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London Mathematical Society Lecture Note Series. 324

Singularities and Computer Algebra

Edited by

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 2RU, UK

www.cambridge.org
Information on this title: www.cambridge.org/9780521683098

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First published 2006

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this book is available from the British Library

Library of Congress Cataloguing in Publication data

ISBN-13: 978-0-521-68309-8 paperback
ISBN-10: 0-521-68309-2 paperback

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Dedicated to
Gert-Martin Greuel
on the Occasion of
His 60^{th} Birthday

Prof. Dr. Gert-Martin Greuel

Contents

Preface

From 18 to 20 October 2004, a conference *"Singularities and Computer Algebra"* was held at the University of Kaiserslautern on the occasion of Gert-Martin Greuel's 60th birthday. It was attended by 70 participants from Europe, Israel, Japan, Canada and the U.S.A. We were particularly happy that Greuel's teacher, Egbert Brieskorn, was among them.

Most of the participants have been influenced by Greuel's work on singularities and their computational aspects over the last 30 years. Among them, one could find colleagues and friends from the early years in Göttingen and Bonn, but also former and present diploma and Ph.D. students of Gert-Martin Greuel at Kaiserslautern. In particular, each of the invited speakers could look retrospectively at cooperating in one way or another with Greuel.

The papers of this volume concern ten of the invited lectures, supplemented by four articles which are written by participants of the conference and focus on computational aspects. Most of the contributions are intended to give an overview on a particular aspect of singularities. They describe the development of important areas of singularity theory over the past years and they discuss open questions.

In the lead text, we include a list of the invited lectures and a list of the participants as well as a picture of the septic with 99 nodes found by Oliver Labs and Duco van Straten, which has acted as a logo for the conference. Further, we include an article focussing on *Aspects of Gert-Martin Greuel's Mathematical Work*.

We would like to thank all the people who have contributed to the sucess of the conference and to this volume.

Christoph Lossen and Gerhard Pfister
(Organizers of the Conference)

Invited Lectures

Helmut A. Hamm: *Depth and De Rham Cohomology*

Lê Dung Trang: *Singularity Invariants in Versal Deformations*

Eugenii Shustin: *The Patchworking Construction and Applications to Tropical Enumerative Geometry*

Ragnar-Olaf Buchweitz: *Free Divisors in the Representation Theory of Algebras*

Wolfgang Ebeling: *Monodromy*

Yuri A. Drozd: *Derived Categories of Modules and Coherent Sheaves*

Antonio Campillo: *Some Aspects and Applications of Singularities in Positive Characteristic*

Jonathan Wahl: *Topology, Geometry, and Equations of Normal Surface Singularities*

Ignacio Luengo: *Superisolated Singularities*

Kyoji Saito: *A Linearization Theorem of the Real Discriminants for Simple Singularities*

Charles T.C. Wall: *Transversality in families of mappings*

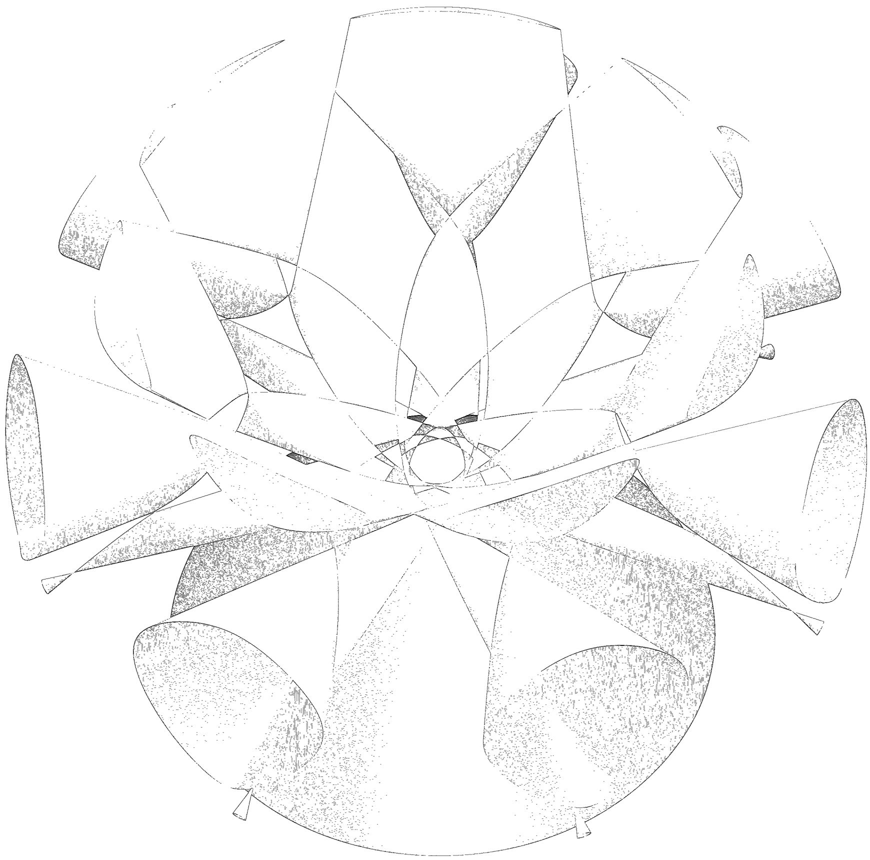
Joseph H.M. Steenbrink: *Adjunction Conditions for 1-Forms on Surfaces in Projective Three-Space*

Duco van Straten: *Lagrangian Singularities*

Bernard Teissier: *On the Structure of the Newton Polyhedra of Certain Discriminants*

Wolfram Decker: *SINGULAR and PLURAL*

Frank-Olaf Schreyer: *An Experimental Approach to Numerical Godeaux Surfaces*



A septic surface S in $\mathbb{P}^3(\mathbb{C})$ with 99 real nodes. It was discovered in 2004 by O. Labs and D. van Straten using Singular *experiments over small finite fields of prime order. If $\alpha \in \mathbb{C}$ satisfies $7\alpha^3 + 7\alpha + 1 = 0$, a defining equation for S over $\mathbb{Q}(\alpha)$ is the following:*

$$
\begin{aligned}
&(z + a_5 w)\Big((z+w)(x^2+y^2) + a_1 z^3 + a_2 z^2 w + a_3 z w^2 + a_4 w^3\Big)^2 \\
&\quad -x^7 + 21x^5y^2 - 35x^3y^4 + 7xy^6 - 7z\left(x^2+y^2\right)^3 + 56z^3\left(x^2+y^2\right)^2 \\
&\quad -112z^5\left(x^2+y^2\right) + 64z^7\,,
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &:= -\tfrac{12}{7}\alpha^2 - \tfrac{384}{49}\alpha - \tfrac{8}{7}, & a_2 &:= -\tfrac{32}{7}\alpha^2 + \tfrac{24}{49}\alpha - 4,\\
a_3 &:= -4\alpha^2 + \tfrac{24}{49}\alpha - 4, & a_4 &:= -\tfrac{8}{7}\alpha^2 + \tfrac{8}{49}\alpha - \tfrac{8}{7},\\
a_5 &:= 49\alpha^2 - 7\alpha + 50.
\end{aligned}
$$

List of Participants

Klaus ALTMANN, Freie Universität Berlin (Germany)
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Aspects of Gert-Martin Greuel's Mathematical Work

Christoph Lossen Gerhard Pfister

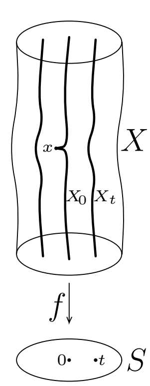
This article emanates from the opening speech of the conference *"Singularities and Computer Algebra"* which took place on October 18–20, 2004 on the occasion of Gert-Martin Greuel's 60th birthday. When preparing the speech, we realized soon that it is impossible to cover in such a speech Gert-Martin's complete work which is documented in more than eighty publications. Not to mention Gert-Martin's organisational work for the mathematical community.

We decided to illuminate only some cornerstones of Gert-Martin's mathematical work: his Ph.D.-Thesis in 1973, his Habilitationsschrift in 1979, the Singular project, the work on moduli spaces, and the work on equisingular families.

Ph.D.-Thesis (1973).

In his Diploma Thesis, titled *"Zur Picard-Lefschetz-Monodromie isolierter Singularitäten von vollständigen Durchschnitten"*, and his Ph.D.-Thesis, titled *"Der Gauß-Manin Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten"*, G.-M. Greuel develops the theory of the Gauß-Manin connection for isolated complete intersection singularities: Let $f : (X, x) \to (S, 0)$ be a map of germs with the following properties:

- X is an m-dimensional complete intersection,
- S is a k-dimensional complex manifold,
- f is flat,
- $x \in X_0 = f^{-1}(0)$ is an isolated singular point,
- the critical set C of f is of dimension $k-1$.



Then (X_0, x) is an isolated complete intersection singularity of dimension $n := m - k$. We introduce $S' := S \setminus D_f$ and $X' := X \setminus f^{-1}(D_f)$, where $D_f = f(C) \subset S$ denotes the discriminant of f.

By a result of Hamm (extending a result of Milnor), we may assume that the restriction $f: X' \to S'$ is a locally trivial differentiable fibre bundle whose fibres are homotopy equivalent to a bouquet of n-spheres. The number of spheres in the bouquet, which equals $\dim_{\mathbb{C}} H^n(X_t, \mathbb{C})$, is called the *Milnor number* of the complete intersection germ (X_0, x) and denoted by $\mu(X_0, x)$.

The fibration $f: X' \to S'$ induces a vector bundle with fibre $H^n(X_t, \mathbb{C})$ over $t \in S'$. Its sheaf of holomorphic sections, $\mathcal{H}^n = R^n f_* \mathbb{C}_{X'} \otimes_{\mathbb{C}_{S'}} \mathcal{O}_{S'}$, has a canonical integrable connection $\nabla: \mathcal{H}^n \to \mathcal{H}^n \otimes_{\mathcal{O}_{S'}} \Omega^1_{S'}$, $\omega \otimes f \mapsto \omega \otimes df$. The monodromy of this connection is the Picard-Lefschetz monodromy of f in x, $\rho_{\mathbb{C}}: \pi_1(S', t) \to \mathrm{Aut}(H^n(X_t, \mathbb{C}))$, induced by the action of $\pi_1(S', t)$ on $H^n(X_t, \mathbb{Z})$. Greuel's main result is now the following theorem (extending Brieskorn's result for hypersurfaces):

Theorem. *The connection ∇ on S' can be extended to a (meromorphic) regular singular connection on S, the Gauß-Manin connection*

$$\nabla_{X/S}: \mathcal{H}^n_{DR}(X/S) \longrightarrow \mathcal{H}^n_{DR}(X/S) \otimes \Omega^1_S(D_f)$$

of coherent $\mathcal{O}_S$-modules, where $\mathcal{H}^n_{DR}(X/S)$ is the hypercohomology $\mathbb{R}^n f_* \Omega^\bullet_{X/S}$ of the complex of relative holomorphic differential forms.

This result is already contained in his diploma thesis. An important ingredient of the proof is a proof of the generalized de Rham lemma saying that, for each holomorphic map $h = (h_1, \ldots, h_t): X \to \mathbb{C}^k$, the morphism

$$\Omega^p_{X/S} \Big/ \sum_{i=1}^{t} dh_i \wedge \Omega^{p-1}_{X/S} \longrightarrow \Omega^{p+t}_{X/S}, \quad [\omega] \longmapsto [dh_1 \wedge \ldots \wedge dh_t \wedge \omega]$$

is injective for $0 \le p < \mathrm{codim}_X \mathrm{Sing}(f, h)$, where $(f, h): X \to S \times \mathbb{C}^k$. Here, X does not need to have an isolated singularity. The de Rham lemma was later formulated by K. Saito in a more algebraic context, but the first proof is due to Greuel [1]. Indeed, Greuel proves a much more general statement, and the proof provides results which were recently used by Gusein-Zade and Ebeling to compute indices of vector fields.

In his Ph.D. thesis [2], published in [4], Greuel proves that $\mathcal{H}^n_{DR}(X/S)$ is locally free for $\dim S \le 2$ and that a different extension $\mathcal{H}'''_{DR}$ (corresponding to Brieskorns $\mathcal{H}''$ is locally free for arbitrary S (of rank $\mu(X_0, x)$). Using these results and applying the index theorem of Malgrange to the Gauß-Manin connection, Greuel gets a purely algebraic formula for the Milnor number of an isolated complete intersection singularity $(X_0, x) \subset (X, x)$ as above:

Theorem. *The Milnor number $\mu(X_0, x)$ has the following properties:*

(1) $\mu(X_0, x) = \dim_{\mathbb{C}} \Omega^n_{X_0,x}/d\Omega^{n-1}_{X_0,x}$ *if* $n > 0$, *and* $\mu(X_0, x) = \dim_{\mathbb{C}} \mathcal{O}_{X_0,x} - 1$ *if* $n = 0$.

In particular, the Milnor number depends only on X_0 (and not on f).

(2) If $\dim S = 1$, then $\mu(X_0, x) + \mu(X, x) = \dim_{\mathbb{C}} \Omega^m_{X/S,x} = \dim_{\mathbb{C}} \mathcal{O}_{X,x}/\mathscr{C}$, where $\mathscr{C}$ denotes the ideal of $\mathcal{O}_{X,x}$ generated by the entries of the Jacobian matrix $\partial(g_1, \ldots, g_r, f)/\partial \boldsymbol{x}$. Here $g_1, \ldots, g_r$ are supposed to generate the ideal of X in $\mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \ldots, x_m\}$.

In particular, we can compute $\mu(X_0, x)$ by recursion:

(3) If $X_i := V(f_1, \ldots, f_{k-i}) \subset \mathbb{C}^m$ and $f_{k-i+1} : X_i \to S_i = \mathbb{C}$, then

$$\mu(X_0, 0) = \sum_{i=1}^{k} (-1)^{k-i} \dim_{\mathbb{C}} \Omega^{n+i}_{X_i/S_i,0} = \sum_{i=1}^{k} (-1)^{k-i} \dim_{\mathbb{C}} \mathbb{C}\{\boldsymbol{x}\}\big/\mathscr{C}_i\,,$$

where $\mathscr{C}_i$ denotes the ideal of $\mathbb{C}\{\boldsymbol{x}\}$ generated by $f_1, \ldots, f_{i-1}$ and the i-minors of the Jacobian matrix $\partial(f_1, \ldots, f_i)/\partial \boldsymbol{x}$.

(4) If $X_0 = V(f_1, \ldots, f_k) \subset \mathbb{C}^m$ is quasihomogeneous, then

$$\mu(X_0, 0) = \dim_{\mathbb{C}} \mathbb{C}\{\boldsymbol{x}\}\big/(\mathscr{C}_k + \langle f_k \rangle)\,.$$

Related Publications 1971–1977

1. Zur Picard-Lefschetz-Monodromie isolierter Singularitäten von vollständigen Durchschnitten. *Diplomarbeit, Göttingen* (1971).
2. Der Gauß-Manin-Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten. *Ph.D. Thesis, Göttingen* (1973).
3. Singularities of complete intersections. *In: A. Hattori: Manifolds, Tokyo 1973. Univ. of Tokyo Press, 123–129* (with E. Brieskorn, 1975).
4. Der Gauß-Manin-Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten. *Math. Ann.* **214***, 235–266* (1975).
5. Spitzen, Doppelpunkte und vertikale Tangenten in der Diskriminante verseller Deformationen von vollständigen Durchschnitten. *Math. Ann.* **222***, 71–88* (with Lê Dung Tráng, 1976).
6. Cohomologie des singularités non isolés. *C.R. Acad. Sci. Paris Sér. A Math.* **284***, 321–322* (1977).
7. Die Zahl der Spitzen und die Jacobi-Algebra einer isolierten Hyperflächensingularität. *Manuscr. Math.* **21** *227–241* (1977).

Habilitationsschrift (1979).

Greuel's Habilitationsschrift, which has the title *"Kohomologische Methoden in der Theorie isolierter Singularitäten"*, consists of three parts:

I. *The Milnor number and deformations of complex curve singularities.*

II. *Deformation spezieller Kurvensingularitäten und eine Formel von Deligne.*

III. *Dualität in der lokalen Kohomologie isolierter Singularitäten.*

Large Parts of the Habilitationsschrift were written during a one year stay for research in France at IHES (Bures-sur Yvette) and at the mathematical institute of the Université de Nice.

In the first part, which has been based on a joint work with R.O. Buchweitz (see [11]), again a Milnor number $\mu(C,0)$ is the main object of investigation. This time for $(C,0) \subset (\mathbb{C}^n,0)$ being an arbitrary reduced complex curve singularity. Buchweitz and Greuel define this new invariant as

$$\mu(C,0) := \dim_{\mathbb{C}} \omega_{C,0}/d\mathcal{O}_{C,0},$$

where $\omega_{C,0} = \mathrm{Ext}^{n-1}_{\mathcal{O}_{\mathbb{C}^n,0}}\big(\mathcal{O}_{C,0}, \Omega^n_{\mathbb{C}^n,0}\big)$ is the dualizing module of Grothendieck, extending in this way the notion of the Milnor number of an isolated complete intersection curve singularity, $\mu(C,0) = \dim_{\mathbb{C}} \Omega^1_{C,0}/d\mathcal{O}_{C,0}$.

According to Greuel, the main results of Part I can be summarized by saying that also the general notion of a Milnor number reflects the topological nature of curve singularities:

> *"Obwohl μ für Kurvensingularitäten in Kodimension ≥ 2 keine topologische Invariante ist, spiegelt sie doch im Wesentlichen den topologischen Charakter der Singularität wider. Das ist der gemeinsame Nenner der Hauptresultate des ersten Teils."*

More precisely, Buchweitz and Greuel obtain the following results:

Theorem (Generalized Milnor formula). *If $(C,0)$ is a reduced complex curve singularity with r branches, then*

$$\mu(C,0) = 2\delta(C,0) - r + 1\,,$$

where $\delta(C,0) = \dim_{\mathbb{C}} \mathcal{O}_{\overline{C},\overline{0}}/\mathcal{O}_{C,0}$ for $(\overline{C},\overline{0}) \to (C,0)$ the normalization.

Theorem. *Let $f : \mathscr{C} \to D \subset \mathbb{C}$ be a good representative of a flat family of reduced curve singularities. Then, for all $t \in D$,*

(1) *The fibre $\mathscr{C}_t$ is connected.*

(2) $\mu(\mathscr{C}_0,0) - \sum\limits_{x\in\mathscr{C}_t} \mu(\mathscr{C}_t,x) = \dim_{\mathbb{C}} H^1(C_t,\mathbb{C})\,.$

(3) $\mu(\mathscr{C}_0,0) - \sum\limits_{x\in\mathscr{C}_t} \mu(\mathscr{C}_t,x) \geq \delta(\mathscr{C}_0,0) - \sum\limits_{x\in\mathscr{C}_t} \delta(\mathscr{C}_t,x)\,.$

(4) $\mu_t := \sum_{x \in \mathscr{C}_t} \mu(\mathscr{C}_t, x)$ *is constant in* t *iff all fibres* $\mathscr{C}_t$ *are contractible.*

Statement (2) shows that the Milnor number is again a measure for the vanishing cohomology.

Theorem ($\boldsymbol{\mu}$-constant is equivalent to topological triviality).
Let $f : \mathscr{C} \to D \subset \mathbb{C}$ *be a good representative of a flat family of reduced curve singularities with section* $\sigma : D \to \mathscr{C}$ *such that* $\mathscr{C}_t \setminus \{\sigma(t)\}$ *is smooth for all* $t \in D$*. Then the following are equivalent:*

(a) $\mu(\mathscr{C}_t, \sigma(t))$ *is constant for* $t \in D$*.*

(b) $\delta(\mathscr{C}_t, \sigma(t))$ *and* $r(\mathscr{C}_t, \sigma(t))$ *are constant for* $t \in D$*.*

(c) $f : \mathscr{C} \to D$ *is topologically trivial.*

Theorem (Generalized Zariski discriminant criterion).
Let $f : \mathscr{C} \to D \subset \mathbb{C}$ *be a sufficiently small representative of a flat deformation of a reduced complete intersection curve singularity* $(C, 0)$*. Then the following are equivalent:*

(a) There exists a finite mapping $\pi = (\pi_1, f) : (\mathscr{C}, 0) \to (\mathbb{C} \times D, 0)$ *such that the multiplicity of the discriminant (with Fitting structure) along* $\{0\} \times D$ *is constant for* $t \in D$ *and equal to*

$$\sum_{x \in \mathscr{C}_t} \big(\mu(\mathscr{C}_t, x) + \operatorname{mult}(\mathscr{C}_t, x) - 1\big)\,, \quad t \neq 0\,.$$

(b) $f : \mathscr{C} \to D$ *admits a holomorphic section* $\sigma : D \to \mathscr{C}$ *such that* $\mathscr{C}_t$ *is smooth outside* $\{\sigma(t)\}$*, and* $\mu(\mathscr{C}_t, \sigma(t))$ *and* $\operatorname{mult}(\mathscr{C}_t, \sigma(t))$ *are constant in* t*.*

Note that condition (b) is stronger than equisingularity, since for a complete intersection in codimension ≥ 2 constant Milnor number does not imply constant multiplicity.

Part II of the Habilitationsschrift deals with smoothable singularities: Let $(C, 0) \subset (\mathbb{C}^n, 0)$ be a reduced complex curve singularity, and let

$$\begin{array}{ccc} (C,0) & \xhookrightarrow{i} & (\mathscr{C},0) \\ \downarrow & & \downarrow{\scriptstyle \phi \text{ flat}} \\ \{0\} & \hookrightarrow & (S,0) \end{array}$$

be the semiuniversal deformation. Then $(C, 0)$ is called *smoothable* if there exists a component E of $(S, 0)$ such that the fibre $\mathscr{C}_t$ over a general point

$t \in E$ is smooth. The component $(E, 0)$ is then referred to as a *smoothing component for* $(C, 0)$.

Deligne's formula (1973) plays an important role in the investigation of smoothable singularities. It states that for a smoothable reduced complex curve singularity $(C, 0)$, each smoothing component has the dimension

$$e(C,0) := 3\delta(C,0) - \underbrace{\dim_{\mathbb{C}} \overline{\Theta}/\Theta}_{=:\, m_1(C,0)} ,$$

where $\Theta = \mathrm{Hom}_{\mathcal{O}_{C,0}}\big(\Omega^1_{C,0}, \mathcal{O}_{C,0}\big)$, and $\overline{\Theta} = \mathrm{Hom}_{\overline{\mathcal{O}}}\big(n_*\Omega^1_{\overline{C},\overline{0}}, \overline{\mathcal{O}}\big)$ for $\overline{\mathcal{O}} = n_*\mathcal{O}_{\overline{C},\overline{0}}$. And, for each reduced curve singularity $(C, 0)$, the codimension $m_1(C, 0)$ can be computed as $m_1(C,0) = r(C,0) + \dim_{\mathbb{C}} \overline{G}/G$, where $\overline{G} = \mathrm{Aut}\big(\overline{\mathcal{O}}/I\big)$ for some ideal $I \subset \overline{\mathcal{O}}$ contained in the conductor, and $G \subset \overline{G}$ the stabilisator of $\mathcal{O}_{C,0}/I$.

The main goal in Part II of Greuel's Habilitationsschrift (published in [13]) is to extend Deligne's Formula to not necessarily smoothable singularities and to express it by means of invariants of $(C, 0)$ that are easier to compute:

Theorem. *(1) If $(C, 0)$ is a quasihomogeneous complex curve singularity,*

$$e(C,0) = \mu(C,0) + t(C,0) - 1 ,$$

where $t(C,0) = \dim_{\mathbb{C}}(\omega_{C,0}/\mathfrak{m}_{C,0}\omega_{C,0})$ is the Cohen-Macaulay type of $(C, 0)$.

(2) If $(C, 0)$ is Gorenstein and irreducible, then

$$e(C,0) \le \mu(C,0) ,$$

and equality holds iff $(C, 0)$ is quasihomogeneous.

(3) For an arbitrary reduced complex curve singularity $(C, 0)$,

$$\begin{aligned} e(C,0) \;\; = \;\; & \mu(C,0) + t(C,0) - 1 \\ & + \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_{C,0}}\big(\Omega^1_{C,0}, \mathcal{O}_{C,0}\big) / \mathrm{Hom}_{\mathcal{O}_{C,0}}\big(\overline{\mathcal{O}}/\mathcal{O}_{C,0}\big) \\ & - \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_{C,0}}\big(\mathfrak{m}_{C,0}, \mathcal{O}_{C,0}\big) / \mathrm{Hom}_{\mathcal{O}_{C,0}}\big(\overline{\mathcal{O}}/\mathcal{O}_{C,0}\big) . \end{aligned}$$

The irreducibility assumption in (2) was later removed (see [17]). While Deligne's proof of the formula for the dimension of a smoothing component was global, a local proof was given as an application of the main result in [16].

Part III of the Habilitationsschrift (published in [12]) is devoted to the comparison of the Milnor and the Tjurina number of an isolated complete intersection singularity. The name "Tjurina number" for $\tau(X, 0) := \dim_{\mathbb{C}} T^1_{X,0}$ (for an arbitrary singularity $(X,0)$) was coined by Greuel and introduced in that paper. If $(X,0)$ is unobstructed (e.g., a complete intersection), then $\tau(X,0)$ equals the dimension of the base space of the semiuniversal deformation. Part III contains the following result:

Theorem. *Let $(X,0)$ be an isolated complete intersection singularity.*

(1) If $(X,0)$ is quasihomogeneous, then $\mu(X,0)=\tau(X,0)$.

(2) If the neighbourhood boundary of $(X,0)$ is a rational homology sphere or if $\dim(X,0)=1$, then $\mu(X,0)\geq\tau(X,0)$.

The last statement has been generalized by Looijenga and Steenbrink to arbitrary complete intersection singularities of dimension ≥ 2.

Related Publications 1978–1990

8. Invarianten quasihomogener vollständiger Durchschnitte. *Invent. Math.* **49**, *67–86* (with H.A. Hamm, 1978).
9. Le nombre de Milnor, équisingularité, et déformations de singularités des courbes réduites. *C.R. Acad. Sci. Paris Sér. A Math.* **288** *35–38 and Sém. sur les Singularités. Publ. Math. Univ. Paris 7, 13–30* (with R.-O. Buchweitz, 1979-80).
10. Kohomologische Methoden in der Theorie isolierter Singularitäten. *Habilitationsschrift, Bonn* (1979).
11. The Milnor number and deformations of complex curve singularities. *Invent. Math.* **58** *241-281* (with R.-O. Buchweitz, 1980).
12. Dualität in der lokalen Kohomologie isolierter Singularitäten. *Math. Ann.* **250** *157–173* (1980).
13. On deformation of curves and a formula of Deligne. *In: J.M. Aroca et al: Algebraic Geometry, La Rábida 1981. Springer LNM 961, 141-168* (1983).
14. On the topology of smoothable singularities. *In: P. Orlik: Singularities, Arcata 1981. Proc. Sympos. Pure Math.* **40**, *535–545* (with J.H.M. Steenbrink, 1983).
15. Einfache Kurvensingularitäten und torsionsfreie Moduln. *Math. Ann.* **270**, *417-425* (with H. Knörrer, 1985).
16. The dimension of smoothing components. *Duke Math. Journ.* **52**, *263–272* (with E. Looijenga, 1985).
17. Numerische Charakterisierung quasihomogener Gorenstein-Kurvensingularitäten. *Math. Nachr.* **124**, *123–131* (with B. Martin, G. Pfister, 1985).
18. Constant Milnor number implies constant multiplicity for quasihomogeneous singularities. *Manuscr. Math.* **56**, *159–166* (1986).
19. Torsion free modules and simple curve singularities. *Canad. Math. Soc. Conf. Proc.* **6**, *91–94* (1986).

20. Deformationen isolierter Kurvensingularitäten mit eingebetteten Komponenten. *Manuscr. Math.* **70**, *93–114* (with C. Brücker, 1990).
21. Simple Singularities in Positive Characteristic. *Math. Z.* **203**, *339–354* (with H. Kröning, 1990).

The SINGULAR Project.

The birth of the SINGULAR project can be dated back to about 1982, when G.-M. Greuel and the second author tried to generalize K. Saito's theorem which states that, *for a germ* $(X,0)$ *of an isolated hypersurface singularity, the following conditions are equivalent:*

(a) $(X,0)$ *is quasi-homogeneous (that is, has a good* $\mathbb{C}^*$*-action).*
(b) $\mu(X,0)=\tau(X,0)$.
(c) *The Poincaré complex of* $(X,0)$ *is exact.*

Trying to extend this theorem to complete intersection curve singularities, they only succeeded in proving the equivalence of (a) and (b) (see [17]). They expected that (b) and (c) are, indeed, not equivalent for general complete intersection curve singularities. They succeeded in expressing the exactness of the Poincaré complex as an equality of dimensions of certain $\mathcal{O}_{X,0}$-modules. In those days, however, there was no computer algebra system available which could compute Milnor numbers, Tjurina numbers and the dimensions of the differential modules in the Poincaré complex. To be able to compute these numbers, such a system requires an implementation of T. Mora's tangent cone algorithm, a modification of Buchberger's Gröbner basis algorithm designed for computations over local rings.

Having implemented this algorithm, the expected counterexamples were found by H. Schönemann and the second author in C.T.C. Wall's list of unimodal complete intersection curve singularities: consider

$$\{xy+z^{\ell-1}=xz+yz^2+y^{k-1}=0\}$$

for $4\le\ell\le k$, $5\le k$.

Motivated by this success, Greuel and the second author tried to attack Zariski's famous multiplicity conjecture by searching for a counterexample. Starting point was the following result of Greuel, confirming Zariski's conjecture for families of quasihomogeneous isolated singularities [18]:

Theorem. *Let* $f\in\mathbb{C}\{x_1,\dots,x_n\}$ *be such that the ideal* $\langle f\rangle$ *can be generated by a semiquasihomogeneous polynomial, and let* $f_t\in\mathbb{C}\{x_1,\dots,x_n,t\}$ *be a* μ*-constant deformation of* f*, then the multiplicity of* f_t *is constant (that is, independent of* t *for* $t\in\mathbb{C}$ *small).*

The method of proof suggested a way to look for possible counterexamples in the non-quasihomogeneous case. However, since Zariski's conjecture holds for curves and semiquasihomogeneous singularities, potential counterexamples have Milnor number > 1000. For these computations, the existing implementation of the tangent cone algorithm was not sufficient. Therefore, Greuel, the second author and H. Schönemann decided to set up such a computer algebra system with improved algorithms and extended functionality.

The result is nowadays known as SINGULAR which has grown to a major specialized computer algebra system used in mathematical research and teaching, and even in industrial applications (see the article on SINGULAR in this volume by H. Schönemann and the first author).

The computational complexity provided by the potential counterexamples to Zariski's conjecture was a big challenge and resulted in sophisticated strategies for the implementation of Buchberger's (resp. Mora's) algorithm. One can say that the hardness of the problem is one of the main reasons for SINGULAR to have one of the fastest implementations of a standard basis algorithm.

Although the search for a counterexample to Zariski's conjecture failed, it was not useless. The experiments with SINGULAR suggested a positive answer to Zariski's conjecture in another special case, proved in [24]. In general, Zariski's conjecture is still open.

Some Publications Related to SINGULAR 1996–2005

22. Standard bases, syzygies and their implementation in SINGULAR. In: *Beiträge zur angewandten Analysis und Informatik.* Shaker, Aachen, 69–96 (with H. Grassmann, B. Martin, W. Neumann, G. Pfister, W. Pohl, H. Schönemann, T. Siebert, 1994).
23. On an implementation of standard bases and syzygies in SINGULAR. *AAECC 7, 235–149* (with H. Grassmann, B. Martin, W. Neumann, G. Pfister, W. Pohl, H. Schönemann, T. Siebert, 1996).
24. Advances and improvements in the theory of standard bases and syzygies. *Arch. Math.* **66***, 163–176* (with G. Pfister, 1996).
25. Description of SINGULAR: A Computer Algebra System for Singularity Theory, Algebraic Geometry and Commutative Algebra. *Euromath Bulletin* **2***, 161–172* (1996).
26. The normalisation: a new algorithm, implementation and comparisons. *In: Proc. EUROCONFERENCE Computational Methods for Representations of Groups and Algebras (1.4.-5.4.1997). Birkhäuser* (with W. Decker, T. de Jong, G. Pfister, 1998).
27. Primary decomposition: algorithms and comparisons. *In: G.-M. Greuel, B.H. Matzat, G. Hiss: Algorithmic Algebra and Number Theory. Springer Verlag, Heidelberg, 187–220* (with W. Decker, G. Pfister, 1998).

28. Gröbner bases and algebraic geometry. *In: B. Buchberger and F. Winkler: Gröbner Bases and Applications. LNS 251 , CUP, 109–143* (with G. Pfister, 1998).
29. Applications of Computer Algebra to Algebraic Geometry, Singularity Theory and Symbolic-Numerical Solving. *In: European Congress of Mathematicians, Barcelona, July 10-14, 2000, Vol. II, 169–188* (2000).
30. Computer Algebra and Algebraic Geometry - Achievements and Perspectives. *Journ. Symb. Comp.* **30**, *253–290* (2000).
31. Three Algorithms in Algebraic Geometry, Coding Theory, and Singularity Theory. *In: C. Ciliberto et al: Application of Algebraic Geometry to Coding Theory, Physics and Computation, Proceedings. Kluwer, 161–194* (with C. Lossen, M. Schulze, 2001).
32. *A* SINGULAR *Introduction to Commutative Algebra*. Springer-Verlag, 605 pp. (with G. Pfister, and with contributions by O. Bachmann, C. Lossen and H. Schönemann, 2002).
33. Two-variable identities for finite solvable groups. *C.R. Acad. Sci. Paris, Ser. I* **337**, *581–586* (with T. Bandman, F. Grunewald, B. Kunyavskii, G. Pfister, E. Plotkin, 2003).
34. Engel-Like Identities Characterizing Finite Solvable Groups. To appear in Compos. Math. (with T. Bandman, F. Grunewald, B. Kunyavskii, G. Pfister, E. Plotkin, 2005).

Applying Computer Algebra Methods in Mathematical Research.

The publications [33,34] are related to a problem from group theory. The solution to this problem may serve as a model for how computer algebra methods may be used for establishing conjectures and for proving theorems in other fields of mathematics.

The was problem addressed to G.-M. Greuel and the second author by B. Kunyavskii. It can be stated as follows:

> *Characterize the class of solvable finite groups G by explicit two-variable identities.*

To explain this problem, note that a group G is Abelian iff the two-variable identity $xy = yx$ is satisfied for all $x, y \in G$. Moreover, Zorn (1930) proved that, setting

$$v_1(x,y) := \big[x,y\big] := xyx^{-1}y^{-1}\,, \qquad v_{k+1}(x,y) := \big[v_k,y\big]\,,$$

a finite group G is nilpotent iff there exists some $n \geq 1$ such that the two-variable identity $v_n(x,y) = 1$ holds for all $x, y \in G$. The identity $v_n(x,y) = 1$ is referred to as an *Engel Identity*.

The existence of two-variable (but non-explicit) identities for finite solvable groups has been proved by R. Brandl and J.S. Wilson (1981,1988). B. Plotkin suggested that there should be an explicit definition for such a two-variable identity $U_n(x,y)=1$, using the recursion $U_{k+1}=[xU_kx^{-1},yU_ky^{-1}]$. A key point has been to find an appropriate candidate for $U_1(x,y)$. Indeed, experimenting with Singular such a candidate was found (see [33,34]):

Theorem. *Define U_k inductively by*

$$U_1(x,y):=x^{-2}y^{-1}x, \qquad U_{k+1}(x,y):=\left[xU_k(x,y)x^{-1},yU_k(x,y)y^{-1}\right].$$

Then a finite group G is solvable iff there exist some n such that the two-variable identity $U_n(x,y)=1$ holds for all $x,y\in G$.

That solvable groups satisfy the identity above is clear by the definition of a solvable group. Thus, it remains to show that for a (minimal) non-solvable finite group no such equality holds. Fortunately, the minimal non-solvable finite groups have been classified by Thompson (1968): his list consists of

1. $\mathrm{PSL}(2,\mathbb{F}_p)$, $p\geq 5$ prime,
2. $\mathrm{PSL}(2,\mathbb{F}_{2^p})$, p prime,
3. $\mathrm{PSL}(2,\mathbb{F}_{3^p})$, p prime,
4. $\mathrm{PSL}(3,\mathbb{F}_3)$,
5. the Suzuki groups $\mathrm{Sz}(2^p)$, p prime.

The key observation that allows one to translate B. Plotkins suggestion to a problem of algebraic geometry is the following[1]: *if $x,y\in G$ satisfy $1\neq U_1(x,y)=U_2(x,y)$, then $U_n(x,y)\neq 1$ for all $n\in\mathbb{Z}$.*

It thus remains to show that for each group in Thompson's list, there are elements $x,y\in G$ such that $1\neq U_1(x,y)=U_2(x,y)$.

It is quite instructive to show how such a problem from group theory can be translated to a problem in algebraic geometry and how to solve it with the help of computer algebra. Let us consider here the family of groups $G=\mathrm{PSL}(2,\mathbb{F}_p)$, $p\geq 5$ a prime. The next two cases of Thomson's list can be handeled similarly, the fourth case is treated by giving an explicit example. The last case, however, has turned out to be much more difficult, with a surprising complexity and involving in addition deep theorems from arithmetic geometry.

We represent two elements x and y of G by two matrices of the following types:

$$x=\begin{pmatrix} t & 1\\ -1 & 0\end{pmatrix}, \qquad y=\begin{pmatrix} 1 & b\\ c & 1+bc\end{pmatrix},$$

[1] This observation is independent of the choice of U_1.

with $b, c, t \in \mathbb{F}_p$.

Clearly we have $y \neq x^{-1}$ for all $(b,c,t) \in \mathbb{F}_p^3$, thus $U_1(x,y) \neq 1$. It remains to show that for each choice of p the equation $U_1(x,y) = U_2(x,y)$ has a solution $(b,c,t) \in \mathbb{F}_p^3$.

The ideal $I \subset \mathbb{Z}[b,c,t]$ spanned by the entries of $U_1(x,y) - U_2(x,y)$ is generated by four polynomials of degree at most 8. For a fixed prime number p, it defines a curve in the three-dimensional space over $\mathbb{F}_p$. To prove that there are $\mathbb{F}_p$-rational points on the curve we use the the Hasse-Weil-Theorem as generalized by Aubry and Perret for singular curves: *If $C \subseteq \overline{\mathbb{F}_q}^n$ is an irreducible affine curve, defined over $\mathbb{F}_q$, $q = p^m$, and if $\overline{C} \subset \mathbb{P}^n$ is its projective closure, then*

$$\#C(\mathbb{F}_q) \geq q + 1 - 2p_a\big(\overline{C}\big)\sqrt{q} - \deg\big(\overline{C}\big)\,.$$

To be able to apply the theorem to our situation, we have to show that the image of the ideal I in $\mathbb{F}_p[b,c,t]$ defines an irreducible curve C over the algebraic closure $\overline{\mathbb{F}_p}$ of $\mathbb{F}_p$. In algebraic terms, we have to show that the image of I generates a prime ideal of $\overline{\mathbb{F}_p}[b,c,t]$.

If this is the case, we may compute the degree and the arithmetic genus of the projective curve $\overline{C} \subset \mathbb{P}^3$ via the Hilbert-polynomial which equals $H(t) = 10t - 11$. Hence, $\deg\big(\overline{C}\big) = 10$ and $p_a\big(\overline{C}\big) = 11 + 1 = 12$, and the Hasse-Weil formula gives $\#C(\mathbb{F}_p) > 0$ for all primes $p > 593$.

As the remaining finitely many cases can be checked directly with a computer, it remains to prove that for any prime $p \geq 5$, the ideal $I \cdot \overline{\mathbb{F}_p}[b,c,t]$ is, indeed, a prime ideal.

We have $I \cdot \overline{\mathbb{F}_p}[b,c,t] = \big(I \cdot \overline{\mathbb{F}_p}(t)[b,c]\big) \cap \overline{\mathbb{F}_p}[b,c,t]$, and $I \cdot \overline{\mathbb{F}_p}(t)[b,c]$ is generated by two polynomials $f_1 \in \mathbb{F}_p[b,t]$, $f_2 \in \mathbb{F}_p[b,c,t]$, as obtained using Singular and verified by hand later on. Thus, it is enough to prove that $I \cdot \overline{\mathbb{F}_p}(t)[b,c]$ is a prime ideal, which is equivalent to showing that f_1 is irreducible in $\overline{\mathbb{F}_q}[t,b]$. As the polynomial f_1 has a small degree (namely 4) in x, this could be proved by making an Ansatz and showing that the resulting systems of polynomial equations have no solution over the algebraic closure (which was done first by the computer, then by hand).

Work on Cohen-Macaulay Modules and Moduli Spaces.

In the joint paper [15] with H. Knörrer, G.-M. Greuel showed that a reduced plane curve singularity is of finite CM-representation type, that is, its analytic local ring has only finitely many isomorphism classes of indecomposable Cohen-Macaulay modules iff it is a simple (ADE-) singularity.

This result has been extended later by Knörrer and Greuel in a joint paper with R.-O. Buchweitz and F.-O. Schreyer to arbitrary isolated hypersurface singularities [41]. It attracted interest by mathematicians working in representation theory of finite dimensional algebras, and the question came up whether the so-called tame–wild dichotomy for finite dimensional algebras (proved by Y. Drozd) also holds for curve singularities w.r.t. Cohen-Macaulay modules. During a workshop in Bielefeld in 1990 organized by C.M. Ringel, Greuel proposed this as a conjecture when he gave a talk about the construction of moduli spaces of CM modules over a fixed local ring of a reduced curve singularity. Y. Drozd, who was in the audience, immediately realized that the sandwiched construction used for the construction of moduli spaces could be used to reduce the question to a matrix problem.

The tame–wild dichotomy for CM-modules over curve singularities was finally proved by Greuel and Drozd in a joint paper. Several other joint papers of Greuel and Drozd were devoted to the classification of tame curve and surface [47] singularities and their CM-modules. Moreover, in [45], the tame–wild dichotomy was shown to hold also for singular projective curves with a particular nice geometric description of the tame curves for which a classification of all indecomposable vector bundles resp. torsion free sheaves was achieved.

In the remaining part of this section, we focus on the general approach to constructing moduli spaces for singularities and related objects developed by G.-M. Greuel and the second author in the 1980s. This approach basically consists of the following steps:

1. Fix some rough invariants.
2. Find the worst object among them you want to classify.
3. Consider the versal deformation $X \to T$ of the worst object with fixed invariants.
4. Prove that this family contains all objects you want to classify.
5. Compute the kernel $\mathcal{L}$ of the Kodaira-Spencer map of the family.
6. Compute a stratification $\{T_\alpha\}$ of T, by fixing suitable invariants such that the geometric quotient $T_\alpha/\mathcal{L}$ exists.
7. Modulo the action of a finite group, we obtain coarse moduli spaces.

