i},\ \prod_{r,s=1}^{n}(a)_{y_r},\ \text{or}\ (\mathrm{a})_{\mathrm{y_1+\cdots+y_n}}. \tag{B.10}$$

The expressions (including powers of q) in the "evaluation point" of our various upper bound sums yield our convergence conditions.

We are now ready to finish the proof of Theorem 4.1.

We begin by showing that (4.3) becomes (4.2a) as $N \to \infty$.

Apply (2.4) to the two products in (4.3c) that involve q^{-N}. We obtain

$$\frac{\left(q^{-N}\right)_{y_1+\cdots+y_n}}{\left(\frac{bcde_1\cdots e_nf}{qa^2}q^{-N}\right)_{y_1+\cdots+y_n}}=\frac{\left(q^{1+N-(y_1+\cdots+y_n)}\right)_{y_1+\cdots+y_n}}{\left(\frac{q^{2+N-(y_1+\cdots+y_n)}a^2}{bcde_1\cdots e_nf}\right)_{y_1+\cdots+y_n}}\left(\frac{a^2q}{bcde_1\cdots e_nf}\right)^{y_1+\cdots+y_n}. \tag{B.11}$$

It now follows from (B.11) and (B.5), for fixed $\{y_1,\ldots,y_n\}$, that the terms of (4.3) become the terms of (4.2a), as $N\to\infty$.

Keeping in mind

$$q^{y_1+2y_2+\cdots+ny_n}=q^{y_1+\cdots+y_n}\cdot q^{y_2+2y_3+\cdots+(n-1)y_n} \tag{B.12}$$

and appealing to (B.8), (B.9), then (B.11), (B.6), (B.7), and the summation condition $0\le y_1+\cdots+y_n\le N$ in (4.3a), the absolute value of the sum (4.3) is bounded by

$$\begin{aligned}M_1\ \Bigg\{\sum_{y_1,\ldots,y_n\ge 0}\Bigg|&\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}d\right)_{y_i}\left(\frac{x_i}{x_n}f\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i}\left(\frac{x_i}{x_n}\frac{aq}{c}\right)_{y_i}}\right]\\
&\times\frac{(b)_{y_1+\cdots+y_n}(c)_{y_1+\cdots+y_n}}{\left(\frac{aq}{d}\right)_{y_1+\cdots+y_n}\left(\frac{aq}{f}\right)_{y_1+\cdots+y_n}}\\
&\times\prod_{i=1}^{n}\left[\frac{\left(\frac{x_i}{x_n}a\right)_{y_1+\cdots+y_n}}{\left(\frac{x_i}{x_n}\frac{aq}{e_i}\right)_{y_1+\cdots+y_n}}\right]\prod_{r,s=1}^{n}\left[\frac{\left(\frac{x_r}{x_s}e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\right]\\
&\times\left(\frac{a^2q^2}{bcde_1\cdots e_nf}\right)^{y_1+\cdots+y_n}\Bigg|\ \Bigg\},\end{aligned} \tag{B.13}$$

where M_1 is a positive constant independent of $\{y_1,\dots,y_n\}$ and N. Note that we first obtained the bound (B.13), with a *finite* sum over $y_1,\dots,y_n \geq 0$ and $0 \leq y_1 + \cdots + y_n \leq N$. Adding additional nonnegative terms by removing the condition $0 \leq y_1 + \cdots + y_n \leq N$ then gave the final infinite upper bound sum in (B.13).

By Lemma B.1, the upper bound sum (B.13) converges absolutely when $0 < |q| < 1$ and

$$\left| \frac{a^2q^2}{bcde_1\cdots e_n f} \right| < 1. \tag{B.14}$$

From the Dominated Convergence Theorem, the sum (4.3) now becomes (4.2a) as $N \to \infty$.