To illustrate this general idea, let us consider an example:

> *Classify all* $R = \mathbb{C}[[t^c, t^{c+1}, \ldots]]$*-modules of rank one with set of values* $\Gamma = \{\gamma_0, \ldots, \gamma_k, c, c{+}1, \ldots\}, \quad 0 = \gamma_0 < \gamma_1 < \ldots < \gamma_k < c\,.$

Following our philosophy, we determine the *worst object*:

$$M_0 = \sum_{i=1}^{k} t^{\gamma_i} + t^c\mathbb{C}[[t]]\,.$$

Its *versal deformation* is given by

$$\mathscr{M}_\Gamma = \sum_{i=1}^{k} m_i \cdot \mathbb{C}[\boldsymbol{\lambda}][[t^c, t^{c+1}, \ldots]] + t^c\mathbb{C}[\boldsymbol{\lambda}][[t]],$$

where $\boldsymbol{\lambda} = \big\{\lambda_{i,j}\big\}_{j+\gamma_i \notin \Gamma}$, and $m_i = t^{\gamma_i} + \sum_{j+\gamma_i\notin\Gamma}\lambda_{i,j}t^{j+\gamma_i}$. The *Kodaira-Spencer map* is a mapping

$$\rho : \mathrm{Der}_{\mathbb{C}}\, \mathbb{C}[\boldsymbol{\lambda}] \longrightarrow \mathrm{Ext}^1_{\mathbb{C}[\boldsymbol{\lambda}][[t^c,\ldots]]}(\mathscr{M}_\Gamma, \mathscr{M}_\Gamma)\,,$$

and $t, t' \in T = \mathrm{Spec}(\mathbb{C}[\boldsymbol{\lambda}])$ define isomorphic modules iff they are in the same integral manifold of the kernel $\mathcal{L}$ of ρ. This kernel is of the form $\sum_\ell \mathbb{C}[\boldsymbol{\lambda}]\delta_\ell$, with $L = \sum_\ell \mathbb{C}\delta_\ell$ an Abelian Lie algebra.

The computation of the moduli spaces as geometric quotients is based on the following theorem:

Theorem. *Let A be a K-algebra, $\mathcal{L} \subset \mathrm{Der}^{\mathrm{nil}}_K(A)$ a Lie algebra, $\delta_1, \ldots, \delta_n \in \mathcal{L}$ such that $\mathcal{L} \subset \sum_{\ell=1}^n A\delta_\ell$, and let $x_1, \ldots, x_n$ be elements of A such that $\det\big(\delta_\ell(x_j)\big)$ is a unit in A and such that for each k-minor M of the first k columns of $\big(\delta_\ell(x_j)\big)$ we have $\boldsymbol{\delta}(M) \in \sum_{j<k} A\boldsymbol{\delta}(x_j)$. Then the following holds:*

(1) $A^{\mathcal{L}}[x_1, \ldots, x_n] = A$ and $x_1, \ldots, x_n$ are algebraically independent over $A^{\mathcal{L}}$. In particular, $\mathrm{Spec}(A) \to \mathrm{Spec}\big(A^{\mathcal{L}}\big)$ is a (trivial) geometric quotient.

(2) If, additionally, $\mathcal{L} = L$ is a finite dimensional nilpotent Lie algebra of dimension n, then $H^1(\mathcal{L}, A) = 0$.

This theorem has as consequence the following corollary which is the basis for the applications:

Corollary. *Let A be a Noetherian K-algebra, $L \subset \mathrm{Der}^{\mathrm{nil}}_K(A)$ a finite dimensional, nilpotent Lie algebra, and let $d : A \to \mathrm{Hom}_K(L, A)$ be the differential, $da(\delta) = \delta(a)$. Assume that the following holds:*

- *$0 = Z_{k+1}(L) \subset Z_k(L) \subset \ldots \subset Z_0(L) = L$ is a finite filtration of L satisfying $[L, Z_j(L)] \subset Z_{j+1}(L)$.*
- *$0 = F^{-1}(A) \subset F^0(A) \subset F^1(A) \subset \ldots$ is a filtration of the K-algebra A such that $\delta(F^i(A)) \subset F^{i-1}(A)$ for all i and all $\delta \in L$.*
- *$\mathrm{Spec}(A) = \bigcup_\alpha U_\alpha$ is the flattening stratification of the modules*

$$\mathrm{Hom}_K(L, A)\big/A \cdot d(F^i(A))\,, \qquad \mathrm{Hom}_K(Z_j(L), A)\big/\pi_j(A \cdot d(A))\,,$$

where $\pi_j : \mathrm{Hom}_K(L, A) \to \mathrm{Hom}_K(Z_{j+1}(L))$ denotes the canonical projection.

Then U_α is L-invariant and admits a locally trivial geometrical quotient with respect to the action of L.

We illustrate the use of this corollary by continuing the example treated above: in this case, L is Abelian, therefore no Z-filtration is needed.

Let a be the multiplicity of the maximal semigroup $\Gamma_0 \subset \Gamma$ acting on Γ. Then we define $F^i(\mathbb{C}[\boldsymbol{\lambda}])$ to be the $\mathbb{C}$-vector space generated by all quasihomogeneous polynomials in $\mathbb{C}[\boldsymbol{\lambda}]$ of degree less than $(i+1) \cdot a$. Here, we assign the degree j to λ_{ij}, which makes the vector fields δ_ℓ homogeneous of degree $-\ell$.

Then the assumptions of the corollary are satisfied for the nilpotent Lie algebra $L^{(0)} := \sum_{\ell \geq a} \mathbb{C}\delta_\ell \subset L$. Hence, if $\mathrm{Spec}(\mathbb{C}[\boldsymbol{\lambda}]) = \bigcup_\alpha U_\alpha$ is the flattening stratification of the modules $\mathrm{Hom}_{\mathbb{C}}(L^{(0)}, \mathbb{C}[\boldsymbol{\lambda}])/\mathbb{C}[\boldsymbol{\lambda}] \cdot d(F^i(\mathbb{C}[\boldsymbol{\lambda}]))$, then $U_\alpha \to U_\alpha/L^{(0)}$ is a geometric quotient. Using an H^1-vanishing argument, one can show that $U_\alpha \to U_\alpha/L$ is a geometric quotient, too (see [37] for details). This quotient turns out to be the moduli space of all modules with a certain Hilbert function fixed.

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Work on Equisingular Families

The short paper [48], actually an appendix to a paper of A. Tannenbaum in Compos. Math. 1984, was, in a sense, the initial point for G.-M. Greuel to start his own research on equisingular families. Tannenbaum observed that Segre's analysis of families with prescribed singularities can be rigorously justified for curves with at most ordinary nodes and cusps as singularities: Segre associated his characteristic linear series to $H^0\big(C, n_*\widetilde{\mathscr{N}}\big)$, where $\widetilde{\mathscr{N}}$ is a certain locally free sheaf on the normalization. *Correct* would have been $H^0\big(C, \mathcal{I}_{Z^{ea}(C)}(C)\big)$, where $Z^{ea}(C)$ is the zero-dimensional scheme locally defined by the Tjurina ideal. Indeed, Tannenbaum proved the existence of an exact sequence

$$0 \longrightarrow \mathcal{I}_{Z^{ea}(C)}(C) \longrightarrow n_*\widetilde{\mathscr{N}} \longrightarrow \mathscr{T} \longrightarrow 0\,,$$

where $\mathscr{T}$ is a torsion sheaf supported at the singular locus of C, with stalk $\mathscr{T}_x = 0$ if (C, x) is a node or a cusp.

As an addendum, Greuel computed the dimension of $\mathscr{T}_x$ as

$$\dim_{\mathbb{C}} \mathscr{T}_x = \tau(C,x) + r(C,x) - \mathrm{mult}(C,x) - \delta(C,x)\,.$$

In particular, at a singular point x of C, $\mathscr{T}_x$ is nonzero unless this singular point is either a node or a cusp.

Greuel's interest in equisingular families of curves with arbitrary singularities (not just nodes and cusps) was stimulated, and more questions came up. To describe some of these, we restrict ourselves to the case of plane curves, using the following definition:

Definition. Let $S_1, \ldots, S_r$ be (analytic or topological) types of plane curve singularities. Then we set

$$V_d(S_1,\ldots,S_r) := \left\{ C \subset \mathbb{P}^2 \;\middle|\; \begin{array}{l} C \text{ is a reduced curve of degree } d, \\ \text{having exactly } r \text{ singular points of} \\ \text{types } S_1,\ldots,S_r \end{array} \right\},$$

respectively $V_d^{irr}(S_1, \ldots, S_r)$ where it is additionally assumed that C is irreducible.

In a joint work first with U. Karras [49], Greuel showed that the set $V_d(S_1,\dots,S_r)$ carries a natural structure as a complex space, even for analytic types of arbitrary isolated singularities, given by deformation theory. Moreover, with U. Karras and then with E. Shustin and the first author, G.-M. Greuel looked for general numerical criteria answering the following questions (for $V = V_d(S_1,\dots,S_r)$, resp. $V_d^{irr}(S_1,\dots,S_r)$):

- Is V non-empty ?
- Is V smooth (that is, is the characteristic linear series complete) ?
- Is V *T-smooth* (that is, smooth and of the expected dimension) ?
- Is V irreducible ?

The general method for answering these questions is based on a translation to a statement about the cohomology of ideal sheaves of zero-dimensional schemes.

For instance, the T-smoothness property for analytic types translates as

$$V \text{ is T-smooth at } C \iff H^1\big(\mathcal{J}_{Z^{ea}(C)}(d)\big) = 0\,,$$

for $Z^{ea}(C) \subset \mathbb{P}^2$ the zero-dimensional scheme which is locally given by the Tjurina ideal for a local equation of C. For topological types, $Z^{ea}(C)$ has to be replaced by the zero-dimensional scheme $Z^{es}(C)$ which is locally given by the equisingularity ideal in the sense of J. Wahl.

The first criteria obtained have been based on the following vanishing theorem of Riemann-Roch-type (see [49,50]):

Theorem. *Let S be a smooth projective surface, $C \subset S$ a reduced curve. Let $\mathcal{F}$ be a torsion-free, coherent $\mathcal{O}_C$–module having rank 1 on each irreducible component C_i of C, $i = 1,\dots,s$. Then $H^1\big(C,\mathcal{F}\big)$ vanishes if*

$$\chi(\omega_C \otimes \mathcal{O}_{C_i}) - \mathrm{isod}_{C_i}(\mathcal{F},\mathcal{O}_C) \;<\; \chi(\mathcal{F}_{C_i})$$

for all $i = 1,\dots,s$. Here, $\mathcal{F}_{C_i} = \mathcal{F} \otimes \mathcal{O}_{C_i}$ (mod torsion). Moreover,

$$\mathrm{isod}_{C_i,x}(\mathcal{F},\mathcal{G}) := \min\left(\dim_{\mathbb{C}} \mathrm{coker}(\varphi_{C_i} : \mathcal{F}_{C_i,x} \hookrightarrow \mathcal{G}_{C_i,x})\right),$$

where the minimum is taken over all φ_{C_i}, which are induced by homomorphisms $\varphi : \mathcal{F}_x \to \mathcal{G}_x$.

As a consequence, it was shown in [49,50] that V is T-smooth at C if the total (equisingular) Tjurina number of C is bounded by a linear function in the degree of d. The resulting criteria are usually referred to as the $3d$-criterion and the $4d$-criterion.

A slightly weaker $4d$-criterion was found before by E. Shustin by different methods. G.-M. Greuel met E. Shustin at the ICM 1990 in Kyoto (Japan) where they discussed the different approaches and realized that joining efforts could result in a major progress in this area.

A major breakthrough in the study of equisingular families was the first asymptotically proper general sufficient condition for the existence of plane curves with prescribed (topological types of) singularities obtained in [53]:

Theorem. *If $S_1, \ldots, S_r$ are topological types of singularities, and if*

$$\sum_{i=1}^{r} \mu(S_i) \leq \frac{1}{392}(d+2)^2,$$

then $V_d^{irr}(S_1, \ldots, S_r)$ is non-empty.

This criterion is referred to as being asymptotically proper, since (from the asymptotical point of view) it differs from the necessary criterion

$$\sum_{i=1}^{r} \mu(S_i) \leq (d-1)^2$$

for non-emptyness only by a constant factor. Later, this factor $\frac{1}{392}$ has been improved to $\frac{1}{9}$, and the statement has been extended to analytic types by E. Shustin.

For the T-smoothness problem, the linear right-hand side (as in the $3d$- and $4d$-criterion) could be replaced by a quadratic function in d, too. So far, the best known general sufficient criterion has been obtained in [55,56]:

Theorem. *Let $d \geq 6$. Then $V_d^{irr}(S_1, \ldots, S_r)$ is T-smooth at C if*

$$\sum_{i=1}^{r} \gamma^{ea}(C, z_i) \leq (d+3)^2, \qquad \left(resp. \;\; \sum_{i=1}^{r} \gamma^{es}(C, z_i) \leq (d+3)^2 \right),$$

for new invariants $\gamma^{ea} \leq (\tau+1)^2$, $\gamma^{es} \leq (\tau^{es}+1)^2$.

In particular, for families of curves with n nodes and k cusps (resp. for families of curves with ordinary m_i-fold points) the sufficient condition reads

$$4n + 9k \leq (d+3)^2 \qquad \left(\text{resp. } \; 4 \cdot \#(\text{nodes}) + \sum_{m_i > 2} 2m_i^2 \leq (d+3)^2 \right).$$

For the irreducibility problem, the best known general sufficient criterion is:

Theorem. *If* $\max_{i=1..r} \tau'(S_i) \leq (2/5)\cdot d - 1$ *and*

$$\frac{25}{2}\cdot \#(nodes) + 18\cdot \#(cusps) + \frac{10}{9}\cdot \sum_{\tau'(S_i)\geq 3} \big(\tau'(S_i)+2\big)^2 \, < \, d^2,$$

then $V_d^{irr}(S_1,\dots,S_r)$ *is non-empty and irreducible.*

Here τ' refers to the Tjurina number, resp. to the equisingular Tjurina number, that is, the codimension of the equisingularity ideal.

In contrast to the conditions for non-emptiness and T-smoothness, this condition seems not to be asymptotically proper. Indeed, for instance, for plane curves with r ordinary m-fold points, the known examples of reducible families (see [56]) satisfy $d^2 \sim r\cdot m^2$ while the left-hand side of our criterion is of type $r\cdot \frac{m^4}{4}$.

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Exterior Algebra Methods for the Construction of Rational Surfaces in the Projective Fourspace

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Abstract

The aim of this paper is to present a construction of smooth rational surfaces in projective fourspace with degree 12 and sectional genus 13. The construction is based on exterior algebra methods, finite field searches and standard deformation theory.

Introduction

This paper is dedicated to Gert-Martin Greuel on the occasion of his sixtieth birthday. The use of computer algebra systems is essential for the proof of the main result of this paper. It will become clear that without computer algebra systems like Singular and Plural developed in Kaiserslautern we could not obtain the main result of this paper at all. We thank the group in Kaiserslautern for their excellent program.

Hartshone conjectured that only finitely many components of the Hilbert scheme of surfaces in $\mathbb{P}^4$ correspond to smooth rational surfaces. In 1989, this conjecture was positively solved by Ellingsrud and Peskine [6]. The exact bound for the degree is, however, still open. This motivates our search for smooth rational surfaces in $\mathbb{P}^4$. Examples of smooth rational surfaces in $\mathbb{P}^4$ prior to this paper were known up to degree 11, see [4]. Our main result is the proof of existence of the following example.

Theorem 0.1. *There exists a family of smooth rational surfaces in $\mathbb{P}^4$ over $\mathbb{C}$ with $d = 12$, $\pi = 13$ and hyperplane class*

$$H \equiv 12L - \sum_{i_1=1}^{2} 4E_i - \sum_{i_2=3}^{11} 3E_i - \sum_{i_3=12}^{14} 2E_i - \sum_{i_4=15}^{21} E_i.$$

1991 *Mathematics Subject Classification.* 14J10, 14J26 (*secondary:* 14Q10)
Key words. Rational surface, monad, exterior algebra, finite field

in terms of a plane model.

Abstractly, these surfaces arise as the blow up of $\mathbb{P}^2$ in 21 points. L and E_i in the Theorem denote the class of a general line and the exceptional divisors.

The 21 points lie in special position due to the fact that we need $\mathrm{h}^0(X, \mathcal{O}(H) = 5$ and $\mathrm{h}^1(X, \mathcal{O}(H)) = 4$. Indeed, it will turn out that the component of the Hilbert scheme corresponding to these surfaces has dimension 38, hence up to projectivities this is a $38 - 24 = 14$ dimensional family of abstract surfaces. This fits with the fact that the 21 points have to satisfy a condition of codimension $\leq 20 = 4 \cdot 5$, which leaves us with a family of collections of points in $\mathbb{P}^2$ of dimension $\geq 2 \cdot 21 - 20 = 22$. Up to automorphism of $\mathbb{P}^2$ this leads to a family of dimension $\geq 22 - 8 = 14$, and hence equality holds. The great difficulty to find points in $\mathbb{P}^2$ in very special positions was one of the sources, which led Hartshorne to his conjecture.

We construct these surfaces via their "Beilinson monad": Let V be an $n+1$-dimensional vector space over a field K and let W be its dual space. The basic idea behind a Beilinson monad is to represent a given coherent sheaf on $\mathbb{P}^n = \mathbb{P}(W)$ as a homology of a finite complex of vector bundles, which are direct sums of exterior powers of the tautological rank n subbundle $U = \ker\big(W \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathcal{O}_{\mathbb{P}(W)}(1)\big)$ on $\mathbb{P}(W)$. (Thus $U \simeq \Omega^1(1)$ is the twisted sheaf of 1-forms. As Beilinson, we will use the notation $\Omega^p(p)$ for the exterior powers of U.)

The differentials in the monad are given by homogeneous matrices over an exterior algebra $E = \bigwedge V$. To construct a Beilinson monad for a given coherent sheaf, we typically take the following steps: Determine the type of the Beilinson monad, that is, determine the vector bundles of the complex, and then find differentials in the monad.

Let X be a smooth rational surface in $\mathbb{P}^4 = \mathbb{P}(W)$ with degree 12 and sectional genus 13. The type of a Beilinson monad for the (suitably twisted) ideal sheaf of X can be derived from the knowledge of its cohomology groups. Such information is partially determined from general results such as the Riemann-Roch formula and the Kodaira vanishing theorem. It is, however, hard to determine the dimensions of all cohomology groups needed to determine the type of the Beilinson monad. For this reason, we assume that the ideal sheaf of X has the so-called "natural cohomology" in some range of twists. In particular, we assume that in each twist $-1 \leq n \leq 6$ at most one of the cohomology groups $\mathrm{H}^i(\mathbb{P}^4, \mathcal{I}_X(n)$ for $i = 0 \ldots 4$ is non-zero. This is an open condition for surfaces in a given component of the Hilbert scheme. Under this assumption the Beilinson monad for the twisted ideal sheaf $\mathcal{I}_X(4)$ of X has the following form:

$$4\Omega^3(3) \xrightarrow{A} 2\Omega^2(2) \oplus 2\Omega^1(1) \xrightarrow{B} 3\mathcal{O}. \tag{1}$$

To detect differentials in (1), we use the following techniques developed recently: (1) the first technique is an exterior algebra method due to Eisenbud, Fløystad and Schreyer [5] and (2), the other one is the method using small finite fields and random trials due to Schreyer [9].

(1) Eisenbud, Fløystad and Schreyer presented an explicit version of the Bernstein-Gel'fand-Gel'fand correspondence. This correspondence is an isomorphism between the derived category of bounded complexes of finitely generated S-graded modules and the derived category of certain "Tate resolutions" of E-modules, where $S = \mathrm{Sym}_K(W)$. As an application, they constructed the Beilinson monad from the Tate resolution explicitly. This enables us to describe the conditions, that the differentials in the Beilinson monad must satisfy in an exterior algebra context.

(2) Let $\mathbb{M}$ be a parameter space for objects in algebraic geometry such as the Hilbert scheme or a moduli space. Suppose that $\mathbb{M}$ is a subvariety of a rational variety $\mathbb{G}$ of codimension c. Then the probability for a point p in $\mathbb{G}(\mathbb{F}_q)$ to lie in $\mathbb{M}(\mathbb{F}_q)$ is about $(1 : q^c)$. This approach will be successful if the codimension c is small and the time required to check $p \notin \mathbb{M}(\mathbb{F}_q)$ is sufficiently small as compared with q^c. This technique was applied first by Schreyer [9] to find four different families of smooth surfaces in $\mathbb{P}^4$ with degree 11 and sectional genus 11 over $\mathbb{F}_3$ by a random search, and he provided a method to establish the existence of lifting these surfaces to characteristic 0. This technique has been successfully applied to solve various problems in constructive algebraic geometry (see [10], [12] and [1]).

The Singular or Macaulay2 scripts used to construct and to analyse these surfaces are available at `http://www.math.uni-sb.de/~ag-schreyer` and `http://www.math.colostate.edu/~abo/programs.html`.

1 The Exterior Algebra Method

Our construction of the rational surfaces uses the "Beilinson monad". A Beilinson monad represents a given coherent sheaf in terms of direct sums of (suitably twisted) bundles of differentials and homomorphisms between these bundles, which are given by homogeneous matrices over an exterior algebra E. Recently, Eisenbud, Fløystad and Schreyer [5] showed that for a given sheaf, one can get the Beilinson monad from its "Tate resolution", that is a free resolution over E, by a simple functor. This enables us to discuss the Beilinson monad in an exterior algebra context. In this section, we take a quick look at the exterior algebra method developed by Eisenbud, Fløystad and Schreyer.

1.1 Tate Resolution of a Sheaf

Let W be a $(n+1)$-dimensional vector space over a field K, let V be its dual space, and let $\{x_i\}_{0\leq i\leq n}$ and $\{e_i\}_{0\leq i\leq n}$ be dual bases of V and W respectively. We denote by S the symmetric algebra of W and by E the exterior algebra $\bigwedge V$ on V. Grading on S and E are introduced by $\deg(x)=1$ for $x\in W$ and $\deg(e)=-1$ for $e\in V$ respectively. The projective space of 1-quotients of W will be denoted by $\mathbb{P}^n=\mathbb{P}(W)$.

Let $M=\bigoplus_{i\in\mathbb{Z}} M_i$ be a finitely generated S-graded module. We set

$$\omega_E := \operatorname{Hom}_K(E,K) = \bigwedge W = E\otimes_K \bigwedge^{n+1} W \simeq E(-n-1)$$

and

$$F^i := \operatorname{Hom}_K(E,M_i) \simeq M_i\otimes_K \omega_E.$$

The morphism $\phi_i : F^i \to F^{i+1}$ takes the map $\alpha\in F^i$ to the map

$$\left[e\mapsto \sum_i x_i\alpha(e_i\wedge e)\right]\in F^{i+1}.$$

Then the sequence

$$\mathbf{R}(M):\ \cdots\to F^{i-1}\xrightarrow{\phi_{i-1}} F^i \xrightarrow{\phi_i} F^{i+1}\to\cdots$$

is a complex. This complex is eventually exact. Indeed, $\mathbf{R}(M)$ is exact at $\operatorname{Hom}_K(E,M_i)$ for all $i\geq s$ if and only if $s>r$, where r is the Castelnouvo-Mumford regularity of M (see [5] for a detailed proof). So starting from $\mathbf{T}(M)^{>r}:=\mathbf{T}(M_{>r})$, we can construct a doubly infinite exact E-free complex $\mathbf{T}(M)$ by adjoining a minimal free resolution of the kernel of ϕ_{r+1}:

$$\mathbf{T}(M):\cdots\to T^r\to T^{r+1}:=\operatorname{Hom}_K(E,M_{r+1})\xrightarrow{\phi_{r+1}}\operatorname{Hom}_K(E,M_{r+2})\to\cdots.$$

This E-free complex is called the *Tate resolution* of M. Since $\mathbf{T}(M)$ can be constructed by starting from $\mathbf{R}(M_{>s})$, $s\geq r$, the Tate resolution depends only on the sheaf $\mathcal{F}=\widetilde{M}$ on $\mathbb{P}(W)$ associated to M. We call $\mathbf{T}(\mathcal{F}):=\mathbf{T}(M)$ the *Tate resolution* of $\mathcal{F}$. The following theorem gives a description of all the terms of a Tate resolution:

Theorem 1.1 ([5]). *Let M be a finitely generated graded S-module and let $\mathcal{F}:=\widetilde{M}$ be the associated sheaf on $\mathbb{P}(W)$. Then the term of the complex $\mathbf{T}(\mathcal{F})$ with cohomological degree i is $\bigoplus_j \mathrm{H}^j\mathcal{F}(i-j)\otimes\omega_E$.*

Important to us is also the fact the dual complex $\operatorname{Hom}_E(\mathbf{T}(\mathcal{F}),E)$ stays exact.

1.2 Beilinson Monad

Eisenbud, Fløystad and Schreyer [5] showed, that applying a simple functor to the Tate resolution $\mathbf{T}(\mathcal{F})$, gives a finite complex of sheaves whose homology is the sheaf $\mathcal{F}$ itself: Given $\mathbf{T}(\mathcal{F})$, we define $\Omega(\mathcal{F})$ to be the complex of vector bundles on $\mathbb{P}(W)$ obtained by replacing each summand $\omega_E(i)$ by the bundle $\Omega^i(i)$. The differentials of the complex are given by using isomorphisms

$$\mathrm{Hom}_E(\omega_E(i), \omega_E(j)) \simeq \bigwedge^{i-j} V \simeq \mathrm{Hom}(\Omega^i(i), \Omega^j(j)).$$

Theorem 1.2 ([5]). *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}(W)$. Then $\mathcal{F}$ is the homology of $\Omega(\mathcal{F})$ in cohomological degree 0, and $\Omega(\mathcal{F})$ has no homology otherwise.*

We call $\Omega(\mathcal{F})$ the *Beilinson monad* for $\mathcal{F}$.

2 Construction

In this section we will construct our family of rational surfaces X in $\mathbb{P}^4$ with degree $d = 12$, sectional genus $\pi = 13$. The construction takes the following four steps:

(1) Analyse the monad and parts of the Tate resolution.

(2) Find a smooth surface X with the prescribed invariants over a finite field of a small characteristic.

(3) Determine the type of the linear system, which embeds X into $\mathbb{P}^4$ to justify that the surface X found in the previous step is rational.

(4) Establish the existence of a lift to characteristic zero.

2.1 Analysis of the Monad and Tate Resolution

Let K be a field, let W be a five-dimensional vector space over K with basis $\{x_i\}_{0\leq i\leq 4}$, and let V be its dual space with dual basis $\{e_i\}_{0\leq i\leq 4}$. Let X be a smooth surface in $\mathbb{P}^4 = \mathbb{P}(W)$ with the invariants given above. The first step is to determine the type of the Beilinson monad for the twisted ideal sheaf of X, which is derived from the partial knowledge of its cohomology groups. Such information can be determined from general results such as the Riemann-Roch formula and Kodaira vanishing theorem (see [2] for more detail). We assume that X has the natural cohomology in the range $-1 \leq j \leq 6$ of twists:

$h^i\mathcal{I}_X(j)$

i \ j	-1	0	1	2	3	4	5	6
4								
3	13							
2			4	2				
1					2	3		
0							5	29

Here a zero is represented by the empty box. By Theorem 1.1, the Tate resolution $\mathbf{T}(\mathcal{I}_X)[4] = \mathbf{T}(\mathcal{I}_X(4))$ includes an exact E-free complex of the following type:

$$\to 4\omega_E(3) \to 2\omega_E(2) \oplus 2\omega_E(1) \to 3\omega_E \oplus 5\omega_E(-1) \to 29\omega_E(-2) \to \cdots . \quad (2)$$

From Theorem 1.2, it follows therefore, that the corresponding Beilinson monad for $\mathcal{I}_X(4)$ is of the following type:

$$0 \to 4\Omega^3(3) \xrightarrow{A} 2\Omega^2(2) \oplus 2\Omega^1(1) \xrightarrow{B} 3\mathcal{O} \to 0. \quad (3)$$

The next step is to describe what maps A and B could be the differentials of the monad (3). The identifications

$$\mathrm{Hom}(\Omega^i(i), \Omega^j(j)) \simeq \mathrm{Hom}_E(\omega_E(i), \omega_E(j)) \simeq \mathrm{Hom}_E(E(i), E(j)),$$

allow us to think of the maps A and B as homomorphisms between E-free modules. Since the Tate resolution and its E-dual are exact, the matrix A determines B up to isormorphism.

However, we start with B in our construction. To ease our calculations, we take the map

$$2\omega_E(1) \xrightarrow{B_1} 3\omega_E$$

to be defined by the matrix

$$B_1 = \begin{pmatrix} e_0 & e_1 \\ e_1 & e_2 \\ e_3 & e_4 \end{pmatrix},$$

Since the $\mathrm{GL}(5, K) \times \mathrm{GL}(2, K) \times \mathrm{GL}(3, K)$ orbit of this matrix is dense in $\mathrm{Hom}_E(2\omega_E(1), 3\omega_E)$ this is a reasonable mild additional assumption. The crucial step in the construction is the choice of the map

$$3\omega_E \xrightarrow{C} 4\omega_E(-2),$$

where the target $4\omega_E(-2)$ is a free summand of the cokernel of the map $5\omega_E(-1) \to 29\omega_E(-2)$. Note that $C \circ B = 0$ must hold in the Tate resolution. The condition $C \circ B_1 = 0$ means, that C corresponds to a 4-dimensional quotient space of

$$T = \mathrm{Coker}(2\Lambda^3 W \xrightarrow{B_1} 3\Lambda^2 W).$$

An exterior algebra computation proves that $\dim T = 10 = 3 * 10 - 2 * 10$ as expected. Indeed the map to T is given by the following 10×3 matrix of two forms in E:

$$\varphi = \begin{pmatrix} 0 & 0 & e_3e_4 \\ 0 & -e_3e_4 & e_2e_3 - e_1e_4 \\ -e_3e_4 & 0 & e_1e_3 - e_0e_4 \\ 0 & e_1e_4 - e_2e_3 & e_1e_2 \\ e_2e_3 - e_1e_4 & e_1e_3 - e_0e_4 & -e_0e_2 \\ e_0e_4 - e_1e_3 & 0 & e_0e_1 \\ 0 & e_1e_2 & 0 \\ e_1e_2 & e_0e_2 & 0 \\ e_0e_2 & e_0e_1 & 0 \\ e_0e_1 & 0 & 0 \end{pmatrix}$$

Thus we obtain C from a point $[c] \in \mathbb{G} = \mathbb{G}(10,4)$ in the Grassmanian as the product $C = \varphi \circ c$, where $c \in K^{4\times 10}$ denotes a representing 4×10 matrix. For these C the condition $C \circ B_1 = 0$ will be satisfied.

Consider

$$\overline{\mathbb{M}} = \{[c] \in \mathbb{G} \mid \exists B_2 \text{ with } C \circ B_2 = 0\}.$$

More precisely, we consider those $[c] \in \mathbb{G}$ such that

$$0 \to 2\Lambda^4 W \xrightarrow{B_1} 3\Lambda^3 W \xrightarrow{C} 4W \to 0$$

has two dimensional homology in the middle. The alternating dimensions of the vector spaces in the complex add to zero $2 * 5 - 3 * 10 + 4 * 5 = 0$. The complex is exact for a general choice of $[c] \in \mathbb{G}$ as we see by a computation in an example. Thus $[c] \in \mathbb{G}$, which give the desired two-dimensional homology in the middle, also give two-dimensional homology at the right. We conclude that $\overline{\mathbb{M}} \subset \mathbb{G}$ has codimension at most $4 = 2 * 2$ at such points $[c]$.

Once we have choosen a $[c] \in \overline{\mathbb{M}}$, we can expect, that $B = (B_1, B_2)$ and C determine the monad and hence the desired surface, due to the following Hilbert function argument:

The alternating sum of the dimensions in the complex

$$0 \to 2\Lambda^3 W \oplus 2\Lambda^4 W \xrightarrow{B} 3\Lambda^2 W \xrightarrow{C} 4K \to 0$$

is $2*10+2*5-3*10+4=4$. Hence we expect a 4 dimensional homology on the right, which gives the matrix A.

In summary, we proved the following proposition.

Proposition 2.1. *There exists a quasi-projective subvariety* $\mathbb{M} \subset \mathbb{G}(10,4)$ *of codimension at most* 4*, whose points define a monad of a smooth rational surface in* $\mathbb{P}^4$*. The* $\mathrm{PGL}(5,\overline{K})$ *orbit of each family corresponding to a component of* $\mathbb{M}$ *is an open part of a component of the Hilbert scheme of surfaces.*

Here $\overline{K}$ denotes the algebraic closure of our ground field K.

Proof. Indeed, apart from the condition $[c] \in \overline{\mathbb{M}}$, all other conditions are open conditions. □

However, this does not prove, that $\mathbb{M}$ is non-empty. Note that $\overline{\mathbb{M}}$ is defined over the integers $\mathbb{Z}$.

2.2 Finite Field Search

If $\mathbb{M}$ is not empty we can expect to find a point in $\mathbb{M}(\mathbb{F}_q) \subset \mathbb{G}(\mathbb{F}_q)$ at a rate of $(1:q^4)$ by Proposition 2.1. The statistics suggests that there are two different components of $\overline{\mathbb{M}}(\mathbb{F}_5) \subset \mathbb{G}(\mathbb{F}_5)$, whose elements have syzygies with Betti table

$$\begin{array}{r|cccc} 2 & 4 & . & . & . \\ 1 & . & 3 & 2 & . \\ 0 & . & . & 2 & 4 \\ \text{-1} & . & . & . & 5 \end{array} \quad \text{and} \quad \begin{array}{r|cccc} 2 & 4 & . & . & . \\ 1 & . & 3 & 2 & . \\ 0 & . & . & 2 & 4 \\ \text{-1} & . & . & . & 10 \end{array}$$

However, we never obtained a Beilinson monad of a surface from an example with the Betti table of the second type. So these points do not belong to $\mathbb{M}(\mathbb{F}_5)$. Examples with the first Betti table appeared 18 times in a test of $5^4 \cdot 10$ examples. It will turn out, that this family has indeed codimension 4.

Proposition 2.2. *There is a smooth surface in* $\mathbb{P}^4$ *over* $\mathbb{F}_5$ *with* $d=12$ *and* $\pi=13$.

Proof. By random search, we can find $C \in \mathbb{M}(\mathbb{F}_5)$ and hence B and A satisfying the desired conditions. Determine the corresponding maps $A : 4\Omega^3(3) \to 2\Omega^2(2) \oplus 2\Omega^1(1)$ and $B=(B_2,B_1) : 2\Omega^2(2) \oplus 2\Omega^1(1) \to 3\mathcal{O}$. Then compute the homology $\ker(B)/\operatorname{im}(A)$. If the homology is isomorphic to the ideal sheaf of a surface with the desired invariants, then check smoothness of the surface with the Jacobian criterion. If we are lucky, the surface is smooth. If not, we search for a further $C \in \mathbb{M}(\mathbb{F}_5)$.

For example, the point $[c] \in \overline{\mathbb{M}}(\mathbb{F}_5)$ represented by the matrix

$$c = \begin{pmatrix} 2 & 2 & -2 & 0 & -2 & 2 & -1 & 1 & -1 & -2 \\ 1 & -1 & 2 & 2 & -1 & 2 & 2 & 0 & 2 & -2 \\ 1 & -2 & 1 & -2 & 0 & -1 & -2 & 2 & 1 & -2 \\ -2 & -1 & -2 & -1 & 0 & 2 & 0 & -1 & 2 & 1 \end{pmatrix}$$

leads to a smooth surface in $\mathbb{P}^4$ defined over $\mathbb{F}_5$ of degree $d = 12$ and sectional genus $\pi = 13$. □

2.3 Adjunction Process

In this subsection, we spot the surface found in the previous step within the Enriques-Kodaira classification and determine the type of the linear system that embeds X into $\mathbb{P}^4$. First of all, we recall a result of Sommese and Van de Ven for a surface over $\mathbb{C}$:

Theorem 2.3 ([11]). *Let X be a smooth surface in $\mathbb{P}^n$ over $\mathbb{C}$ with degree d, sectional genus π, geometric genus p_g and irregularity q, let H be its hyperplane class, let K be its canonical divisor and let $N = \pi - 1 + p_g - 1$. Then the adjoint linear system $|H + K|$ defines a birational morphism*

$$\Phi = \Phi_{|H+K|} : X \to \mathbb{P}^{N-1}$$

onto a smooth surface X_1, which blows down precisely all (-1)-curves on X, unless

(i) *X is a plane, or Veronese surface of degree 4, or X is ruled by lines;*

(ii) *X is a Del Pezzo surface or a conic bundle;*

(iii) *X belongs to one of the following four families:*

  (a) *$X = \mathbb{P}^2(p_1, \ldots, p_7)$ embedded by $H \equiv 6L - \sum_{i=0}^{7} 2E_i$;*

  (b) *$X = \mathbb{P}^2(p_1, \ldots, p_8)$ embedded by $H \equiv 6L - \sum_{i=0}^{7} 2E_i - E_8$;*

  (c) *$X = \mathbb{P}^2(p_1, \ldots, p_8)$ embedded by $H \equiv 9L - \sum_{i=0}^{8} 3E_i$;*

  (d) *$X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is an indecomposable rank 2 bundle over an elliptic curve and $H \equiv B$, where B is a section $B^2 = 1$ on X.*

Proof. See [11] for the proof. □

Setting $X = X_1$ and performing the same operation repeatedly, we obtain a sequence

$$X \to X_1 \to X_2 \to \cdots \to X_k.$$

This process will be terminated if $N-1 \leq 0$. For a surface with nonnegative Kodaira dimension, one obtains the minimal model at the end of the adjunction process. If the Kodaira dimension equals $-\infty$, we end up with a ruled surface, a conic bundle, a Del Pezzo surface, $\mathbb{P}^2$, or one of the few exceptions of Sommese and Van de Ven.

It is not known, whether the adjunction theory holds over a finite field. However, we have the following proposition:

Proposition 2.4 ([4], Prop. 8.3). *Let X be a smooth surface over a field of arbitrary characteristic. Suppose that the adjoint linear system $|H+K|$ is base point free. If the image X_1 in $\mathbb{P}^N$ under the adjunction map $\Phi_{|H+K|}$ is a surface of the expected degree $(H+K)^2$, the expected sectional genus $\frac{1}{2}(H+K)(H+2K)+1$ and with $\chi(\mathcal{O}_X)=\chi(\mathcal{O}_{X_1})$, then X_1 is smooth and $\Phi : X \to X_1$ is a simultaneous blow down of the $K_1^2-K^2$ many exceptional lines on X.* □

Remark 2.5. The union of the exceptional divisors contracted in each step is defined over the base field.

In [2] and [4], it is described how to compute the adjunction process for a smooth surface given by explicit equations (see [4] for the computational details). Let X be the smooth surface found in the previous step. The computation for the adjunction process in characteristic 5 gives

$$H \equiv 12L - \sum_{i_1=1}^{2} 4E_{i_1} - \sum_{i_2=3}^{11} 3E_{i_2} - \sum_{i_3=12}^{14} 2E_{i_3} - \sum_{i_4=15}^{21} E_{i_4}, \tag{4}$$

where L is the class of a line in $\mathbb{P}^2$. This process ends with a Del Pezzo surface of degree 7, which is the blowing up of $\mathbb{P}^2$ in two points. Therefore we can conclude that X is rational.

2.4 Lift to Characteristic Zero

In the previous step, we constructed a smooth surface in $\mathbb{P}^4$ defined over $\mathbb{F}_5$. However, our main interest is the field of complex numbers $\mathbb{C}$. In this section, we show the existence of a lift to characteristic 0 as follows: Let $\mathbb{M}$ and $\mathbb{G}$ be given as in the previous subsections.

Proposition 2.6 ([9]). *Let $[c] \in \mathbb{M}(\mathbb{F}_p)$ be a point, where $\mathbb{M} \subset \mathbb{G}$ has codimension 4. Then there exist a number field $\mathbb{L}$, a prime $\mathfrak{p}$ in $\mathbb{L}$ with residue field $\mathcal{O}_{\mathbb{L},\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathbb{L},\mathfrak{p}} \simeq \mathbb{F}_p$ and a family of surfaces $\mathcal{X}$ defined over $\mathcal{O}_{\mathbb{L},\mathfrak{p}}$ with special fiber the surface X defined over $\mathbb{F}_p$ corresponding to $[c]$. Furthermore, since the surface $X/\mathbb{F}_p$ corresponding to $[c]$ is smooth, the surface $X/\mathbb{L}$ corresponding to the generic point of $\operatorname{Spec}\mathbb{L} \subset \operatorname{Spec}\mathcal{O}_{\mathbb{L},\mathfrak{p}}$ is also smooth.*

Proof. Let p be a prime number. If this is not the case, $\mathbb{Z}$ has to be replaced by the ring of integers in a number field which has $\mathbb{F}_p$ as the residue field.

Since $\mathbb{M}$ has pure codimension 4 in $[c]$, there are four hyperplanes $H_1, \dots, H_4$ in $\mathbb{G}$, such that $[c]$ is an isolated point of $\mathbb{M}(\overline{\mathbb{F}}_p) \cap H_1 \cap \cdots \cap H_4$. We may assume that $H_1, \dots, H_4$ are defined over $\operatorname{Spec} \mathbb{Z}$ and that they meet transversally in $[c]$. This allows us to think that $\mathbb{M} \cap H_1 \cap \cdots \cap H_4$ is defined over $\mathbb{Z}$. Let Z be an irreducible component of $\mathbb{M}_{\mathbb{Z}}$ containing C. Then $\dim Z = 1$.

The residue class field of the generic point of Z is a number field $\mathbb{L}$ that is finitely generated over $\mathbb{Q}$, because $\mathbb{M}$ is projective over $\mathbb{Z}$. Let $\mathcal{O}_{\mathbb{L}}$ be the ring of integers of $\mathbb{L}$ and let $\mathfrak{p}$ be a prime ideal corresponding to $[c] \in Z$. Then $\operatorname{Spec} \mathcal{O}_{\mathbb{L},\mathfrak{p}} \to Z \subset \mathbb{M}$ is an $\mathcal{O}_{\mathbb{L},\mathfrak{p}}$-valued point which lifts $[c]$.

Performing the construction of the surface over $\mathcal{O}_{\mathbb{L},\mathfrak{p}}$ gives a flat family $\mathcal{X}$ of surfaces over $\mathcal{O}_{\mathbb{L},\mathfrak{p}}$. Since smoothness is an open property, and since the special fiber $X = \mathcal{X}_{\mathfrak{p}}$ is smooth, the general fiber $\mathcal{X}_{\mathbb{L}}$ is also smooth. □

Next, we argue that the adjunction process of the surface over the number field $\mathbb{L}$ has the same numerical behavior:

Proposition 2.7 ([4], Cor. 8.4). *Let* $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_{\mathbb{L},\mathfrak{p}}$ *be a family as in Proposition 2.6. If the Hilbert polynomial of the first adjoint surface of* $X = \mathcal{X} \otimes \mathbb{F}_q$ *is as expected, and if* $\mathrm{H}^1(X, \mathcal{O}_X(-1)) = 0$*, then the adjunction map of the general fiber* $\mathcal{X}_{\mathbb{L}}$ *blows down the same number of exceptional lines as the adjunction map of the special fiber* X*.* □

Last step in the proof of Theorem 0.1. Let $[c]$ be the element of $\mathbb{M}(\mathbb{F}_5)$, which gives the surface in Proposition 2.2. We check, that $[c]$ satisfies the condition of Proposition 2.6 by computing the Zariski tangent space $T_{\mathbb{M},[c]}$ at $[c]$. Our computation shows that $\operatorname{codim} T_{\mathbb{M},[c]} = 4$. So $\mathbb{M}$ is smooth of codimension 4 at $[c]$, and $[c]$ and hence the surface lift to a number field.