In order to show that (4.14) becomes (4.2c) as $N \to \infty$, we first apply (B.4) to expand $q^{y_1+2y_2+\cdots+ny_n}$ times the Vandermonde determinate in the sum (4.14). An interchange of summation then gives

$$\prod_{i=1}^{n} x_i^{1-i} \sum_{\sigma\in\mathcal{S}_n} \epsilon(\sigma) \prod_{i=1}^{n} x_{\sigma(i)}^{i-1} \sum_{\substack{0\leq y_1,\dots,y_n\leq N/2 \\ 0\leq y_1+\cdots+y_n\leq N}} \left[\prod_{i=1}^{n} q^{\sigma^{-1}(i) y_i} \right] F(y_1,\dots,y_n,N), \tag{B.15}$$

where $\epsilon(\sigma)$ is the sign of the permutation σ, and $F(y_1,\dots,y_n,N)$ is all the factors of the term in (4.14) except for (B.4a).

From an interchange of summation and then (B.4), we will be done once we show, as $N \to \infty$, that the inner multiple sum

$$\sum_{\substack{0\leq y_1,\dots,y_n\leq N/2 \\ 0\leq y_1+\cdots+y_n\leq N}} \left[\prod_{i=1}^{n} q^{\sigma^{-1}(i) y_i} \right] F(y_1,\dots,y_n,N) \tag{B.16}$$

in (B.15) becomes

$$\sum_{y_1,\dots,y_n\geq 0} \left[\prod_{i=1}^{n} q^{\sigma^{-1}(i) y_i} \right] G(y_1,\dots,y_n), \tag{B.17}$$

where $\epsilon(\sigma)$ is the sign of the permutation σ, and $G(y_1,\dots,y_n)$ is all the factors of the term in (4.2c) except for (B.4a).

It follows from (B.5), for fixed $\{y_1,\dots,y_n\}$, that $F(y_1,\dots,y_n,N)$ in (B.16) becomes $G(y_1,\dots,y_n)$ in (B.17), as $N \to \infty$.

In order to obtain our upper bound sum for (B.16), we first factor

$$\prod_{i=1}^{n} \left[1 - \frac{x_n}{x_i} \frac{aq}{de_1\cdots e_n f} q^{N-(y_i+(y_1+\cdots+y_n))} \right] \tag{B.18}$$

from $F(y_1,\ldots,y_n,N)$ and then write (B.16) as

$$\text{(B.19a)} \qquad \sum_{\substack{0\le y_1,\ldots,y_n\le N/2\\ 0\le y_1+\cdots+y_n\le N}} \left[\prod_{i=1}^{n} q^{\sigma^{-1}(i)y_i}\right]$$

$$\text{(B.19b)} \qquad \prod_{i=1}^{n}\left[1-\frac{x_n}{x_i}\frac{aq}{de_1\cdots e_nf}q^{N-(y_i+(y_1+\cdots+y_n))}\right]$$

$$\text{(B.19c)} \qquad H(y_1,\ldots,y_n,N),$$

where $H(y_1,\ldots,y_n,N)$ is all the factors of $F(y_1,\ldots,y_n,N)$ except (B.18).

For each $\sigma\in\mathcal{S}_n$, rewrite the factor in (B.19a) as

$$\text{(B.20a)} \qquad \prod_{i=1}^{n} q^{\sigma^{-1}(i)y_i}$$

$$\text{(B.20b)} \qquad = q^{\frac{1}{2}y_k}\prod_{\substack{i=1\\ i\neq k}}^{n} q^{(\sigma^{-1}(i)-\frac{3}{2})y_i}$$

$$\text{(B.20c)} \qquad \times q^{\frac{1}{2}(y_1+\cdots+y_n)}\prod_{\substack{i=1\\ i\neq k}}^{n} q^{y_i},$$

where $\sigma^{-1}(k)=1$.

It is not hard to see that the product of (B.18) and (B.20c) is

$$\text{(B.21a)} \qquad \left(q^{\frac{1}{2}(y_1+\cdots+y_n)}-\frac{x_n}{x_k}\frac{aq}{de_1\cdots e_nf}q^{N-y_k-\frac{1}{2}(y_1+\cdots+y_n)}\right)$$

$$\text{(B.21b)} \qquad \times\prod_{\substack{i=1\\ i\neq k}}^{n}\left[q^{y_i}-\frac{x_n}{x_i}\frac{aq}{de_1\cdots e_nf}q^{N-(y_1+\cdots+y_n)}\right],$$

where $\sigma^{-1}(k)=1$.