Finally we count dimension. Our component $\mathbb{M} \subset \mathbb{G}(10,4)$ containing $[c]$ has codimension 4, hence dimension $4*(10-4)-4 = 20$. The normalization of B_1 (up to conjugation) gives additional 18 parameters, because the Hilbert scheme of cubic scrolls in $\mathbb{P}(V)$ has dimension 18. So the component of the Hilbert scheme, that contains our surface, has dimension 38. □

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Superisolated Surface Singularities

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Dedicated to Gert-Martin Greuel on the occasion of his 60th birthday

Abstract

In this survey, we review part of the theory of superisolated surface singularities (SIS) and its applications including some new and recent developments. The class of SIS singularities is, in some sense, the simplest class of germs of normal surface singularities. Namely, their tangent cones are reduced curves and the geometry and topology of the SIS singularities can be deduced from them. Thus this class *contains*, in a canonical way, all the complex projective plane curve theory, which gives a series of nice examples and counterexamples. They were introduced by I. Luengo to show the non-smoothness of the μ-constant stratum and have been used to answer negatively some other interesting open questions. We review them and the new results on normal surface singularities whose link are rational homology spheres. We also discuss some positive results which have been proved for SIS singularities.

Introduction

A superisolated surface, SIS for short, singularity $(V,0) \subset (\mathbb{C}^3,0)$ is a generic perturbation of the cone over a (singular) reduced projective plane curve C of degree d, $C = \{f_d(x,y,z) = 0\} \subset \mathbb{P}^2$, by monomials of higher degree. The geometry, resolution and topology of $(V,0)$ is determined by the singularities of C and the pair $(\mathbb{P}^2, C)$. This provides a canonical way to *embed* the classical and rich theory of complex projective plane curves into the theory of normal surface singularities of $(\mathbb{C}^3,0)$. In this way one can use properties of plane curves to get interesting properties of SIS singularities. They were introduced by I. Luengo [45], and were used to answer several questions and

2001 *Mathematics Subject Classification.* 32Sxx, 4B05,14H20,14J17
Key words: normal surface singularities, plane curves, monodromy, deformations

conjectures, like the fact that the μ-constant stratum in the semiuniversal deformation space of an isolated hypersurface singularity is, in general, not smooth. Using Zariski pairs as tangent cones of SIS singularities, E. Artal [2] found also counterexamples for a S. S.-T. Yau's conjecture [91] relating the link of the singularity, the characteristic polynomial and the embedded topology. A *Zariski pair* is a set of two curves $C_1, C_2 \subset \mathbb{P}^2$ with the same combinatorial type but such that $(\mathbb{P}^2, C_1)$ is not homeomorphic to $(\mathbb{P}^2, C_2)$.

In a recent paper [47], A. Némethi and the last two authors have found counterexamples to several conjectures on normal surface singularities whose link is a rational homology sphere. For doing this there were used SIS singularities whose tangent cone is a rational cuspidal curve. It was shown that the *Seiberg-Witten invariant conjecture* (of L.I. Nicolaescu and A. Némethi [56]), the *universal abelian cover conjecture* (of W. Neumann and J. Wahl [64]) and the *geometric genus conjecture* ([62, Question 3.2], see also [55, Problem 9.2]) fail (at least at that generality in which they were formulated).

On the other hand, from the positive point of view, SIS singularities have been used by Pi. Cassou-Noguès and the authors [7] to confirm the Monodromy Conjecture for the topological zeta function introduced by J. Denef and F. Loeser [15]. We review these results in Section 4.

It is interesting to point out that the relationship between plane curves and normal surface singularities can be used also in the other direction: to use results and ideas from normal surface singularities to get new results about curves. In this way, the results in [7] allow to find necessary conditions for the existence of an arrangement of rational plane curves. Even more, J. Fernández de Bobadilla, A. Némethi and the last two authors [20] have found a compatibility property for a rational cuspidal projective plane curve to exist based on a heavily study of the failure of the Seiberg-Witten invariant conjecture of the corresponding SIS singularities.

Since the class of SIS singularities continue being useful we have decided to write down this survey, dedicated to our friend Gert-Martin, where we present known results and open problems on SIS singularities.

1 Superisolated Surface Singularities

1.1 Isolated Hypersurface Singularities

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an analytic function and the corresponding germ $(V, 0) := (f^{-1}(0), 0) \subset (\mathbb{C}^{n+1}, 0)$ of a hypersurface singularity. The *Milnor fibration* of the holomorphic function f at 0 is the C^∞ locally trivial fibration $f| : B_\varepsilon(0) \cap f^{-1}(\mathbb{D}^*_\eta) \to \mathbb{D}^*_\eta$, where $B_\varepsilon(0)$ is the open ball of radius ε centered at 0, $\mathbb{D}_\eta = \{z \in \mathbb{C} : |z| < \eta\}$ and $\mathbb{D}^*_\eta$ is the open punctured disk ($0 < \eta \ll \varepsilon$ and ε small enough). Milnor's classical result also shows that the topology of the germ $(V,0)$ in $(\mathbb{C}^{n+1},0)$ is determined by the pair $(S_\varepsilon^{2n+1}, L_V^{2n-1})$, where $S^{2n+1} = \partial B_\varepsilon(0)$ and $L_V^{2n-1} := S_\varepsilon^{2n+1} \cap V$ is the *link* of the singularity.

Any fiber $F_{f,0}$ of the Milnor fibration is called the *Milnor fiber* of f at 0. The *monodromy transformation* $h : F_{f,0} \to F_{f,0}$ is the well-defined (up to isotopy) diffeomorphism of $F_{f,0}$ induced by a small loop around $0 \in \mathbb{D}_\eta$. The *complex algebraic monodromy of* f *at* 0 is the corresponding linear transformation $h_* : H_*(F_{f,0},\mathbb{C}) \to H_*(F_{f,0},\mathbb{C})$ on the homology groups.

If $(V,0)$ defines a germ of isolated hypersurface singularity then we have that $\tilde{H}_j(F_{f,0},\mathbb{C}) = 0$ but for $j = n$. In particular the non-trivial complex algebraic monodromy will be denoted by $h : H_n(F_{f,0},\mathbb{C}) \to H_n(F_{f,0},\mathbb{C})$ and $\Delta_V(t)$ will denote its characteristic polynomial. The Monodromy Theorem describes the main properties of the monodromy operator, see for instance the references in [19]:

(a) $\Delta_V(t)$ is a product of cyclotomic polynomials.

(b) Let N be the maximal size of the Jordan blocks of h, then $N \leq n+1$.

(c) Let N_1 be the maximal size of the Jordan blocks of h for the eigenvalue 1, then $N_1 \leq n$.

1.2 Normal Surface Singularities

Let $(V,0) = (\{f_1 = \ldots = f_m = 0\},0) \subset (\mathbb{C}^N,0)$ be a normal surface singularity with link L_V. One of the main problems is to determine which analytical properties of $(V,0)$ can be read from the topology of the singularity, see the very nice survey paper by A. Némethi [55]. Since $V \cap B_\varepsilon$ is a cone over the link then L_V characterizes the topological type of $(V,0)$.

The resolution graph $\Gamma(\pi)$ of a resolution $\pi : \tilde{V} \to V$ allows to relate analytical and topological properties of V. Via plumbing construction, W. Neumann [61] proved that the information carried in any resolution graph is the same as the information carried by the link L_V. Let $\pi : \tilde{V} \to V$ be a *good* resolution of the singular point $0 \in V$. Good means that $E = \pi^{-1}(0)$ is a normal crossing divisor. Let $\Gamma(\pi)$ be the dual graph of the resolution (each vertex decorated with the genus $g(E_i)$ and the self-intersection E_i^2 of E_i in $\tilde{V}$). Mumford proved that the intersection matrix $I = (E_i \cdot E_j)$ is negative definite and Grauert proved the converse, i.e., any such graph corresponds to the link of a normal surface singularity.

1.3 Superisolated Surface Singularities

Definition 1.1. A hypersurface surface singularity $(V,0) \subset (\mathbb{C}^3,0)$ defined as the zero locus of $f = f_d + f_{d+1} + \cdots \in \mathbb{C}\{x,y,z\}$ (where f_j is homogeneous of degree j) is *superisolated*, SIS for short, if the singular points of the complex projective plane curve $C := \{f_d = 0\} \subset \mathbb{P}^2$ are not situated on the projective curve $\{f_{d+1} = 0\}$, that is $\operatorname{Sing}(C) \cap \{f_{d+1} = 0\} = \emptyset$ in $\mathbb{P}^2$. Note that C must be a reduced curve.

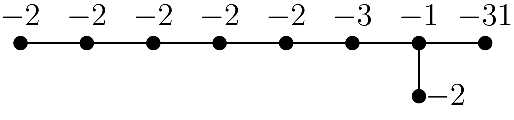
The SIS singularities were introduced by I. Luengo in [45] to study the μ-constant stratum, see Section 2. The main idea is that for a SIS singularity $(V, 0)$, the embedded topological type (and the equisingular type) of $(V, 0)$ does not depend on the choice of f_j's (for $j > d$, as long as f_{d+1} satisfies the above requirement), e.g. one can take $f_j = 0$ for any $j > d+1$ and $f_{d+1} = l^{d+1}$ where l is a linear form not vanishing at the singular points [46].

The minimal resolution. Let $\pi : \tilde{V} \to V$ be the monoidal transformation centered at the maximal ideal $\mathfrak{m} \subset \mathcal{O}_V$ of the local ring of V at 0. Then $(V, 0)$ is a SIS singularity if and only if $\tilde{V}$ is a non-singular surface. Thus π is the *minimal resolution* of $(V, 0)$. To construct the resolution graph $\Gamma(\pi)$ consider $C = D_1 + \ldots + D_r$ the decomposition in irreducible components of the reduced curve C in $\mathbb{P}^2$. Let d_i be the degree of the curve D_i in $\mathbb{P}^2$. Then $\pi^{-1}(0) \cong C = D_1 + \ldots + D_r$ and the self-intersection of D_i in $\tilde{V}$ is $D_i \cdot D_i = -d_i(d - d_i + 1)$, [45, Lemma 3]. Since the link L_V can be identified with the boundary of a regular neighbourhood of $\pi^{-1}(0)$ in $\tilde{V}$ then the topology of the tangent cone determines the topology of the abstract link L_V [45].

The minimal good resolution of $(V, 0)$ is obtained from π by doing the minimal embedded resolution of each plane curve singularity $(C, P) \subset (\mathbb{P}^2, P)$, $P \in \operatorname{Sing}(C)$, which is not an ordinary double point whose branches belong to different global irreducible components. Let D_j be an irreducible component of C such that $P \in D_j$ and with multiplicity $n \geq 1$ at P. After blowing-up at P, the new self-intersection of the (strict transform of the) curve D_j in the (strict transform of the) surface $\tilde{V}$ is $D_j^2 - n^2$. In this way one constructs the minimal good resolution graph Γ of $(V, 0)$.

In particular the theory of hypersurface superisolated surface singularities "contains" in a canonical way the theory of complex projective plane curves.

Example 1.2. Let $f = f_5 + z^6$ be given by the equation $f_5 = z(yz - x^2)^2 - 2xy^2(yz - x^2) + y^5$. The curve C is irreducible with unique singularity at $[0 : 0 : 1]$ (of type $\mathbb{A}_{12}$). The minimal good resolution graph Γ of the superisolated singularity $(V, 0)$ is

Γ :


Here all the curves have genus zero.

The embedded resolution. In [1], the first author studied the Mixed Hodge Structure of the cohomology of the Milnor fibre of a SIS singularity. For that, he constructed in an effective way an embedded resolution of a SIS singularity.

The germ $(V,0) \subset (\mathbb{C}^3,0)$ is an isolated surface singularity. Hence, $H_0(F,\mathbb{C})$ and $H_2(F,\mathbb{C})$ are the only non-vanishing homology vector spaces on which the monodromy acts (we denote the Milnor fiber by F). The only eigenvalue of the action of the monodromy on $H_0(F,\mathbb{C})$ is equal to 1. The Jordan form of the complex monodromy on $H_2(F,\mathbb{C})$ is computed for a SIS singularity. Let $\Delta_V(t)$ be the corresponding characteristic polynomial of the complex monodromy on $H_2(F,\mathbb{C})$. Denote by $\mu(V,0) = \deg(\Delta_V(t))$ the Milnor number of $(V,0) \subset (\mathbb{C}^3,0)$.

Let $\Delta^P(t)$ be the characteristic polynomial (or Alexander polynomial) of the action of the complex monodromy of the germ (C,P) on $H_1(F_{g^P},\mathbb{C})$, (where g^P is a local equation of C at P and F_{g^P} denotes the corresponding Milnor fiber). Let μ^P be the Milnor number of g^P at P.

Let H be a $\mathbb{C}$-vector space and $\varphi : H \to H$ an endomorphism of H. The i-th Jordan polynomial of φ, denoted by $\Delta_i(t)$, is the monic polynomial such that for each $\zeta \in \mathbb{C}$, the multiplicity of ζ as a root of $\Delta_i(t)$ is equal to the number of Jordan blocks of size $i+1$ with eigenvalue equal to ζ.

Let $\Delta_1(t)$ and $\Delta_2(t)$ be the first and the second Jordan polynomials of the complex monodromy on $H_2(F,\mathbb{C})$ of V and let $\Delta_1^P(t)$ be the first Jordan polynomial of the complex monodromy of the local plane singularity (C,P). After the Monodromy Theorem these polynomials joint with $\Delta_V(t)$ and $\Delta^P(t)$, $P \in \operatorname{Sing}(C)$, determine the corresponding Jordan form of the complex monodromy. The Alexander polynomial $\Delta_C(t)$ of the projective plane curve $C \subset \mathbb{P}^2$ was introduced by A. Libgober [37, 38] and F. Loeser and M. Vaquié [42].

Theorem 1.3 ([1]). *Let $(V,0)$ be a SIS singularity whose tangent cone C has r irreducible components. Then the Jordan form of the complex monodromy on $H_2(F,\mathbb{C})$ is determined by the following polynomials*

(i) *The characteristic polynomial $\Delta_V(t)$ is equal to*

$$\Delta_V(t) = \frac{(t^d-1)^{\chi(\mathbb{P}^2\setminus C)}}{(t-1)} \prod_{P\in\operatorname{Sing}(C)} \Delta^P(t^{d+1}).$$

(ii) *The first Jordan polynomial is equal to*

$$\Delta_1(t) = \frac{1}{\Delta_C(t)(t-1)^{r-1}} \prod_{P\in\operatorname{Sing}(C)} \frac{\Delta_1^P(t^{d+1})\Delta_{(d)}^P(t)}{\Delta_{1,(d)}^P(t)^3},$$

where

$$\Delta_{(d)}^P(t) := \gcd(\Delta^P(t),(t^d-1)^{\mu^P}) \text{ and } \Delta_{1,(d)}^P(t) := \gcd(\Delta_1^P(t),(t^d-1)^{\mu^P}).$$

(iii) *The second Jordan polynomial is equal to*

$$\Delta_2(t) = \prod_{P\in \mathrm{Sing}(C)} \Delta^P_{1,(d)}(t).$$

The first part of the theorem was stated by J. Stevens in [79]. A general formula for the zeta function of the monodromy was proved by D. Siersma [76] (see also [51],[28]). In particular the Milnor number $\mu(V,0)$ of a SIS singularity verifies the identity

$$\mu(V,0) = (d-1)^3 + \sum_{P\in \mathrm{Sing}\, C} \mu^P.$$

Yomdin Singularities, Series of Singularities and Spectrum.

The first natural generalization of superislolated singularities are Yomdin singularities, where $d+1$ is replaced by $d+k$.

Let $(V,0) \subset (\mathbb{C}^{n+1},0)$ be the germ of hypersurface defined by $f = 0$, $f = f_d + f_{d+k} + \ldots \in \mathbb{C}\{x_0,\ldots,x_n\}$. The singularity $(V,0)$ is called of *Yomdin type* if $\mathrm{Sing}(\{f_d = 0\}) \cap \{f_{d+k} = 0\} = \emptyset$ in $\mathbb{P}^n$.

For each $P \in \mathrm{Sing}(\{f_d = 0\})$, let g^P be a local equation of $\{f_d = 0\} \subset \mathbb{P}^n$ at P. Formulæ for the Milnor number (see [92, 35, 46]) and for the zeta function $\zeta_f(t)$ of the complex monodromy can be written as follows [79, 76, 28, 51]:

$$\mu(V,0) = (d-1)^{n+1} + k \sum_{P\in \mathrm{Sing}\{f_d=0\}} \mu^P,$$

and

$$\zeta_f(t) = (1-t^d)^{\chi(\mathbb{P}^n\setminus\{f_d=0\})} \left(\prod_{P\in \mathrm{Sing}\{f_d=0\}} (1-t^{d+k})(\zeta_g^P)^k(t^{d+k}) \right)^{-1}.$$

Here, $(\zeta_g^P)^k(t)$ is the monodromy zeta function of the k-power of the corresponding monodromy $\zeta_g^P(t)$ of g at P.

Let H be a hyperplane such that $\mathrm{Sing}(\{f_d = 0\}) \cap H = \emptyset$, H being the zero locus of a linear form g. Then the family

$$F(x_0,\ldots,x_n,t) = f_d + (1-t)(f-f_d) + tg^{d+k}$$

is a μ-constant family (in fact a μ^*-constant family), see [46]. It means that to study properties of Yomdin type singularities which are preserved under μ-constant deformations is equivalent to study *series* of singularities of type $f_d + g^{d+k}$. Notice that in such a case the singular locus of f_d is 1-dimensional.

Let f be a germ of an analytic function at zero whose singular locus is 1-dimensional. Let g be a generic linear function such that $g(0) = 0$. Y. Yomdin [92] compared the vanishing cohomologies of their Milnor fibres (and then its Milnor numbers) of f and $f + g^N$, for N big enough. Later on D. Siersma [76] compared the zeta functions of their monodromies. Finally it was J. Steenbrink [78] who conjectured a relationship between the spectrum $\mathrm{Sp}(f, 0)$ of f and the spectrum $\mathrm{Sp}(f + g^N, 0)$ of $f + g^N$. This conjecture was proved by M. Saito [74] using his theory of mixed Hodge modules. Another proof has been given by A. Némethi and J. Steenbrink, [59]. Recently G. Guibert, F. Loeser and M. Merle [27] have proved Steenbrink's conjecture without any condition on the singular locus of f and g being any function vanishing at 0.

The notion of a spectrum $\mathrm{Sp}(f, x)$ at x of a function f on a smooth complex algebraic variety was introduced by J. Steenbrink in [77] and by A. Varchenko in [84]. It is a fractional Laurent polynomial $\sum_{\alpha\in\mathbb{Q}} n_\alpha t^\alpha$, $n_\alpha \in \mathbb{Z}$ defined using the semi-simple part of the action of the monodromy on the mixed Hodge structure on the cohomology of the Milnor fibre of f at x. Here we use the convention given by M. Saito in [74] (denoted by $\mathrm{Sp}'(f, x)$) which differs from that in [78] by multiplication by t (see Remark 2.3 in [74]).

Let $f = f_d + f_{d+k} + \ldots \in \mathbb{C}\{x_0, \ldots, x_n\}$ define a Yomdin type singularity $(V, 0) \subset (\mathbb{C}^{n+1}, 0)$ and, for each $P \in \mathrm{Sing}(\{f_d = 0\})$, let g^P be a local equation of $\{f_d = 0\} \subset \mathbb{P}^n$ at P. Since the spectrum does not change under μ-constant deformations, see [82, 83], then the spectrum $\mathrm{Sp}(f, 0)$ of $(V, 0)$ can be computed via [78, Theorem 6.1] and [74, Theorem 5.7] in terms of the spectral numbers (also called exponents) $\{\alpha_i^P\}_P$ of g^P at P.

Theorem 1.4 ([46, 74, 78]). *With the previous notations, the spectrum* $\mathrm{Sp}(f, 0)$ *of a Yomdin singularity* $(V, 0) \subset (\mathbb{C}^{n+1}, 0)$ *defined by* $f = f_d + f_{d+k} + \ldots \in \mathbb{C}\{x_0, \ldots, x_n\}$ *is equal to*

$$\begin{aligned}\mathrm{Sp}(f,0) = \;& \left(\frac{t^{1/d} - t}{1 - t^{1/d}}\right)^n - \left(\frac{1-t}{1-t^{1/d}}\right) \sum_{P\in\mathrm{Sing}(C)} \sum_{\alpha_i^P\in\mathrm{Sp}(g^P,P)} t^{\beta_i^P} \\ & + \left(\frac{1-t}{1-t^{1/(d+k)}}\right) \sum_{P\in\mathrm{Sing}(C)} \sum_{\alpha_i^P\in\mathrm{Sp}(g^P,P)} t^{\gamma_i^{P,k}},\end{aligned}$$

where $\gamma_i^{P,k} := \dfrac{k\alpha_i^P + \lfloor d(\alpha_i^P - 1)\rfloor + d + 1}{d+k}$ *and* $\beta_i^P := \dfrac{\lfloor d(\alpha_i^P - 1)\rfloor + d + 1}{d}$.

The study of non-isolated singularities defined by an analytic complex function f using perturbation $f + g^k$, g a generic linear form, has been extensively studied mostly using polar methods, Lê cycles and other methods, see [48, 49, 80] and references therein.

2 Deformations

Let $p : \mathcal{V} \to \mathcal{T}$ be a deformation of a SIS singularity $(V, 0) \subset (\mathbb{C}^3, 0)$ with a section σ such that $(V_t, \sigma(t))$ is an isolated singularity (one may assume that $\mathcal{T}$ is one-dimensional and smooth). In general $(V_t, \sigma(t))$ is not a SIS singularity but if the corresponding multiplicities coincides, that is $\operatorname{mult}(V_t, \sigma(t)) = \operatorname{mult}(V, 0)$, then $(V_t, \sigma(t))$ is a SIS singularity, because one can take local coordinates such that $F(x, y, z, t) = 0$ is the equation of $(\mathcal{V}, 0) \subset (\mathbb{C}^3 \times \mathbb{C}, 0)$, $\sigma(t) = (0, t)$, and $F_d(x, y, z, t) = 0$ gives the tangent cone of $(V_t, \sigma(t))$ for all t. Thus p induces a deformation $P : \mathcal{C} \to \mathcal{T}$ of the tangent cone $C \subset \mathbb{P}^2$ and since the condition of being a SIS singularity is open then $\operatorname{Sing}(C_t) \cap \{f_{d+1,t} = 0) = \emptyset$ for t close to 0.

Assume now that p is a μ-constant deformation, that is $\mu(V_t, \sigma(t)) = \mu(V, 0)$ along the family. Even in this case it is not known that the multiplicity is constant. In fact the following well-known problems are still open also for SIS singularities.

Problem 2.1. Let $p : \mathcal{V} \to \mathcal{T}$ be a deformation of a SIS singularity $(V, 0) \subset (\mathbb{C}^3, 0)$ such that $\mu(V_t, \sigma(t)) = \mu(V, 0)$. Is it true that the multiplicity is constant?

Problem 2.2. Let $p : \mathcal{V} \to \mathcal{T}$ be a deformation of a SIS singularity $(V, 0) \subset (\mathbb{C}^3, 0)$ such that $\mu(V_t, \sigma(t)) = \mu(V, 0)$. Is it a topologically constant deformation?

In [45, Theorem 1], the second author gives an affirmative answer for Problem 2.1 using B. Perron results (see [70]) whose proof turned out to be incomplete. By putting together [45, Theorem 2] and the correct part of [45, Theorem 1] one gets:

Theorem 2.3. *Let $p : \mathcal{V} \to \mathcal{T}$ be a deformation of a SIS singularity $(V, 0) \subset (\mathbb{C}^3, 0)$. Then the following conditions are equivalent:*

(a) $(V, 0)$ *is topologically equivalent to* $(V_t, \sigma(t))$,

(b) $\mu(V_t, \sigma(t)) = \mu(V, 0)$ *and* $\operatorname{mult}(V_t, \sigma(t)) = \operatorname{mult}(V, 0)$,

(c) *the family* $\{(V_t, \sigma(t))\}_t$ *is* μ^**-constant,*

(d) $\operatorname{mult}(V_t, \sigma(t)) = \operatorname{mult}(V, 0)$ *and the induced deformation* $P : \mathcal{C} \to \mathcal{T}$ *of the tangent cone* $C \subset \mathbb{P}^2$ *of* $(V, 0)$ *is equisingular.*

In [44], it was shown how to compute, in an effective way, equations for the equisingularity stratum Σ_C of C in the family of all projective plane curves of degree d, giving examples in which Σ_C is not smooth. Thus if one considers the SIS singularity with such tangent cone, then one gets that the μ^*-constant stratum in the versal deformation is not smooth.

The simplest example is $f = y(xy^3 + z^4)^2 + x^9 + y^{10}$. Then C has only one singular point with an $\mathbb{A}_{35}$ singularity, and Σ_C is singular. J. Stevens [79] using the V-filtration proved that the μ^*-constant stratum is a component of the μ-constant stratum giving the non-smoothness of the μ-constant stratum.

The nice construction by V. A. Vasil'iev and V.V. Serganova in [85], using matroids, gives another examples with non-smooth μ^*-constant stratum. The study of the properties of the equisingularity stratum Σ_C of curves is a classical subject which gained a great impulse with the work of Gert-Martin Greuel, Ch. Lossen and E. Shustin. See [25] for a detailed account of the subject and references.

Let $(V, 0) \subset (\mathbb{C}^3, 0)$ be an isolated surface singularity. B. Teissier asked whether $(V, 0)$ can be put in a μ^*-constant family such that there exists a member of the family which is defined over $\mathbb{Q}$ (resp. $\mathbb{R}$). Using SIS singularities one can answer negatively to this question. Namely, it is known that there are many curves $C \subset \mathbb{P}^2$ such that no element of the equisingularity stratum Σ_C can be defined over $\mathbb{Q}$ (or $\mathbb{R}$), see [93] and the end of next section. For such a curve not defined over $\mathbb{R}$ see [4]. If one takes a SIS singularity over such a curve, Theorem 2.3 gives us that no member of a μ^*-constant deformation can be defined over $\mathbb{Q}$ (or $\mathbb{R}$).

3 Zariski Pairs

Let us consider $C \subset \mathbb{P}^2$ a reduced projective curve of degree d defined by an equation $f_d(x, y, z) = 0$. If $(V, 0) \subset (\mathbb{C}^3, 0)$ is a SIS singularity with tangent cone C, then the link L_V of the singularity is completely determined by C. Let us recall, that L_V is a Waldhausen manifold and its plumbing graph is the dual graph of the good minimal resolution. In order to determine L_V we do not need the embedding of C in $\mathbb{P}^2$, but only its embedding in a regular neighborhood. The needed data can be encoded in a combinatorial way.

Definition 3.1. Let $\operatorname{Irr}(C)$ be the set of irreducible components of C. For $P \in \operatorname{Sing}(C)$, let $B(P)$ be the set of local irreducible components of C. The *combinatorial type* of C is given by:

- A mapping $\deg : \operatorname{Irr}(C) \to \mathbb{Z}$, given by the degrees of the irreducible components of C.
- A mapping $\operatorname{top} : \operatorname{Sing}(C) \to \operatorname{Top}$, where Top is the set of topological types of singular points. The image of a singular point is its topological type.
- For each $P \in \operatorname{Sing}(C)$, a mapping $\beta_P : T(P) \to \operatorname{Irr}(C)$ such that if γ is a branch of C at P, then $\beta_P(\gamma)$ is the global irreducible component containing γ.

Remark 3.2. There is a natural notion of isomorphism of combinatorial types. It is easily seen that combinatorial type determines and is determined by any of the following graphs (with vertices decorated with self-intersections):

- The dual graph of the preimage of C by the minimal resolution of $\mathrm{Sing}'(C)$. The set $\mathrm{Sing}'(C)$ is obtained from $\mathrm{Sing}(C)$ by forgetting ordinary double points whose branches belong to distinct global irreducible components. We need to mark in the graph the r vertices corresponding to $\mathrm{Irr}(C)$.
- The dual graph of the minimal good minimal of V. Since the minimal resolution is unique, it is not necessary to mark vertices.

Note also that the combinatorial type determine the Alexander polynomial $\Delta_V(t)$ of V (see Theorem 1.3).

Definition 3.3. A *Zariski pair* is a set of two curves $C_1, C_2 \subset \mathbb{P}^2$ with the same combinatorial type but such that $(\mathbb{P}^2, C_1)$ is not homeomorphic to $(\mathbb{P}^2, C_2)$. An *Alexander-Zariski pair* $\{C_1, C_2\}$ is a Zariski pair such that the Alexander polynomials of C_1 and C_2 do not coincide.

In [1], (see here Theorem 1.3) it is shown that Jordan form of complex monodromy of a SIS singularity is determined by the combinatorial type and the Alexander polynomial of its tangent cone. The first example of Zariski pair was given by Zariski, [94, 95]; there exist sextic curves with six ordinary cusps. If these cusps are (resp. not) in a conic then the Alexander polynomial equals $t^2 - t + 1$ (resp. 1). Then, it gives an Alexander-Zariski pairs. Many other examples of Alexander-Zariski pairs have been constructed (Artal,[2], Degtyarev [13]). We state the main result in [1].

Theorem 3.4. *Let V_1, V_2 be two SIS singularities such that their tangent cones form an Alexander-Zariski pair. Then V_1 and V_2 have the same abstract topology and characteristic polynomial of the monodromy but not the same embedded topology.*

Recall that the Jordan form of the monodromy is an invariant of the embedded topology of a SIS singularity (see Theorem 1.3); since it depends on the Alexander polynomial $\Delta_C(t)$ of the tangent cone, we deduce this theorem.

Remark 3.5. Every SIS singularity of Theorem 3.4 provides a counter-example to a Conjecture by S.S.T Yau stated in [91]: *abstract topology and characteristic polynomial of the monodromy determine embedded topology.*

There are also examples of Zariski pairs which are not Alexander-Zariski pairs (see [68, 3, 5]). Some of them are distinguished by the so-called characteristic varieties introduced by Libgober [39]. These are subtori of $(\mathbb{C}^*)^r$,

$r := \#\operatorname{Irr}(C)$, which measure the excess of Betti numbers of finite Abelian coverings of the plane ramified on the curve (as Alexander polynomial does it for cyclic coverings).

Problem 3.6. How can one translate characteristic varieties of a projective curve in terms of invariants of the SIS singularity associated to it?

Though Alexander polynomial and characteristic varieties are topological invariants, they are in fact arithmetic invariants in the following sense. Let us suppose that a curve C is defined by a polynomial with coefficients in a number field K; then Alexander polynomial and characteristic varieties can be computed *inside* K, i.e., they do not depend on the embedding $K \hookrightarrow \mathbb{C}$.

Definition 3.7. An *arithmetic Zariski pair* is a Zariski pair such that its elements are defined with coefficients in a number field and with conjugate equations by the action of a Galois element.

The existence of arithmetic Zariski pairs is a consequence of a work of Serre [75] and Chisini's conjecture [33]. Explicit examples have been found in [4]; moreover, there are a lot of candidates to be arithmetic Zariski pairs, for example, sextic curves with an $\mathbb{A}_{19}$ singularity (discovered by Yoshihara [93]).

Problem 3.8. Let C_1, C_2 be an arithmetic Zariski pair and let V_1, V_2 SIS singularities such that C_1, C_2 are their respective tangent cones. Do they have the same embedded topological type?

4 Monodromy Conjecture

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function and let

$$(V, 0) := (f^{-1}(0), 0) \subset (\mathbb{C}^{n+1}, 0)$$

be the germ of hypersurface singularity defined by the zero locus of f.

Let $\pi : (Y, \mathcal{D}) \to (\mathbb{C}^{n+1}, 0)$ be an *embedded resolution* of $(V, 0)$, that is, a proper analytic map on a non-singular complex manifold Y such that:

(1) the analytic subspace $\mathcal{D} := \pi^{-1}(0)$ of Y is the union of non-singular n-dimensional manifolds in Y which are in general position;

(2) the map $\pi|_{Y\setminus\mathcal{D}}$ is an analytic isomorphism: $Y \setminus \mathcal{D} \to \mathbb{C}^{n+1} \setminus 0$;

(3) in a neighbourhood of any point of $\mathcal{D}$ there exist a local system of coordinates $y_0, \ldots, y_n$ such that $f \circ \pi(y_0, \ldots, y_n) = y_0^{N_0} \cdots y_n^{N_n}$.

Let $E_i, i \in I$, be the irreducible components of the divisor $\pi^{-1}(f^{-1}(0))$. For each subset $J \subset I$ we set

$$E_J := \bigcap_{j \in J} E_j\,, \quad \text{and} \quad \check{E}_J := E_J \setminus \bigcup_{j \notin J} E_{J \cup \{j\}}.$$

For each $j \in I$, let us denote by N_j the multiplicity of E_j in the divisor of $f \circ \pi$ and by $\nu_j - 1$ the multiplicity of E_j in the divisor of $\pi^*(\omega)$ where ω is a non-vanishing holomorphic $(n+1)$-form in $\mathbb{C}^{n+1}$.

The invariant we are interested in is the *local topological zeta function* $Z_{top,0}(f,s) \in \mathbb{Q}(s)$, which is an analytic (but not topological, see [6]) subtle invariant associated with any germ of an analytic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. This rational function was first introduced by J. Denef and F. Loeser as a sort of limit of the p-adic Igusa zeta function, see [15, 16]. The original definition was written in terms of an embedded resolution of its zero locus germ $(V, 0) \subset (\mathbb{C}^{n+1}, 0)$ (although it does not depend on any particular resolution). In [16], J. Denef and F. Loeser gave an intrinsic definition of $Z_{top,0}(f,s)$ using arc spaces and the motivic Igusa zeta function, −see also [17] and the Séminaire Bourbaki talk of E. Looijenga [43].

The *local topological zeta function* of f is:

$$Z_{top,0}(f,s) := \sum_{J \subset I} \chi(\check{E}_J \cap \pi^{-1}(0)) \prod_{j \in J} \frac{1}{\nu_j + N_j s} \in \mathbb{Q}(s),$$

where χ denotes the Euler-Poincaré characteristic. Each exceptional divisor E_j of an embedded resolution $\pi : (Y, \mathcal{D}) \to (\mathbb{C}^{n+1}, 0)$ of the germ $(V, 0)$ gives a candidate pole $-\nu_j/N_j$ of the rational function $Z_{top,0}(f,s)$. Nevertheless only a few of them give an actual pole of $Z_{top,0}(f,s)$. There are several conjectures related to the topological zeta functions. We focus our attention in the *Monodromy Conjecture*, see [14, 15].

Conjecture 4.1 (Local Monodromy). *If s_0 is a pole of the topological zeta function $Z_{top,0}(f,s)$ of the local singularity defined by f, then* $\exp(2i\pi s_0)$ *is an eigenvalue of the local monodromy at some complex point of $f^{-1}(0)$.*

If f defines an isolated hypersurface singularity, then $\exp(2i\pi s_0)$ has to be an eigenvalue of the complex algebraic monodromy of the germ $(f^{-1}(0), 0)$.

There are three general problems to consider when trying to prove (or disprove) the conjecture using resolution of singularities:

(i) Explicit computation of an embedded resolution of the hypersurface $(V, 0) \subset (\mathbb{C}^{n+1}, 0)$.

(ii) Determination of the poles $\{-\nu_j/N_j\}$ of $Z_{top,0}(f,s)$.

(iii) Explicit computation of the eigenvalues of the complex algebraic monodromy (or computing the characteristic polynomials of the corresponding action of the complex algebraic monodromy) in terms of the resolution data.

The Monodromy Conjecture, which was first stated for the Igusa zeta function, has been proved for curve singularities by F. Loeser [40]. F. Loeser actually proved a stronger version of the Monodromy Conjecture: *any pole of the topological zeta function gives a root of the Bernstein polynomial of the singularity.* The behaviour of the topological zeta function for germs of curves is rather well understood once an explicit embedded resolution $\pi : (Y, \mathcal{D}) \to (\mathbb{C}^2, 0)$ of curve singularities is known, e.g. the minimal one. Basically, the poles are the $\{-\nu_j/N_j\}$ coming from rupture components in the minimal resolution, see the proof by Veys [86, 87] . The case of curves was proved in consecutive works by Strauss, Meuser, Igusa and Loeser for Igusa's local zeta function, but the same proof works for the topological zeta function. There are other recent proofs of the conjecture for the case of curves by Cassou-Noguès and the authors [8], Nicaise [67] and Rodrigues [72].

There are other classes of singularities where the embedded resolution is known. For example, for any singularity of hypersurface defined by an analytic function which is non-degenerated with respect to its Newton polytope, problems (i) and (iii) above are solved. Nevertheless, (ii) seems to be a hard combinatorial problem. This problem was partially solved by Loeser in the case where f has a non-degenerate Newton polytope and satisfies certain extra technical conditions, –[41].

Even in one of the simplest cases where f has non-isolated singularities, namely the cone over a curve, problems (i) and (iii) are solved, but (ii) presents serious difficulties. B. Rodrigues and W. Veys proved in [73] the Monodromy Conjecture for any homogeneous polynomial $f_d \in \mathbb{C}[x_1, x_2, x_3]$ satisfying $\chi(\mathbb{P}^2 \setminus \{f_d = 0\}) \neq 0$. In [7] the authors complete the proof of this case studying homogeneous polynomials $f_d \in \mathbb{C}[x_1, x_2, x_3]$ satisfying $\chi(\mathbb{P}^2 \setminus \{f_d = 0\}) = 0$.

As we mentioned before, an embedded resolution is also known for superisolated surface singularities, –see [1]. This allow Pi. Cassou-Noguès and the authors to solve problems (ii) and (iii) for SIS singularities, namely the main result of [7] is to prove:

Theorem 4.2 ([7]). *The local Monodromy Conjecture is true for superisolated surface singularities.*

The local topological zeta function of a SIS singularity satisfies the following equality, see [7, Corollary 1.12]:

$$
\begin{aligned}
Z_{top,0}(V,s) = {} & \frac{\chi(\mathbb{P}^2\setminus C)}{t-s} + \frac{\chi(\check{C})}{(t-s)(s+1)} + \\
& + \sum_{P\in\operatorname{Sing}(C)} \left(\frac{1}{t} + (t+1)\left(\frac{1}{(t-s)(s+1)} - \frac{1}{t}\right) Z_{top,P}(g^P,t)\right),
\end{aligned}
$$

where g^P is a local equation of C at P, $\check{C} := C\setminus\operatorname{Sing}(C)$ and $t := 3+(d+1)s$. The following properties can be easily described from the previous equalities:

Proposition 4.3. *Let $\mathcal{P}$ be the set of poles of $Z_{top,0}(V,s)$.*

(i) $\mathcal{P} \subset \{-1, -\frac{3}{d}\} \cup \bigcup_{P\in\operatorname{Sing}(C)} \left\{ -\frac{3+t_0}{(d+1)} \,\middle|\, t_0 \text{ pole of } Z_{top,P}(g^P,t)\right\}$.

(ii) *If $-\frac{3}{d} \neq s_0 \in \mathcal{P}$ then* $\exp(2i\pi s_0)$ *is an eigenvalue of the complex algebraic monodromy of V.*

(iii) *Let $s_0 = -\frac{3}{d}$. If s_0 is a pole of $Z_{top,P}(C,s)$ at some point $P \in \operatorname{Sing}(C)$ and either $\chi(\mathbb{P}^2\setminus C) > 0$ or $\chi(\mathbb{P}^2\setminus C) = 0$, then* $\exp(2i\pi s_0)$ *is an eigenvalue of the complex algebraic monodromy of V.*

(iv) *If $s_0 = -\frac{3}{d}$ is a multiple pole of $Z_{top,0}(V,s)$ then* $\exp(2i\pi s_0)$ *is an eigenvalue of the local algebraic monodromy at some singular point of C.*

(v) *If $s_0 = -\frac{3}{d}$ is not a pole of $Z_{top,P}(C,s)$, the residue of $Z_{top,0}(V,s)$ at $-\frac{3}{d}$ equals $d\rho(C)$ where*

$$
\rho(C) := \chi(\mathbb{P}^2\setminus C) + \chi(\check{C})\frac{d}{d-3} + \sum_{P\in\operatorname{Sing}(C)} Z_{top,P}\left(C, -\frac{3}{d}\right).
$$

Following Proposition 4.3, the Monodromy Conjecture for SIS singularities is proved in all but two cases:

(N-1) $\chi(\mathbb{P}^2\setminus C) = 0$, $s_0 = -\frac{3}{d}$ is not a pole for the local topological zeta function at any singular point in C and $\rho(C) \neq 0$.

(N-2) $\chi(\mathbb{P}^2\setminus C) < 0$.

The *bad divisors* are the degree d effective divisor D on $\mathbb{P}^2$ ($d > 3$) such that $\chi(\mathbb{P}^2\setminus D) \leq 0$ and $s_0 = -\frac{3}{d}$ is not a pole of $Z_{top,P}(g_D^P, s)$, for any singular point P in its support D_{red}, where g_D^P is the local equation of the divisor D at P. The main part of [7, §2] is devoted to determine the bad divisors D on $\mathbb{P}^2$ such that $\rho(D) \neq 0$ and finally to prove the Monodromy Conjecture.

Note that the Euler-Poincaré characteristic condition on a bad divisor D implies that D has at least two irreducible components, all of them rational curves, see [31, 30, 32, 9]. In fact, the main result in [7] can be used to study arrangements $C = C_1 + \ldots + C_s$ of rational plane curves such that $\chi(\mathbb{P}^2 \setminus C) \leq 0$. In particular, some necessary conditions on the combinatorial type of C (see Section 3) are obtained in order to the curve C exists.

The authors and S.M. Gusein-Zade have computed in an unpublished work the topological zeta function for Yomdin surface singularities, obtaining also a similar formula to the one for SIS singularities.

To avoid problems (i) and (ii) one can compute the so called motivic Igusa zeta function using motivic integration. In particular Pi. Cassou-Noguès and the authors in [8] have verified the conjecture (even the original conjecture by Igusa) for *quasi-ordinary hypersurface singularities* in arbitrary dimension measuring arcs and using Newton maps [8].

5 SIS Singularities with Rational Homology Sphere Links and Rational Cuspidal Curves

Superisolated surface singularities can be used to construct normal surface singularities whose link are rational homology spheres.

Let $(V, 0) = (\{f_1 = \ldots = f_m = 0\}, 0) \subset (\mathbb{C}^N, 0)$ be a normal surface singularity with link L_V. One of the main problems is to determine which analytical properties of $(V, 0)$ can be read from the topology of the singularity, see [55]. Let $\pi : \tilde{V} \to V$ be a resolution of V.

The link L_V is called a *rational homology sphere* ($\mathbb{Q}$HS) if $H_1(L_V, \mathbb{Q}) = \{0\}$, and L_V is called an *integer homology sphere* ($\mathbb{Z}$HS) if $H_1(L_V, \mathbb{Z}) = \{0\}$. In general the first Betti number $b_1(L_V) = b_1(\Gamma(\pi)) + 2\sum_i g(E_i)$, where $b_1(\Gamma(\pi))$ is the number of independent cycles of the graph. In fact L_V is a $\mathbb{Q}$HS if and only if $\Gamma(\pi)$ is a tree and every E_i is a rational curve. If additionally the intersection matrix has determinant ± 1 then L_V is an $\mathbb{Z}$HS.

Example 5.1. If $(V, 0) \subset (\mathbb{C}^3, 0)$ is a SIS singularity with an irreducible tangent cone $C \subset \mathbb{P}^2$ then L_V is a rational homology sphere if and only if C is a rational curve and each of its singularities (C, p) is locally irreducible, i.e a cusp.

In [47] A. Némethi and the last two authors have used SIS singularities whose link is a rational homology sphere to disprove several conjectures made during last years, see loc. cit. for a series of counterexamples. In Example 5.2 we present one of them.

For instance, it is shown that in the $\mathbb{Q}$HS link case the geometric genus p_g (analytical property of $(V, 0)$) does not depend only on its link L_V, even if we work only with Gorenstein singularities (cf. [62, Question 3.2], see also [55, Problem 9.2]). Moreover for $\mathbb{Q}$-Gorenstein singularities (with $b_1(L_V) = 0$) analytical properties like the multiplicity, embedded dimension, Hilbert-Samuel function are not topological properties.

It is also shown that the universal abelian cover conjecture by Neumann and Wahl in [64] did not held with the generality they stated it. The starting point of the conjecture was Neumann's result [60] that the universal abelian cover of a singularity with a good $\mathbb{C}^*$-action and with $b_1(L_V) = 0$ is a Brieskorn complete intersection whose weights can be determined from the Seifert invariants of the link. Their original conjecture was:

Assume that $(V, 0)$ *is* $\mathbb{Q}$*-Gorenstein singularity satisfying* $b_1(L_V) = 0$*. Then there exists an equisingular and equivariant deformation of the universal abelian cover of* $(V, 0)$ *to an isolated complete intersection singularity. Moreover, the equations of this complete intersection, together with the action of* $H_1(L_V, \mathbb{Z})$*, can be recovered from* L_V *via the "splice equations".*

The semigroup condition as stated in [64] does not hold in general. Thus Neumann and Wahl restrict themselves to a very interesting class of complete intersection normal complex surface singularities called *splice type singularities*, see [65, 66]. In [66] the authors conjectured that rational singularities and $\mathbb{Q}$HS link minimally elliptic singularities belong to the class of *splice type singularities*. Just recently T. Okuma in [69] has given a proof of this result. See the paper by J. Wahl [89] in these proceedings.

Another conjecture that was disproved in [47] was the Seiberg-Witten invariant conjecture (SWC). A. Némethi and L. Nicolaescu [56] offered a candidate as a topological bound for the geometric genus of a rational homology sphere link of a normal normal surface singularity: *Let* L_V *be the link of a normal surface singularity.*

(a) *If* L_V *is a rational homology sphere then*

$$p_g \leq \mathbf{sw}(L_V) - (Z_K^2 + s)/8.$$

(b) *Additionally, if the singularity is* $\mathbb{Q}$*-Gorenstein, then in* (a) *the equality holds.*

Here Z_K is the *canonical cycle* associated with $\Gamma(\pi)$, and s the number of vertices in $\Gamma(\pi)$. Then $Z_K^2 + s$ does not depend on the choice of $\Gamma(\pi)$, it is a topological invariant of L_V. Set $H := H_1(L_V, \mathbb{Z})$.

The *Seiberg-Witten invariant* $\mathbf{sw}(L_V)$ of the link L_V (associated with the canonical *spin*^{c} structure) is

$$\mathbf{sw}(L_V) := -\frac{\lambda(L_V)}{|H|} + \mathcal{T}(L_V),$$

where $\mathcal{T}(M)$ is the sign-refined Reidemeister-Turaev torsion $\mathcal{T}(M)$ (associated with the canonical *spin*^{c} structure) [81] and $\lambda(L_V)$ is the normalized by the Casson-Walker invariant, using the convention of [36] (cf. also with [56, 57, 58, 55]). Both invariants $\mathcal{T}(L_V)$ and $\lambda(L_V)$ can be determined from the graph (for details, see [56] or [55]).

The SWC-conjecture was verified by Némethi and Nicolaescu for quotient singularities [56], for singularities with good $\mathbb{C}^*$-actions [57] and hypersurface suspension singularities $g(u,v)+w^n$ with g irreducible [58].

Let $(V,0) \subset (\mathbb{C}^3,0)$ be a SIS singularity whose tangent cone $C \subset \mathbb{P}^2$ is an irreducible rational cuspidal curve (each singularity of C is locally irreducible).

We denote by Δ^P the characteristic polynomial of $(C,P) \subset (\mathbb{P}^2,P)$, set $\Delta(t) := \prod_{P\in\mathrm{Sing}(C)} \Delta^P(t)$ and $2\delta := \deg \Delta(t)$. By the rationality of C one has

$$(d-1)(d-2) = 2\delta = \sum_{P\in\mathrm{Sing}(C)} \mu^P,$$

where δ is the sum of the delta-invariants of the germs (C,P), $P \in \mathrm{Sing}(C)$.

The *minimal resolution* of V was described in Section 1. Since $\Delta^P(1) = 1$, this implies that $|H| = \Delta_V(1) = d$. In fact, one can verify easily that $H = \mathbb{Z}_d$, and a possible generator of H is an elementary loop in a transversal slice to C.

The other invariants which are involved in the SWC can be computed from the *minimal resolution* of V and using Laufer's formula [34]:

$$\left\{\begin{array}{l} Z_K^2 + s = -(d-1)(d^2-3d+1); \quad p_g = d(d-1)(d-2)/6; \quad \text{and} \\ \\ \mathbf{sw}(L_V) = \dfrac{1}{d}\displaystyle\sum_{\xi^d=1\neq\xi} \dfrac{\Delta(\xi)}{(\xi-1)^2} + \dfrac{1}{2d}\Delta(t)''(1) - \dfrac{\delta(6\delta-5)}{12d}. \end{array}\right. \tag{1}$$

Example 5.2. Let us continue with Example 1.2. The link of such SIS singularity is a rational homology sphere because the curve C is irreducible, rational and cuspidal. The plumbing graph is star-shaped, in particular it can be realized by a weighted homogeneous singularity $(V_w,0)$.

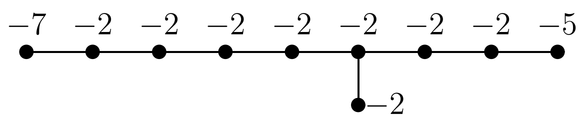
In this case, $p_g(V,0) = 10$ by the previous formula and $p_g(V_w,0) = 10$ by Pinkham's formula [71]. In particular, using [62] (3.3), $(V,0)$ is in an equisingular deformation of $(V_w,0)$. This deformation, found with the help of J. Stevens, can be described as follows. The weights of the variables $(a,\dots,f,\lambda)$ are (62,26,30,28,93,91,-3):

$$V(\lambda) = \left\{\begin{array}{lll} ab - c^2d = \lambda f\,, & bc - d^2 = \lambda^2 a\,, & ad - c^3 = \lambda e\,, \\ be - df = -\lambda ac^2\,, & de - cf = -\lambda a^2\,, & af - c^2e = -\lambda b^6\,, \\ e^2 + a^3 + b^6c = 0\,, & ef + a^2c^2 + b^6d = 0\,, & f^2 + ac^4 + b^7 = 0 \end{array}\right\}.$$

Here, $(V(0),0)=(V_w,0)\subset(\mathbb{C}^6,0)$ is Gorenstein, but it is not a complete intersection. Moreover, *the two singularities* $(V,0)$ *and* $(V_w,0)$ *have the same topological types (the same graphs* Γ*), but their embedded dimensions are not the same*: they are 3 and 6 respectively. It is even more surprising that *their multiplicities are also different*: $\mathrm{mult}(V,0)=5$ and $\mathrm{mult}(V_w,0)=6$ (the second computed by Singular [26]).

In [60] it was proved that the universal abelian cover $(V_w^{ab},0)$ of $(V_w,0)$ is $\Sigma(13,31,2)$, the Brieskorn hypersurface singularity $\{u^{13}+v^{31}+w^2=0\}$. The corresponding resolution graph Γ^{ab} (of both $(V^{ab},0)$ and $(V_w^{ab},0)$) is

Γ^{ab} :



Even more, *there is no equisingular deformation of the universal abelian covers.* Both $(V^{ab},0)$ and $(V_w^{ab},0)$ have the same graph Γ^{ab} but one can show (see [47]) that $(V^{ab},0)$ is not in the equisingular deformation of $(V_w^{ab},0)$.

Thus, the only possible "splice equation" which defines $(V^{ab},0)$ is $(V_w^{ab},0)$ but the universal abelian cover $(V^{ab},0)$ is not in the equisingular deformation of $(V_w^{ab},0)$. Therefore, the universal abelian cover conjecture is not true. Moreover, one has two Gorenstein singularities (one of them is even a hypersurface Brieskorn singularity) with the same rational homology sphere link, but with different geometric genus. This provides counterexample for both SWC and geometric genus conjecture.

Looking at the identity (1), one considers now the (a priori) rational function

$$R(t):=\frac{1}{d}\sum_{\xi^d=1}\frac{\Delta(\xi t)}{(1-\xi t)^2}-\frac{1-t^{d^2}}{(1-t^d)^3}. \tag{2}$$

J.F. Fernández de Bobadilla, A. Némethi and the last two authors in [20] proved that $R(t)\in\mathbb{Z}[t]$ and it can be written as

$$R(t)=\sum_{l=0}^{d-3}\Big(c_l-\frac{(l+1)(l+2)}{2}\Big)\,t^{d(d-3-l)}\in\mathbb{Z}[t]. \tag{3}$$

Moreover

$$R(1)=\mathbf{sw}(L_V)-\frac{Z_K^2+s}{8}-p_g.$$

In particular, the (SWC) is equivalent to $R(1)\geq 0$.

In fact it is rather curious that in all examples, based on SIS singularities, studied in [47] one gets $R(1)\leq 0$. Motivated by these examples, in [20] there were worked out many examples discovering that the coefficients of

$R(t)$ are always non-positive. This gives strong necessary conditions on the singularities of C. It is know that in the problem of classification of rational cuspidal curves one of the key points is to find necessary conditions on the singularities. We state this *compatibility property* on $R(t)$ that we have found as a conjecture.

Conjecture 5.3 ((CP) [20]). *Let $(C,p_i)_{i=1}^{\nu}$ be a collection of local plane curve singularities, all of them locally irreducible, such that $2\delta=(d-1)(d-2)$ for some integer d. Then if $(C,p_i)_{i=1}^{\nu}$ can be realized as the local singularities of a degree d (automatically rational and cuspidal) projective plane curve of degree d then*

$$c_l \leq (l+1)(l+2)/2 \text{ for all } \; l=0,\dots,d-3. \qquad (*_l)$$

In fact the coefficients c_l can be compute from the polynomial $Q(t)$ defined in terms of $\Delta(t)$:

$$\Delta(t)=1+(t-1)\delta+(t-1)^2Q(t)=\sum_{l\nmid d} b_l t^l+\sum_{l=0}^{d-3} c_l t^{(d-3-l)d}.$$

The main result in [20] is the proof of the following theorem:

Theorem 5.4. *If the logarithmic Kodaira dimension $\bar{\kappa}:=\bar{\kappa}(\mathbb{P}^2\setminus C)$ is less than 2, then (CP) is true. In fact, in all these cases $c_l=\frac{(l+1)(l+2)}{2}$ for any $l=0,\dots,d-3$.*

Corollary 5.5 ([20]). *Let $f=f_d+f_{d+1}+\cdots\;:(\mathbb{C}^3,0)\to(\mathbb{C},0)$ be a hypersurface superisolated singularity with $\bar{\kappa}(\mathbb{P}^2\setminus\{f_d=0\})<2$. Then the Seiberg-Witten invariant conjecture is true for $(V,0)=(\{f=0\},0)$.*

It is even more interesting to study the compatibility property when $\nu=1$. In this case one can prove that all the inequalities $(*_l)$ are indeed identities. These identities are equivalent (via a theorem by A. Campillo, F. Delgado and S.M. Gusein-Zade in [12]) to a very remarkable distribution of the elements of the semigroup $\Gamma_{(C,P)}$ of the singularity (C,P) in intervals of length d. It is shown there that (CP) in this case is equivalent to the following conjectural identity:

$$\sum_{k\in\Gamma_{(C,P)}} t^{\lceil k/d\rceil}=\frac{1-t^d}{(1-t)^2}=1+2t+\cdots+(d-1)t^{d-2}+d(t^{d-1}+t^d+t^{d+1}+\cdots).$$

6 Final Remarks

6.1 Weighted-Yomdin Singularities

The second natural generalization of SIS singularities is obtained if one considers a weighted version of this singularities.

Definition 6.1. Let $\omega := (p_x, p_y, p_z) \in \mathbb{N}^3$ be such that $\gcd(p_x, p_y, p_z) = 1$. A polynomial f is *ω-weighted homogeneous of degree d* if

$$f(t^{p_x}x, t^{p_y}y, t^{p_z}z) = t^d f(x, y, z)\,.$$

It then defines a curve in the *weighted projective plane* $\mathbb{P}^2_\omega := \mathbb{C}^3 \setminus \{0\}/\sim$, $(x, y, z) \sim (t^{p_x}x, t^{p_y}y, t^{p_z}z)$ for all $t \in \mathbb{C}^*$. If $P \in \mathbb{P}^2_\omega$, we define its *order* ν_P as the gcd of the weights of the non-zero coordinates of P.

Definition 6.2. If $C \subset \mathbb{P}^2_\omega$ is a curve defined by a weighted homogeneous polynomial f and $P \in C$ we define the *weighted Milnor number* $\mu^\omega(C, P)$ as $\frac{\mu}{\nu_P}$ where μ is defined as follows; let us suppose that P is the equivalence class of $(x_0, y_0, 1)$ and consider the Milnor number of $f(x, y, 1) = 0$ at (x_0, y_0). A *singular point* of C is a point such that $\mu^\omega(C, P) > 0$.

Let us consider a germ $(W, 0) \subset (\mathbb{C}^3, 0)$ defined by a power series g; let $g = g_d + g_{d+k} + \ldots$ be the weighted homogeneous decomposition of f with respect to ω and let $C^\omega_m \subset \mathbb{P}^2_\omega$ be the weighted projective locus of zeroes of g_m.

Definition 6.3. We say that $(W, 0) \subset (\mathbb{C}^3, 0)$ is a *weighted Yomdin singularity with respect to ω* if $\operatorname{Sing}(C^\omega) \cap C^\omega_{d+k} = \emptyset$.

In a forthcoming joint work with J. Fernández de Bobadilla, we will give a proof of a formula which was suggested to us by C. Hertling.

Proposition 6.4. *The Milnor number μ of a* weighted Yomdin singularity $(W, 0) \subset (\mathbb{C}^3, 0)$ with respect to ω *satisfies the following equality:*

$$\mu(W, 0) = \left(\frac{d}{p_x} - 1\right)\left(\frac{d}{p_y} - 1\right)\left(\frac{d}{p_z} - 1\right) + k \sum_{P \in \operatorname{Sing}(C^\omega)} \mu(C^\omega, P).$$

6.2 $*$-Polynomials

The theory of (local) SIS or Yomdin singularities has an analogous global counterpart defined by polynomials of type $f = f_d + f_{d-k} + \ldots$ and the same geometric condition. For instance the formula for the global Milnor number is done by the authors in [10] and for the zeta-function of the monodromy at infinity by S.M. Gusein-Zade and the last two authors in [29]. A finer study has been done in a series of works A. Némethi and R. García López [22, 23, 24] for $*$-polynomials $f = f_d + f_{d-1} + \ldots$. The behaviour of these polynomials at infinity imitates in some way the local behaviour of SIS singularities. They computed formulæ for the global Milnor number, monodromy at infinity, Mixed Hodge structure at infinity...

6.3 Intersection form of a SIS singularity

In the topological study of singularities, we are interested in invariants living in the complex setting (like the Jordan form of the monodromy) but also in invariants living in the integers, like monodromy over $\mathbb{Z}$, Seifert form or the intersection form in a distinguished basis of vanishing cycles.

It is well-known how to compute these invariants for local germs of curves. In his thesis, M. Escario computes these invariants for polynomials in two variables which are generic at infinity (in fact, for the more general concept of *tame* polynomials), using a generic polar mapping Φ and the braid monodromy of the discriminant of Φ.

Combining these techniques with Gabriélov's method (see [21]), M. Escario gives a method to compute the intersection form of the Milnor fiber in a distinguished basis of vanishing cycles for SIS singularities. In fact, this method works also for Yomdin singularities.

6.4 Durfee's conjecture for SIS singularities

A. Durfee [18] conjectured that the signature of the Milnor fibre of an hypersurface surface singularity is negative. In fact, Durfee's conjecture is the stronger inequality

$$6p_g \leq \mu. \tag{*}$$

Y. Xu and S.S.T. Yau proved $(*)$ for weighted homogeneous surface singularities, [90]. A. Neméthi [52] verified the inequality $(*)$ in the case $f(x, y) + z^n$ with $f(x, y) \in \mathbb{C}\{x, y\}$ irreducible, see also T. Ashikaga [11].

Using SIS singularities, A. Melle Hernández [50] proved $(*)$ for *absolutely isolated* surface singularities. A surface hypersurface $(V, 0) \subset (\mathbb{C}^3, 0)$ is absolutely isolated if there exists a resolution $\pi : \tilde{V} \to V$ such that π is a composition of blowing-ups at points.

Acknowledgments

The authors would like to thank the persons who have collaborated with them in their works on SIS singularities whose ideas and comments are cited in this survey, in special to Pi. Cassou-Noguès, J. Carmona, J.I. Cogolludo, M. Escario, J. Fernández de Bobadilla, S.M. Gusein-Zade, C. Hertling, M. Marco, A. Némethi, J. Stevens, H. Tokunaga and W. Veys.

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Linear Free Divisors and Quiver Representations

Ragnar-Olaf Buchweitz David Mond

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Abstract

Linear free divisors are free divisors, in the sense of K. Saito, with linear presentation matrix. Using techniques of deformation theory on representations of quivers, we exhibit families of such linear free divisors as discriminants in representation varieties for real Schur roots of a finite quiver. Along the way we review some basic results on representation varieties of quivers, their associated fundamental exact sequence and semi-invariants; explain in detail how to verify the occurring discriminant as a free divisor and how to determine its components and their equations. As an illustration, the linear free divisors that arise as the discriminant from the highest root of a Dynkin quiver are treated explicitly.

1 Introduction

Let X be a non-singular n-dimensional complex manifold (or algebraic variety over an algebraically closed field k of characteristic zero), and let $D \subset X$ be a hypersurface with reduced defining ideal I_X. We denote by $\mathrm{Der}(-\log D)$ the sheaf of vector fields $\chi \in \mathrm{Der}_X$ such that $\chi \cdot I_X \subset I_X$, or, equivalently, such that χ is tangent to D at its regular points. It is clearly an $\mathcal{O}_X$-module.

Definition 1.1. The hypersurface $D \subseteq X$ is a *free divisor* if $\mathrm{Der}(-\log D)$ is a locally free $\mathcal{O}_X$-module.

1991 *Mathematics Subject Classification.* 14D15, 16G20, 58K60 (*secondary:* 16G60, 13A 50, 14C20)

Key words. Free divisor, deformation theory, quiver representation, Schur root, representation variety, discriminant, semi-invariant

Free divisors were introduced by K. Saito in [28]. The simplest example is the normal crossing divisor, but the main source of examples, motivating Saito's definition, has been the deformation theory of singularities, where discriminants and bifurcation sets are frequently free divisors. If D is the discriminant hypersurface in the base S of a versal deformation of an isolated hypersurface singularity, the module $\mathrm{Der}(-\log D)$ is the kernel of the Kodaira-Spencer map from Der_S onto the relative T^1 of the deformation, and from this freeness follows by an easy homological argument, due initially to Teissier. Variants on this argument show the freeness of the discriminant in the base of a versal deformation in a number of cases: isolated complete interesection singularities ([20]), space-curve singularities ([32]), functions on space curves ([14], [22]), Gorenstein surface singularities in 5-space ([5]), Hilbert schemes of a smooth surface ([4]). Damon, in his paper "The legacy of free divisors" ([7]), has shown, by an essentially similar argument, how the bifurcation set in the base space of a versal deformation of a non-linear section of a free divisor is once again a free divisor, provided a natural condition, namely, the existence of "Morse-type singularities", is met. Another significant source of examples is the theory of hyperplane arrangements, where many examples of free arrangements have been constructed by combinatorial means (see e.g. [25] Chapter 4).

Saito's original paper [28] contained the following criterion, now known by his name, for a divisor D to be free:

Proposition 1.2 (Saito's Criterion). *The hypersurface $D \subset X$ is a free divisor in the neighbourhood of a point x if and only if there are germs of vector fields $\chi_1, \ldots, \chi_n \in \mathrm{Der}(-\log D)_x$, such that the determinant of the matrix of coefficients $[\chi_1, \ldots, \chi_n]$, with respect to some, or any, $\mathcal{O}_{X,x}$-basis of $\mathrm{Der}_{X,x}$, is a reduced equation for D at x. In this case, $\chi_1, \ldots, \chi_n$ form a basis for $\mathrm{Der}(-\log D)_x$.* □

Note that it is clear that the determinant of the matrix of coefficients of any n-tuple of vector fields in $\mathrm{Der}(-\log D)$ must vanish identically on D, since at any regular point $x \in D$ all n vectors lie in the $n-1$-dimensional vector space T_xD. Moreover, since $\mathrm{Der}(-\log D)$ coincides with Der_X outside D, the determinant of the matrix of coefficients of any set of generators of $\mathrm{Der}(-\log D)$ must vanish only on D.

In practice, one uses often the following concrete algebraic version of this criterion that does not refer to vector fields directly, rather characterizes the Taylor series of the function f defining a free divisor at some point $x \in X$:

Proposition 1.3. *A formal power series $f \in P = k[\![z_1, ..., z_n]\!]$ defines a (formal) free divisor, if it is reduced, that is, squarefree, and there is an $(n \times n)$–matrix A with entries from P such that*

$$\det A = f \quad \textit{and} \quad (\nabla f)A \equiv (0, ..., 0) \bmod f\,,$$

where $\nabla f = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$ *is the gradient of* f*, and the last condition just expresses that each entry of the (row) vector* $(\nabla f)A$ *is divisible by* f *in* P*. The columns of* A *can then be viewed as the coefficients of a basis, with respect to the partial derivatives* $\partial/\partial z_i$*, of the logarithmic vectorfields along the divisor* $f = 0$*.* □

The normal crossing divisor $D = \{x_1 \cdots x_n = 0\}$ provides a simple example: Saito's criterion shows that the vector fields $x_1\partial/\partial x_1, \ldots, x_n\partial/\partial x_n$ form a basis for $\mathrm{Der}(-\log D)$. This free divisor has the striking property that $\mathrm{Der}(-\log D)$ has a basis consisting of vector fields that are homogeneous of weight zero with respect to the natural grading. Among free hyperplane arrangements it is the only one with this property ([25, Chapter 4]). Until recently, the only other free divisor with this property known to either of the authors of this paper was the "*bracelet*", the discriminant in the space of binary cubics (see [13], and [23], where it is described in some detail, though not under this name).

Definition 1.4. The free divisor D is *linear* if $\mathrm{Der}(-\log D)$ has a basis consisting of vector fields of weight zero — that is, all of whose coefficients are linear functions of the variables.

Here we show that far from being uncommon, linear free divisor are abundant. We show that the set of degenerate, or non-generic, orbits in the representation space $\mathrm{Rep}(Q, \mathbf{d})$ of a quiver with dimension vector $\mathbf{d}$, is a linear free divisor whenever $\mathbf{d}$ is a real Schur root (definition in Section 3) of Q, and provided that a natural condition on the existence of "codimension 1" degeneracies holds — a condition which is closely related to Damon's condition on the existence of "Morse-type singularities" mentioned above.

Since we hope that our paper will be read by singularity theorists, we include some background on quiver representations.

2 Linear Free Divisors

Suppose that D is a linear free divisor, and let $\chi_1, \ldots, \chi_n$ be a basis consisting of weight-zero vector fields. Since the weight of the Lie bracket of any two homogeneous vector fields is the sum of their weights, $\chi_1, \ldots, \chi_n$ form the basis of an n-dimensional Lie algebra L_D over k, as well as a basis of the free $\mathcal{O}$-module $\mathrm{Der}(-\log D)$. Consider the standard action of $\mathrm{Gl}_n(k)$ on k^n. The vector field $x_i\partial/\partial x_j$ is the infinitesimal generator of this action corresponding to the elementary matrix E_{ij} (1 in the i-th row and j-th column, zeroes elsewhere). It follows that L_D is the image, under the infinitesimal action, of an n-dimensional Lie subalgebra of $\mathfrak{gl}_n(k)$, which we denote $\mathfrak{g}_D$. In the complex case, if the exponential of $\mathfrak{g}_D$ is a closed subgroup G_D of $\mathrm{Gl}_n(\mathbb{C})$, then G_D has an open orbit in $\mathbb{C}^n$ and D is its complement. This follows easily from Mather's lemma on Lie group actions ([21] Lemma 3.1), which gives sufficient conditions for a connected submanifold of a manifold to lie in a single orbit of the action of a Lie group G: that

(i) at each point of X, T_xX should be contained in the tangent space to the G orbit of x, and

(ii) the dimension of this orbit should be constant for $x \in X$.

Taking $X = \mathbb{C}^n \setminus D$, both conditions evidently hold here.

In all examples known, this indeed applies. To find linear free divisors one may thus look for n–dimensional Lie groups acting on k^n with an open orbit. It is precisely these that the representation theory of quivers offers in abundance. Indeed, in that situation, the Lie groups G_D are reductive. Examples of some nonreductive groups that also give rise to linear free divisors will be presented in [15]. Here we mention just one series.

Example 2.1. The group $B_n(k)$ of upper triangular $n \times n$ matrices acts on the space $\mathrm{Sym}_n(k)$ of symmetric matrices by

$$B \cdot S = {}^tB\, S\, B.$$

There is an open orbit; the equation of the complement is the product of n nested symmetric determinants, beginning with the top left hand entry (1×1 determinant) in the symmetric matrix S and continuing with the determinant of the top left hand 2×2 block, the determinant of the top left hand 3×3 block, etc.

3 Representations of Quivers

A *quiver* is a finite directed graph. That is, it consists of a finite set Q_0 of nodes (or vertices), and a finite set of arrows Q_1 equipped with two maps $h, t : Q_1 \to Q_0$ that assign to each arrow $\varphi \in Q_1$ its head $h\varphi$ and tail $t\varphi$ in Q_0. A *representation* V of a quiver Q consists of a choice of vector space V_x for each node x, and a k-linear map $V(\varphi) : V_{t\varphi} \to V_{h\varphi}$ for each arrow $\varphi \in Q_1$. The representation is *finite dimensional* if each V_x is a finite dimensional vector space.

If W is a second such representation, then a *morphism of representations* $\psi : W \to V$ is a family of k–linear maps $\psi_x : W_x \to V_x, x \in Q_0$, such that for each $\varphi \in Q_1$ the square

$$\begin{tikzcd} W_{t\varphi} \arrow[r, "W(\varphi)"] \arrow[d, "\psi_{t\varphi}"'] & W_{h\varphi} \arrow[d, "\psi_{h\varphi}"] \\ V_{t\varphi} \arrow[r, "V(\varphi)"] & V_{h\varphi} \end{tikzcd}$$

commutes. The k–vector space of all morphisms of representations from W to V is denoted $\mathrm{Hom}_Q(W,V)$. The so-defined category of (finite dimensional) representations of Q is *abelian*. Moreover, it is *hereditary*, which means that the extension groups in this abelian category — denoted $\mathrm{Ext}^i_Q(W,V)$, or $\mathrm{Ext}^i_{kQ}(W,V)$ if we wish to specify the coefficients — vanish whenever $i \geq 2$.

Once we fix the dimensions of the spaces at each node, by assigning to Q a *dimension vector* $\mathbf{d} \in \mathbb{N}^{Q_0}$, we can consider the k-vector space of representations

$$\mathrm{Rep}(Q,\mathbf{d}) = \prod_{\varphi\in Q_1} \mathrm{Hom}_k(V_{t\varphi}, V_{h\varphi}) \simeq \prod_{\varphi\in Q_1} \mathrm{Hom}_k\big(k^{d(t\varphi)}, k^{d(h\varphi)}\big).$$

The group $\mathrm{Gl}(Q,\mathbf{d}) = \prod_{x\in Q_0} \mathrm{Gl}_{d(x)}(k)$ acts on $\mathrm{Rep}(Q,\mathbf{d})$ by

$$(g_x)_{x\in Q_0} \cdot (V(\varphi))_{\varphi\in Q_1} = \big(g_{h\varphi} \circ V(\varphi) \circ g_{t\varphi}^{-1}\big)_{\varphi\in Q_1}.$$

The orbits of this group action are the isomorphism classes of Q–representations with the prescribed dimension vector. It will be from this action that we obtain the generators of $\mathrm{Der}(-\log D)$ for the linear free divisors we construct.

Given $V' \in \mathrm{Rep}(Q,\mathbf{d}')$ and $V'' \in \mathrm{Rep}(Q,\mathbf{d}'')$, the *direct sum* $V' \oplus V'' \in \mathrm{Rep}(Q,\mathbf{d}'+\mathbf{d}'')$ is the representation with $(V'\oplus V'')_x = V'_x \oplus V''_x$ for $x \in Q_0$ and

$$(V'\oplus V'')(\varphi) = \begin{pmatrix} V'(\varphi) & 0 \\ 0 & V''(\varphi) \end{pmatrix}.$$

A given representation $V \in \mathrm{Rep}(Q,\mathbf{d})$ is *decomposable* if it is the direct sum of subrepresentations — that is, if there are representations $V' \in \mathrm{Rep}(Q,\mathbf{d}')$ and $V'' \in \mathrm{Rep}(Q,\mathbf{d}'')$ such that $V = V' \oplus V''$. In this case, of course, $\mathbf{d} = \mathbf{d}' + \mathbf{d}''$.

A quiver Q is a *Dynkin quiver* if the underlying undirected graph $\overline{Q}$ is a disjoint union of Dynkin diagrams of type A_n, D_n, E_6, E_7 or E_8. Dynkin quivers are ubiquitous in the theory of representations of quivers, and central in this paper.

Example 3.1. Let Q be the following Dynkin quiver of type A_3

$$\bullet \xrightarrow{A} \bullet \xrightarrow{B} \bullet$$

(i) With dimension vector $(1,1,1)$ any representation in which each of the morphisms is non-zero is indecomposable.
(ii) Indeed, these are the only indecomposable representations whose dimension vector is *sincere*, meaning that it is nonzero at each node. For example, if $\mathbf{d} = (1,2,1)$ there is no indecomposable representation. Representations in which $BA \neq 0$ decompose as the direct sum

$$k \xrightarrow{A} \text{im } A \cong k \xrightarrow{B|_{\text{im } A}} k \quad \oplus \quad 0 \longrightarrow \ker B \cong k \longrightarrow 0$$

Representations in which $BA = 0$ and $A \neq 0$ decompose as

$$k \xrightarrow{A} \text{im } A \cong k \longrightarrow 0 \quad \oplus \quad 0 \longrightarrow k^2/\,\text{im } A \cong k \xrightarrow{B} k$$

where the middle term in the second summand can be viewed as a complement to im A. Representations in which $A = 0$ decompose as

$$k \longrightarrow 0 \longrightarrow 0 \quad \oplus \quad 0 \longrightarrow k^2 \xrightarrow{B} k.$$

Similarly, any representation with $\mathbf{d} = (l,m,n)$ for $l,m,n \geq 0$ decomposes as:

$$\begin{gathered}(k \longrightarrow 0 \longrightarrow 0)^{\oplus a} \oplus (0 \longrightarrow k \longrightarrow 0)^{\oplus b} \oplus (0 \longrightarrow 0 \longrightarrow k)^{\oplus c} \\ \oplus\, (k \xrightarrow{1} k \longrightarrow 0)^{\oplus d} \oplus (0 \longrightarrow k \xrightarrow{1} k)^{\oplus e} \oplus (k \xrightarrow{1} k \xrightarrow{1} k)^{\oplus f},\end{gathered}$$

where

$$\begin{aligned}&a = \dim \ker A \quad , \quad b = \dim \ker B/(\text{im } A \cap \ker B) \quad , \quad c = \dim \text{cok } B\,,\\ &d = \dim \ker BA/\ker A \quad , \quad e = \dim \text{im } B/\,\text{im } BA\,,\\ &f = \dim \text{im } BA = l - a - d = m - b - d - e = n - c - e\,.\end{aligned}$$

Definition 3.2. The dimension vector $\mathbf{d}$ is a *root* of Q if $\mathrm{Rep}(Q,\mathbf{d})$ contains an indecomposable representation. The root is *real* if $\mathrm{Rep}(Q,\mathbf{d})$ contains exactly one orbit of, necessarily isomorphic, indecomposable representations. It is *imaginary* if there is a family of non-isomorphic indecomposable representations. If a general representation in $\mathrm{Rep}(Q,\mathbf{d})$ is indecomposable, then $\mathbf{d}$ is a *Schur* root. [1]

The frequent use of the term "root" in these definitions is no coincidence, as we will see below.

[1]Some authors define a Schur root as a root $\mathbf{d}$ for which $\mathrm{Rep}(Q,\mathbf{d})$ contains a 'brick' — a representation V for which $\mathrm{End}_Q(V) = k$. If $\mathbf{d}$ is a Schur root in this sense, then by the upper semicontinuity of $\dim \mathrm{End}_Q(V)$ with respect to V, the general representation also has endomorphism ring k, and so is indecomposable. Conversely, 2.7 of [19] shows that if the general representation is indecomposable then it is a brick. So the two versions of the definition are equivalent.

A crucial role in the representation theory of quivers is played by the *Euler form*, a bilinear form on the space $\mathbb{N}^{Q_0}$ of dimension vectors. It is defined by

$$\begin{aligned} \langle \mathbf{e}, \mathbf{d} \rangle &= \sum_{x \in Q_0} e_x d_x - \sum_{\varphi \in Q_1} e_{t\varphi} d_{h\varphi} \\ &= \dim \prod_{x \in Q_0} \operatorname{Hom}(W_x, V_x) - \dim \prod_{\varphi \in Q_1} \operatorname{Hom}_k\big(W(t(\varphi)), V(h(\varphi))\big) \end{aligned}$$

for any $W \in R(Q, \mathbf{e})$ and $V \in R(Q, \mathbf{d})$, and accordingly we sometimes denote $\langle \mathbf{e}, \mathbf{d} \rangle$ by $\langle W, V \rangle$.

The *Tits form* on the space of dimension vectors is the associated quadratic form, $q(\mathbf{d}) = \langle\, \mathbf{d}, \mathbf{d} \rangle$. Observe that the Tits form does not depend on the orientation of the arrows. Indeed, it is used to calculate the members of the root system of the Kac–Moody Lie algebra attached to the underlying graph $\overline{Q}$, and those roots with nonnegative components are precisely the roots for Q, regardless of the orientation of the arrows, see [17]. For example, if $\overline{Q}$ is a Dynkin diagram then $\mathbf{d} \in \mathbb{Q}^{|Q_0|}$ is a root of the corresponding semi-simple Lie algebra, in the classical sense, if and only if $q(\mathbf{d}) = 1$. In particular, all roots are real in this case.

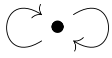
Choosing an ordering of the nodes in Q_0, we may write $\langle \mathbf{e}, \mathbf{d} \rangle = \mathbf{e} E \mathbf{d}^T$, where $\mathbf{e}, \mathbf{d}$ are thought of as row vectors, and E is the corresponding *Euler matrix*. Its entries are $E_{x,y} = \delta_x^y - \#\{\varphi \in Q_1 \mid t\varphi = x, h\varphi = y\}$, with δ_x^y denoting the Kronecker delta. Put differently, $E = I_{|Q_0|} - A$, where $I_{|Q_0|}$ is the identity matrix of the indicated size and the matrix entry $A_{x,y}$ records the number of arrows from x to y in Q_1. The matrix associated to the Tits form is then $C = E + E^T$, the *Cartan matrix* of Q, which coincides with the usual Cartan matrix of the associated Dynkin diagram $\overline{Q}$, in case Q is a Dynkin quiver[2].

The following simple result is useful for the actual calculation of the linear free divisors below.

Lemma 3.3. *If Q is a finite quiver without oriented cycles, then its Euler matrix is* invertible. *The inverse is given by $E^{-1} = I_{|Q_0|} + A'$, where $A'_{x,y}$ equals the number of* directed paths *from x to y.* □

Now we recall the trichotomy of the representation theory of quivers:

Definition 3.4. A quiver Q is of *finite representation type* if Q has only finitely many indecomposable representations, up to isomorphism. The quiver is *wild* if its representation theory is at least as complicated as that of the quiver



[2]More generally, C is the Cartan matrix of the Kac–Moody Lie algebra associated to $\overline{Q}$, for an arbitrary finite quiver Q without oriented cycles, see [17] again.

The quiver is *tame* if it is neither of finite representation type, nor wild [3].

Gabriel ([11],[12]) showed

Theorem 3.5. *A connected quiver Q is of finite representation type if and only if it is a Dynkin quiver. Assigning to an isomorphism class of indecomposable representations of Q its dimension vector induces then a bijection between these classes and the positive roots of the underlying Dynkin diagram.* □

The last part of this result can be restated thus: if $\mathbf{d}$ is a positive root of the underlying Dynkin diagram (as listed, for example, in the appendix to [3]) then $\mathbf{d}$ is also a root of any associated Dynkin quiver Q, in the sense of Definition 3.2. Moreover, in this case each root is a real Schur root: there is a (unique) open orbit in $\mathrm{Rep}(Q, \mathbf{d})$ whose points correspond to indecomposable representations. A good account of all this can be found in [2].

The class of tame quivers has a similar characterisation:

Theorem 3.6 ([9],[24]). *A connected quiver is tame if and only if the underlying undirected graph is an extended Dynkin diagram.* □

Finally, in what follows we will need a result of V.Kac ([17]):

Proposition 3.7. *Let Q be a connected quiver whose proper subquivers are all either of finite or tame type. Then a dimension vector $\mathbf{d}$ is a real root if and only if $q(\mathbf{d}) = 1$, and it is an imaginary root if and only if $q(\mathbf{d}) \leq 0$.* □

4 The Fundamental Exact Sequence

Let V and W be representations of the quiver Q. In [26], Ringel introduced the following exact sequence $\mathscr{E}_V^W$ of vector spaces:

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_Q(W, V) \longrightarrow & \prod_{x \in Q_0} \mathrm{Hom}_k(W_x, V_x) \\ \xrightarrow{d_V^W} & \prod_{\varphi \in Q_1} \mathrm{Hom}_k\big(W(t(\varphi)), V(h(\varphi))\big) \xrightarrow{e_V^W} \mathrm{Ext}^1_Q(W, V) \longrightarrow 0\,. \end{aligned} \tag{1}$$

The morphism d_V^W is defined by

$$d_V^W\big((\psi(x))_{x\in Q_0}\big) = \big(\psi_{h(\varphi)} \circ W(\varphi) - V(\varphi) \circ \psi_{t(\varphi)}\big)_{\varphi \in Q_1};$$

[3] The reader should be aware that the definition often is "tame"="not wild", thus, different from our usage here.

the component of $d_V^W((\psi))$ corresponding to $\varphi \in Q_1$ measures non-commutativity of the diagram

$$\begin{tikzcd} W_{t\varphi} \arrow[r, "W(\varphi)"] \arrow[d, "\psi_{t\varphi}"'] & W_{h\varphi} \arrow[d, "\psi_{h\varphi}"] \\ V_{t\varphi} \arrow[r, "V(\varphi)"] & V_{h\varphi} \end{tikzcd}$$

whence it is clear that $\ker d_V^W$ is indeed equal to $\mathrm{Hom}_Q(W,V)$.

To define e_V^W, from $\theta = (\theta_\varphi)_{\varphi\in Q_1}$ we construct a new representation Z of Q and an exact sequence,

$$e_V^W(\theta) \quad \equiv \quad 0 \to V \xrightarrow{i} Z \xrightarrow{j} W \to 0\,,$$

by the following recipe: $Z_x = V_x \oplus W_x$ for each $x \in Q_0$, $i_x : V_x \to V_x \oplus W_x$ and $j_x : V_x \oplus W_x \to W_x$ are the standard inclusion and projection, and for each $\varphi \in Q_1$, $Z(\varphi) : V_{t\varphi} \oplus W_{t\varphi} \to V_{h\varphi} \oplus W_{h\varphi}$ has matrix

$$\begin{pmatrix} W(\varphi) & \theta_\varphi \\ 0 & V(\varphi) \end{pmatrix}.$$

It is straightforward to check that the short exact sequence $e_V^W(\theta)$ of representations of Q is split iff $\theta = d_V^W(\psi)$ for some $\psi \in \prod_{x\in Q_0}\mathrm{Hom}(W_x,V_x)$, and that e_V^W is onto.

Exactness of the sequence $\mathscr{E}_V^W$ implies that

$$\langle \mathbf{e}, \mathbf{d}\rangle = \dim_k \mathrm{Hom}_Q(W,V) - \dim_k \mathrm{Ext}^1_Q(W,V)$$

for any $W \in R(Q,\mathbf{e})$ and $V \in R(Q,\mathbf{d})$, so that the expression on the right hand side depends only on the dimension vectors and not on the choice of representations, although evidently the dimensions of $\mathrm{Ext}^1_Q(W,V)$ and $\mathrm{Hom}_Q(W,V)$ do depend on the choice of $V \in R(Q,\mathbf{d})$ and $W \in R(Q,\mathbf{e})$.

The fundamental sequence $\mathscr{E}_V^W$ plays two roles in what follows. In the next section we show how to reinterpret it in terms of the deformation theory of representations, where it may become more familiar to singularity-theorists. From this we will see how free divisors appear naturally in this context.

Second, following Schofield [30], we use it to generate semi-invariants of the representation space $R(Q,\mathbf{d}) = \prod_{\varphi\in Q_1}\mathrm{Hom}_k(k^{d(t\varphi)}, k^{d(h\varphi)})$, and thereby find explicit equations for the free divisors, in Sections 8 and 10.

5 Deformations of Representations

Recall that the group $\mathrm{Gl}(Q,\mathbf{d})$ acts on $\mathrm{Rep}(Q,\mathbf{d})$ by

$$(g_x)_{x\in Q_0}\cdot (V(\varphi))_{\varphi\in Q_1} = \big(g_{h\varphi}\circ V(\varphi)\circ g_{t\varphi}^{-1}\big)_{\varphi\in Q_1}.$$

The orbit of V in $\mathrm{Rep}(Q, \mathbf{d})$ is open if and only if the associated map

$$\alpha_V : \mathrm{Gl}(Q, \mathbf{d}) \to \mathrm{Rep}(Q, \mathbf{d})$$

sending g to $g \cdot V$ is a submersion, and for this it is enough that it be a submersion at the identity. The Lie algebra $\mathfrak{gl}(Q, \mathbf{d})$ of $\mathrm{Gl}(Q, \mathbf{d})$ is

$$\prod_{x \in Q_0} \mathrm{End}(k^{\mathbf{d}_x}) = \prod_{x \in Q_0} \mathrm{Hom}(k^{\mathbf{d}_x}, k^{\mathbf{d}_x}),$$

and the tangent space to $\mathrm{Rep}(Q, \mathbf{d})$ at V is $\mathrm{Rep}(Q, \mathbf{d})$ itself, that is, $\prod_{x \in Q_0} \mathrm{Hom}_k(k^{\mathbf{d}_{t\varphi}}, k^{\mathbf{d}_{h\varphi}})$. The derivative of α_V at the identity in $\mathrm{Gl}(Q, \mathbf{d})$ is precisely the map d_V^V of the exact sequence $\mathscr{E}_V^V$. In fact we may canonically identify $\mathrm{Ext}^1_Q(V, V)$ with $T^1(V)$ for the associated deformation theory, though we will not make any formal use of this identification.