The indices of summation conditions $0\le y_1,\ldots,y_n\le N/2$ and $0\le y_1+\cdots+y_n\le N$ in (B.19a) imply that all of the powers of q in (B.21) are nonnegative. Just note that $y_i\ge 0$, $N-(y_1+\cdots+y_n)\ge 0$, $\frac{1}{2}(y_1+\cdots+y_n)\ge 0$, and $N-y_k-\frac{1}{2}(y_1+\cdots+y_n)=(N/2-y_k)+(\frac{1}{2}N-\frac{1}{2}(y_1+\cdots+y_n))\ge 0$.

The triangle inequality and $0<|q|<1$ then immediately gives that the absolute value of (B.21) is bounded by

$$\text{(B.22)} \qquad \prod_{i=1}^{n}\left[1+\left|\frac{x_n}{x_i}\frac{aq}{de_1\cdots e_nf}\right|\right].$$

Keeping in mind (B.19)–(B.22), (B.20b), and the summation conditions in (B.19a), applying (B.6) and (B.7) to the factors of $H(y_1,\ldots,y_n,N)$ in (B.19c) involving q^N, it follows that the absolute value of the sum (B.16) is bounded by

$$M_2 \left\{ \sum_{y_1,\ldots,y_n \geq 0} \left| \prod_{i=1}^{n} \left[\frac{\left(\frac{x_i}{x_n}\frac{aq}{bc}\right)_{y_i} \left(\frac{x_i}{x_n}d\right)_{y_i} \left(\frac{x_i}{x_n}f\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i} \left(\frac{x_i}{x_n}\frac{aq}{c}\right)_{y_i} \left(\frac{x_i}{x_n}\frac{de_1\cdots e_n f}{a}\right)_{y_i}} \right] \right. \right.$$

$$\text{(B.23)} \qquad \left. \left. \times \prod_{r,s=1}^{n} \left[\frac{\left(\frac{x_r}{x_s}e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}} \right] q^{\frac{1}{2}y_k} \prod_{\substack{i=1\\ i\neq k}}^{n} q^{(\sigma^{-1}(i)-\frac{3}{2})y_i} \right| \right\},$$

where $\sigma \in \mathcal{S}_n$, $\sigma^{-1}(k) = 1$, and M_2 is a positive constant independent of $\{y_1,\ldots,y_n\}$ and N. Note that we first obtained the bound (B.23), with a *finite* sum over $0 \leq y_1,\ldots,y_n \leq N/2$ and $0 \leq y_1+\cdots+y_n \leq N$. Adding additional nonnegative terms by removing the conditions $y_1,\ldots,y_n \leq N/2$ and $0 \leq y_1+\cdots+y_n \leq N$ then gave the final infinite upper bound sum in (B.23).

By Lemma B.1, the upper bound sum (B.23) converges absolutely when $0 < |q| < 1$. Just observe that $\sigma^{-1}(i) \geq 2$, for $i = 1,2,\ldots,n$, and $i \neq k$.

From the Dominated Convergence Theorem, the sum (B.16) now becomes (B.17) as $N \to \infty$.

Remark. The above proof that (4.14) becomes (4.2c), as $N \to \infty$, can be presented in a slightly different form. First, it follows from (B.5), for fixed $\{y_1,\ldots,y_n\}$, that the terms in (4.14) become those in (4.2c), as $N \to \infty$. From (B.15), (B.16), (B.23), and an interchange of summation, we have that the absolute value of the sum (4.14) is bounded by