A deformation, in the analytic category, of a representation V is, by definition, the germ of an analytic map $(B, 0) \to (\mathrm{Rep}(Q, \mathbf{d}), V)$. If $(B, 0)$ is smooth, a deformation $\mathscr{V} : (B, 0) \to (\mathrm{Rep}(Q, \mathbf{d}), V)$ is *versal* if and only if it is *complete*, that is, if every other deformation $\mathscr{V}' : (B', 0) \to (\mathrm{Rep}(Q, \mathbf{d}), V)$ is equivalent to one induced from it by base-change $\eta : (B', 0) \to (B, 0)$. The equivalence here is the existence of a map-germ $g : (B, 0) \to (\mathrm{Gl}(Q, \mathbf{d}), 1)$ such that

$$\mathscr{V}'(b') = g(b') \cdot \mathscr{V}(\eta(b')).$$

Thus it is evident that $\mathrm{Rep}(Q, \mathbf{d})$ itself, or more precisely the identity map $(\mathrm{Rep}(Q, \mathbf{d}), V) \to (\mathrm{Rep}(Q, \mathbf{d}), V)$, is a versal deformation; for any other deformation $\mathscr{V}'$, the base change map η is simply $\mathscr{V}'$ itself, and g is the constant map taking the value 1. The slice theorem from the theory of smooth group actions is now enough to establish the versality of any deformation obtained from this one by restricting its domain to any smooth space-germ transverse to the orbit of V, or indeed by pulling it back by any map-germ $(B, 0) \to (\mathrm{Rep}(Q, \mathbf{d}), V)$ transverse to the orbit of V. These considerations imply the Artin–Voigts Lemma: that the dimension of $\mathrm{Ext}^1_Q(V, V) \cong T^1(V)$ equals the codimension of the orbit of V in $\mathrm{Rep}(Q, \mathbf{d})$. In particular, if there is an open orbit, then the representations therein have no self-extensions: they are *rigid* as representations.

Now we consider the relative T^1, obtained by regarding the coefficients of the morphisms $V(\varphi)$ as variables. This can be done in the analytic, formal or algebraic category, and amounts to no more than tensoring the exact sequence $\mathscr{E}_V^V$ with the appropriate ring, or sheaf, of functions — $\mathscr{O}_{\mathrm{Rep}(Q,\mathbf{d})}$, $k[\mathrm{Rep}(Q, \mathbf{d})^*]$ or $k[[\mathrm{Rep}(Q, \mathbf{d})^*]]$. We refer to these indistinctly as R. The module (sheaf) of vector fields on $\mathrm{Rep}(Q, \mathbf{d})$ is $\theta_R = \mathrm{Der}_k(R) \cong \mathrm{Rep}(Q, \mathbf{d}) \otimes_k R$, and the k-linear map $\mathfrak{gl}(Q, \mathbf{d})) \to \mathrm{Rep}(Q, \mathbf{d})$ extends to a morphism of R-modules $\mathfrak{gl}(Q, \mathbf{d}) \otimes_k R \to \theta_R$ whose cokernel can be viewed both as $\mathrm{Ext}^1_{RQ}(M,M)$ for the universal representation M of the quiver Q with coefficients in R, and as the relative T^1 of the versal deformation $\mathfrak{i} : \mathrm{Rep}(Q,\mathbf{d}) \to \mathrm{Rep}(Q,\mathbf{d})$, denoted $T^1(\mathfrak{i}/\mathrm{Rep}(Q,\mathbf{d}))$. The surjection $\theta_R \to T^1(\mathfrak{i}/\mathrm{Rep}(Q,\mathbf{d}))$ is the *Kodaira-Spencer map* of the versal deformation $\mathfrak{i}$.

The kernel of this projection is the space of simultaneous endomorphisms of the representations $V \in \mathrm{Rep}(Q,\mathbf{d})$, or, in other words, the endomorphism ring of the universal representation M. Provided the general representation in $\mathrm{Rep}(Q,\mathbf{d})$ is indecomposable, this ring is isomorphic to R. Let us understand why this is so. It is clear that if $V \in \mathrm{Rep}(Q,\mathbf{d})$ is any representation then for each $\lambda \in k^*$ we have $(\lambda I_{d_x})_{x\in Q_0} \in \mathrm{Aut}_k(V)$, and similarly $(\lambda I_{d_x})_{x\in Q_0} \in \mathrm{End}_Q(V)$ for $\lambda \in k$. If V is stably indecomposable (that is, if there is a neighbourhood of V in $\mathrm{Rep}(Q,\mathbf{d})$ consisting of indecomposable representations) then this copy of k accounts for all of $\mathrm{End}_Q(V)$ (see e.g. [19], 2.7). Now if the general representation in $\mathrm{Rep}(Q,\mathbf{d})$ is indecomposable — which means that $\mathbf{d}$ is a Schur root — then at each of these representations, any endomorphism of the universal representation M must be a scalar. By density, the same must be true everywhere, and so $\mathrm{End}_R(M)$ can be identified with R.

The cokernel of the inclusion of Lie algebras $0 \to k \to \mathfrak{gl}(Q,\mathbf{d})$ is, by definition, $\mathfrak{pgl}(Q,\mathbf{d})$, and we can identify the cokernel of the inclusion of free R-modules $0 \to R \to \mathfrak{gl}(Q,\mathbf{d})) \otimes R$ with $\mathfrak{pgl}(Q,\mathbf{d}) \otimes R$. Thus, provided the generic representation in $\mathrm{Rep}(Q,\mathbf{d})$ is indecomposable, we have a short exact sequence

$$0 \to \mathfrak{pgl}(Q,\mathbf{d}) \otimes_k R \xrightarrow{\tilde{d}^M_M} \theta_R \to \mathrm{Ext}^1_{RQ}(M,M) \to 0. \tag{2}$$

Even without generic indecomposability, we still have an exact sequence

$$\mathfrak{pgl}(Q,\mathbf{d}) \otimes_k R \xrightarrow{\tilde{d}^M_M} \theta_R \to \mathrm{Ext}^1_{RQ}(M,M) \to 0. \tag{3}$$

Let D be the support of $\mathrm{Ext}^1_{RQ}(M,M) = T^1(\mathfrak{i}/\mathrm{Rep}(Q,\mathbf{d}))$, with (possibly non-reduced) coordinate ring $R[D] = R/\mathscr{F}_0\big(\mathrm{Ext}^1_{RQ}(M,M)\big)$, where $\mathscr{F}_0$ means zero'th Fitting ideal.

Proposition 5.1. (i) *D is the set of non-rigid representations. Its open complement is the set of rigid representations*[4].

If $q(\mathbf{d}) = 1$ and the general representation in $\mathrm{Rep}(Q,\mathbf{d})$ *is indecomposable, thus, $\mathbf{d}$ is a Schur root, then*

[4]Singularity theorists might prefer 'stable' to 'rigid'; however in representation theory the term 'stable' often refers to its meaning in geometric invariant theory, so here we use 'rigid'.

(ii) *D is a divisor in* $\mathrm{Rep}(Q, \mathbf{d})$.

(iii) $\mathrm{Ext}^1_{RQ}(M, M)$ *is a maximal Cohen-Macaulay $R[D]$-module.*

(iv) *The image of* $\tilde{d}^M_M : \mathfrak{pgl}(Q, \mathbf{d}) \otimes_k R \to \theta_R$ *is contained in* $\mathrm{Der}(-\log D)$.

Proof. (i) Let m_V be the maximal ideal of R corresponding to $V \in \mathrm{Rep}(Q, \mathbf{d})$. By right-exactness of tensor product, tensoring the sequence (3) with R/m_V gives the exact sequence

$$\mathfrak{pgl}(Q, \mathbf{d}) \xrightarrow{d^V_V} \mathrm{Rep}(Q, \mathbf{d}) \to \mathrm{Ext}^1_Q(V, V) = T^1(V) \to 0.$$

This establishes (i).

(ii) Since now $\mathfrak{pgl}(Q, \mathbf{d}) \otimes_k R$ and θ_R are free R-modules of the same rank, $\mathscr{F}_0(\mathrm{Ext}^1_{RQ}(M, M))$ is generated by $\det(\tilde{d}^M_M)$, and so

$$D = \mathrm{supp}(\mathrm{Ext}^1_{RQ}(M, M)) = V\big(\mathscr{F}_0(\mathrm{Ext}^1_{RQ}(M, M))\big) = V(\det(\tilde{d}^M_M))\,.$$

(iii) Exactness of the sequence (2) implies, by the Auslander-Buchsbaum formula, that

$$\mathrm{depth}_R(\mathrm{Ext}^1_{RQ}(M, M)) = \dim\, R - 1 = \dim\, \mathrm{Ext}^1_{RQ}(M, M)\,,$$

where “dim” here refers to Krull dimension.

Hence $\mathrm{Ext}^1_{RQ}(M, M)$ is a Cohen-Macaulay R-module. It is annihilated by $\mathscr{F}_0\big(\mathrm{Ext}^1_{RQ}(M, M)\big)$, so is an $R[D]$-module, and as such, a maximal Cohen-Macaulay module.

(iv) The vector fields in $\tilde{d}^M_M\big(\mathfrak{pgl}(Q, \mathbf{d}) \otimes_k R\big)$ are infinitesimal generators of the action of $\mathrm{Gl}(Q, \mathbf{d})$ on $\mathrm{Rep}(Q, \mathbf{d})$, and are thus tangent to all its orbits. So they are tangent to D, which is a union of orbits. □

Note that by 3.5, if Q is a Dynkin quiver then in order for (ii)-(iv) to hold we need only require that $q(\mathbf{d}) = 1$.

If the conditions of 5.1(ii)-(iv) hold, and moreover the vector fields in $\tilde{d}^M_M\big(\mathfrak{pgl}(Q, \mathbf{d}) \otimes_k R\big)$ *generate* $\mathrm{Der}(-\log D)$ then D is a linear free divisor, since by exactness of (2), $\mathrm{Der}(-\log D)$ is free over R. Saito's criterion (1.2 above) shows that in order that they do generate, it is enough that $\det\big(\tilde{d}^M_M\big)$ be reduced.

Thus, we obtain the following result:

Corollary 5.2. *With the conditions of* 5.1(ii)-(iv), *suppose in addition that D is reduced. Then it is a linear free divisor.* □

From now on we will refer to the divisor D of non-rigid representations in $\mathrm{Rep}(Q, \mathbf{d})$ as the *discriminant* and call $\Delta = \det\big(\tilde{d}^M_M\big)$ its *canonical equation*.

Suppose that D is reduced at V. Then by Saito's criterion, the vector fields in $\tilde{d}^M_M\big(\mathfrak{pgl}(Q, \mathbf{d}) \otimes_k R\big)$ generate the stalk at V of the sheaf $\mathrm{Der}(-\log D)$. If V is a regular point of D, then the tangent space $T_V D$ is equal to $d^V_V\big(\mathfrak{pgl}(Q, \mathbf{d})\big) \subseteq \mathrm{Rep}(Q, \mathbf{d})$. It follows that the deformation of V obtained by following any smooth curve transverse to D is versal. For the same reason, any deformation in a direction tangent to D is infinitesimally trivial at V. Since the same holds at any nearby point, any deformation of V in the smooth part of D is globally trivial. In terms of the group action, this means that then each irreducible component of D contains a dense open orbit, and for each representation V in such an orbit, $T^1(V)$ will be one-dimensional. We now investigate further the relation between the dimension of $T^1(V)$ for a generic representation on such a component and the multiplicity with which that component occurs in the discriminant.

Lemma 5.3. *Let D_j be an irreducible component of D, and h_j its reduced equation, m_j the multiplicity of h_j in* $\det\big(\tilde{d}^M_M\big)$*, and V_j a generic representation on D_j. One has then $m_j \geq \dim_k T^1(V_j)$ and equality holds if and only if h_j annihilates* $\mathrm{Ext}^1_{RQ}(M, M)$*. In particular, $m_j = 1$ forces* $\dim T^1(V_j)$ *to be one-dimensional and the orbit generated by V_j to be dense in D_j.*

Proof. Let p be the ideal (h_j). Then the localisation R_p is a discrete valuation ring. We must have

$$\mathrm{Ext}^1_{RQ}(M, M) \otimes_R R_p \simeq \bigoplus_1^{\ell} R_p/(pR_p)^{\alpha_t} \tag{4}$$

for some positive integers α_t; it follows that the matrix $\tilde{d}^M_M$ is equivalent, over R_p, to a matrix of the form $\mathrm{diag}(h_j^{\alpha_1}, \ldots, h_j^{\alpha_\ell}) \oplus I_{r-\ell}$, a block matrix formed of the indicated diagonal matrix and the identity matrix $I_{r-\ell}$, where $r = \dim_k \mathfrak{pgl}(Q, \mathbf{d})$. Evidently $\det \tilde{d}^M_M = (h_j)^{\sum_{t=1}^{\ell} \alpha_t}$ in R_p, and so $\sum_{t=1}^{\ell} \alpha_t = m_j$. Moreover, by (4), $\sum_{t=1}^{\ell} \alpha_t$ is also the rank of $\mathrm{Ext}^1_{RQ}(M, M)$ at a generic point V_j of D_j. Dividing by the maximal ideal m_{V_j}, we see that then ℓ is equal to $\dim_k \mathrm{Ext}^1_Q(V_j, V_j)$. Therefore, $m_j = \sum_{t=1}^{\ell} \alpha_t \geq \ell = \dim_k \mathrm{Ext}^1_Q(V_j, V_j) = \dim T^1(V_j)$. Clearly, $m_j = \ell$ if and only if each $\alpha_t = 1$ if and only if h_j annihilates $\mathrm{Ext}^1_{RQ}(M, M)$. $\square$

In the case of Dynkin quivers, it follows that the discriminant is indeed reduced, as we show next.

Proposition 5.4. *Let $\mathbf{d}$ be a real Schur root of a Dynkin quiver Q and assume that $V \in \mathrm{Rep}(Q, \mathbf{d})$ satisfies* $\dim T^1(V) = 1$*. If $D' \subseteq D$ denotes the irreducible* *component of the discriminant that contains V and $h' = 0$ is its reduced equation, then h' divides $\Delta = \det\big(\tilde{d}_M^M\big)$ with multiplicity one.*

Proof. We begin by clarifying in general what it means that D' appears with multiplicity one, if we know already that the generic representation on it has one-dimensional T^1: As $T^1(V) = \operatorname{Ext}^1_Q(V,V)$ is one-dimensional, the semi-universal deformation of V as a representation of Q has a one-dimensional base. Because V deforms into a rigid representation generically, its reduced discriminant consists just of the origin. By Openess of Versality, it suffices to prove that the discriminant in that semi-universal deformation is indeed reduced. If $\mathfrak{V}$ is the universal module over $k[\![t]\!]$, the (formal) base ring of the semi-universal deformation, it suffices to show that $\operatorname{Ext}^1_{k[\![t]\!]Q}(\mathfrak{V},\mathfrak{V})$ is a one-dimensional vector space. Now $\operatorname{Ext}^1_{k[\![t]\!]Q}(\mathfrak{V},\mathfrak{V})$ is concentrated on the discriminant, thus a finite dimensional vector space. Moreover, $\operatorname{Ext}^1_{k[\![t]\!]Q}(\mathfrak{V},\mathfrak{V}) \otimes_{k[\![t]\!]} k \cong \operatorname{Ext}^1_Q(V,V) \cong k$, whence as $k[\![t]\!]$–module $\operatorname{Ext}^1_{k[\![t]\!]Q}(\mathfrak{V},\mathfrak{V}) \cong k[\![t]\!]/(t^m)$ for some m. We need to show that $m = 1$, and this can be achieved by establishing that the following natural projection, in its various guises:

$$\begin{tikzcd}
\operatorname{Ext}^1_{k[\![t]\!]Q}(\mathfrak{V},\mathfrak{V}) \otimes_{k[\![t]\!]} k[\![t]\!]/(t^2) \arrow[r, "\cong"] \arrow[d] & \operatorname{Ext}^1_{k[\![t]\!]Q}(\mathfrak{V},\mathfrak{V}/(t^2)\mathfrak{V}) \arrow[r, "\cong"] \arrow[d] & k[\![t]\!]/(t^m,t^2) \arrow[d] \\
\operatorname{Ext}^1_{k[\![t]\!]Q}(\mathfrak{V},\mathfrak{V}) \otimes_{k[\![t]\!]} k \arrow[r, "\cong"] & \operatorname{Ext}^1_{k[\![t]\!]Q}(\mathfrak{V},\mathfrak{V}/(t)\mathfrak{V}) \cong \operatorname{Ext}^1_{kQ}(V,V) \arrow[r, "\cong"] & k
\end{tikzcd}$$

is an isomorphism. To this end, let

$$0 \to V \xrightarrow{i} W \xrightarrow{p} V \to 0 \tag{5}$$

represent a nontrivial element in $\operatorname{Ext}^1_Q(V,V)$. Define an action of t on W through $t(w) = ip(w)$. Clearly, $t^2 = ipip = 0$ on W, whence the Q–representation W becomes as well a $k[\![t]\!]/(t^2)$–module. Infinitesimal deformation theory says that indeed $W \cong \mathfrak{V}/t^2\mathfrak{V}$, and that the extension above can be viewed as an extension of $k[\![t]\!]$–modules,

$$0 \to V \cong \mathfrak{V}/t\mathfrak{V} \xrightarrow{i \cong t\times -} W \cong \mathfrak{V}/t^2\mathfrak{V} \xrightarrow{p \cong -\otimes_{k[\![t]\!]} k} V \cong \mathfrak{V}/t\mathfrak{V} \to 0\,.$$

Now apply $\operatorname{Hom}_{k[\![t]\!]Q}(\mathfrak{V},-)$ to this exact sequence to obtain the following long exact sequence of $k[\![t]\!]$–modules, with δ denoting the connecting homomorphism:

$$\begin{aligned}
0 \longrightarrow \operatorname{Hom}_{k[\![t]\!]Q}(\mathfrak{V},V) \longrightarrow \operatorname{Hom}_{k[\![t]\!]Q}(\mathfrak{V},W) \longrightarrow \operatorname{Hom}_{k[\![t]\!]Q}(\mathfrak{V},V) \xrightarrow{\delta} \\
\longrightarrow \operatorname{Ext}^1_{k[\![t]\!]Q}(\mathfrak{V},V) \longrightarrow \operatorname{Ext}^1_{k[\![t]\!]Q}(\mathfrak{V},W) \xrightarrow{\operatorname{Ext}^1_{k[\![t]\!]Q}(\mathfrak{V},p)} \operatorname{Ext}^1_{k[\![t]\!]Q}(\mathfrak{V},V) \longrightarrow 0
\end{aligned}$$

The map $\pi = \mathrm{Ext}^1_{k[\![t]\!]Q}(\mathfrak{V}, p)$ is the same as the projection alluded to above, which we wish to show is an isomorphism. Using the various identifications, we may rewrite this long exact sequence as

$$0 \longrightarrow \mathrm{End}_Q(V) \longrightarrow \mathrm{End}_{(k[\![t]\!]/(t^2))Q}(W) \longrightarrow \mathrm{End}_Q(V) \xrightarrow{\delta}$$

$$\longrightarrow \mathrm{Ext}^1_Q(V,V) \longrightarrow \mathrm{Ext}^1_{k[\![t]\!]Q}(\mathfrak{V}, W) \xrightarrow{\pi} \mathrm{Ext}^1_Q(V,V) \longrightarrow 0$$

As $\mathbf{d}$ is a Schur root, and $\dim T^1(V) = \dim \mathrm{Ext}^1_Q(V,V) = 1$, we see that π is an isomorphism if and only if $\delta \neq 0$ if and only if there exists a Q–endomorphism of V that cannot be lifted to a $k[\![t]\!]$–linear Q–endomorphism of W. While these considerations apply to any quiver, we now show that $\delta \neq 0$, thereby establishing that π is indeed an isomorphism, for any Schur root of a Dynkin quiver.

By assumption, $q(V) = 1$ and $\dim \mathrm{Ext}^1_Q(V,V) = 1$, whence V is decomposable, say, $V = V' \oplus V''$ for nonzero Q–representations V', V''. It follows from $\dim \mathrm{End}_Q(V) = 2$ that $\mathrm{End}_Q(V) \cong \mathrm{End}_Q(V') \oplus \mathrm{End}_Q(V'')$, and that the endomorphism rings of V', V'' are one-dimensional, in particular these representations are indecomposable. This means that their dimension vectors are real Schur roots as well, and so the representations are rigid. From $\mathrm{Ext}^1_Q(V,V) \cong \mathrm{Ext}^1_Q(V' \oplus V'', V' \oplus V'')$, it then follows that exactly one of the groups $\mathrm{Ext}^1_Q(V', V'')$ or $\mathrm{Ext}^1_Q(V'', V')$ is nonzero — and then one-dimensional. Assume $\mathrm{Ext}^1_Q(V', V'') \neq 0$. The associated nontrivial extension

$$0 \to V'' \xrightarrow{i} W' \xrightarrow{p} V' \to 0 \tag{6}$$

gives rise to the following nonzero extension class in $\mathrm{Ext}^1_Q(V,V)$:

$$\begin{array}{ccccccccc}
 & & V' & \xrightarrow{\quad = \quad} & V' & & & & \\
 & & \oplus & & \oplus & & & & \\
0 & \to & V'' & \xrightarrow{\quad i \quad} & W' & \xrightarrow{\quad p \quad} & V' & \to & 0 \\
 & & & & \oplus & & \oplus & & \\
 & & & & V'' & \xrightarrow{\quad = \quad} & V'' & &
\end{array}$$

Note that W' has dimension vector $\mathbf{d}$, as that is the sum of the dimension vectors of V'' and V', equal to the dimension vector of V. It is now a general fact that $V = V' \oplus V''$ deforms into the middle term W', for any extension. As the sequence does not split, $W' \not\cong V$, and, as V has a onedimensional semi-universal deformation, W' must be the indecomposable representation of dimension vector $\mathbf{d}$. Using the observation following (5), the $k[\![t]\!]$–module structure on the middle term $W = V' \oplus W' \oplus V''$ is as follows:

$$t(V') = 0 \quad , \quad t|_{W'} = p \quad , \quad t|_{V''} = i\,.$$

With W' an indecomposable Q–representation and the action of t as described, it follows easily that $W = V'' \oplus W' \oplus V'$ is indecomposable as a Q–representation over $k[\![t]\!]$.

Accordingly, its endomorphism ring $\mathrm{End}_{(k[\![t]\!]/(t^2))Q}(W)$ contains only the trivial idempotents, thus none of the idempotents in $\mathrm{End}_Q(V)$ that corresponds to the projections onto the indecomposable factors of V can be lifted, and the natural ring homomorphism $\mathrm{End}_{(k[\![t]\!]/(t^2))Q}(W) \to \mathrm{End}_Q(V)$ is not surjective. This yields the claim. □

Corollary 5.5. *If Q is a Dynkin quiver and $\mathbf{d}$ is a real root of Q then the discriminant in $\mathrm{Rep}(Q, \mathbf{d})$ is a linear free divisor.*

Proof. By Gabriel's theorem Q is of finite representation type. Therefore at a generic point V on each irreducible component of D, any deformation of V inside D is trivial. Thus $T^1(V)$ is 1-dimensional. □

Everything we have said so far only depends on the *support* of the dimension vector $\mathbf{d}$, that is, the full subquiver whose nodes are those $x \in Q_0$ with $\mathbf{d}(x) \neq 0$. A dimension vector is *sincere* if its support is all of Q_0.

6 A Criterion for D to be a Linear Free Divisor

The group $\mathrm{Gl}(Q, \mathbf{d})$ acts on the ring R of polynomial functions on $\mathrm{Rep}(Q, \mathbf{d})$ by the contragredient action, as described earlier in Section 5. A polynomial $f \in R$ is a *semi-invariant of weight* χ, where χ is a character of $\mathrm{Gl}(Q, \mathbf{d})$, if for all $g \in \mathrm{Gl}(Q, \mathbf{d})$ we have $g \cdot f = \chi(g)f$. As the characters of $\mathrm{Gl}_n(k)$ are just integral powers of det, the characters of $\mathrm{Gl}(Q, \mathbf{d})$ are in bijection with elements of $\mathbb{Z}^{Q_0}$. The *weight* $w(f)$ of a semi-invariant f is usually identified with the image in $\mathbb{Z}^{Q_0}$ of its associated character.

Theorem 6.1. (Sato-Kimura [29]) *Let the connected algebraic group G act on the vector space V. If there is an open orbit then the ring $SI(G, V)$ spanned by the semi-invariants is a polynomial ring:*

$$SI(G, V) = k[f_1, \ldots, f_s]$$

for some collection of algebraically independent and irreducible semi-invariants $f_1, \ldots, f_s$. Moreover if $f_i \in SI(G, V)_{\chi_i}$ then the χ_i are linearly independent in the space of characters of G. □

Corollary 6.2. *Under the assumptions of the theorem, the set of characters χ such that $SI(G, V)_\chi \neq 0$ forms a free abelian semigroup, isomorphic to $\mathbb{N}^s$.*

In particular, if f is any semi-invariant, of weight χ, then $f = uf_1^{a_1} \cdots f_s^{a_s}$, where u is a unit in k and the $a_i \geq 0$ are the unique integers such that $\chi = \sum_{i=1}^{s} a_i \chi_i$ in the space of characters of G. □

Suppose that $\mathbf{d}$ is a real Schur root of Q, and let D be the discriminant in $\mathrm{Rep}(Q, \mathbf{d})$. As D is preserved under the action of $\mathrm{Gl}(Q, \mathbf{d})$, its canonical equation Δ is a semi-invariant. If $V \notin D$, and f is a non-zero semi-invariant, then $f(V)$ cannot vanish; if it did, then it would vanish everywhere on the orbit of V, which is dense. In other words, the zero locus of any semi-invariant must be contained in the discriminant. In particular, with the f_i as in 6.1, $f_1 \cdots f_s$ is necessarily a reduced equation for D, and so $\Delta = uf_1^{a_1} \cdots f_s^{a_s}$, with u a unit in k, and uniquely determined integers $a_i > 0$.
Moreover, Kac has shown in [18, p.153] that the discriminant for a real Schur root $\mathbf{d}$ contains precisely $n-1$ irreducible components, where n is the number of nodes in the support of $\mathbf{d}$, thus, there are $s = n - 1$ fundamental semi-invariants f_i in $SI(\mathrm{Gl}(Q, \mathbf{d}), \mathrm{Rep}(Q, \mathbf{d}))$. This gives us a first combinatorial criterion for D to be a linear free divisor.

Proposition 6.3. *Suppose that $\mathbf{d}$ is a real Schur root of Q, supported on n nodes. Assume further that $g_1, ..., g_{n-1}$ are semi-invariants on $\mathrm{Rep}(Q, \mathbf{d})$ with linearly independent weights $w_i = w(g_i)$. If the weight of the discriminant D satisfies $w(D) = \sum_{i=1}^{n-1} a_i w_i$, for integers $a_i \geq 1$, then $\Delta = ug_1^{a_1} \cdots g_{n-1}^{a_{n-1}}$ for some unit $u \in k$. If we know further that the weights w_i generate the semigroup of all weights occurring in $SI(\mathrm{Gl}(Q, \mathbf{d}), \mathrm{Rep}(Q, \mathbf{d}))$, then the g_i constitute the reduced equations of the components of D, and D is a linear free divisor if and only if each $a_i = 1$.* □

Derksen and Weyman in [8] describe in general the semigroup of weights occurring in $SI(\mathrm{Gl}(Q, \mathbf{d}), \mathrm{Rep}(Q, \mathbf{d}))$ through a single equation[5] and integral inequalities that depend upon the dimension vectors of generic subrepresentations, whence the criterion can be applied, at least in principle. We may as well turn the criterion around to determine all semi-invarants if we already know that D is a linear free divisor, such as for real roots whose support is a Dynkin quiver:

Corollary 6.4. *Assume the discriminant D in $\mathrm{Rep}(Q, \mathbf{d})$, for $\mathbf{d}$ a real Schur root, is a free divisor and its canonical equation factors as $\Delta = g_1 \cdots g_{n-1}$ for semi-invariant polynomials g_i with linearly independent weights. If n is the number of nodes in the support of $\mathbf{d}$, then the factors g_i are algebraically independent and irreducible polynomials that generate the ring of semi-invariants $SI(\mathrm{Gl}(Q, \mathbf{d}), \mathrm{Rep}(Q, \mathbf{d}))$.* □

[5] Namely that the ordinary scalar product of the weight of a semi-invariant with the dimension vector $\mathbf{d}$ has to vanish, that is, $w \cdot \mathbf{d} = 0$.

Using yet another result of Schofield [30], one may find the weights of all semi-invariants — indeed the semi-invariants themselves, as we will discuss in more detail later, see Section 8. Suppose that $\mathbf{e}$ is a dimension vector such that $\langle \mathbf{e}, \mathbf{d}\rangle = 0$. In the exact sequence (1), the matrix d_V^W is now square. We define a polynomial function $c : \mathrm{Rep}(Q, \mathbf{e}) \times \mathrm{Rep}(Q, \mathbf{d}) \to k$ by $c(W, V) = \det d_V^W$. The map

$$\begin{aligned}&\mathrm{Rep}(Q, \mathbf{e}) \times \mathrm{Rep}(Q, \mathbf{d})\\ &\qquad \to \mathrm{Hom}_k\left(\prod_{x\in Q_0} \mathrm{Hom}\big(k^{e(x)}, k^{d(x)}\big), \prod_{\varphi\in Q_1} \mathrm{Hom}_k\big(k^{e(t(\varphi))}, k^{d(h(\varphi))}\big)\right)\end{aligned}$$

sending (W, V) to d_V^W is $\mathrm{Gl}(\mathbf{e}) \times Gl(\mathbf{d})$-equivariant, and it follows that for fixed W, the map $c^W := c(W, \ )$ represents a semi-invariant polynomial on $\mathrm{Rep}(Q, \mathbf{d})$.

Theorem 6.5. ([30] 4.3) *Let Q be a quiver without oriented cycles, and let $\mathbf{d}$ be a sincere real Schur root for Q. The polynomials c^W with $\langle W, V\rangle = 0$ span the ring of semi-invariants $SI(\mathrm{Gl}(\mathbf{d}), \mathrm{Rep}(Q, \mathbf{d}))$.* □

The weights of these semi-invariants, as well as that of the discriminant, are then easily established, using essentially the same argument as in [30, 1.4]. To formulate it succinctly, we introduce the *in–degree* $\mathbf{in_d}$ and the *out–degree* $\mathbf{out_d}$ of $\mathbf{d}$ as the dimension vectors

$$\mathbf{in_d}(x) = \sum_{\varphi\in Q_1 : h\varphi = x} \mathbf{d}(t\varphi)\,, \quad \mathbf{out_d}(x) = \sum_{\varphi\in Q_1 : t\varphi = x} \mathbf{d}(h\varphi)\,, \quad \text{for } x \in Q_0. \tag{7}$$

In terms of the Euler matrix E of Q (see Section 3) one has

$$\mathbf{in_d} = \mathbf{d} - \mathbf{d}E \quad , \quad \mathbf{out_d} = \mathbf{d} - \mathbf{d}E^T\,.$$

Lemma 6.6. *Let $\mathbf{d}, \mathbf{e}$ be dimension vectors for the quiver Q with $\langle \mathbf{e}, \mathbf{d}\rangle = 0$. The weight of the $\mathrm{Gl}(\mathbf{e}) \times \mathrm{Gl}(\mathbf{d})$ semi-invariant polynomial $c(W, V)$ in the character group $\mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0}$ is*

$$w(c(W, V)) = (\mathbf{d} - \mathbf{out_d}, -\mathbf{e} + \mathbf{in_e}) = (\mathbf{d}E^T, -\mathbf{e}E)\,,$$

while that of the $\mathrm{Gl}(\mathbf{d})$ semi-invariant c^W in $\mathbb{Z}^{Q_0}$ is

$$w(c^W) = -\mathbf{e} + \mathbf{in_e} = -\mathbf{e}E$$

and the weight of the discriminant in $\mathrm{Rep}(Q, \mathbf{d})$ equals

$$w(\Delta) = \mathbf{in_d} - \mathbf{out_d} = \mathbf{d}(E^T - E)\,.$$

Proof. Let V, W be two representations with dimension vectors $\mathbf{d}, \mathbf{e}$ such that $\langle \mathbf{e}, \mathbf{d} \rangle = 0$. The map d_V^W can be viewed as a linear map

$$d_V^W : \bigoplus_{x \in Q_0} V_x \otimes W_x^* \longrightarrow \bigoplus_{\varphi \in Q_1} V(h\varphi) \otimes W(t\varphi)^* ,$$

where $(-)^*$ denotes the k–dual. Denoting by $\Lambda(-)$ the highest exterior power of a vector space, and observing that

$$\Lambda(U^*) \cong \Lambda(U)^* , \quad \Lambda(U \oplus U') \cong \Lambda(U) \otimes \Lambda(U') ,$$

$$\Lambda(U \otimes U') \cong \Lambda(U)^{\dim U'} \otimes \Lambda(U')^{\dim U} ,$$

for vector spaces U, U', the determinant $\det d_V^W$ of d_V^W can be represented as

$$\Lambda(d_V^W) : \bigotimes_{x \in Q_0} \Lambda(V_x)^{\mathbf{e}(x)} \otimes \Lambda(W_x^*)^{\mathbf{d}(x)} \to \bigotimes_{\varphi \in Q_1} \Lambda(V_{h\varphi})^{\mathbf{e}(t\varphi)} \otimes \Lambda(W_{t\varphi}^*)^{\mathbf{d}(h\varphi)} .$$

One reads off that as a semi-invariant for $\mathrm{Gl}(\mathbf{e}) \times \mathrm{Gl}(\mathbf{d})$ the determinant of d_V^W transforms according to

$$\left(\prod_{\varphi \in Q_1} \det\big(\mathrm{Gl}(\mathbf{d}(h\varphi))\big)^{\mathbf{e}(t\varphi)} \det\big(\mathrm{Gl}(\mathbf{e}(t\varphi))\big)^{-\mathbf{d}(h\varphi)} \right) \cdot \left(\prod_{x \in Q_0} \det\big(\mathrm{Gl}(\mathbf{d}(x))\big)^{-\mathbf{e}(x)} \det\big(\mathrm{Gl}(\mathbf{e}(x))\big)^{\mathbf{d}(x)} \right) .$$

Thus, its weight, in the character group $\mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0}$ of $\mathrm{Gl}(\mathbf{e}) \times \mathrm{Gl}(\mathbf{d})$, is given on a pair of nodes (y, x) by

$$w(\det d_V^W)(y, x) = \mathbf{d}(y) - \sum_{t\varphi = y} \mathbf{d}(h\varphi) - \mathbf{e}(x) + \sum_{h\varphi = x} \mathbf{e}(t\varphi)$$

thus,

$$w(\det d_V^W) = (\mathbf{d} - \mathbf{out}_{\mathbf{d}}, -\mathbf{e} + \mathbf{in}_{\mathbf{e}}) = (\mathbf{d}E^T, -\mathbf{e}E) \ \in \ \mathbb{Z}^{Q_0 \times Q_0}.$$

For $V = W$, the diagonal summand $k \subseteq \oplus_{x \in Q_0} \mathrm{Hom}(V_x, V_x)$ does not contribute to the weight of the determinant, and restricting $w(\det d_V^V)$ to the diagonal $y = x$ yields the claimed formula for the discriminant. $\square$

Now we are ready to study some examples.

7 Examples

To illustrate the results and to exhibit explicit linear free divisors arising from Dynkin quivers, we concentrate mainly on the most complicated ones, those corresponding to the *highest root* of a Dynkin diagram viewed as the dimension vector of some Dynkin quiver. Recall that the connected Dynkin diagrams are in natural bijection with the *binary polyhedral groups*, the congruence classes of finite subgroups of $\mathrm{Sl}(2, \mathbb{C})$. One has the following simple relation between the dimension of the representation variety associated to the highest root and the order of the corresponding finite group.

Lemma 7.1. *Let Q be a connected Dynkin quiver, $\mathbf{d}$ the highest root of the underlying Dynkin diagram, and Γ the associated binary polyhedral group. The dimension of* $\mathrm{Rep}(Q, \mathbf{d})$*, equal to the degree of the discriminant D, is then* $\dim \mathrm{Rep}(Q, \mathbf{d}) = |\Gamma| - 2$.

Proof. By the McKay correspondence, the components $\mathbf{d}(x)$ of the highest root are in bijection with the dimensions of the isomorphism classes of irreducible and nontrivial representations of Γ. Accordingly,

$$|\Gamma| = 1 + \sum_{x \in Q_0} \mathbf{d}(x)^2 = 2 + \dim \mathfrak{pgl}(\mathbf{d}) = 2 + \dim \mathrm{Rep}(Q, \mathbf{d})\,.$$

□

Example 7.2. Let Q be a Dynkin quiver of type A_n with any orientation, and let $\mathbf{d}$ be its highest root, the dimension vector assigning 1 at each vertex. Then $\mathrm{Rep}(Q, \mathbf{d})$ can be identified with $k^{|Q_1|} = k^{n-1}$ by associating to each morphism its 1×1 matrix. Each of the coordinates is a semi-invariant, and D is the normal crossing divisor in $n - 1$ variables. Notice that D is independent of the orientation of the arrows.

Example 7.3. Consider the two Dynkin quivers $Q^{(1)}$ and $Q^{(2)}$ of type E_6 with the highest root as dimension vector as shown. Each space $\mathrm{Rep}(Q^{(i)}, \mathbf{d})$ has dimension $22 = 24 - 2$, as the corresponding binary tetrahedral group has order 24.

$$\begin{tikzcd} & & 2 & & \\ 1 \arrow[r, "A"'] & 2 \arrow[r, "B"'] & 3 \arrow[u, "E"'] & 2 \arrow[l, "C"] & 1 \arrow[l, "D"] \end{tikzcd} \qquad \begin{tikzcd} & & 2 & & \\ 1 \arrow[r, "A"'] & 2 \arrow[r, "B"'] & 3 \arrow[u, "E"'] \arrow[r, "C"'] & 2 \arrow[r, "D"'] & 1 \end{tikzcd}$$

One sees easily that codimension 1 degeneracies are given, for $Q^{(1)}$, by the vanishing of any of[6]

$$\det[EB],\ \det[EC],\ \det[B|CD],\ \det[BA|C],\ \det[EBA|ECD]\,.$$

[6] We indicate by $X|Y$ the concatenation of two matrices X, Y with the same number of rows.

The third of these measures the independence of the images of B and CD in the 3-dimensional space attached to the central node; the fourth and fifth are to be understood similarly. The degrees of the corresponding equations, equal to $4, 4, 4, 4$, and 6, add to 22, and their weights are easily seen to be linearly independent. Thus these form a complete list of the factors, and the linear free divisor D is the union of these five, necessarily irreducible components.

For $Q^{(2)}$, four codimension 1 degeneracies are defined by the vanishing of

$$\det[EB], \det[CB], \det \begin{bmatrix} E \\ DC \end{bmatrix}, DCBA\,.$$

One further degeneracy is easier to describe verbally than by an equation (however, see Section 8 and in particular Example 8.1 below): it is the failure of general position, in the 3-dimensional space at the central node, of the three lines $\mathrm{im}(BA), \ker(E), \ker(C)$.

In both cases, each equation of degree 4 has 12 monomials and the equation of degree 6 has 48. Moreover, the complements of the discriminants $D^{(1)} \subset \mathrm{Rep}(Q^{(1)}, \mathbf{d})$ and $D^{(2)} \subset \mathrm{Rep}(Q^{(2)}, \mathbf{d})$ are isomorphic to one another, being orbits, with trivial isotropy, of the groups $\mathbb{P}\mathrm{Gl}(Q^{(i)}, \mathbf{d})$, which are themselves isomorphic to one another. However, the two discriminants are not isomorphic. Essentially, this is because the equations involve different numbers of variables. In the first case, the five equations involve, respectively, $12, 12, 14, 14$, and 22 variables, while in the second the five equations involve $12, 12, 14, 16$ and 20 variables. A *Macaulay* calculation confirms that the spaces of vector fields with constant coefficients tangent to the germs at $0 \in \mathrm{Rep}(Q^{(i)}, \mathbf{d})$ of the five components have dimensions $10, 10, 8, 8$, and 0 in the first case, and $10, 10, 8, 6$, and 2 in the second. Any isomorphism $D^{(1)} \cong D^{(2)}$ must map 0 to 0, since because of the presence of the Euler field, in each case 0 is the only point where all of the vector fields in $\mathrm{Der}(-\log D^{(i)})$ vanish. It follows that these dimensions are geometrical invariants: the just exhibited dimension corresponding to the irreducible component $D_j^{(i)}$ of $D^{(i)}$ is the maximum dimension of a non-singular factor in a product decomposition $(D_j^{(i)}, 0) \cong (E_j^{(i)}, 0) \times (F_j^{(i)}, 0)$ ([10]).

Proposition 7.4. *Let Q be the quiver whose nodes consist of $n+1$ sources surrounding one sink, with an arrow going from each source to the sink. The discriminant with respect to the dimension vector that assigns 1 to each of the sources and n to the sink is a linear free divisor. It is of the form $\Delta_1 \cdots \Delta_{n+1}$, where the Δ_i are the maximal minors of a generic $n \times (n+1)$–matrix.*

Proof. We can identify $\mathrm{Rep}(Q, \mathbf{d})$ with the space of $n \times (n+1)$–matrices, with the matrix of each of the arrows determining a column. The degree of the discriminant D equals $n(n+1)$. The generic representation describes $n+1$ distinct lines in a vector space of dimension n, with no n of them lying in a hyperplane. Such a representation is indecomposable and lies in an open orbit, with the group $\mathrm{Gl}(n)$ acting transitively on the set of such line configurations in general position. Accordingly, the dimension vector is a real Schur root. There are $n+1$ codimension 1 degeneracies, each one determined by the vanishing of an $n \times n$ minor of the $n \times (n+1)$ matrix. The product of these minors has degree $n(n+1)$, equal to the degree of D, and the weights, assigning -1 to each source contributing to the minor, 0 to the remaining source, and 1 to the sink, are clearly linearly independent. Thus each is present in det $\tilde{d}_M^M$ with multiplicity 1. $\square$

Note that from Theorem 6.1 we recover the classical result that these maximal minors are algebraically independent.

Example 7.5. Consider the four quivers shown below, in which the underlying undirected graph is the extended Dynkin diagram of type $\widetilde{D}_4$. Assign to each the dimension vector with 1 at each outer node and 3 at the central node. According to Kac's result quoted as Proposition 3.7 above, the dimension vector shown is a real root. In (i)–(iii), it is also a Schur root, but in case (iv), it is not. In case (i), the discriminant is a linear free divisor, according to Proposition 7.4 above, but in cases (ii) and (iii) this fails. In case (iv), the discriminant is the whole space, and there is no rigid representation.

$$\begin{tikzcd} & \bullet \arrow[d, "A"] & \\ \bullet \arrow[r, "B"] & \bullet & \bullet \arrow[l, "D"] \\ & \bullet \arrow[u, "C"] & \end{tikzcd} \qquad \begin{tikzcd} & \bullet & \\ \bullet \arrow[r, "B"] & \bullet \arrow[u, "A"'] & \bullet \arrow[l, "D"] \\ & \bullet \arrow[u, "C"] & \end{tikzcd} \qquad \begin{tikzcd} & \bullet \arrow[d, "A"] & \\ \bullet & \bullet \arrow[l, "B"'] \arrow[r, "D"'] \arrow[d, "C"'] & \bullet \\ & \bullet & \end{tikzcd} \qquad \begin{tikzcd} & \bullet \arrow[d, "A"] & \\ \bullet & \bullet \arrow[l, "B"'] \arrow[r, "D"'] & \bullet \\ & \bullet \arrow[u, "C"] & \end{tikzcd}$$

(i) (ii) (iii) (iv)

In case (ii), there is a modulus attached to the codimension 1 degeneracy in which the images of B, C and D lie in a plane P; these three lines, together with the fourth line $P \cap \ker A$, determine a cross-ratio. Any representation V of this type therefore has T_V^1 of dimension (at least) 2, and so the multiplicity of the corresponding component in D is also at least 2. In fact it is exactly 2: the remaining three components of D are $\det AB, \det AC, \det AD$, each of degree 2. Together with twice the degree of $\det[B|C|D]$ these add up to 12, the degree of the (non-reduced) equation det $\tilde{d}_M^M$ of D. As the four components described have linearly independent weights, the multiplicity of the non-reduced component is exactly 2.

Case (iii), obtained by reversing all of the arrows, is dual to (ii): here the non-reduced component of D is where the kernels of the three outgoing arrows B, C, D meet along a line L. Together with the plane $L + \text{im } A$, these make four planes in the pencil of planes containing L, and thus once again determine a cross ratio.

In the fourth quiver, the given dimension vector is not a Schur root. For there is no open orbit. In a general representation V, im A and im C span a plane P. The intersections with P of ker B and ker D determine two further lines in P, and thus a cross-ratio. Since thus dim $\text{Ext}^1_Q(V, V) \geq 1$, it follows that dim $\text{Hom}_Q(V, V) \geq 2$, and V must be decomposable. Indeed, it is easily verified that the intersection ker $B \cap$ ker D, concentrated on the central node, splits off. By Kac's theorem, there is exactly one orbit of indecomposable representations. We invite the reader to find it.

Proposition 7.6. *Suppose that* $\mathbf{d}$ *is a real Schur root of the connected quiver* Q*, and let* Q^{opp} *be obtained from* Q *by reversing all of the arrows. If the discriminant in* $\text{Rep}(Q, \mathbf{d})$ *is a linear free divisor then the same holds in* $\text{Rep}(Q^{opp}, \mathbf{d})$.

Proof. This is essentially projective duality. Transposition determines an isomorphism of representation spaces $\text{Rep}(Q, \mathbf{d}) \to \text{Rep}(Q^{\text{opp}}, \mathbf{d})$ which maps orbits to orbits. □

Example 7.7. Suppose Q is a quiver and $x \in Q_0$ is a node. Construct a new quiver Q_x by replacing the node x by a pair of nodes x', x'' connected by an arrow F from x' to x'', and attaching the arrows previously attached to x either to x' or to x''. Two possible outcomes of this process are shown in the figure below. If $\mathbf{d}$ is any dimension vector for Q, we define a dimension vector $\mathbf{d}_x$ for Q_x by setting $\mathbf{d}_x(y) = \mathbf{d}(y)$ if $y \neq x', x''$, $\mathbf{d}_x(x') = \mathbf{d}_x(x'') = \mathbf{d}(x)$. Then $\langle \mathbf{d}_x, \mathbf{d}_x \rangle = \langle \mathbf{d}, \mathbf{d} \rangle$. If the generic representation in $\text{Rep}(Q, \mathbf{d})$ is indecomposable, then the same is true in $\text{Rep}(Q_x, \mathbf{d}_x)$, since generically $V(F)$ is an isomorphism. So it is reasonable to hope that if $\langle \mathbf{d}, \mathbf{d} \rangle = 1$ and $D \subset \text{Rep}(Q, \mathbf{d})$ is a linear free divisor, then the discriminant in $\text{Rep}(Q_x, \mathbf{d}_x)$ is also a linear free divisor. The following examples show that this is sometimes but not always the case.

The quivers Q_2 and Q_3 shown below are obtained from Q_1 by the operation just described. Assign to Q_1 the dimension vector $\mathbf{d}$ with 1's at all the sources and 4 at the central sink, and define $\mathbf{d}_x$ accordingly. By 7.4, the discriminant $D_1 \subset \text{Rep}(Q_1, \mathbf{d})$ is a linear free divisor with components given by the vanishing of

$$\det[A|B|C|D],\ \det[A|B|C|E],\ \det[A|B|D|E],\ \det[A|C|D|E],\ \det[B|C|D|E]\,.$$

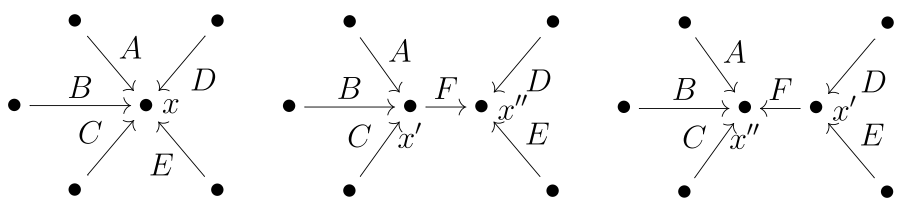
In $\text{Rep}(Q_2, \mathbf{d}_x)$, these become

$$\det[FA|FB|FC|D],\ \det[FA|FB|FC|E],\ \det[FA|FB|D|E],$$
$$\det[FA|FC|D|E],\ \det[FB|FC|D|E],\ \det F\,.$$

In $\mathrm{Rep}(Q_3, \mathbf{d}_x)$, they become

$$\det[A|B|C|FD],\ \det[A|B|C|FE],\ \det[A|B|FD|FE],$$
$$\det[A|C|FD|FE],\ \det[B|C|FD|FE],\ \det F\,.$$

The degrees of the (reduced) discriminants $D_2 \subset \mathrm{Rep}(Q_2, \mathbf{d}_x)$ and $D_3 \subset \mathrm{Rep}(Q_3, \mathbf{d}_x)$ are thus 36 and 32 respectively. So D_2 is a linear free divisor, whereas D_3 is not. The exponent of det F in the canonical equation Δ_3 is 2.



Q_1 $\qquad$ Q_2 $\qquad$ Q_3

One can easily show, by the same technique of counting degrees, that if one performs this operation on the central node in the quiver of Proposition 7.4, then one obtains a linear free divisor if and only if just two of the arrows coming from the outer nodes are attached to x'', and the rest are attached to x'.

By applying the same construction to Dynkin quivers and their roots, one can obtain further examples of linear free divisors. In particular, one easily deals with the case D_n in this way:

Proposition 7.8. *Let Q be the Dynkin quiver of type D_n with the following orientation*

$$\begin{tikzcd} 1 \arrow[dr, "A"] \\ & 2 \arrow[r, "C_1"] & 2 & \cdots & 2 \arrow[r, "C_{n-4}"] & 2 \arrow[r, "D"] & 1 \\ 1 \arrow[ur, "B"'] \end{tikzcd}$$

The indicated dimension vector $\mathbf{d}$ *is the highest root of* $\overline{D}_n$*. The discriminant in* $\mathrm{Rep}(Q, \mathbf{d})$ *is a linear free divisor of degree* $4n - 10$ *with* $n - 1$ *factors*

$$\det[A|B],\ \det C_1, \ldots, \det C_{n-4},\ DC_{n-4}\cdots C_1A,\ DC_{n-4}\cdots C_1B\,,$$

where the first $n-3$ *factors are of degree* 2*, the last two of degree* $n-2$*.*

Changing the orientation of arrows in Q *results in an isomorphic linear free divisor.*

Proof. The criterion 6.3 shows immediately that the factors are correct, as they represent semi-invariants with linearly independent weights. For the last assertion, note that changing the direction of the arrow underlying the matrix C_i, say, results in the same linear free divisor as the one already established, provided one replaces C_i by its adjoint matrix. Similarly, changing, say, the direction of the arrow underlying A, amounts to replacing $A = (a_1, a_2)$ by $A' = (a_2, -a_1)$ in the above factors, and the situation for B, D is analogous. □

8 Equations for D

To find equations for D in general, one can use the following recipe due to Schofield [30] that is based on his result 6.5 above. We quote it in the slightly simplified form that is all that we require here. Assume that Q is a finite connected quiver without oriented cycles and fix the sincere real Schur root $\mathbf{d}$ and a generic representation $V \in \mathrm{Rep}(Q, \mathbf{d})$.

To apply 6.5, one looks for roots $\mathbf{e}$ of Q such that $\langle \mathbf{e}, \mathbf{d} \rangle = 0$, and computes, for generic W in $\mathrm{Rep}(Q, \mathbf{e})$, the polynomial c^W. If $\mathrm{Hom}_Q(W, V) \neq 0$, then the square matrix underlying c^W_V has a nontrivial kernel and c^W vanishes on the open orbit, thus, identically. In view of this, one needs only to consider representations W that lie in the *left*[7] *orthogonal* category ${}^{\perp}V$, the full subcategory of all those finite dimensional representations W of Q that satisfy

$$\mathrm{Hom}_Q(W, V) = \mathrm{Ext}^1_Q(W, V) = 0\,.$$

Schofield shows that this left orthogonal category is equivalent to the category of finite dimensional representations of some new quiver Q' that has $n - 1$ nodes and contains no oriented cycles. In [8] (Lemma 1) it is pointed out that a short exact sequence

$$0 \to W' \to W \to W'' \to 0$$

of representations of Q leads either to the factorisation

$$c^W = c^{W''} c^{W'}$$

if $\langle W', V \rangle = \langle W'', V \rangle = 0$, or to the conclusion that $c^W = 0$ if $\langle W', V \rangle < 0$. So, if the generic representation in $\mathrm{Rep}(Q, \mathbf{e})$ is not simple in ${}^{\perp}V$, the semi-invariant we obtain will either be zero or a non-trivial product of others. Accordingly, one needs to consider only the $n - 1$ simple objects W in

[7] One may as well work throughout with the right orthogonal category $V^{\perp}$, the treatment is symmetric.

${}^{\perp}V$, and those must provide the factors of the discriminant via the associated determinants c^W. Indeed, the dimension vectors $\mathbf{e}_i$ of the simple objects W_i, for $i = 1, ..., n-1$ form the unique basis of the free abelian semigroup of dimension vectors $\mathbb{N}^{Q'_0}$ for ${}^{\perp}V$, and their associated characters $\langle \mathbf{e}_i, ?\rangle = w(c^{W_i}) = -\mathbf{e}_i + \mathbf{in}_{\mathbf{e}_i} = -\mathbf{e}_i E$, see 6.6, form the unique basis of the free abelian semigroup of weights for the semi-invariants of $\mathrm{Rep}(Q, \mathbf{d})$. Conversely, knowing the weights w_i of the generating semi-invariants, one may calculate the dimension vectors $\mathbf{e}_i$ through $\mathbf{e}_i = -w_i(E^{-1})$, with E^{-1} as exhibited in 3.3.

The map $\mathbb{N}^{Q'_0} \to \mathbb{N}^{Q_0}$ that maps the ith basis vector to $\mathbf{e}_i$ is an isometry with respect to the Euler forms on Q' and Q, and as the simple representations for Q' have real Schur roots as their dimension vectors, the same must hold true for the dimension vectors $\mathbf{e}_i$. Thus, in case of a Dynkin quiver Q, we simply need to go through the list of positive roots that are perpendicular to $\mathbf{d}$ and find among them the uniquely determined basis for the semigroup $\mathbb{N}^{Q'_0}$.

More generally, if $\mathbf{d}$ is the dimension vector of a *preprojective or pre-injective* representation, as is the case for any Schur root of a Dynkin quiver, (see, e.g. [1, VIII.1] for the definitions and result), then one can read off the roots $\mathbf{e}_i$ from the Auslander–Reiten quiver of Q, as explained in [16, Proof of Proposition 2.1]. In that case, the quiver Q' is obtained from Q by deletion of a node along with its incident arrows and possibly some changes in the orientation of the remaining arrows. It is noteworthy that conversely for any quiver, any dimension vector of a preprojective or pre-injective representation is a real Schur root, thus providing a huge reservoir for potentially linear free divisors. Given that in this situation one can easily determine the simple objects of the orthogonal category from the Auslander–Reiten quiver, it seems reasonable to expect that one should be able to decide in general which of these roots give rise to linear free divisors.

We now turn to the two most complex Dynkin quivers, those of type E_7 and E_8, and demonstrate how the algorithm described here works in practice.

Example 8.1. Consider the Dynkin quiver of type E_7 with Schur root as shown - the highest root of E_7.

$$\begin{tikzcd} & & & 2 \arrow[d, "F"'] & & & \\ 1 \arrow[r, "A"] & 2 \arrow[r, "B"] & 3 \arrow[r, "C"] & 4 & 3 \arrow[l, "D"'] & 2 \arrow[l, "E"'] & V \end{tikzcd}$$

The representation space has dimension $46 = 48 - 2$ as the associated binary polyhedral group, the binary octahedral group, is a double cover of the symmetric group on four letters.

By [18, p.153] the discriminant D has 6 irreducible components. Of these, five may be found by inspection: they are the four described by the equations

$$\det[CBA|D],\ \det[CB|DE],\ \det[F|DE],\ \det[CB|F],$$

and the component corresponding to the degeneracy $\mathrm{im}\ C \cap \mathrm{im}\ D \cap \mathrm{im}\ F \neq 0$, for which an equation is less obvious. One further component remains to be found. We obtain all of them using Schofield's recipe. Consider first

$$\begin{tikzcd}
 & & & 2 \arrow[d, "F"'] & & & \\
1 \arrow[r, "A"] & 2 \arrow[r, "B"] & 3 \arrow[r, "C"] & 4 & 3 \arrow[l, "D"'] & 2 \arrow[l, "E"'] & \mathrm{V} \\
1 \arrow[r, "a"] \arrow[u, dashed, "S_1"] \arrow[ur, dashed, "T_1"] & 1 \arrow[r, "b"] \arrow[u, dashed, "S_2"] \arrow[ur, dashed, "T_2"] & 1 \arrow[r, "c"] \arrow[u, dashed, "S_3"] \arrow[ur, dashed, "T_3"] & 1 \arrow[u, dashed, "S_4"] & 1 \arrow[l, "d"] \arrow[u, dashed, "S_5"'] \arrow[ul, dashed, "T_4"'] & & \mathrm{W}
\end{tikzcd}$$

and note that the dimension vector $\mathbf{e}$ of W is a root with support a Dynkin diagram of type A_5, the "type" of $\mathbf{e}$, that satisfies $\langle \mathbf{e}, \mathbf{d} \rangle = 0$. We have

$$d_V^W(S_1, \ldots, S_5) = (AS_1 - S_2 a, BS_2 - TS_3 b, CS_3 - S_4 c, DS_5 - S_4 d).$$

Thus d_V^W has matrix

$$\begin{array}{|c|c|c|c|c|}
\hline
A & -aI_2 & 0 & 0 & 0 \\
\hline
0 & B & -bI_3 & 0 & 0 \\
\hline
0 & 0 & C & -cI_4 & 0 \\
\hline
0 & 0 & 0 & -dI_4 & D \\
\hline
\end{array}$$

where the five columns refer to the five maps $S_1, \ldots, S_5$ and the four rows to the four maps $T_1, \ldots, T_4$. Here for each p, q we have ordered the natural basis vectors $E_{ij}, 1 \leq i \leq q, 1 \leq j \leq p$, of $\mathrm{Hom}(k^p, k^q)$ lexicographically. Assuming $abc \neq 0$, row operations transform this successively to

$$\begin{array}{|c|c|c|c|c|}
\hline
A & -aI_2 & 0 & 0 & 0 \\
\hline
\frac{1}{a}BA & 0 & -bI_3 & 0 & 0 \\
\hline
0 & 0 & C & -cI_4 & 0 \\
\hline
0 & 0 & 0 & -dI_4 & D \\
\hline
\end{array}\,, \quad
\begin{array}{|c|c|c|c|c|}
\hline
A & -aI_2 & 0 & 0 & 0 \\
\hline
\frac{1}{a}BA & 0 & -bI_3 & 0 & 0 \\
\hline
\frac{1}{ab}CBA & 0 & 0 & -cI_4 & 0 \\
\hline
0 & 0 & 0 & -dI_4 & D \\
\hline
\end{array}\,,$$

$$\begin{array}{|c|c|c|c|c|}
\hline
A & -aI_2 & 0 & 0 & 0 \\
\hline
\frac{1}{a}BA & 0 & -bI_3 & 0 & 0 \\
\hline
\frac{1}{ab}CBA & 0 & 0 & -cI_4 & 0 \\
\hline
\frac{-d}{abc}CBA & 0 & 0 & 0 & D \\
\hline
\end{array}$$

so that $C(V, W) = \pm d \det[CBA|D]$, and fixing $d \neq 0$ we obtain the first of the degeneracies listed above. Note that the indicated root $\mathbf{e}$ underlying W

predicts, by 6.6, the following weight of the semi-invariant c^W:

$$w(c^W) = -\mathbf{e} + \mathbf{in_e} : \quad \begin{array}{cccccc} & & & 0 & & \\ -1 & 0 & 0 & 1 & -1 & 0 \end{array}$$

which is indeed the weight of $\det[CBA|D]$. The reader will have no difficulty checking that the next three semi-invariants listed above can be obtained, by the same procedure, from the first three roots in the diagram

$$\begin{array}{cccccc} & & & \circ\,0 & & \\ & & & \downarrow & & \\ \circ & \longrightarrow \bullet & \longrightarrow \bullet & \longrightarrow \bullet & \longleftarrow \bullet & \longleftarrow \bullet \\ 0 & 1 & 1 & 1 & 1 & 1 \end{array} \qquad \begin{array}{cccccc} & & & \bullet\,1 & & \\ & & & \downarrow & & \\ \circ & \longrightarrow \circ & \longrightarrow \circ & \longrightarrow \bullet & \longleftarrow \bullet & \longleftarrow \bullet \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}$$

$$\begin{array}{cccccc} & & & \bullet\,1 & & \\ & & & \downarrow & & \\ \circ & \longrightarrow \bullet & \longrightarrow \bullet & \longrightarrow \bullet & \longleftarrow \circ & \longleftarrow \circ \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array} \qquad \begin{array}{cccccc} & & & \bullet\,1 & & \\ & & & \downarrow & & \\ \circ & \longrightarrow \circ & \longrightarrow \bullet & \longrightarrow \bullet & \longleftarrow \bullet & \longleftarrow \circ \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}$$

The last root gives rise to the matrix

$$\begin{array}{|c|c|c|c|} \hline C & -cI_4 & 0 & 0 \\ \hline 0 & -dI_4 & D & 0 \\ \hline 0 & -fI_4 & 0 & F \\ \hline \end{array}$$

and assuming $c \neq 0$, column and row operations transform this into

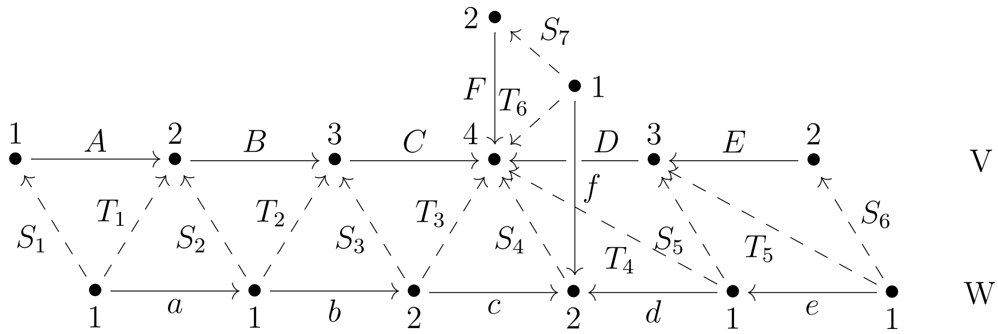
$$\begin{array}{|c|c|c|c|} \hline 0 & -cI_4 & 0 & 0 \\ \hline -\frac{d}{c}C & 0 & D & 0 \\ \hline -\frac{f}{c}C & 0 & 0 & F \\ \hline \end{array}$$

If also $df \neq 0$, then this determinant vanishes if and only if that of

$$\begin{array}{|c|c|c|} \hline -C & D & 0 \\ \hline -C & 0 & F \\ \hline \end{array}$$

vanishes, which is the case when $\operatorname{im} C \cap \operatorname{im} D \cap \operatorname{im} F \neq 0$; this can be seen by noting that if $Cu = Dv = Fw$ then the vector $(u, v, w)^t$ lies in its kernel, and vice versa.

The sixth and last component of D is given by the vanishing of the semi-invariant arising from the root represented by W in the diagram



The resulting determinant is

A	$-aI_2$	0	0	0	0	0	0	0
0	B	$-b_{11}I_3$	$-b_{21}I_3$	0	0	0	0	0
0	0	C	0	$-c_{11}I_4$	$-c_{21}I_4$	0	0	0
0	0	0	C	$-c_{12}I_4$	$-c_{22}I_4$	0	0	0
0	0	0	0	$-d_{11}I_4$	$-d_{21}I_4$	D	0	0
0	0	0	0	0	0	$-eI_3$	E	0
0	0	0	0	$-f_{11}I_4$	$-f_{21}I_4$	0	0	F

where the columns and rows refer, in this order, to the maps $S_1 \ldots, S_7$ and $T_1, \ldots, T_6$ respectively. Row and column operations, and the deletion of rows and columns containing only an invertible matrix, transform this to the matrix

$\frac{1}{b_{11}a}CBA$	0	$(c_{12}b_{21} - c_{11}b_{11})I_4$	$(c_{22}b_{21} - c_{21}b_{11})I_4$	0	0
0	C	$-c_{12}I_4$	$-c_{22}I_4$	0	0
0	0	$-d_{11}I_4$	$-d_{21}I_4$	$\frac{1}{e}DE$	0
0	0	$-f_{11}I_4$	$-f_{21}I_4$	0	F

and now permuting columns brings it to the form

CBA	0	0	0	$\lambda_1 I_4$	$\mu_1 I_4$
0	C	0	0	$\lambda_2 I_4$	$\mu_2 I_4$
0	0	DE	0	$\lambda_3 I_4$	$\mu_3 I_4$
0	0	0	F	$\lambda_4 I_4$	$\mu_4 I_4$

where the λ_i and μ_j are polynomials in the coefficients $a, b, \ldots$ of the representation W, and we have multiplied some rows and columns by other such polynomials to simplify the expression (since we choose a generic W

in $\mathrm{Rep}(Q, \mathbf{e})$ to obtain the polynomial C^W, this multiplication has the effect only of multiplying C^W by a scalar).

The geometrical significance of the vanishing of the determinant is that the three lines $\mathrm{im}\, DE \cap \mathrm{im}\, C$, $\mathrm{im}\, F \cap \mathrm{im}\, C$ and $\mathrm{im}\, CBA$ fail to span $\mathrm{im}\, C$. The reader will note the similarity in the geometric description of the last two semi-invariant factors. This can be understood by looking at their weights. They are given by

$$\begin{array}{cccccc} & & & -1 & & \\ 0 & 0 & -1 & 2 & -1 & 0 \end{array} \qquad \text{and} \qquad \begin{array}{cccccc} & & & -1 & & \\ -1 & 0 & -1 & 2 & 0 & -1 \end{array}$$

According to Derksen and Weyman [8, p.477, Step 2], if the weight of W is not sincere, as in these cases, one may simplify the calculation by removing successively nodes not in the support, adding instead one arrow for each pair of ingoing and outgoing arrows. In the first case at hand, this produces a weight with support a Dynkin quiver of type D_4, in the second a weight of type D_5. For the first four orthogonal roots listed, the type of the weight equals A_3, explaining the similarity in the description of the corresponding semi-invariants. Once one has modified the quiver in this fashion, one can then simply calculate the corresponding semi-invariant on the new quiver, where one drops from $\mathbf{d}$ as well the nodes not in the support of the weight, and substituting at the end the actual composition of the maps along each pair of ingoing and outgoing arrow into the resulting semi-invariant. Revisiting, for example, the first orthogonal root considered above and its corresponding weight of type A_3; see e.g. the table below; it becomes thus transparent that the semi-invariant obtained, $\det[CBA|D]$, has indeed to be a polynomial in the entries of CBA and D.

We can summarize the information gathered so far for the discriminant in the representation variety of the highest root of E_7 in the given orientation through the following table, where we list the opposite of the weights to display fewer minus signs:

Polynomial	Deg	Root$^{\perp\mathbf{d}}$	$-$Weight	Type (Root, Weight)
$P_1 = \det[CBA\|D]$	6	$\begin{array}{cccccc} & & & 0 & & \\ 1 & 1 & 1 & 1 & 1 & 0 \end{array}$	$\begin{array}{cccccc} & & & 0 & & \\ 1 & 0 & 0 & -1 & 1 & 0 \end{array}$	(A_5, A_3)
$P_2 = \det[CB\|DE]$	8	$\begin{array}{cccccc} & & & 0 & & \\ 0 & 1 & 1 & 1 & 1 & 1 \end{array}$	$\begin{array}{cccccc} & & & 0 & & \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}$	(A_5, A_3)
$P_3 = \det[F\|DE]$	6	$\begin{array}{cccccc} & & & 1 & & \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}$	$\begin{array}{cccccc} & & & 1 & & \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array}$	(A_4, A_3)
$P_4 = \det[CB\|F]$	6	$\begin{array}{cccccc} & & & 1 & & \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}$	$\begin{array}{cccccc} & & & 1 & & \\ 0 & 1 & 0 & -1 & 0 & 0 \end{array}$	(A_4, A_3)

$P_5 = \det \begin{bmatrix} -C & D & 0 \\ -C & 0 & F \end{bmatrix}$	8	1 $0\ 0\ 1\ 1\ 1\ 0$	1 $0\ 0\ 1\ -2\ 1\ 0$	(D_4, D_4)
P_6	12	1 $1\ 1\ 2\ 2\ 1\ 1$	1 $1\ 0\ 1\ -2\ 0\ 1$	(E_7, D_5)
$\Delta = (\text{unit})P_1 \cdots P_6$	46		4 $2\ 2\ 2\ -8\ 2\ 3$	

The following interlude will allow us to find the equations for semi-invariants such as P_5 or P_6 above in a more direct form, using some commutative algebra.

9 An Interlude from Commutative Algebra

Let $0 \to M \xrightarrow{j} R^{m+a} \xrightarrow{\varphi} R^a \xrightarrow{p} T \to 0$ be an exact sequence of modules over a commutative normal (and noetherian) domain R, with integers $m, a > 0$, and T a torsion R–module. Assume given moreover an R–linear map $\psi : R^{m+a} \to R^m$. The module M has a (constant) rank, equal to m, and its *determinant* is by definition the reflexive R–module $\det M = (\Lambda^m_R M)^{\vee\vee}$, where $(-)^\vee$ denotes the R–dual module. In words, $\det M$ is the reflexive hull of the m^{th} exterior power of M over R. It is isomorphic to R, and the composition ψj induces an R–linear map $\det(\psi j) : R \cong \det M \to \det R^m \cong R$. At issue now is to find a closed form for that determinant.

Lemma 9.1. *The determinant of ψj satisfies* $\det(\psi j) = \det(\psi, \varphi)$.

Proof. Consider the following commutative diagram whose rows are exact

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "j"] \arrow[d, "\psi j"'] & R^{m+a} \arrow[r, "\varphi"] \arrow[d, "(\psi,\varphi)"'] & R^a \arrow[r, "p"] \arrow[d, equal] & T \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R^m \arrow[r, "in_1"] & R^m \oplus R^a \arrow[r, "pr_2"] & R^a \arrow[r] & 0 &
\end{tikzcd}$$

The multiplicativity of the determinant shows first that $\det M \cong R$ and then yields $\det(\psi j) = \det(\psi, \varphi)$. □

Example 9.2. We use this result to find a closed form for the semi-invariant P_6 for the highest weight of E_7 described in the last section. Namely, with the same notations as there, that invariant measures whether the three lines $\operatorname{im} DE \cap \operatorname{im} C$, $\operatorname{im} F \cap \operatorname{im} C$ and $\operatorname{im} CBA$ span $\operatorname{im} C$. To translate this into multilinear algebra, note that it is equivalent to say that the fibre product

X of DE with C over their common target, the fibre product Y of F with C over the common target, and the image Z of BA do not span the domain of C. Each of X, Y, Z is a rank one submodule of the domain of C, which is a free module of rank 3 over R, the ring of the representation variety. We thus expect the corresponding invariant to be $\det[X|Y|Z]$, and the preceding lemma lets us make this precise: In the following diagram, the top row is a direct sum of three short exact sequences of graded R–modules

$$\begin{tikzcd}
0 \arrow[r] & \begin{matrix} X \\ \oplus \\ Y \\ \oplus \\ R(-2) \end{matrix} \arrow[rr, "\begin{smallmatrix} (i_1,j_1) \\ \oplus \\ (i_2,j_2) \\ \oplus \\ \mathrm{id}_R \end{smallmatrix}"] \arrow[d, "{(i_1,i_2,BA)}"'] & & \begin{matrix} R^3 \oplus R^2 \\ \oplus \\ R^3 \oplus R(-1)^2 \\ \oplus \\ R(-2) \end{matrix} \arrow[rr, "\begin{smallmatrix} (C,-F) \\ \oplus \\ (C,-DE) \\ \oplus \\ 0 \end{smallmatrix}"] \arrow[d, "M"] & & \begin{matrix} R(1)^4 \\ \oplus \\ R(1)^4 \\ \oplus \\ 0 \end{matrix} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & R^3 \arrow[rr, "in_1"] & & \begin{matrix} R^3 \\ \oplus \\ R(1)^4 \\ \oplus \\ R(1)^4 \end{matrix} \arrow[rr, "pr_{23}"] & & \begin{matrix} R(1)^4 \\ \oplus \\ R(1)^4 \end{matrix} \arrow[r] & 0
\end{tikzcd}$$

where the maps i_1, i_2, j_1, j_2, in_1 are the natural inclusions, pr_{23} the projection onto the sum of second and third factor, and the matrix M is of the form:

	R^3	R^2	R^3	$R(-1)^2$	$R(-2)$
R^3	I	0	I	0	BA
$R(1)^4$	C	$-F$	0	0	0
$R(1)^4$	0	0	C	$-DE$	0

The desired semi-invariant is now $\det(i_1, i_2, BA)$, which equals the determinant of M in view of the lemma above. Subtracting (a multiple of) the first column from the third and fifth results in the following simpler form

	R^3	R^2	R^3	$R(-1)^2$	$R(-2)$
R^3	I	0	0	0	0
$R(1)^4$	C	$-F$	$-C$	0	$-CBA$
$R(1)^4$	0	0	C	$-DE$	0

whence the desired semi-invariant is seen to be the determinant of an 8×8 matrix,

$$P_6 = \det \begin{bmatrix} F & C & 0 & CBA \\ 0 & C & -DE & 0 \end{bmatrix}$$

whose degree can be read off to be 12 as stated in the table above.

10 The Case of E_8 with the Centre as Only Sink

As our final example, we determine the discriminant in the representation variety that belongs to the highest root of the Dynkin quiver of type E_8 with all arrows oriented towards the central trivalent vertex:

$$\begin{tikzcd}
\overset{2}{\bullet} \arrow[r, "A"] & \overset{4}{\bullet} \arrow[r, "B"] & \overset{6}{\bullet} & \overset{5}{\bullet} \arrow[l, "D"'] & \overset{4}{\bullet} \arrow[l, "E"'] & \overset{3}{\bullet} \arrow[l, "F"'] & \overset{2}{\bullet} \arrow[l, "G"'] \\
 & & \underset{3}{\bullet} \arrow[u, "C"] & & & &
\end{tikzcd}$$

The capital letters $A, ..., G$ stand for the corresponding matrices of independent indeterminates, and the coordinate ring of $\mathrm{Rep}(E_8.\mathbf{d})$ is $R = K[A, B, C, D, E, F, G]$, a polynomial ring in $118 = 120 - 2$ variables, where 120 is the order of the binary icosahedral group.

We will also need below three additional auxiliary vertices, denoted by $\circ$, and corresponding maps X, Y, Z, as indicated by the dashed arrows here:

$$\begin{tikzcd}
\overset{2}{\bullet} \arrow[r, "A"] & \overset{4}{\bullet} \arrow[r, "B"] & \overset{6}{\bullet} & \overset{5}{\bullet} \arrow[l, "D"'] & \overset{4}{\bullet} \arrow[l, "E"'] & \overset{3}{\bullet} \arrow[l, "F"'] & \overset{2}{\bullet} \arrow[l, "G"'] \\
 & \underset{1}{\circ} \arrow[u, dotted] \arrow[r, dotted] \arrow[ur, dashed, "X"] & \underset{3}{\bullet} \arrow[u, "C"] & \underset{2}{\circ} \arrow[l, dotted] \arrow[u, dotted] \arrow[ul, dashed, "Y"'] & \underset{1}{\circ} \arrow[l, dotted] \arrow[u, dotted] \arrow[ull, dashed, "Z"'] & &
\end{tikzcd}$$

The map X is the natural one from the fibre product of B and C to the central node. The fibre product itself is an R–module of rank 1. The map Y indicated above is the natural one from the fibre product of D and C to the central node. This fibre product has rank 2. Finally, the map Z is the natural one from the fibre product of C and DE to the central node. Again, the fibre product has rank 1.

The discriminant D in question is of degree 118 and has 7 irreducible components, thus, its canonical equation Δ is a product of 7 irreducible polynomials P_i in the entries of the 7 matrices A through G. Moreover, we obtain from 6.6 that it is a semi-invariant belonging to the weight

$$\begin{matrix} -4 & -4 & 12 & -2 & -2 & -2 & -3 \\ & & -6 & & & & \end{matrix}$$

We can spot immediately three semi-invariants:

$$P_1 = \det[BA|DE]\,, P_2 = \det[C|DEF]\,, P_3 = \det[B|DEFG]\,,$$

each of degree 12 and belonging to weights of type A_3. The remaining four can be described thus

- The failure of $\operatorname{im} X = \operatorname{im} B \cap \operatorname{im} C$ and $\operatorname{im} D$ to generate the vector space at the central node. According to 9.1, the corresponding polynomial is $P_4 = \det[X|D]$, the determinant of

	$R(-1)^4$	$R(-1)^3$	$R(-1)^5$
R^6	B	0	D
R^6	B	$-C$	0

  It is of degree 12.

- The failure of $\operatorname{im} BA$, $\operatorname{im} X$, $\operatorname{im} DEF$ to generate the vector space at the central node. Again using 9.1, the corresponding polynomial is $P_5 = \det[BA|X|DEF]$, the determinant of

	$R(-2)^2$	$R(-1)^4$	$R(-1)^3$	$R(-3)^3$
R^6	BA	B	0	DEF
R^6	0	B	$-C$	0

  It is of degree 20.

- The failure of $\operatorname{im} BA$, $\operatorname{im} Y$, $\operatorname{im} DEFG$ to generate the vector space at the central node. In this case, the corresponding polynomial is $P_6 = \det[BA|Y|DEFG]$, the determinant of

	$R(-2)^2$	$R(-1)^3$	$R(-1)^5$	$R(-4)^2$
R^6	BA	C	0	$DEFG$
R^6	0	C	$-D$	0

  It is also of degree 20. The three semi-invariants P_4 through P_6 belong to weights of type D, as can easily be seen from the geometric description. Now we turn to the last and biggest one:

- The rank of BA and $DEFG$ is 2, that of X, Z is 1. Their images in the central vector space are thus expected to generate. The failure will be measured by the polynomial $P_7 = \det[BA|X|Z|DEFG]$, which is the determinant of

	$R(-2)^2$	$R(-1)^4$	$R(-1)^3$	$R(-1)^3$	$R(-2)^4$	$R(-4)^2$
R^6	BA	C	0	C	0	$DEFG$
R^6	0	C	$-D$	0	0	0
R^6	0	0	0	C	$-DE$	0

  It is of degree 30 and its weight is of type E_6.

We summarize the results again in a table:

POLYNOMIAL	DEG	ROOT$^{\perp \mathbf{d}}$	−WEIGHT	TYPE (Root, Wt)
$P_1 = \det(BA\|DE)$	12	1 1 1 1 1 0 0 0	1 0 −1 0 1 0 0 0	(A_5, A_3)
$P_2 = \det(C\|DEF)$	12	0 0 1 1 1 1 0 1	0 0 −1 0 0 1 0 1	(A_5, A_3)
$P_3 = \det(B\|DEFG)$	12	0 1 1 1 1 1 1 0	0 1 −1 0 0 0 1 0	(A_6, A_3)
$P_4 = \det(X\|D)$	12	0 1 1 1 0 0 0 1	0 1 −2 1 0 0 0 1	(D_4, D_4)
$P_5 = \det(BA\|X\|DEF)$	20	1 2 2 1 1 1 0 1	1 1 −2 0 0 1 0 1	(E_7, D_5)
$P_6 = \det(BA\|Y\|DEFG)$	20	1 1 2 2 1 1 1 1	1 0 −2 1 0 0 1 1	(E_8, D_5)
$P_7 = \det(BA\|X\|Z\|DEFG)$	30	1 2 3 2 2 1 1 2	1 1 −3 0 1 0 1 2	(E_8, E_6)
$\Delta = (\mathrm{unit})P_1 \cdots P_7$	118		4 4 −12 2 2 2 3 6	

Theorem 10.1. *The above table is correct.*

Proof. Inspection shows that each of the polynomials P_i is a semi-invariant and that its weight is as listed. The indicated weights are easily seen to be linearly independent, and add up to the weight of Δ. Thus, their product must describe the discriminant up to a unit. □

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Derived Categories of Modules and Coherent Sheaves

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Abstract

We present recent results on derived categories of modules and coherent sheaves, namely, tame–wild dichotomy and semi-continuity theorem for derived categories over finite dimensional algebras, as well as explicit calculations for derived categories of modules over nodal rings and of coherent sheaves over projective configurations of types A and $\tilde{A}$.

This paper is a survey of some recent results on the structure of derived categories obtained by the author in collaboration with Viktor Bekkert and Igor Burban [6, 11, 12]. The origin of this research was the study of Cohen–Macaulay modules and vector bundles by Gert-Martin Greuel and myself [27, 28, 29, 30] and some ideas from the work of Huisgen-Zimmermann and Saorín [42]. Namely, I understood that the technique of "matrix problems," briefly explained below in subsection 2.3, could be successfully applied to the calculations in derived categories, almost in the same way as it was used in the representation theory of finite-dimensional algebras, in study of Cohen–Macaulay modules, etc. The first step in this direction was the semi-continuity theorem for derived categories [26] presented in subsection 2.1. Then Bekkert and I proved the tame–wild dichotomy for derived categories over finite dimensional algebras (see subsection 2.2). At the same time, Burban and I described the indecomposable objects in the derived categories over nodal rings (see Section 3) and projective configurations of types A and $\tilde{A}$ (see Section 4). Note that it follows from [23, 29] that these are the only cases, where such a classification is possible; for all other pure noetherian rings (or projective curves) even the categories of modules (respectively, of vector bundles) are wild. In both cases the description reduces to a special class of matrix problems ("bunches of chains" or "clans"), which also arises in a wide range of questions from various areas of mathematics.

1991 *Mathematics Subject Classification.* 18E30, 16G60, 16G50, 15A21, 16G30, 14H60

Key words. Derived category, tame–wild dichotomy, semi-continuity, nodal rings, projective configurations, vector bundles, Cohen-Macaulay modules

I tried to explain the backgrounds, but, certainly, only sketched proofs, referring for the details to the original papers cited above.

1 Generalities

We first recall some definitions. Let $\mathbb{S}$ be a commutative ring. An $\mathbb{S}$-*category* is a category $\mathscr{A}$ such that all morphism sets $\mathscr{A}(A,B)$ are $\mathbb{S}$-modules and the multiplication of morphisms is $\mathbb{S}$-bilinear. We call $\mathscr{A}$

- *local* if every object $A \in \mathscr{A}$ decomposes into a finite direct sum of objects with local endomorphism rings;
- ω-*local* if every object $A \in \mathscr{A}$ decomposes into a finite or countable direct sum of objects with local endomorphism rings;
- *fully additive* if any idempotent morphism in $\mathscr{A}$ splits, that is defines a decomposition into a direct sum;
- *locally finite* (over $\mathbb{S}$) if all morphism spaces $\mathscr{A}(A,B)$ are finitely generated $\mathbb{S}$-modules. If $\mathbb{S}$ is a field, a locally finite category is often called *locally finite dimensional.* If, moreover, $\mathscr{A}$ has finitely many objects, we call it *finite* (over $\mathbb{S}$). Especially, if $\mathscr{A}$ is an $\mathbb{S}$-algebra (i.e. a $\mathbb{S}$-category with one object), we call it a *finite* $\mathbb{S}$-*algebra.*
- If $\mathscr{A}$ is fully additive and locally finite over $\mathbb{S}$, we shall call it a *falf* ($\mathbb{S}$-) category.

Mostly the ring $\mathbb{S}$ will be local and complete noetherian ring. Then, evidently, every falf $\mathbb{S}$-category is local; moreover, an endomorphism algebra $\mathscr{A}(A,A)$ in a falf category is a finite $\mathbb{S}$-algebra. It is known that any local (or ω-local) category is fully additive; moreover, a decomposition into a direct sum of objects with local endomorphism rings is always unique; in other words, any local (or ω-local) category is a *Krull–Schmidt* one, cf. [4, Theorem 3.6].

For a local category $\mathscr{A}$ we denote by $\operatorname{rad}\mathscr{A}$ its *radical*, that is the set of all morphisms $f: A \to B$, where $A, B \in \operatorname{Ob}\mathscr{A}$, such that no component of the matrix presentation of f with respect to some (hence any) decomposition of A and B into a direct sum of indecomposable objects is invertible. Note that if $f \notin \operatorname{rad}\mathscr{A}$, there is a morphism $g: B \to A$ such that $fgf = f$ and $gfg = g$. Hence both gf and fg are nonzero idempotents, which define decompositions $A \simeq A_1 \oplus A_2$ and $B \simeq B_1 \oplus B_2$ such that the matrix presentation of f with respect to these decompositions is diagonal: $\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$, and f_1 is invertible. Obviously, if $\mathscr{A}$ is locally finite dimensional, then $\operatorname{rad}\mathscr{A}(A,B)$ coincide with the set of all morphisms $f: A \to B$ such that gf (or fg) is nilpotent for any morphism $g: B \to A$.

We denote by $\mathscr{C}(\mathscr{A})$ the category of complexes over $\mathscr{A}$, i.e. that of diagrams

$$(A_\bullet, d_\bullet): \qquad \dots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \dots,$$

where $A_n \in \operatorname{Ob}\mathscr{A}$, $d_n \in \mathscr{A}(A_n, A_{n-1})$, with relations $d_n d_{n+1} = 0$ for all n. Sometimes we omit $d_\bullet$ denoting this complex by $A_\bullet$. Morphisms between two such complexes, $(A_\bullet, d_\bullet)$ and $(A'_\bullet, d'_\bullet)$ are, by definition, commutative diagrams of the form

$$\phi_\bullet: \quad \begin{array}{ccccccccc} \dots & \longrightarrow & A_{n+1} & \xrightarrow{d_{n+1}} & A_n & \xrightarrow{d_n} & A_{n-1} & \xrightarrow{d_{n-1}} & \dots \\ \dots & & \Big\downarrow{\scriptstyle \phi_{n+1}} & & \Big\downarrow{\scriptstyle \phi_n} & & \Big\downarrow{\scriptstyle \phi_{n-1}} & & \dots \\ \dots & \longrightarrow & A'_{n+1} & \xrightarrow{d'_{n+1}} & A'_n & \xrightarrow{d'_n} & A'_{n-1} & \xrightarrow{d'_{n-1}} & \dots \end{array}$$

Note that we use "homological" notations (with down indices) instead of more usual "cohomological" ones (with upper indices). Two morphisms, $\phi_\bullet$ and $\psi_\bullet$, between $(A_\bullet, d_\bullet)$ and $(A'_\bullet, d'_\bullet)$ are called *homotopic* if there are morphisms $\sigma_n : A_n \to A'_{n+1}$ $(n \in \mathbb{N})$ such that $\phi_n - \psi_n = d'_{n+1}\sigma_n + \sigma_{n-1}d_n$ for all n. We denote it by $\phi \sim \psi$. We also often omit evident indices and write, for instance, $\phi - \psi = d'\sigma + \sigma d$. The *homotopy category* $\mathscr{H}(\mathscr{A})$ is, by definition, the factor category $\mathscr{C}(\mathscr{A})/\mathscr{C}_{\sim 0}$, where $\mathscr{C}_{\sim 0}$ is the ideal of morphisms homotopic to zero.