$$M_3 \left\{ \sum_{y_1,\ldots,y_n \geq 0} \left| \prod_{i=1}^{n} \left[\frac{\left(\frac{x_i}{x_n}\frac{aq}{bc}\right)_{y_i} \left(\frac{x_i}{x_n}d\right)_{y_i} \left(\frac{x_i}{x_n}f\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i} \left(\frac{x_i}{x_n}\frac{aq}{c}\right)_{y_i} \left(\frac{x_i}{x_n}\frac{de_1\cdots e_n f}{a}\right)_{y_i}} \right] \right. \right.$$

$$\text{(B.24)} \qquad \left. \left. \times \prod_{r,s=1}^{n} \left[\frac{\left(\frac{x_r}{x_s}e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}} \right] q^{\frac{1}{2}(y_1+\cdots+y_n)} \right| \right\},$$

where M_3 is a positive constant independent of $\{y_1,\ldots,y_n\}$ and N. Just observe, for each $0 < |q| < 1$, that $|q|^x$ is a decreasing function of x, and $\sigma^{-1}(i) \geq 2$, for $i = 1,2,\ldots,n$, and $i \neq k$. Appealing to Lemma B.1 and the Dominated Convergence Theorem completes the proof.

Remark. If, in deriving an upper bound for (4.14), we do not first expand $q^{y_1+2y_2+\cdots+ny_n}$ times the Vandermonde determinate as in (B.15), but instead, keeping in mind (B.12), multiply $q^{y_1+\cdots+y_n}$ times (B.18), appeal to (B.8), then (B.6), (B.7), the expression (B.22), and the summation conditions $0 \leq y_1, \ldots, y_n \leq N/2$ and $0 \leq y_1 + \cdots + y_n \leq N$ in (4.14a), we would attempt to bound the absolute value of the sum (4.14) by

$$M_4 \left\{ \sum_{y_1,\ldots,y_n \geq 0} \left| \prod_{i=1}^{n} \left[\frac{\left(\frac{x_i}{x_n}\frac{aq}{bc}\right)_{y_i} \left(\frac{x_i}{x_n}d\right)_{y_i} \left(\frac{x_i}{x_n}f\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i} \left(\frac{x_i}{x_n}\frac{aq}{c}\right)_{y_i} \left(\frac{x_i}{x_n}\frac{de_1\cdots e_n f}{a}\right)_{y_i}} \right] \times \prod_{r,s=1}^{n} \left[\frac{\left(\frac{x_r}{x_s}e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}} \right] \right| \right\},$$
(B.25)

where M_4 is a positive constant independent of $\{y_1, \ldots, y_n\}$ and N. However, applying Lemma B.1 to the sum (B.25) fails as the limits in (B.3) are all 1.

Remark. If, as a short-cut to the upper bound (B.24), we attempt to replace (B.20) by

$$\prod_{i=1}^{n} q^{\sigma^{-1}(i)y_i} \tag{B.26a}$$

$$= q^{\frac{1}{2}(y_1+\cdots+y_n)} \tag{B.26b}$$

$$\times \prod_{i=1}^{n} q^{(\sigma^{-1}(i)-\frac{1}{2})y_i}, \tag{B.26c}$$

it is immediate that the product of (B.18) and (B.26c) is

$$\prod_{i=1}^{n} \left[q^{(\sigma^{-1}(i)-\frac{1}{2})y_i} - \frac{x_n}{x_i}\frac{aq}{de_1\cdots e_n f} q^{N-(y_1+\cdots+y_n)} q^{(\sigma^{-1}(i)-\frac{3}{2})y_i} \right]. \tag{B.27}$$

The summation conditions $0 \leq y_1, \ldots, y_n \leq N/2$ and $0 \leq y_1 + \cdots + y_n \leq N$ in (4.14a) imply that all the powers of q in (B.27) are nonnegative, except for $(\sigma^{-1}(k) - \frac{3}{2})y_k = -\frac{1}{2}y_k$, where $\sigma^{-1}(k) = 1$. Moreover, for $n \geq 2$, and $y_1 = \cdots = y_n = y$, if $\frac{1}{n+1/2}N < y \leq \frac{N}{n}$, then $N - (y_1 + \cdots + y_n) - \frac{1}{2}y_k < 0$. For fixed $n \geq 2$, as $N \to \infty$, the number of such $\{y_1, \ldots, y_n\}$'s goes to infinity as well. An upper bound such as (B.22) for (B.27) does not hold, and the analysis here breaks down.