Suppose now that $\mathscr{A}$ is an abelian category. Then, for every complex $(A_\bullet, d_\bullet)$, its *homologies* $\mathrm{H}_\bullet = \mathrm{H}_\bullet(A_\bullet, d_\bullet)$ are defined, namely $\mathrm{H}_n(A_\bullet, d_\bullet) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$. Every morphism $\phi_\bullet$ as above induces morphisms of homologies $\mathrm{H}_n(\phi_\bullet) : \mathrm{H}_n(A_\bullet, d_\bullet) \to \mathrm{H}_n(A'_\bullet, d'_\bullet)$. It is convenient to consider $\mathrm{H}_\bullet(\mathscr{A}_\bullet, d_\bullet)$ as a complex with zero differential and we shall usually do so. Then $\mathrm{H}_\bullet$ becomes an endofunctor inside $\mathscr{C}(\mathscr{A})$. If $\phi_\bullet \sim \psi_\bullet$, then $\mathrm{H}_\bullet(\phi_\bullet) = \mathrm{H}_\bullet(\psi_\bullet)$, so $\mathrm{H}_\bullet$ can be considered as a functor $\mathscr{H}(\mathscr{A}) \to \mathscr{C}(\mathscr{A})$. We call $\phi_\bullet$ a *quasi-isomorphism* if $\mathrm{H}_\bullet(\phi_\bullet)$ is an isomorphism. Then we write $\phi_\bullet : (A_\bullet, d_\bullet) \approx (A'_\bullet, d'_\bullet)$ or sometimes $(A_\bullet, d_\bullet) \approx (A'_\bullet, d'_\bullet)$ if $\phi_\bullet$ is not essential. The *derived category* $\mathscr{D}(\mathscr{A})$ is defined as the category of fractions (in the sense of [34]) $\mathscr{H}(\mathscr{A})[\mathcal{Q}^{-1}]$, where $\mathcal{Q}$ is the set of quasi-isomorphisms. In particular, the functor of homologies $\mathrm{H}_\bullet$ becomes a functor $\mathscr{D}(\mathscr{A}) \to \mathscr{C}(\mathscr{A})$. Note that a morphism between two complexes with zero differential is homotopic to zero if and only if it is zero, and is a quasi-isomorphism if and only if it is an isomorphism. Moreover, any morphism between such complexes in the derived category is equal (in this category) to the image of a real morphism between these complexes in $\mathscr{C}(\mathscr{A})$. Thus we can consider the category $\mathscr{C}^0(\mathscr{A})$ of complexes with zero differential as a full subcategory of $\mathscr{H}(\mathscr{A})$ or of $\mathscr{D}(\mathscr{A})$. In particular, we can (and shall) identify every object $A \in \mathscr{A}$ with the complex $A_\bullet$ such that $A_0 = A$, $A_n = 0$ for $n \neq 0$. It gives a full embedding of $\mathscr{A}$ into $\mathscr{H}(\mathscr{A})$ or $\mathscr{D}(\mathscr{A})$.

We denote by $\mathscr{C}^-(\mathscr{A})$ (respectively, $\mathscr{C}^+(\mathscr{A})$, $\mathscr{C}^b(\mathscr{A})$) the categories of *right bonded* (respectively, *left bounded*, (*two-side*) *bounded*) complexes, i.e. such that $A_n = 0$ for $n \ll 0$ (respectively, $n \gg 0$ or both). Correspondingly, we consider the right (left, two-side) bounded homotopy categories $\mathscr{H}^-(\mathscr{A}), \mathscr{H}^+(\mathscr{A}), \mathscr{H}^b(\mathscr{A})$ and right (left, two-side) bounded derived categories $\mathscr{D}^-(\mathscr{A}), \mathscr{D}^+(\mathscr{A}), \mathscr{D}^b(\mathscr{A})$.

The categories $\mathscr{C}(\mathscr{A}), \mathscr{H}(\mathscr{A}), \mathscr{D}(\mathscr{A})$, as well as their bounded subcategories, are *triangulated categories* [38]. Namely, the *shift* maps a complex $A_\bullet$ to the complex $A_\bullet[1]$, where $A_n[1] = A_{n-1}$.[1] A *triangle* is a sequence isomorphic (as a diagram in the corresponding category) to a sequence of the form

$$A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} \mathsf{C}f_\bullet \xrightarrow{h_\bullet} A_\bullet[1],$$

where $f_\bullet$ is a morphism of complexes, $\mathsf{C}f_\bullet$ is the *cone* of this morphism, i.e. $\mathsf{C}f_n = A_{n-1} \oplus B_n$, the differential $\mathsf{C}f_n \to \mathsf{C}f_{n-1} = A_{n-2} \oplus B_{n-1}$ is given by the matrix $\begin{pmatrix} -d_{n-1} & 0 \\ f_{n-1} & d_n \end{pmatrix}$; $g(b) = (0, b)$ and $h(a, b) = a$.

If $\mathscr{A} = \mathscr{R}$-Mod, the category of modules over a pre-additive category $\mathscr{R}$ (for instance, over a ring), the definition of the right (left) bounded derived category can be modified. Namely, $\mathscr{D}^-(\mathscr{R}\text{-Mod})$ is equivalent to the homotopy category $\mathscr{H}^-(\mathscr{R}\text{-Proj})$, where $\mathscr{R}$-Proj is the category of projective $\mathscr{R}$-modules. Recall that a *module* over a pre-additive category $\mathscr{R}$ is a functor $M : \mathscr{R} \to \mathsf{Ab}$, the category of abelian groups. Such a module is projective (as an object of the category $\mathscr{R}$-Mod) if and only if it is isomorphic to a direct summand of a direct sum of *representable* modules $\mathscr{A}^A = \mathscr{A}(A, \_)$ ($A \in \operatorname{Ob} \mathscr{A}$). Just in the same way, the left bounded category $\mathscr{D}^+(\mathscr{R}\text{-Mod})$ is equivalent to the homotopy category $\mathscr{H}^+(\mathscr{R}\text{-Inj})$, where $\mathscr{R}$-Inj is the category of injective $\mathscr{R}$-modules. If the category $\mathscr{R}$ is *noetherian*, i.e. every submodule of every representable module is finitely generated, the right bounded derived category $\mathscr{D}^-(\mathscr{R}\text{-mod})$, where $\mathscr{R}$-mod denotes the category of finitely generated $\mathscr{R}$-modules, is equivalent to $\mathscr{H}^-(\mathscr{R}\text{-proj})$, where $\mathscr{R}$-proj is the category of finitely generated projective $\mathscr{R}$-modules.

In general, it is not true that $\mathscr{D}^b(\mathscr{R}\text{-Mod})$ is equivalent to $\mathscr{H}^b(\mathscr{R}\text{-Proj})$ (or to $\mathscr{H}^b(\mathscr{R}\text{-Inj})$). For instance, a projective resolution of a module M, which is isomorphic to M in $\mathscr{D}(\mathscr{R}\text{-Mod})$, can be left unbounded. Nevertheless, there is a good approximation of the two-side derived category by finite complexes of projective modules. Namely, consider the full subcategory $\mathscr{C}^{(N)} = \mathscr{C}^{(N)}(\mathscr{R}) \subseteq \mathscr{C}^b(\mathscr{R}\text{-proj})$ consisting of all bounded complexes $P_\bullet$ such that $P_n = 0$ for $n > N$ (note that we do not fix the right bound). We say that two morphisms, $\phi_\bullet, \psi_\bullet : P_\bullet \to P'_\bullet$, from $\mathscr{C}^{(N)}$ are *almost homotopic* and write $\phi \overset{N}{\sim} \psi$ if there are morphisms $\sigma_n : P_n \to P'_{n+1}$ such that

[1] Note again the *homological* (down) indices here.

$\phi_n - \psi_n = d'_{n+1}\sigma_n + \sigma_{n-1}d_n$ for all $n < N$ (not necessarily for $n = N$). We denote by $\mathscr{H}^{(N)} = \mathscr{H}^{(N)}(\mathscr{A})$ the factor category $\mathscr{C}^{(N)}/\mathscr{C}_{\underset{\sim}{N}0}$, where $\mathscr{C}_{\underset{\sim}{N}0}$ is the ideal consisting of all morphisms almost homotopic to zero. There are natural functors $\mathcal{I}_N : \mathscr{H}^{(N)} \to \mathscr{H}^{(N+1)}$. Namely, for a complex $P_\bullet \in \mathscr{H}^{(N)}$ find a homomorphism $d_{N+1} : P_{N+1} \to P_N$, where P_{N+1} is projective and $\operatorname{Im} d_{N+1} = \operatorname{Ker} d_N$. Then the complex

$$\mathcal{I}_N P_\bullet : \; P_{N+1} \xrightarrow{d_{N+1}} P_N \xrightarrow{d_N} P_{N-1} \to \dots$$

is uniquely defined up to isomorphism in $\mathscr{H}^{(N+1)}$. Moreover, any morphism $\phi_\bullet : P_\bullet \to P'_\bullet$ from $\mathscr{H}^{(N)}$ induces a morphism $\mathcal{I}_N\phi_\bullet : \mathcal{I}_N P_\bullet \to \mathcal{I}_N P'_\bullet$ which coincides with $\phi_\bullet$ for all places $n \le N$. This morphism is also uniquely defined as a morphism from $\mathscr{H}^{(N+1)}$. It gives the functor $\mathcal{I}_N$. One can easily verify that actually all these functors are full embeddings and $\mathscr{D}^b(\mathscr{R}\text{-Mod}) \simeq \varinjlim_N \mathscr{H}^{(N)}(\mathscr{R})$. If $\mathscr{R}$ is noetherian, the same is true for the category $\mathscr{D}^b(\mathscr{R}\text{-mod})$ if we replace everywhere $\mathscr{R}$-Proj by $\mathscr{R}$-proj.

One can also consider the projection $\mathcal{E}_N : \mathscr{H}^{(N+1)} \to \mathscr{H}^{(N)}$ which is defined by erasing the term P_{N+1} in a complex $P_\bullet \in \mathscr{H}^{(N+1)}$ and show that $\mathscr{D}^-(\mathscr{R}\text{-Mod}) \simeq \varprojlim_N \mathscr{H}^{(N)}(\mathscr{R})$.

Suppose now that $\mathscr{A}$ is a falf category over a *complete local noetherian* ring $\mathbb{S}$. Then, evidently, the bounded categories $\mathscr{C}^b(\mathscr{A})$ and $\mathscr{H}^b(\mathscr{A})$ are also falf categories, hence Krull-Schmidt categories. In [11, Appendix A] it is proved that the same is true for *unbounded* categories $\mathscr{C}(\mathscr{A})$ and $\mathscr{H}(\mathscr{A})$. The proof is based on the following analogue of the Hensel lemma (cf. [11, Corollary A.5]).

Lemma 1.1. *Let Λ be a finite algebra over a local noetherian ring $\mathbb{S}$ with maximal ideal $\mathfrak{m}$ and $a \in \Lambda$. For every $n \in \mathbb{N}$ there is a polynomial $g(x) \in \mathbb{S}[x]$ such that*

- $g(a)^2 \equiv g(a) \mod \mathfrak{m}^{n+1}$*;*
- $g(e) \equiv e \mod \mathfrak{m}^n$ *for every element e of an arbitrary finite $\mathbb{S}$-algebra such that $e^2 \equiv e \mod \mathfrak{m}^n$;*
- $g(a) \equiv 1 \mod \mathfrak{m}$ *if and only if a is invertible;*
- $g(a) \equiv 0 \mod \mathfrak{m}$ *if and only if a is nilpotent modulo $\mathfrak{m}$.*

Theorem 1.2. *Suppose that $\mathbb{S}$ is a complete local noetherian ring with maximal ideal $\mathfrak{m}$. If $\mathscr{A}$ is a falf category over $\mathbb{S}$, the categories $\mathscr{C}(\mathscr{A})$ and $\mathscr{H}(\mathscr{A})$ are ω-local (in particular, Krull–Schmidt). Moreover, a morphism $f_\bullet : A_\bullet \to B_\bullet$ from one of these categories belongs to the radical if and only if all components $f_n g_n$ (or $g_n f_n$) are nilpotent modulo $\mathfrak{m}$ for any morphism $g_\bullet : B_\bullet \to A_\bullet$.*

Proof. Let $a_\bullet$ be an endomorphism of a complex $A_\bullet$ from $\mathscr{C}(\mathscr{A})$. Consider the sets $I_n \subset \mathbb{Z}$ defined as follows: $I_0 = \{0\}$, $I_{2k} = \{ l \in \mathbb{Z} \mid -k \le l \le k \}$ and $I_{2k-1} = \{ l \in \mathbb{Z} \mid -k < l \le k \}$. Obviously, $\bigcup_n I_n = \mathbb{Z}$, $I_n \subset I_{n+1}$ and $I_{n+1} \setminus I_n$ consists of a unique element l_n. Using corollary 1.1, we can construct a sequence of endomorphisms $a_\bullet^{(n)}$ such that, for each $i \in I_n$,

- $(a_i^{(n)})^2 \equiv a_i^{(n)} \mod \mathfrak{m}^n$;
- $a_i^{(n+1)} \equiv a_i^{(n)} \mod \mathfrak{m}^n$;
- $a_i^{(n)}$ is invertible or nilpotent modulo $\mathfrak{m}$ if and only if so is a_i.

Then one easily sees that setting $u_i = \lim_{n\to\infty} a_i^{(n)}$, we get an idempotent endomorphism $u_\bullet$ of $A_\bullet$, such that $u_i \equiv 0 \mod \mathfrak{m}$ ($u_i \equiv 1 \mod \mathfrak{m}$) if and only if a_i is nilpotent modulo $\mathfrak{m}$ (respectively a_i is invertible).

Especially, if either one of a_l is neither nilpotent nor invertible modulo $\mathfrak{m}$, or one of a_l is nilpotent modulo $\mathfrak{m}$ while another one is invertible, then $u_\bullet$ is neither zero nor identity. Hence the complex $A_\bullet$ decomposes. Thus $A_\bullet$ is indecomposable if and only if, for any endomorphism $a_\bullet$ of $A_\bullet$, either $a_\bullet$ is invertible or all components a_n are nilpotent modulo $\mathfrak{m}$. Since all algebras $\operatorname{End} A_n / \mathfrak{m} \operatorname{End} A_n$ are finite dimensional, neither product $\alpha\beta$, where $\alpha, \beta \in \operatorname{End} A_n$ and one of them is nilpotent modulo $\mathfrak{m}$, can be invertible. Therefore, the set of endomorphisms $a_\bullet$ of an indecomposable complex $A_\bullet$ such that all components a_n are nilpotent modulo $\mathfrak{m}$ form an ideal R of $\operatorname{End} A_\bullet$ and $\operatorname{End} A_\bullet / R$ is a skew field. Hence $R = \operatorname{rad}(\operatorname{End} A_\bullet)$ and $\operatorname{End} A_\bullet$ is local.

Now we want to show that any complex from $\mathscr{C}(\mathscr{A})$ has an indecomposable direct summand. Consider an arbitrary complex $A_\bullet$ and suppose that $A_0 \neq 0$. For any idempotent endomorphism $e_\bullet$ of $A_\bullet$ at least one of the complexes $e(A_\bullet)$ or $(1-e)(A_\bullet)$ has a non-zero component at the zero place. On the set of all endomorphisms of $A_\bullet$ we can introduce a partial ordering by writing $e_\bullet \ge e'_\bullet$ if and only if $e'_\bullet = e_\bullet e'_\bullet e_\bullet$ and both e_0 and e'_0 are non-zero. Let $e_\bullet \ge e'_\bullet \ge e''_\bullet \ge \dots$ be a chain of idempotent endomorphisms of $A_\bullet$. As all endomorphism algebras $\operatorname{End} A_l$ are finitely generated $\mathbf{S}$-modules, the sequences $e_l, e'_l, e''_l, \dots \in \operatorname{End} A_l$ stabilize for all l, so this chain has a lower bound (formed by the limit values of components). By Zorn's lemma, there is a minimal non-zero idempotent of $A_\bullet$, which defines an indecomposable direct summand.

Again, since all $\operatorname{End} A_l$ are finitely generated, for every n there is a decomposition $A_\bullet = B_\bullet^{(n)} \oplus \bigoplus_{i=1}^{r_n} B_{in\bullet}$ where all $B_{in\bullet}$ are indecomposable and $B_l^{(n)} = 0$ for $l \in I_n$. Moreover, one may suppose that $r_n \le r_m$ for $m > n$ and $B_{in\bullet} = B_{im\bullet}$ for $i \le r_n$. Evidently, it implies that $A_\bullet = \bigoplus_{i=1}^{r} B_{i\bullet}$ where $r = \sup_n r_n$ and $B_{i\bullet} = B_{in\bullet}$ for $i \le r_n$, which accomplishes the proof of the Theorem 1.2 for $\mathscr{C}(\mathscr{A})$.

Note now that the endomorphism ring of each complex $B_{i\bullet}$ in the category $\mathscr{H}(\mathscr{A})$ is a factor ring of its endomorphism ring in $\mathscr{C}(\mathscr{A})$. Hence it is either local or zero; in the latter case the image of $B_{i\bullet}$ in $\mathscr{H}(\mathscr{A})$ is a zero object. Therefore, the claim is also valid for $\mathscr{H}(\mathscr{A})$. □

Since the derived category $\mathscr{D}^-(\mathscr{R}\text{-mod})$ is equivalent to $\mathscr{H}^-(\mathscr{R}\text{-proj})$, we get the following corollary.

Corollary 1.3. *Let $\mathbb{S}$ be $\mathscr{R}$ be a locally finite $\mathbb{S}$-category (e.g. a finite $\mathbb{S}$-algebra). Then the derived category $\mathscr{D}^-(\mathscr{R}\text{-mod})$ is ω-local, in particular, Krull–Schmidt.*

2 Finite Dimensional Algebras

2.1 Semi-Continuity

In this section we suppose that $\mathbb{S} = \Bbbk$ is an algebraically closed field and $\mathbf{A}$ is a finite dimensional $\Bbbk$-algebra with radical $\mathbf{J}$. In this case one can define, following the pattern of [28], the *number of parameters* for objects of the bounded derived category $\mathscr{D}^b(\mathbf{A}\text{-mod})$. First of all, every object M in the category $\mathbf{A}$-mod has a *projective cover*, i.e. an epimorphism $f : P \to M$, where P is a projective module, such that $\operatorname{Ker} f \subseteq \mathbf{J}P$. Moreover, this projective cover is unique up to an isomorphism. It implies that every right bounded complex of $\mathbf{A}$-modules is isomorphic in the homotopy category $\mathscr{H}^-(\mathbf{A}\text{-mod})$ to a *minimal complex*, i.e. such a complex of projective modules

$$P_\bullet : \quad \cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \dots ,$$

that $\operatorname{Im} d_n \subseteq \mathbf{J}P_{n-1}$ for all n.

Consider now the full subcategory $\mathscr{H}_0^{(N)} = \mathscr{H}_0^{(N)}(\mathbf{A})$ of $\mathscr{H}^{(N)}(\mathbf{A})$ consisting of minimal complexes. Then again $\mathscr{D}^b(\mathbf{A}\text{-mod}) \simeq \varinjlim \mathscr{H}_0^{(N)}$. Moreover, two complexes from $\mathscr{H}_0^{(N)}$ are isomorphic in $\mathscr{D}^b(\mathbf{A}\text{-mod})$ if and only if they are isomorphic as complexes. Using this approximation, we can prescribe a *vector rank* to every object $M_\bullet \in \mathscr{D}^b(\mathbf{A}\text{-mod})$. Namely, let $\{ A_1, A_2, \dots, A_s \}$ be a set of representatives of isomorphism classes of indecomposable projective $\mathbf{A}$-modules. Every finitely generated projective $\mathbf{A}$-module P uniquely decomposes as $P \simeq \bigoplus_{i=1}^s r_i A_i$. We call the vector $\mathbf{r}(P) = (r_1, r_2, \dots, r_s)$, the *rank* of the projective module P and for every vector $\mathbf{r} = (r_1, r_2, \dots, r_s)$ set $\mathbf{rA} = \bigoplus_{i=1}^s r_i A_i$. Given a finite complex $P_\bullet$ of projective modules, we define its *vector rank* as the function $\mathbf{rk}(P_\bullet) : \mathbb{Z} \to \mathbb{N}^s$ mapping $n \in \mathbb{Z}$ to $\mathbf{r}(P_i)$. It is a function with finite support. Let Δ be the set of all functions $\mathbb{Z} \to \mathbb{N}^s$ with finite support. For every function $\mathbf{r}_\bullet \in \Delta$, let $\mathcal{C}(\mathbf{r}_\bullet) = \mathcal{C}(\mathbf{r}_\bullet, \mathbf{A})$ be the set of all minimal complexes $P_\bullet$ such that $P_n = \mathbf{r}_n\mathbf{A}$ (we write $\mathbf{r}_n$ for $\mathbf{r}_\bullet(n)$). This set can be considered as an affine algebraic variety over $\Bbbk$, namely, $\mathcal{C}(\mathbf{r}_\bullet)$ is isomorphic to the subvariety of the affine space $\mathcal{H} = \prod_n \mathrm{Hom}_{\mathbf{A}}(P_n, \mathbf{J}P_{n-1})$ consisting of all sequences (f_n) such that $f_n f_{n+1} = 0$ for all n. Set also $\mathbf{G}(\mathbf{r}_\bullet) = \prod_n \mathrm{Aut}\, P_n$. It is an affine algebraic group acting on $\mathcal{C}(\mathbf{r}_\bullet)$ and its orbits are just isomorphism classes of minimal complexes of vector rank $\mathbf{r}_\bullet$. It is convenient to replace affine varieties by projective ones, using the obvious fact that the sequences (f_n) and (λf_n), where $\lambda \in \Bbbk$ is a nonzero scalar, belong to the same orbit. So we write $\mathbb{H}(\mathbf{r}_\bullet)$ for the projective space $\mathbb{P}(\mathcal{H})$ and $\mathbb{D}(\mathbf{r}_\bullet)$ for the image in $\mathbb{H}(\mathbf{r}_\bullet)$ of $\mathcal{C}(\mathbf{r}_\bullet)$. Actually, we exclude the complexes with zero differential, but such a complex is uniquely defined by its vector rank, so they play a negligible role in classification problems.

We consider now *algebraic families* of $\mathbf{A}$-complexes, i.e. flat families over an algebraic variety X. Such a family is a complex $\mathcal{F}_\bullet = (\mathcal{F}_n, d_n)$ of flat coherent $\mathbf{A} \otimes \mathcal{O}_X$-modules. We always assume this complex bounded and *minimal*; the latter means that $\mathrm{Im}\, d_n \subseteq \mathbf{J}\mathcal{F}_{n-1}$ for all n. We also assume that X is connected; it implies that the vector rank $\mathbf{rk}(\mathcal{F}_\bullet(x))$ is constant, so we can call it the *vector rank of the family* $\mathcal{F}$ and denote it by $\mathbf{rk}(\mathcal{F}_\bullet)$ Here, as usually, $\mathcal{F}(x) = \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$, where $\mathfrak{m}_x$ is the maximal ideal of the ring $\mathcal{O}_{X,x}$. We call a family $\mathcal{F}_\bullet$ *non-degenerate* if, for every $x \in X$, at least one of $d_n(x) : \mathcal{F}_n(x) \to \mathcal{F}_{n-1}(x)$ is non-zero. Having a family $\mathcal{F}_\bullet$ over X and a regular map $\phi : Y \to X$, one gets the inverse image $\phi^*(\mathcal{F})$, which is a family of $\mathbf{A}$-complexes over the variety Y such that $\phi^*(\mathcal{F})(y) \simeq \mathcal{F}(\phi(y))$. If $\mathcal{F}_\bullet$ is non-degenerate, so is $\phi^*(\mathcal{F})$. Given an ideal $\mathbf{I} \subseteq \mathbf{J}$, we call a family $\mathcal{F}_\bullet$ an $\mathbf{I}$-*family* if $\mathrm{Im}\, d_n \subseteq \mathbf{I}\mathcal{F}_{n-1}$ for all n. Then any inverse image $\phi^*(\mathcal{F})$ is an $\mathbf{I}$-family as well. Just as in [29], we construct some "almost versal" non-degenerate $\mathbf{I}$-families.

For each vector $\mathbf{r} = (r_1, r_2, \dots, r_s)$ denote $\mathbf{I}(\mathbf{r}, \mathbf{r}') = \mathrm{Hom}_{\mathbf{A}}(\mathbf{r}\mathbf{A}, \mathbf{I} \cdot \mathbf{r}'\mathbf{A})$, where $\mathbf{I}$ is an ideal contained in $\mathbf{J}$. Fix a vector rank of bounded complexes $\mathbf{r}_\bullet = (\mathbf{r}_k) \in \Delta$ and set $\mathcal{H}(\mathbf{r}_\bullet, \mathbf{I}) = \bigoplus_k \mathbf{I}(\mathbf{r}_k, \mathbf{r}_{k-1})$. Consider the projective space $\mathbb{P}(\mathbf{r}_\bullet, \mathbf{I}) = \mathbb{P}(\mathcal{H}(\mathbf{r}_\bullet, \mathbf{I}))$ and its closed subset $\mathbb{D}(\mathbf{r}_\bullet, \mathbf{I}) \subseteq \mathbb{P}$ consisting of all sequences (h_k) such that $h_{k+1}h_k = 0$ for all k. Because of the universal property of projective spaces [40, Theorem II.7.1], the embedding $\mathbb{D}(\mathbf{r}_\bullet, \mathbf{I}) \to \mathbb{P}(\mathbf{r}_\bullet, \mathbf{I})$ gives rise to a non-degenerate $\mathbf{I}$-family $\mathcal{V}_\bullet = \mathcal{V}_\bullet(\mathbf{r}_\bullet, \mathbf{I})$:

$$\mathcal{V}_\bullet : \quad \mathcal{V}_n \xrightarrow{d_n} \mathcal{V}_{n-1} \xrightarrow{d_{n-1}} \dots \longrightarrow \mathcal{V}_m, \tag{1}$$

where $\mathcal{V}_k = \mathcal{O}_{\mathbb{D}(\mathbf{r}_\bullet, \mathbf{I})}(n-k) \otimes \mathbf{r}_k\mathbf{A}$ for all $m \le k \le n$. We call $\mathcal{V}_\bullet(\mathbf{r}_\bullet, \mathbf{I})$ the *canonical* $\mathbf{I}$-*family of* $\mathbf{A}$-*complexes* over $\mathbb{D}(\mathbf{r}_\bullet, \mathbf{I})$. Moreover, regular maps $\phi : X \to \mathbb{D}(\mathbf{r}_\bullet, \mathbf{I})$ correspond to non-degenerate $\mathbf{I}$-families $\mathcal{F}_\bullet$ with $\mathcal{F}_k = 0$ for $k > n$ or $k < m$ and $\mathcal{F}_k = \mathcal{L}^{\otimes(n-k)} \otimes \mathbf{r}_k\mathbf{A}$ for some invertible sheaf $\mathcal{L}$ over X. Namely, such a family can be obtained as $\phi^*(\mathcal{V}_\bullet)$ for a uniquely defined regular map ϕ. Moreover, the following result holds, which shows the "almost versality" of the families $\mathcal{V}_\bullet(\mathbf{r}_\bullet, \mathbf{I})$.

Proposition 2.1. *For every non-degenerate family of* $\mathbf{I}$*-complexes* $\mathcal{F}_\bullet$ *of vector rank* $\mathbf{r}_\bullet$ *over an algebraic variety* X*, there is a finite open covering* $X = \bigcup_j U_j$ *such that the restriction of* $\mathcal{F}_\bullet$ *onto each* U_j *is isomorphic to* $\phi_j^* \mathcal{V}_\bullet(\mathbf{r}_\bullet, \mathbf{I})$ *for a regular map* $\phi_j : U_j \to \mathbb{D}(\mathbf{r}_\bullet, \mathbf{I})$*.*

Proof. For each $x \in X$ there is an open neighbourhood $U \ni x$ such that all restrictions $\mathcal{F}_k|_U$ are isomorphic to $\mathcal{O}_U \otimes \mathbf{r}_k\mathbf{A}$; so the restriction $\mathcal{F}_\bullet|_U$ is obtained from a regular map $U \to \mathbb{D}(\mathbf{r}_\bullet, \mathbf{I})$. Evidently it implies the assertion. $\square$

Note that the maps ϕ_j are not canonical, so we cannot glue them into a "global" map $X \to \mathbb{D}(\mathbf{r}_\bullet, \mathbf{I})$.

The group $\mathbf{G} = \mathbf{G}(\mathbf{r}_\bullet) = \prod_k \operatorname{Aut}(\mathbf{r}_k\mathbf{A})$ acts on $\mathcal{H}(\mathbf{r}_\bullet, \mathbf{I})$: $(g_k) \cdot (h_k) = (g_{k-1} h_k g_k^{-1})$. It induces the action of $\mathbf{G}(\mathbf{r}_\bullet)$ on $\mathbb{P}(\mathbf{R}_\bullet, \mathbf{I})$ and on $\mathbb{D}(\mathbf{r}_\bullet, \mathbf{I})$. The definitions immediately imply that $\mathcal{V}_\bullet(\mathbf{r}_\bullet, \mathbf{I})(x) \simeq \mathcal{V}_\bullet(\mathbf{r}_\bullet, \mathbf{I})(x')$ $(x, x' \in \mathbb{D})$ if and only if x and x' belong to the same orbit of $\mathbf{G}$. Consider the sets

$$\mathbb{D}_i = \mathbb{D}_i(\mathbf{r}_\bullet, \mathbf{I}) = \{\, x \in \mathbb{D} \mid \dim \mathbf{G}x \le i \,\}.$$

It is known that they are closed (it follows from the theorem on dimensions of fibres, cf. [40, Exercise II.3.22] or [48, Ch. I, § 6, Theorem 7]). We set

$$\operatorname{par}(\mathbf{r}_\bullet, \mathbf{I}, \mathbf{A}) = \max_i \{\, \dim \mathbb{D}_i(\mathbf{r}_\bullet, \mathbf{I}) - i \,\}$$

and call this integer the *parameter number* of $\mathbf{I}$-complexes of vector rank $\mathbf{r}_\bullet$. Obviously, if $\mathbf{I} \subseteq \mathbf{I}'$, then $\operatorname{par}(\mathbf{r}_\bullet, \mathbf{I}, \mathbf{A}) \le \operatorname{par}(\mathbf{r}_\bullet, \mathbf{I}', \mathbf{A})$. Especially, the number $\operatorname{par}(\mathbf{r}_\bullet, \mathbf{A}) = \operatorname{par}(\mathbf{r}_\bullet, \mathbf{J}, \mathbf{A})$ is the biggest one.

Proposition 2.1, together with the theorem on the dimensions of fibres and the Chevalley theorem on the image of a regular map (cf. [40, Exercise II.3.19] or [48, Ch. I, § 5, Theorem 6]), implies the following result.

Corollary 2.2. *Let* $\mathcal{F}_\bullet$ *be an* $\mathbf{I}$*-family of vector rank* $\mathbf{r}_\bullet$ *over a variety* X*. For each* $x \in X$ *set* $X_x = \{\, x' \in X \mid \mathcal{F}_\bullet(x') \simeq \mathcal{F}_\bullet(x) \,\}$ *and denote*

$$X_i = \{\, x \in X \mid \dim X_x \le i \,\},$$
$$\operatorname{par}(\mathcal{F}_\bullet) = \max_i \{\, \dim X_i - i \,\}.$$

Then all subsets X_x *and* X_i *are* constructible *(i.e. finite unions of locally closed sets) and* $\operatorname{par}(\mathcal{F}_\bullet) \le \operatorname{par}(\mathbf{r}_\bullet, \mathbf{I}, \mathbf{A})$*.*

Note that the bases $\mathbb{D}(\mathbf{r}_\bullet, \mathbf{I})$ of our almost versal families are *projective*, especially *complete* varieties. We shall exploit this property while studying the behaviour of parameter numbers in families of algebras. Since decompositions of algebras in families into direct sums of projective modules can differ, we restrict our considerations to the complexes of *free* modules. Namely, let $\mathbf{a} = \mathbf{r}(\mathbf{A})$. For every sequence $\mathbf{b} = (b_n, \dots, b_m)$ of integers we set $\mathbf{ba} = (b_n\mathbf{a}, \dots, b_m\mathbf{a})$ and write $\operatorname{par}(\mathbf{b}, \mathbf{I}, \mathbf{A})$ instead of $\operatorname{par}(\mathbf{ba}, \mathbf{I}, \mathbf{A})$.

A (flat) *family of algebras* over an algebraic variety X is a sheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras, which is coherent and flat (thus locally free) as a sheaf of $\mathcal{O}_X$-modules. For such a family and every sequence $\mathbf{b} = (b_m, b_{m+1}, \dots, b_n)$ one can define the function $\operatorname{par}(\mathbf{b}, \mathcal{A}, x) = \operatorname{par}(\mathbf{b}, \mathcal{A}(x))$. Our main result is the upper semi-continuity of these functions.

Theorem 2.3. *Let $\mathcal{A}$ be a flat family of finite dimensional algebras over an algebraic variety X. For every vector $\mathbf{b} = (b_m, b_{m+1}, \dots, b_n)$ the function* $\operatorname{par}(\mathbf{b}, \mathcal{A}, x)$ *is upper semi-continuous, i.e. all sets*

$$X_j = \{\, x \in X \mid \operatorname{par}(\mathbf{b}, \mathcal{A}, x) \geq j \,\}$$

are closed.

Proof. We may assume that X is irreducible. Let $\mathbf{K}$ be the field of rational functions on X. We consider it as a constant sheaf on X. Set $\mathbf{J} = \operatorname{rad}(\mathcal{A} \otimes_{\mathcal{O}_X} \mathbf{K})$ and $\mathcal{J} = \mathbf{J} \cap \mathcal{A}$. It is a sheaf of nilpotent ideals. Moreover, if ξ is the generic point of X, the factor algebra $\mathcal{A}(\xi)/\mathcal{J}(\xi)$ is semisimple. Hence there is an open set $U \subseteq X$ such that $\mathcal{A}(x)/\mathcal{J}(x)$ is semisimple, thus $\mathcal{J}(x) = \operatorname{rad}\mathcal{A}(x)$ for every $x \in U$. Therefore, $\operatorname{par}(\mathbf{b}, \mathcal{A}, x) = \operatorname{par}(\mathbf{b}, \mathcal{J}(x), \mathcal{A}(x))$ for $x \in U$; so $X_j = X_j(\mathcal{J}) \cup X'_j$, where

$$X_j(\mathcal{J}) = \{\, x \in X \mid \operatorname{par}(\mathbf{b}, \mathcal{J}(x), \mathcal{A}(x)) \geq j \,\}$$

and $X' = X \setminus U$ is a closed subset in X. Using noetherian induction, we may suppose that X'_j is closed, so we only have to prove that $X_j(\mathcal{J})$ is closed too.

Consider the locally free sheaf $\mathcal{H} = \bigoplus_{k=m+1}^{n} \mathcal{H}om(b_k\mathcal{A}, b_{k-1}\mathcal{J})$ and the projective space bundle $\mathbb{P}(\mathcal{H})$ [40, Section II.7]. Every point $h \in \mathbb{P}(\mathcal{H})$ defines a set of homomorphisms $h_k : b_k\mathcal{A}(x) \to b_{k-1}\mathcal{J}(x)$ (up to a homothety), where x is the image of h in X, and the points h such that $h_k h_{k+1} = 0$ form a closed subset $\mathbb{D}(\mathbf{b}, \mathcal{A}) \subseteq \mathbb{P}(\mathcal{H})$. We denote by π the restriction onto $\mathbb{D}(\mathbf{b}, \mathcal{A})$ of the projection $\mathbb{P}(\mathcal{H}) \to X$; it is a projective, hence closed map. Moreover, for every point $x \in X$ the fibre $\pi^{-1}(x)$ is isomorphic to $\mathbb{D}(\mathbf{b}, \mathcal{A}(x), \mathcal{J}(x))$. Consider also the group variety $\mathcal{G}$ over X: $\mathcal{G} = \prod_{k=m}^{n} \mathrm{GL}_{b_k}(\mathcal{A})$. There is a natural action of $\mathcal{G}$ on $\mathbb{D}(\mathbf{b}, \mathcal{A})$ over X, and the sets $\mathbb{D}_i = \{\, z \in \mathbb{D}(\mathbf{b}, \mathcal{A}) \mid \dim \mathcal{G}z \leq i \,\}$ are closed in $\mathbb{D}(\mathbf{b}, \mathcal{A})$. Therefore, the sets $Z_i = \pi(\mathbb{D}_i)$ are closed in X, as well as $Z_{ij} = \{\, x \in Z_i \mid \dim \pi^{-1}(x) \geq i + j \,\}$. But $X_j(\mathcal{J}) = \bigcup_i Z_{ij}$, thus it is also a closed set. $\square$

2.2 Derived Tame and Wild Algebras

We are going to define derived tame and derived wild algebras. To do it, we consider families of complexes with non-commutative bases.

Definition 2.4. 1. Let $\mathbf{R}$ be a $\Bbbk$-algebra. A *family of* $\mathbf{A}$-*complexes based on* $\mathbf{R}$ is a complex of finitely generated projective $\mathbf{A} \otimes \mathbf{R}^{\mathrm{op}}$-modules $P_\bullet$. We denote by $\mathscr{C}^{(N)}(\mathbf{A}, \mathbf{R})$ the category of all bounded families with $P_n = 0$ for $n > N$ (again we do not prescribe the right bound). For such a family $P_\bullet$ and an $\mathbf{R}$-module L we denote by $P_\bullet(L)$ the complex $(P_n \otimes_{\mathbf{R}} L, d_n \otimes 1)$. If L is finite dimensional, $P_\bullet(L) \in \mathscr{C}^{(N)}(\mathbf{A}) = \mathscr{C}^{(N)}(\mathbf{A}, \Bbbk)$.

Obviously, if the algebra $\mathbf{R}$ is *affine*, i.e. commutative, finitely generated over $\Bbbk$ and without nilpotents, such families coincide in fact with families of complexes over the algebraic variety $\operatorname{Spec} \mathbf{R}$. Especially, if $\mathbf{R}$ is also connected (i.e. contains no nontrivial idempotents), the *vector rank* of such a family $\mathbf{rk}(P_\bullet)$ is defined as $\mathbf{rk}(P_\bullet \otimes_{\mathbf{R}} S)$, where S is a simple $\mathbf{R}$-module (no matter which one).

2. We call a family $P_\bullet$ *strict* if for every finite dimensional $\mathbf{R}$-modules L, L'

(a) $P_\bullet(L) \simeq P_\bullet(L')$ if and only if $L \simeq L'$;

(b) $P_\bullet(L)$ is indecomposable if and only if so is L.

3. We call $\mathbf{A}$ *derived wild* if it has a strict family of complexes over every finitely generated $\Bbbk$-algebra $\mathbf{R}$.

The following useful fact is well known.

Proposition 2.5. *An algebra* $\mathbf{A}$ *is derived wild if and only if it has a strict family over one of the following algebras:*

- *free algebra* $\Bbbk\langle x, y\rangle$ *in two variables;*
- *polynomial algebra* $\Bbbk[x, y]$ *in two variables;*
- *power series algebra* $\Bbbk[[x, y]]$ *in two variables.*

Definition 2.6. 1. A *rational algebra* is a $\Bbbk$-algebra $\Bbbk[t, f(t)^{-1}]$ for a non-zero polynomial $f(t)$. A *rational family* of $\mathbf{A}$-complexes is a family over a rational algebra $\mathbf{R}$. Equivalently, a rational family is a family over an open subvariety of the affine line.

2. An algebra $\mathbf{A}$ is called *derived tame* if there is a set of rational families of bounded $\mathbf{A}$-complexes $\mathfrak{P}$ such that:

(a) for each $\mathbf{r}_\bullet \in \Delta$, the set $\mathfrak{P}(\mathbf{r}_\bullet) = \{\, P_\bullet \in \mathfrak{P} \mid \mathbf{rk}(P_\bullet) = \mathbf{r}_\bullet \,\}$ is finite.

(b) for every $\mathbf{r}_\bullet$ all indecomposable complexes from $\mathcal{C}(\mathbf{r}_\bullet, \mathbf{A})$, except finitely many of them (up to isomorphism), are isomorphic to a complex $P_\bullet(L)$ for some $P_\bullet \in \mathfrak{P}$ and some finite dimensional L.

We call $\mathfrak{P}$ a *parameterizing set* of $\mathbf{A}$-complexes.

These definitions do not formally coincide with other definitions of derived tame and derived wild algebras, for instance, those proposed in [36, 37], but all of them are evidently equivalent. It is obvious (and easy to prove, like in [20]) that neither algebra can be both derived tame and derived wild. The following result ("tame–wild dichotomy for derived categories") has recently been proved by V. Bekkert and the author [6].

Theorem 2.7. *Every finite dimensional algebra over an algebraically closed field is either derived tame or derived wild.*

2.3 Sliced Boxes

The proof of Theorem 2.7 rests on the technique of representations of boxes ("matrix problems"). We recall now the main related notions. A *box* is a pair $\mathfrak{A} = (\mathscr{A}, \mathscr{V})$, where $\mathscr{A}$ is a category and $\mathscr{V}$ is an $\mathscr{A}$-coalgebra, i.e. an $\mathscr{A}$-bimodule supplied with *comultiplication* $\mu : \mathscr{V} \to \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V}$ and *counit* $\iota : \mathscr{V} \to \mathscr{A}$, which are homomorphisms of $\mathscr{A}$-bimodules and satisfy the usual coalgebra conditions

$$(\mu \otimes 1)\mu = (1 \otimes \mu)\mu, \quad i_l(\iota \otimes 1)\mu = i_r(1 \otimes \iota)\mu = \mathrm{Id},$$

where $i_l : \mathscr{A} \otimes_{\mathscr{A}} \mathscr{V} \simeq \mathscr{V}$ and $i_r : \mathscr{V} \otimes_{\mathscr{A}} \mathscr{A} \simeq \mathscr{V}$ are the natural isomorphisms. The kernel $\overline{\mathscr{V}} = \operatorname{Ker} \iota$ is called the *kernel of the box.* A *representation* of such a box in a category $\mathscr{C}$ is a functor $M : \mathscr{A} \to \mathscr{C}$. Given another representation $N : \mathscr{A} \to \mathscr{C}$, a *morphism* $f : M \to N$ is defined as a homomorphism of $\mathscr{A}$-modules $\mathscr{V} \otimes_{\mathscr{A}} M \to N$, The composition gf of $f : M \to N$ and $g : N \to L$ is defined as the composition

$$\mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{\mu\otimes 1} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{1\otimes f} \mathscr{V} \otimes_{\mathscr{A}} N \xrightarrow{g} L,$$

while the identity morphism Id_M of M is the composition

$$\mathscr{V} \otimes_{\mathscr{A}} M \xrightarrow{\iota\otimes 1} \mathscr{A} \otimes_{\mathscr{A}} M \xrightarrow{i_l} M.$$

Thus we obtain the *category of representations* $\operatorname{Rep}(\mathfrak{A}, \mathscr{C})$. If $\mathscr{C} = \textsc{vec}$, the category of finite dimensional vector spaces, we just write $\operatorname{Rep}(\mathfrak{A})$. If f is a morphism and $\gamma \in \mathscr{V}(a, b)$, we denote by $f(\gamma)$ the morphism $f(b)(\gamma \otimes \_) : M(a) \to N(a)$. A box $\mathfrak{A}$ is called *normal* (or *group-like*) if there is a set of elements $\omega = \{\, \omega_a \in \mathscr{V}(a, a) \mid a \in \operatorname{Ob} \mathscr{A} \,\}$ such that $\iota(\omega_a) = 1_a$ and $\mu(\omega_a) = \omega_a \otimes \omega_a$ for every $a \in \operatorname{Ob} \mathscr{A}$. In this case, if f is an isomorphism, all morphisms $f(\omega_a)$ are isomorphisms $M(a) \simeq N(a)$. This set is called a *section* of $\mathfrak{A}$. For a normal box, one defines the *differentials* $\partial_0 : \mathscr{A} \to \overline{\mathscr{V}}$ and $\partial_1 : \overline{\mathscr{V}} \to \overline{\mathscr{V}} \otimes_{\mathscr{A}} \overline{\mathscr{V}}$ setting

$$\begin{aligned} \partial_0(\alpha) &= \alpha\omega_a - \omega_b\alpha \quad \text{ for } \alpha \in \mathscr{A}(a, b); \\ \partial_1(\gamma) &= \mu(\gamma) - \gamma \otimes \omega_a - \omega_b \otimes \gamma \quad \text{ for } \gamma \in \overline{\mathscr{V}}(a, b). \end{aligned}$$

Usually we omit indices, writing $\partial\alpha$ and $\partial\gamma$.