We now show that (4.16) becomes (4.17) as $N \to \infty$.

First, it follows from (B.5), for fixed $\{y_1, \ldots, y_n\}$, that the terms of (4.16) become the terms of (4.17), as $N \to \infty$. Note that we moved some resulting infinite products (that were independent of $\{y_1, \ldots, y_n\}$) outside the sum in (4.17).

In order to obtain an upper bound for (4.16), we begin by writing the power of q in (4.16l) as

(B.28a) $$q^{y_1+\cdots+y_j} q^{y_1+2y_2+\cdots+(j-1)y_{j-1}+(j+1)y_{j+1}+\cdots+ny_n}$$

(B.28b) $$= q^{y_1+y_2+\cdots+y_{j-1}+y_{j+1}+\cdots+y_n}$$

(B.28c) $$\times q^{y_2+2y_3+\cdots+(j-2)y_{j-1}+(j-1)y_{j+1}+\cdots+(n-2)y_n}$$

(B.28d) $$\times q^{y_1+\cdots+y_n}.$$

The summation conditions $y_1,\ldots,y_n \geq 0$ and $0 \leq y_1+\cdots+y_n < N/2$ in (4.16a) imply that $0 \leq y_1,\ldots,y_n < N/2$. All the conditions $0 \leq y_1,\ldots,y_n < N/2$ and $0 \leq y_1+\cdots+y_n < N/2$ now give that the powers of q in the numerators of (4.16b) and (4.16i) are nonnegative. Furthermore, when multiplied by (B.28b) and (B.28c), respectively, the powers of q in the numerators of (4.16a) and (4.16j) are also nonnegative.

The triangle inequality immediately gives that the absolute value of these (revised) numerators is bounded by

(B.29a) $$\prod_{\substack{1\leq i\leq n\\ i\neq j}} \left[1+\left|\frac{x_n}{x_i}\frac{aq}{de_1\cdots e_n f}\right|\right] \times \left[1+\left|\frac{x_j}{x_n}\frac{de_1\cdots e_n f}{aq}\right|\right]$$

(B.29b) $$\times \prod_{\substack{1\leq i\leq n\\ i\neq j}} \left[1+\left|\frac{x_j}{x_i}\right|\right] \times \prod_{\substack{1\leq r<s\leq n\\ r,s\neq j}} \left[1+\left|\frac{x_r}{x_s}\right|\right].$$

Keeping in mind (B.28) and appealing to (B.29), then (B.6), and the summation conditions $0 \leq y_1,\ldots,y_n < N/2$ and $0 \leq y_1+\cdots+y_n < N/2$ from (4.16a), the absolute value of the sum (4.16) is bounded by

$$M_5 \left\{ \sum_{y_1,\ldots,y_n\geq 0} \left| \prod_{\substack{1\leq i\leq n\\ i\neq j}} \left[\frac{\left(\frac{x_i}{x_n}\frac{aq}{bc}\right)_{y_i} \left(\frac{x_i}{x_n}d\right)_{y_i} \left(\frac{x_i}{x_n}f\right)_{y_i}}{\left(\frac{x_i}{x_n}\frac{aq}{b}\right)_{y_i} \left(\frac{x_i}{x_n}\frac{aq}{c}\right)_{y_i} \left(\frac{x_i}{x_n}\frac{de_1\cdots e_n f}{a}\right)_{y_i}} \right] \right.\right.$$

$$\times \left[\frac{\left(\frac{a^2q^2}{bcde_1\cdots e_n f}\right)_{y_j} \left(\frac{aq}{de_1\cdots e_n}\right)_{y_j} \left(\frac{aq}{e_1\cdots e_n f}\right)_{y_j}}{\left(\frac{a^2q^2}{bde_1\cdots e_n f}\right)_{y_j} \left(\frac{a^2q^2}{cde_1\cdots e_n f}\right)_{y_j} (q)_{y_j}} \right] \times \prod_{i=1}^{n} \left[\frac{\left(\frac{x_n}{x_i}\frac{ae_i q}{de_1\cdots e_n f}\right)_{y_j}}{\left(\frac{x_n}{x_i}\frac{aq^2}{de_1\cdots e_n f}\right)_{y_j}} \right]$$

(B.30) $$\left.\left. \times \prod_{\substack{1\leq r,s\leq n\\ r\neq j}} \left[\frac{\left(\frac{x_r}{x_s}e_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}} \right] q^{y_1+\cdots+y_n} \right| \right\},$$

where M_5 is a positive constant independent of $\{y_1,\ldots,y_n\}$ and N. Note that we first obtained the bound (B.30), with a *finite* sum over $0 \le y_1,\ldots,y_n < N/2$ and $0 \le y_1+\cdots+y_n < N/2$. Adding additional nonnegative terms by removing the conditions $y_1,\ldots,y_n < N/2$ and $0 \le y_1+\cdots+y_n < N/2$ then gave the final infinite upper bound sum in (B.30).

By Lemma B.1, the upper bound sum (B.30) converges absolutely when $0 < |q| < 1$.

From the Dominated Convergence Theorem, the sum (4.16) now becomes (4.17) as $N \to \infty$.

The proof of Theorem 4.1 is now complete.

The application of Lemma B.1 and the Dominated Convergence Theorem to the proof of Theorem 4.2 is very similar to that of Theorem 4.1. We just need the (additional) simple bound

$$\left| q^{e_2(y_1,\ldots,y_n)} \right| \le 1, \tag{B.31}$$

where $e_2(y_1,\ldots,y_n)$ is the second elementary symmetric function of $\{y_1,\ldots,y_n\}$.

Our derivation of Theorem A.3 from Theorem A.2 by means of Lemma B.1 and the Dominated Convergence Theorem is very similar to the above proof that (4.3) becomes (4.2a) as $N \to \infty$. In particular, an analysis such as that in (B.11) through (B.14) yields the sum in (A.4a), while the infinite products in (A.4b) are immediate from the products in (A.2b).

The derivation of Theorem A.4 is similar to that of Theorem A.3. In obtaining the sum in (A.6a), we also need the relation (2.7) and the bound in (B.31). Apply (2.4), as needed, to the products in (A.2b), and simplify, before taking $N \to \infty$.

The convergence condition $0 < |q| < 1$ for the sums in Theorems 5.1 and 5.2 is a consequence of applying Lemma B.1 to suitable absolutely convergent upper bound sums, each with evaluation point $q^{y_1+\cdots+y_n}$.

For the sums in (5.1a) and (5.5a), keep in mind (B.12) and appeal to (B.8), to put together upper bound sums with evaluation point $q^{y_1+\cdots+y_n}$.

On the other hand, for the sums in (5.1c) and (5.5c), utilize (B.28), (B.28b), (B.28c), and the bounds

$$\prod_{\substack{1\le i\le n\\ i\ne j}} \left| \left[q^{y_i} - \frac{x_n}{x_i}\frac{1}{e} q^{1+y_j} \right] \right| \le \prod_{\substack{1\le i\le n\\ i\ne j}} \left[1 + \left| \frac{x_n}{x_i}\frac{1}{e} \right| \right] \tag{B.32}$$

and

$$\prod_{\substack{1\le r<s\le n\\ r,s\ne j}} \left| \left[q^{y_s} - \frac{x_r}{x_s} q^{y_r} \right] \right| \le \prod_{\substack{1\le r<s\le n\\ r,s\ne j}} \left[1 + \left| \frac{x_r}{x_s} \right| \right], \tag{B.33}$$

where the $\{y_1,\ldots,y_n\}$ are nonnegative integers, to arrive at suitable upper bound sums with evaluation point $q^{y_1+\cdots+y_n}$.

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