Recall that a *free category* $\Bbbk\Gamma$, where Γ is an oriented graph, has the vertices of Γ as its objects and the paths from a to b (a, b being two vertices) as a basis of the vector space $\Bbbk\Gamma(a, b)$.If Γ has no oriented cycles, such a category is locally finite dimensional. A *semi-free* category is a category of fractions $\Bbbk\Gamma[S^{-1}]$, where $S = \{\, g_\alpha(\alpha) \mid \alpha \in \mathfrak{L} \,\}$ and $\mathfrak{L}$ is a subset of the set of loops in Γ (called *marked loops*). The arrows of Γ are called the *free* (respectively, *semi-free*) generators of the free (semi-free) category. A *normal box* $\mathfrak{A} = (\mathscr{A}, \mathscr{V})$ is called *free* (*semi-free*) if such is the category $\mathscr{A}$, moreover, the kernel $\overline{\mathscr{V}} = \operatorname{Ker}\iota$ of the box is a free $\mathscr{A}$-bimodule and $\partial\alpha = 0$ for each marked loop α. A set of *free* (respectively, *semi-free*) generators of such a box is a union $\mathbf{S} = \mathbf{S}_0 \cup \mathbf{S}_1$, where $\mathbf{S}_0$ is a set of free (semi-free) generators of the category $\mathscr{A}$ and $\mathbf{S}_1$ is a set of free generators of the $\mathscr{A}$-bimodule $\mathscr{V}$.

We call a category $\mathscr{A}$ *trivial* if it is a free category generated by a trivial graph (i.e. one with no arrows); thus $\mathscr{A}(a, b) = 0$ if $a \neq b$ and $\mathscr{A}(a, a) = \Bbbk$. We call $\mathscr{A}$ *minimal*, if it is a semi-free category with a set of semi-free generators consisting of loops only, at most one loop at each vertex. Thus $\mathscr{A}(a, b) = 0$ again if $a \neq b$, while $\mathscr{A}(a, a)$ is either $\Bbbk$ or a rational algebra. We call a normal box $\mathfrak{A} = (\mathscr{A}, \mathscr{V})$ *so-trivial* if $\mathscr{A}$ is trivial, and *so-minimal* if $\mathscr{A}$ is minimal and all its loops α are minimal too (i.e. with $\partial\alpha = 0$).

A *layered box* [15] is a semi-free box $\mathfrak{A} = (\mathscr{A}, \mathscr{V})$ with a section ω, a set of semi-free generators $\mathbf{S} = \mathbf{S}_0 \cup \mathbf{S}_1$ and a function $\rho : \mathbf{S}_0 \to \mathbb{N}$ satisfying the following conditions:

- A morphism ϕ from $\operatorname{Rep}(\mathfrak{A})$ is an isomorphism if all maps $\phi(\omega_a)$ ($a \in \operatorname{Ob}\mathscr{A}$)) are isomorphisms.
- There is at most one marked loops at each vertex.
- For each $\alpha \in \mathbf{S}_0$ the differential $\partial\alpha$ belongs to the $\mathscr{A}_\alpha$-sub-bimodule of $\overline{\mathscr{V}}$ generated by $\mathbf{S}_1$, where $\mathscr{A}_\alpha$ is the semi-free subcategory of $\mathscr{A}$ with the set of semi-free generators $\{\, \beta \in \mathbf{S}_0 \mid \rho(\beta) < \rho(\alpha) \,\}$.

Obviously, we may suppose, without loss of generality, that $\rho(\alpha) = 0$ for every marked loop α. The set $\{\, \omega, \mathbf{S}, \rho \,\}$ is called a *layer* of the box $\mathfrak{A}$.

In [21] (cf. also [15, 25]) the classification of representations of an arbitrary finite dimensional algebra was reduced to representations of a free layered box. To deal with derived categories we have to consider a wider class of boxes. First, a *factor-box* of a box $\mathfrak{A} = (\mathscr{A}, \mathscr{V})$ modulo an ideal $\mathscr{I} \subseteq \mathscr{A}$ is defined as the box $\mathfrak{A}/\mathscr{I} = (\mathscr{A}/\mathscr{I}, \mathscr{V}/(\mathscr{I}\mathscr{V} + \mathscr{V}\mathscr{I}))$ (with obvious comultiplication and counit). Note that if $\mathfrak{A}$ is normal, so is $\mathfrak{A}/\mathscr{I}$.

Definition 2.8. A *sliced box* is a factor-box $\mathfrak{A}/\mathscr{I}$, where $\mathfrak{A} = (\mathscr{A}, \mathscr{V})$ is a free layered box such that the set of its objects $\mathsf{V} = \operatorname{Ob}\mathscr{A}$ is a disjoint union $\mathsf{V} = \bigcup_{i \in \mathbb{Z}} \mathsf{V}_i$ so that the following conditions hold:

- $\mathscr{A}(a,a) = \Bbbk$ for each object $a \in \mathscr{A}$;
- $\mathscr{A}(a,b) = 0$ if $a \neq b$, $a \in \mathsf{V}_i$, $b \in \mathsf{V}_j$ with $j \geq i$;
- $\mathscr{V}(a,b) = 0$ if $a \in \mathsf{V}_i$, $b \in \mathsf{V}_j$ with $i \neq j$.

The partition $\mathsf{V} = \bigcup_i \mathsf{V}_i$ is called a *slicing*.

Certainly, in this definition we may assume that the elements of the ideal $\mathscr{I}$ are linear combinations of paths of length at least 2. Otherwise we can just eliminate one of the arrows from the underlying graph without changing the factor $\mathfrak{A}/\mathscr{I}$.

Note that for every representation $M \in \operatorname{Rep}(\mathfrak{A})$, where $\mathfrak{A}$ is a free (semi-free, sliced) box with the set of objects V, one can consider its *dimension* $\mathbf{dim}(M)$, which is a function $\mathsf{V} \to \mathbb{N}$, namely $\mathbf{dim}(M)(a) = \dim M(a)$. We call such a representation *finite dimensional* if its support $\operatorname{supp} M = \{\, a \in \mathsf{V} \mid M(a) \neq 0 \,\}$ is finite and denote by $\operatorname{rep}(\mathfrak{A})$ the category of finite dimensional representations. Having these notions, one can easily reproduce the definitions of families of representations, especially strict families, wild and tame boxes; see [21, 25] for details. The following procedure, mostly copying that of [21], allows to model derived categories by representations of sliced boxes.

Let $\mathbf{A}$ be a finite dimensional algebra, $\mathbf{J}$ be its radical. As far as we are interested in $\mathbf{A}$-modules and complexes, we can replace $\mathbf{A}$ by a Morita equivalent reduced algebra, thus suppose that $\mathbf{A}/\mathbf{J} \simeq \Bbbk^s$ [31]. Let $1 = \sum_{i=1}^{s} e_i$, where e_i are primitive orthogonal idempotents; set $\mathbf{A}_{ji} = e_j \mathbf{A} e_i$ and $\mathbf{J}_{ji} = e_j \mathbf{J} e_i$; note that $\mathbf{J}_{ji} = \mathbf{A}_{ji}$ if $i \neq j$. We denote by $\mathscr{S}$ the trivial category with the set of objects $\{\, (i,n) \mid n \in \mathbb{N},\ i = 1,2,\dots,s \,\}$ and consider the $\mathscr{S}$-bimodule $\mathscr{J}$ such that

$$\mathscr{J}\big((i,n),(j,m)\big) = \begin{cases} 0 & \text{if } m \neq n-1, \\ \mathbf{J}^*_{ji} & \text{if } m = n-1. \end{cases}$$

Let $\mathscr{B} = \mathscr{S}[\mathscr{J}]$ be the tensor category of this bimodule; equivalently, it is the free category having the same set of objects as $\mathscr{S}$ and the union of bases of all $\mathscr{J}\big((i,n),(j,m)\big)$ as a set of free generators. Denote by $\mathscr{U}$ the $\mathscr{S}$-bimodule such that

$$\mathscr{U}\big((i,n),(j,m)\big) = \begin{cases} 0 & \text{if } n \neq m, \\ \mathbf{A}^*_{ji} & \text{if } n = m \end{cases}$$

and set $\widetilde{\mathscr{W}} = \mathscr{B} \otimes_{\mathscr{S}} \mathscr{U} \otimes_{\mathscr{S}} \mathscr{B}$. Dualizing the multiplication $\mathbf{A}_{kj} \otimes \mathbf{A}_{ji} \to \mathbf{A}_{ki}$, we get homomorphisms

$$\lambda_r : \mathscr{B} \to \mathscr{B} \otimes_{\mathscr{S}} \widetilde{\mathscr{W}}, \quad \lambda_l : \mathscr{B} \to \widetilde{\mathscr{W}} \otimes_{\mathscr{S}} \mathscr{B}, \quad \tilde{\mu} : \widetilde{\mathscr{W}} \to \widetilde{\mathscr{W}} \otimes_{\mathscr{S}} \widetilde{\mathscr{W}}.$$

In particular, $\tilde{\mu}$ defines on $\widetilde{\mathscr{W}}$ a structure of $\mathscr{B}$-coalgebra. Moreover, the subbimodule $\mathscr{W}_0$ generated by $\mathrm{Im}(\lambda_r - \lambda_l)$ is a coideal in $\widetilde{\mathscr{W}}$, i.e. $\tilde{\mu}(\mathscr{W}_0) \subseteq \mathscr{W}_0 \otimes_{\mathscr{B}} \widetilde{\mathscr{W}} \oplus \widetilde{\mathscr{W}} \otimes_{\mathscr{B}} \mathscr{W}_0$. Therefore, $\mathscr{W} = \widetilde{\mathscr{W}}/\mathscr{W}_0$ is also a $\mathscr{B}$-coalgebra, so we get a box $\mathfrak{B} = (\mathscr{B}, \mathscr{W})$. One easily checks that it is free and triangular.

Dualizing multiplication also gives a map

$$\nu : \mathbf{J}_{ji}^* \to \bigoplus_{k=1}^{s} \mathbf{J}_{jk}^* \otimes \mathbf{J}_{ki}^*. \tag{2}$$

Namely, if we choose bases $\{\,\alpha\,\}$, $\{\,\beta\,\}$ $\{\,\gamma\,\}$ in the spaces, respectively, $\mathbf{J}_{ji}$, $\mathbf{J}_{jk}$, $\mathbf{J}_{ki}$, and dual bases $\{\,\alpha^*\,\}$, $\{\,\beta^*\,\}$, $\{\,\gamma^*\,\}$ in their duals, then $\beta^* \otimes \gamma^*$ occurs in $\nu(\alpha^*)$ with the same coefficient as α occurs in $\beta\gamma$. Note that the right-hand space in (2) coincide with each $\mathscr{B}\big((i,n),(j,n-2)\big)$. Let $\mathscr{I}$ be the ideal in $\mathscr{B}$ generated by the images of ν in all these spaces and $\mathfrak{D} = \mathfrak{B}/\mathscr{I} = (\mathscr{A}, \mathscr{V})$, where $\mathscr{A} = \mathscr{B}/\mathscr{I}$, $\mathscr{V} = \mathscr{W}/(\mathscr{I}\mathscr{W} + \mathscr{W}\mathscr{I})$. If necessary, we write $\mathfrak{D}(\mathbf{A})$ to emphasize that this box has been constructed from a given algebra $\mathbf{A}$. Certainly, $\mathfrak{D}$ is a sliced box, and the following result holds.

Theorem 2.9. *The category of finite dimensional representations* $\mathrm{rep}(\mathfrak{D}(\mathbf{A}))$ *is equivalent to the category* $\mathscr{C}^b_{\min}(\mathbf{A})$ *of bounded minimal projective* $\mathbf{A}$-*complexes.*

Proof. Let $A_i = \mathbf{A}e_i$. They form a complete list of non-isomorphic indecomposable projective $\mathbf{A}$-modules. Further, set $J_i = \operatorname{rad} A_i = \mathbf{J}e_i$. Then $\mathrm{Hom}_{\mathbf{A}}(A_i, J_j) \simeq \mathbf{J}_{ji}$. A representation $M \in \mathrm{rep}(\mathfrak{D})$ is given by vector spaces $M(i,n)$ and linear maps

$$M_{ji}(n) : \mathbf{J}_{ji}^* = \mathscr{A}\big((i,n),(j,n-1)\big) \to \mathrm{Hom}\,\big(M(i,n), M(j,n-1)\big)$$

subject to the relations

$$\sum_{k=1}^{s} \mathbf{m}\big(M_{jk}(n) \otimes M_{ki}(n+1)\big)\nu(\alpha) = 0 \tag{3}$$

for all i,j,k,n and all $\alpha \in \mathbf{J}_{ji}$, where $\mathbf{m}$ denotes the multiplication of maps

$$\mathrm{Hom}\,\big(M(k,n), M(j,n-1)\big) \otimes \mathrm{Hom}\,\big(M(i,n+1), M(k,n)\big) \to \mathrm{Hom}\,\big(M(i,n+1), M(j,n-1)\big).$$

For such a representation, set $P_n = \bigoplus_{i=1}^{s} A_i \otimes M(i,n)$. Then

$$\operatorname{rad} P_n = \bigoplus_{i=1}^{n} J_i \otimes M(i,n)$$

and

$$\begin{aligned}\operatorname{Hom}_{\mathbf{A}}(P_n, \operatorname{rad} P_{n-1}) &\simeq \bigoplus_{i,j} \operatorname{Hom}_{\mathbf{A}}\big(A_i \otimes M(i,n), J_j \otimes M(j,n-1)\big)\\ &\simeq \bigoplus_{ij} \operatorname{Hom}\big(M(i,n), \operatorname{Hom}_{\mathbf{A}}\big(A_i, J_j \otimes M(j,n-1)\big)\big)\\ &\simeq \bigoplus_{ij} M(i,n)^* \otimes \mathbf{J}_{ji} \otimes M(j,n-1)\\ &\simeq \bigoplus_{ij} \operatorname{Hom}\big(\mathbf{J}_{ji}^*, \operatorname{Hom}\big(M(i,n), M(j,n-1)\big)\big).\end{aligned}$$

Thus the set $\{\, M_{ji}(n) \mid i,j = 1,2,\dots,s \,\}$ defines a homomorphism $d_n : P_n \to P_{n-1}$ and vice versa. Moreover, one easily verifies that the condition (3) is equivalent to the relation $d_n d_{n+1} = 0$. Since every projective $\mathbf{A}$-module can be given in the form $\bigoplus_{i=1}^s A_i \otimes V_i$ for some uniquely defined vector spaces V_i, we get a one-to-one correspondence between finite dimensional representations of $\mathfrak{D}$ and bounded minimal complexes of projective $\mathbf{A}$-modules. In the same way one also establishes one-to-one correspondence between morphisms of representations and of the corresponding complexes, compatible with their multiplication, which accomplishes the proof. □

Corollary 2.10. *An algebra* $\mathbf{A}$ *is derived tame (derived wild) if and only if so is the box* $\mathfrak{D}(\mathbf{A})$.

2.4 Proof of Dichotomy

Now we are able to prove Theorem 2.7. Namely, according to Corollary 2.10, it follows from the analogous result for sliced boxes.

Theorem 2.11. *Every sliced box is either tame or wild.*

Actually, just as in [21] (see also [15, 25]), we shall prove this theorem in the following form.

Theorem 2.11a. *Suppose that a sliced box* $\mathfrak{A} = (\mathscr{A}, \mathscr{V})$ *is not wild. For every dimension* $\mathbf{d}$ *of its representations there is a functor* $F_{\mathbf{d}} : \mathscr{A} \to \mathscr{M}$, *where* $\mathscr{M}$ *is a minimal category, such that every representation* $M : \mathscr{A} \to \text{VEC}$ *of* $\mathfrak{A}$ *of dimension* $\mathbf{dim}(M) \leq \mathbf{d}$ *is isomorphic to the inverse image* $F^*N = N \circ F$ *for some functor* $N : \mathscr{M} \to \text{VEC}$. *Moreover,* F *can be chosen* strict, *which means that* $F^*N \simeq F^*N'$ *implies* $N \simeq N'$ *and* F^*N *is indecomposable if so is* N.

Remark. We can consider the induced box $\mathfrak{A}^F = (\mathscr{M}, \mathscr{M} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{M})$. It is a so-minimal box, and F^* defines a full and faithful functor $\operatorname{rep}(\mathfrak{A}^F) \to \operatorname{rep}(\mathfrak{A})$. Its image consists of all representations $M : \mathscr{A} \to \text{VEC}$ that factorize through F.

Proof. As we fix the dimension $\mathbf{d}$, we may assume that the set of objects is finite (namely, $\operatorname{supp}\mathbf{d}$). Hence the slicing $\mathsf{V} = \bigcup_i \mathsf{V}_i$ (see Definition 2.8) is finite too: $\mathsf{V} = \bigcup_{i=1}^m \mathsf{V}_i$ and we use induction by m. If $m = 1$, $\mathfrak{A}$ is free, and our claim follows from [21, 15]. So we may suppose that the theorem is true for smaller values of m, especially, it is true for the restriction $\mathfrak{A}' = (\mathscr{A}', \mathscr{V}')$ of the box $\mathfrak{A}$ onto the subset $\mathsf{V}' = \bigcup_{i=2}^m \mathsf{V}_i$. Thus there is a strict functor $F' : \mathscr{A}' \to \mathscr{M}$, where $\mathscr{M}$ is a minimal category, such that every representation of $\mathfrak{A}'$ of dimension smaller than $\mathbf{d}$ is of the form F'^*N for $N : \mathscr{M} \to \text{VEC}$. Consider now the amalgamation $\mathscr{B} = \mathscr{A} \bigsqcup^{\mathscr{A}'} \mathscr{M}$ and the box $\mathfrak{B} = (\mathscr{B}, \mathscr{W})$, where $\mathscr{W} = \mathscr{B} \otimes_{\mathscr{A}} \mathscr{V} \otimes_{\mathscr{A}} \mathscr{B}$. The functor F' extends to a functor $F : \mathscr{A} \to \mathscr{B}$ and induces a homomorphism of $\mathbf{A}$-bimodules $\mathscr{V} \to \mathscr{W}$; so it defines a functor $F^* : \operatorname{rep}(\mathfrak{B}) \to \operatorname{rep}(\mathfrak{A})$, which is full and faithful. Moreover, every representation of $\mathfrak{A}$ of dimension smaller than $\mathbf{d}$ is isomorphic to F^*N for some N, and all possible dimensions of such N are restricted by some vector $\mathbf{b}$. Therefore, it is enough to prove the claim of the theorem for the box $\mathfrak{B}$.

Note that the category $\mathscr{B}$ is generated by the loops from $\mathscr{M}$ and the images of arrows from $\mathscr{A}(a, b)$ with $b \in \mathsf{V}_1$ (we call them *new arrows*). It implies that all possible relations between these morphisms are of the form $\sum_\beta \beta g_\beta(\alpha) = 0$, where $\alpha \in \mathscr{B}(a, a)$ is a loop (necessarily minimal, i.e. with $\partial\alpha = 0$), g_β are some polynomials, and β runs through the set of new arrows from a to b for some $b \in \mathsf{V}_1$. Consider all of these relations for a fixed b; let them be $\sum_\beta \beta g_{\beta,k}(\alpha) = 0$ $(k = 1, \dots, r)$. Their coefficients form a matrix $\big(g_{\beta,k}(\alpha)\big)$. Using linear transformations of the set $\{\,\beta\,\}$ and of the set of relations, we can make this matrix diagonal, i.e. make all relations being $\beta f_\beta(\alpha) = 0$ for some polynomials f_β. If one of f_β is zero, the box $\mathfrak{B}$ has a sub-box

$$\begin{tikzcd} a \arrow[loop left, "\alpha"] \arrow[rr, "\beta"] & & b \end{tikzcd},$$

with $\partial\alpha = \partial\beta = 0$, which is wild; hence $\mathfrak{B}$ and $\mathfrak{A}$ are also wild. Otherwise, let $f(\alpha) \neq 0$ be a common multiple of all $f_\beta(\alpha)$, $\Lambda = \{\,\lambda_1, \lambda_2, \dots, \lambda_r\,\}$ be the set of roots of $f(\alpha)$. If $N \in \operatorname{rep}(\mathfrak{B})$ is such that $N(\alpha)$ has no eigenvalues from Λ, then $f(N(\alpha))$ is invertible; thus $N(\beta) = 0$ for all $\beta : a \to b$. So we can apply the *reduction of the loop* α with respect to the set Λ and the dimension $d = \mathbf{b}(a)$, as in [21, Propositions 3,4] or [25, Theorem 6.4]. It gives a new box that has the same number of loops as $\mathfrak{B}$, but the loop corresponding to α is "isolated," i.e. there are no more arrows starting or ending at the same vertex. In the same way we are able to isolate all loops, obtaining a semi-free layered box $\mathfrak{C}$ and a morphism $G : \mathfrak{B} \to \mathfrak{C}$ such that G^* is full and faithful and all representations of $\mathfrak{B}$ of dimensions smaller than $\mathbf{b}$ are of the form G^*L. As the theorem is true for semi-free boxes, it accomplishes the proof. $\square$

Remark. Applying reduction functors, like in the proof above, we can also extend to sliced boxes (thus to derived categories) other results obtained before for free boxes. For instance, we mention the following theorem, quite analogous to that of Crawley-Boevey [17].

Theorem 2.12. *If an algebra* $\mathbf{A}$ *is derived tame, then, for any vector rank* $\mathbf{r}_\bullet \in \Delta = (\mathbf{r}_n \mid n \in \mathbb{Z})$*, there is at most finite set of* generic $\mathbf{A}$-complexes *of endolength* $\mathbf{r}_\bullet$*, i.e. such indecomposable minimal bounded complexes* $P_\bullet$ *of projective* $\mathbf{A}$*-modules, not all of which are finitely generated, that* $\mathrm{length}_{\mathbf{E}}(P_n) = \mathbf{r}_n$ *for all* n*, where* $\mathbf{E} = \mathrm{End}_{\mathbf{A}}(P_\bullet)$*.*

Its proof reproduces again that of [17], with obvious changes necessary to include sliced boxes into consideration.

2.5 Deformations of Derived Tame Algebras

Combining the semi-continuity properties with tame–wild dichotomy, we can prove the results on deformations of derived tame algebras, analogous to those of [28, 35]. Note first the following easy observation.

Proposition 2.13. *Let* $\mathbf{A}$ *be a finite dimensional algebra. For every vector* $\mathbf{r} = (r_1, r_2, \dots, r_s)$ *set* $|\mathbf{r}| = \sum_{i=1}^s r_i$*. For every vector rank* $\mathbf{r}_\bullet \in \Delta(\mathbf{A})$ *set* $|\mathbf{r}_\bullet| = \sum_n \mathbf{r}_n$*.*

1. $\mathbf{A}$ *is derived tame if and only if* $\mathrm{par}(\mathbf{r}_\bullet, \mathbf{A}) \le |\mathbf{r}_\bullet|$ *for every* $\mathbf{r}_\bullet \in \Delta$*.*
2. $\mathbf{A}$ *is derived wild if and only if there is a vector rank* $\mathbf{r}_\bullet$ *such that* $\mathrm{par}(k\mathbf{r}_\bullet, \mathbf{A}) \ge k^2$ *for every* $k \in \mathbb{N}$*.*

Proof. The necessity of these conditions follows from the definitions of derived tameness and wildness. Certainly, they exclude each other. Since every algebra is either derived tame or derived wild, the sufficiency follows. □

This proposition together with Theorem 2.3 immediately implies the following result.

Corollary 2.14. *For a family of algebras* $\mathcal{A}$ *over* X *denote*

$$X_{\mathrm{tame}} = \{\, x \in X \mid \mathcal{A}(x) \text{ is } \textit{derived tame} \,\},$$
$$X_{\mathrm{wild}} = \{\, x \in X \mid \mathcal{A}(x) \text{ is } \textit{derived wild} \,\}.$$

Then X_{tame} *is a countable intersection of open subsets and* X_{wild} *is a countable union of closed subsets.*

Proof. By Theorem 2.3 the set $Z(\mathbf{r}_\bullet) = \{\, x \in X \mid \mathrm{par}(\mathbf{r}_\bullet, \mathbf{A}) \le |\mathbf{r}_\bullet| \,\}$ is open. But $X_{\mathrm{tame}} = \bigcap_{\mathbf{r}} Z(\mathbf{r})$ and hence $X_{\mathrm{wild}} = \bigcup_{\mathbf{r}} (X \setminus Z(\mathbf{r}))$. □

The following conjecture seems very plausible, though even its analogue for usual tame algebras has not yet been proved. (Only for *representation finite* algebras the corresponding result was proved in [33].)

Conjecture 2.15. For any (flat) family of algebras over an algebraic variety X the set X_{tame} is open.

Recall that an algebra $\mathbf{A}$ is said to be a (flat) *degeneration* of an algebra $\mathbf{B}$, and $\mathbf{B}$ is said to be a (flat) *deformation* of $\mathbf{A}$, if there is a (flat) family of algebras $\mathcal{A}$ over an algebraic variety X and a point $p \in X$ such that $\mathcal{A}(x) \simeq \mathbf{B}$ for all $x \neq p$, while $\mathcal{A}(p) \simeq \mathbf{A}$. One easily verifies that we can always assume X to be a *non-singular curve*. Corollary 2.14 obviously implies

Corollary 2.16. *Suppose that an algebra* $\mathbf{A}$ *is a (flat) degeneration of an algebra* $\mathbf{B}$*. If* $\mathbf{B}$ *is derived wild, so is* $\mathbf{A}$*. If* $\mathbf{A}$ *is derived tame, so is* $\mathbf{B}$*.*

If we consider non-flat families, the situation can completely change. The reason is that the dimension is no more constant in these families. That is why it can happen that such a "degeneration" of a derived wild algebra may become derived tame, as the following example due to Brüstle [10] shows.

Example 2.17. There is a (non-flat) family of algebras $\mathcal{A}$ over an affine line $\mathbb{A}^1$ such that all of them except $\mathcal{A}(0)$ are isomorphic to the derived wild algebra $\mathbf{B}$ given by the quiver with relations

$$\begin{tikzcd} & & \bullet & \\ \bullet \arrow[r, "\alpha"] & \bullet \arrow[r, "\beta_1"] & \bullet \arrow[u, "\gamma_1"'] \arrow[d, "\gamma_2"] & \bullet \arrow[l, "\beta_2"'] \\ & & \bullet & \end{tikzcd} \qquad \beta_1\alpha = 0,$$

while $\mathcal{A}(0)$ is isomorphic to the derived tame algebra $\mathbf{A}$ given by the quiver with relations

$$\begin{tikzcd} & & \bullet & \\ \bullet \arrow[r, "\alpha"] & \bullet \arrow[r, "\beta_1"] \arrow[ur, "\xi_1"] & \bullet \arrow[u, "\gamma_1"'] \arrow[d, "\gamma_2"'] & \bullet \arrow[l, "\beta_2"'] \arrow[dl, "\xi_2"] \\ & & \bullet & \end{tikzcd} \qquad \beta_1\alpha = \gamma_1\beta_1 = \gamma_2\beta_2 = 0. \tag{4}$$

Namely, one has to define $\mathcal{A}(\lambda)$ as the factor algebra of the path algebra of the quiver as in (4), but with the relations $\beta_1\alpha = 0,\ \gamma_1\beta_1 = \lambda\xi_1,\ \gamma_2\beta_2 = \lambda\xi_2$. Note that $\dim \mathbf{A} = 16$ and $\dim \mathbf{B} = 15$, which shows that this family is not flat.

Actually, in such a situation the following result always holds.

Proposition 2.18. *Let $\mathcal{A}$ be a family (not necessarily flat) of algebras over a non-singular curve X such that $\mathcal{A}(x) \simeq \mathbf{B}$ for all $x \neq p$, where p is a fixed point, while $\mathcal{A}(p) \simeq \mathbf{A}$. Then there is a flat family $\mathcal{B}$ over X such that $\mathcal{B}(x) \simeq \mathbf{B}$ for all $x \neq p$ and $\mathcal{B}(p) \simeq \mathbf{A}/I$ for some ideal I.*

Proof. Note that the restriction of $\mathcal{A}$ onto $U = X \setminus \{ p \}$ is flat, since $\dim \mathcal{A}(x)$ is constant there. Let $n = \dim \mathbf{B}$, Γ be the quiver of the algebra $\mathbf{B}$ and $\mathbf{G} = \Bbbk\Gamma$ be the path algebra of Γ. Consider the Grassmannian $\mathbf{Gr}(n, \mathbf{G})$, i.e. the variety of subspaces of codimension n of $\mathbf{G}$. The ideals form a closed subset $\mathrm{Alg} = \mathrm{Alg}(n, \mathbf{G}) \subset \mathbf{Gr}(n, \mathbf{G})$. The restriction of the canonical vector bundle $\mathcal{V}$ over the Grassmannian onto Alg is a sheaf of ideals in $\mathcal{G} = \mathbf{G} \otimes \mathcal{O}_{\mathrm{Alg}}$, and the factor $\mathcal{F} = \mathcal{G}/\mathcal{V}$ is a universal family of factor algebras of $\mathbf{G}$ of dimension n. Therefore, there is a morphism $\phi : U \to \mathrm{Alg}$ such that the restriction of $\mathcal{A}$ onto U is isomorphic to $\phi^*(\mathcal{F})$. Since Alg is projective and X is non-singular, ϕ can be continued to a morphism $\psi : X \to \mathrm{Alg}$. Let $\mathcal{B} = \psi^*(\mathcal{F})$; it is a flat family of algebras over X. Moreover, $\mathcal{B}$ coincides with $\mathcal{A}$ outside p. Since both of them are coherent sheaves on a non-singular curve and $\mathcal{B}$ is locally free, it means that $\mathcal{B} \simeq \mathcal{A}/\mathcal{T}$, where $\mathcal{T}$ is the torsion part of $\mathcal{A}$, and $\mathcal{B}(p) \simeq \mathcal{A}(p)/\mathcal{T}(p)$. □

Corollary 2.19. *If a degeneration of a derived wild algebra is derived tame, the latter has a derived wild factor algebra.*

In Brüstle's example 2.17, to obtain a derived wild factor algebra of $\mathbf{A}$, one has to add the relation $\xi_1\alpha = 0$, which obviously holds in $\mathbf{B}$.

By the way, as a factor algebra of a tame algebra is obviously tame (which is no more true for derived tame algebras!), we get the following corollary (cf. also [18, 29]).

Corollary 2.20. *Any deformation (not necessarily flat) of a tame algebra is tame. Any degeneration of a wild algebra is wild.*

3 Nodal Rings

3.1 Backström Rings

We consider a class of rings, which generalizes in a certain way local rings of *ordinary multiple points* of algebraic curves. Following the terminology used in the representations theory of orders, we call them *Backström rings*. In this section we suppose all rings being *noetherian* and *semi-perfect* in the sense of [3]; the latter means that all idempotents can be lifted modulo radical, or, equivalently, that every finitely generated module M has a projective cover, i.e. such an epimorphism $f : P \to M$, where P is projective and $\operatorname{Ker} f \subseteq$

$\operatorname{rad} P$. Hence, just as for finite dimensional algebras, the derived category $\mathscr{D}^-(\mathbf{A}\text{-mod})$ is equivalent to the homotopy category of right bounded *minimal complexes*, i.e. such complexes of finitely generated projective modules

$$\cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \to \cdots$$

that $\operatorname{Im} d_n \subseteq \operatorname{rad} P_{n-1}$ for all n.

Definition 3.1. A ring $\mathbf{A}$ (noetherian and semi-perfect) is called a *Backström ring* if there is a hereditary ring $\mathbf{H} \supseteq \mathbf{A}$ (also semi-perfect and noetherian) and a (two-sided, proper) $\mathbf{H}$-ideal $\mathbf{J_A}$ such that both $\mathbf{R} = \mathbf{H}/\mathbf{J}$ and $\mathbf{S} = \mathbf{A}/\mathbf{J}$ are semi-simple.

For Backström rings there is a convenient way to the calculations in derived categories. Recall that for a hereditary ring $\mathbf{H}$ every object $C_\bullet$ from $\mathscr{D}^-(\mathbf{H}\text{-mod})$ is isomorphic to the direct sum of its homologies. Especially, any indecomposable object from $\mathscr{D}^-(\mathbf{H}\text{-mod})$ is isomorphic to a shift $N[n]$ for some $\mathbf{H}$-module N, or, the same, to a "short" complex $0 \to P' \xrightarrow{\alpha} P \to 0$, where P and P' are projective modules and α is a monomorphism with $\operatorname{Im}\alpha \subseteq \operatorname{rad} P$ (maybe $P' = 0$). Thus it is natural to study the category $\mathscr{D}^-(\mathbf{A}\text{-mod})$ using this information about $\mathscr{D}^-(\mathbf{H}\text{-mod})$ and the functor $T : \mathscr{D}^-(\mathbf{A}\text{-mod}) \to \mathscr{D}^-(\mathbf{H}\text{-mod})$ mapping $C_\bullet$ to $\mathbf{H} \otimes_\mathbf{A} C_\bullet$. (Of course, we mean here the left derived functor of $\otimes$, but when we consider complexes of projective modules, it restricts indeed to the usual tensor product.)

Consider a new category $\mathscr{T} = \mathscr{T}(\mathbf{A})$ (the *category of triples*) defined as follows:

- Objects of $\mathscr{T}$ are triples $(A_\bullet, B_\bullet, \iota)$, where
  - $A_\bullet \in \mathscr{D}^-(\mathbf{H}\text{-mod})$;
  - $B_\bullet \in \mathscr{D}^-(\mathbf{S}\text{-mod})$;
  - ι is a morphism $B_\bullet \to \mathbf{R} \otimes_\mathbf{H} A_\bullet$ from $\mathscr{D}^-(\mathbf{S}\text{-mod})$ such that the induced morphism $\iota^R : \mathbf{R} \otimes_\mathbf{S} B_\bullet \to \mathbf{R} \otimes_\mathbf{H} A_\bullet$ is an isomorphism in $\mathscr{D}^-(\mathbf{R}\text{-mod})$.
- A morphism from a triple $(A_\bullet, B_\bullet, \iota)$ to a triple $(A'_\bullet, B'_\bullet, \iota')$ is a pair (Φ, ϕ), where
  - $\Phi : A_\bullet \to A'_\bullet$ is a morphism from $\mathscr{D}^-(\mathbf{H}\text{-mod})$;
  - $\phi : B_\bullet \to B'_\bullet$ is a morphism from $\mathscr{D}^-(\mathbf{S}\text{-mod})$;
  - the diagram

$$\begin{tikzcd} B_\bullet \arrow[r, "\iota"] \arrow[d, "\phi"'] & \mathbf{R} \otimes_\mathbf{H} A_\bullet \arrow[d, "1\otimes\Phi"] \\ B'_\bullet \arrow[r, "\iota'"] & \mathbf{R} \otimes_\mathbf{H} A'_\bullet \end{tikzcd} \tag{5}$$

    commutes in $\mathscr{D}^-(\mathbf{S}\text{-mod})$.

One can define a functor $\mathbf{F} : \mathscr{D}^{-}(\mathbf{A}\text{-mod}) \to \mathscr{T}(\mathbf{A})$ setting

$$\mathbf{F}(C_\bullet) = (\mathbf{H} \otimes_{\mathbf{A}} C_\bullet, \mathbf{S} \otimes_{\mathbf{A}} C_\bullet, \iota)\,,$$

where $\iota : \mathbf{S} \otimes_{\mathbf{A}} C_\bullet \to \mathbf{R} \otimes_{\mathbf{H}} (\mathbf{H} \otimes_{\mathbf{A}} C_\bullet) \simeq \mathbf{R} \otimes_{\mathbf{A}} C_\bullet$ is induced by the embedding $\mathbf{S} \to \mathbf{R}$. The values of $\mathbf{F}$ on morphisms are defined in an obvious way.

Theorem 3.2. *The functor* $\mathbf{F}$ *is a* full representation equivalence*, i.e. it is*

- dense*, i.e. every object from* $\mathscr{T}$ *is isomorphic to an object of the form* $\mathbf{F}(C_\bullet)$*;*
- full*, i.e. each morphism* $\mathbf{F}(C_\bullet) \to \mathbf{F}(C'_\bullet)$ *is of the form* $\mathbf{F}(\gamma)$ *for some* $\gamma : C_\bullet \to C'_\bullet$*;*
- conservative*, i.e.* $\mathbf{F}(\gamma)$ *is an isomorphism if and only if so is* γ*;*

As a consequence, $\mathbf{F}$ *maps non-isomorphic objects to non-isomorphic and indecomposable to indecomposable.*

Note that in general $\mathbf{F}$ is not *faithful*: it is possible that $\mathbf{F}(\gamma) = 0$ though $\gamma \neq 0$ (cf. Example 3.10.3 below).

Proof (sketched). Consider any triple $T = (A_\bullet, B_\bullet, \iota)$. We may suppose that $A_\bullet$ is a minimal complex from $\mathscr{C}^{-}(\mathbf{A}\text{-proj})$, while $B_\bullet$ is a complex with zero differential (since $\mathbf{S}$ is semi-simple), and the morphism ι is a usual morphism of complexes. Note that $\mathbf{R} \otimes_{\mathbf{H}} A_\bullet$ is also a complex with zero differential. We have an exact sequence of complexes:

$$0 \longrightarrow \mathbf{J}A_\bullet \longrightarrow A_\bullet \longrightarrow \mathbf{R} \otimes_{\mathbf{H}} A_\bullet \longrightarrow 0.$$

Together with the morphism $\iota : B_\bullet \to \mathbf{R} \otimes_{\mathbf{H}} A_\bullet$ it gives rise to a commutative diagram in the category of complexes $\mathscr{C}^{-}(\mathbf{A}\text{-mod})$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{J}A_\bullet & \longrightarrow & C_\bullet & \longrightarrow & B_\bullet & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\iota} & & \\
0 & \longrightarrow & \mathbf{J}A_\bullet & \longrightarrow & A_\bullet & \longrightarrow & \mathbf{R} \otimes_{\mathbf{H}} A_\bullet & \longrightarrow & 0,
\end{array}$$

where $C_\bullet$ is the preimage in $A_\bullet$ of $\operatorname{Im}\iota$. The lower row is also an exact sequence of complexes and α is an embedding. Moreover, since ι^R is an isomorphism, $\mathbf{J}A_\bullet = \mathbf{J}C_\bullet$. It implies that $C_\bullet$ consists of projective $\mathbf{A}$-modules and $\mathbf{H} \otimes_{\mathbf{A}} C_\bullet \simeq A_\bullet$, wherefrom $T \simeq \mathbf{F}C_\bullet$.

Let now $(\Phi, \phi) : \mathbf{F}C_\bullet \to \mathbf{F}C'_\bullet$. We suppose again that both $C_\bullet$ and $C'_\bullet$ are minimal, while $\Phi : \mathbf{H} \otimes_{\mathbf{A}} C_\bullet \to \mathbf{H} \otimes_{\mathbf{A}} C'_\bullet$ and $\phi : \mathbf{S} \otimes_{\mathbf{A}} C_\bullet \to \mathbf{S} \otimes_{\mathbf{A}} C'_\bullet$ are morphisms of complexes. Then the diagram (5) is commutative in the category of complexes, so $\Phi(C_\bullet) \subseteq C'_\bullet$ and Φ induces a morphism $\gamma : C_\bullet \to C'_\bullet$. It is evident from the construction that $\mathbf{F}(\gamma) = (\Phi, \phi)$. Moreover, if (Φ, ϕ) is an isomorphism, so are Φ and ϕ (since our complexes are minimal). Therefore, $\Phi(C_\bullet) = C'_\bullet$, i.e. $\operatorname{Im}\gamma = C'_\bullet$. But $\ker\gamma = \ker\Phi \cap C_\bullet = 0$, thus γ is an isomorphism too. $\square$

Evident examples of Backström rings are completions of local rings of ordinary multiple points of algebraic curves. If $\mathbf{A}$ is such a ring, $\mathbf{H}$ is its *normalization* (i.e. integral closure in the full ring of fractions) and $\mathbf{J}$ is the radical of $\mathbf{A}$ (or, the same, of $\mathbf{H}$). If the field $\Bbbk$ is algebraically closed, $\mathbf{A}$ is actually isomorphic to a *bouquet* of power series rings $\Bbbk[[t]]$, i.e. to the subring in $\Bbbk[[t]]^m$, where m is the multiplicity of the singularity, consisting of all sequences $(f_1, f_2, \dots, f_m)$ such that all $f_i(t)$ have the same constant term. Backström rings also include important classes of finite dimensional algebras, such as *gentle, skew-gentle* and others (cf. [13]). Certainly, most of Backström rings are actually wild (hence derived wild). Nevertheless, some of them are derived tame and their derived categories behave very well. An important class of such rings, called *nodal rings*, will be considered in the next subsection.

3.2 Nodal Rings: Strings and Bands

Definition 3.3. A Backström ring $\mathbf{A}$ is called a *nodal ring* if it is *pure noetherian*, i.e. has no minimal ideals, while the hereditary ring $\mathbf{H}$ and the ideal $\mathbf{J}$ from Definition 3.1 satisfy the following conditions:

1. $\mathbf{J} = \operatorname{rad}\mathbf{A} = \operatorname{rad}\mathbf{H}$.
2. $\operatorname{length}_{\mathbf{A}}(\mathbf{H} \otimes_{\mathbf{A}} U) \le 2$ for every simple left $\mathbf{A}$-module U and $\operatorname{length}_{\mathbf{A}}(V \otimes_{\mathbf{A}} \mathbf{H}) \le 2$ for every simple right $\mathbf{A}$-module V.

Note that condition 2 must be imposed both on left and on right modules.

In this situation the hereditary ring $\mathbf{H}$ is also pure noetherian. It is known (cf. e.g. [9]) that such a hereditary ring is Morita equivalent to a direct product of rings $\mathbf{H}(\mathbf{D}, n)$, where $\mathbf{D}$ is a discrete valuation ring (maybe non-commutative) and $\mathbf{H}(\mathbf{D}, n)$ is the subring of $\operatorname{Mat}(n, \mathbf{D})$ consisting of all matrices (a_{ij}) with non-invertible entries a_{ij} for $i < j$. Especially, $\mathbf{H}$ and $\mathbf{A}$ are *semi-prime* (i.e. without nilpotent ideals). For the sake of simplicity we shall only consider the *split case*, when the factor $\mathbf{H}/\mathbf{J}$ is a finite dimensional algebra over a field $\Bbbk$ and $\mathbf{A}/\mathbf{J}$ is its subalgebra.

Remark. In [23] the author showed that if $\mathbf{A}$ is pure noetherian, but not a nodal ring, then the category of $\mathbf{A}$-modules of finite length is wild. All the more so are the categories $\mathbf{A}$-mod and $\mathscr{D}^b(\mathbf{A}\text{-mod})$.

Example 3.4. 1. The first example of a nodal ring is the completion of the local ring of a *simple node* (or a simple double point) of an algebraic curve over a field $\Bbbk$. It is isomorphic to $\mathbf{A} = \Bbbk[[x,y]]/(xy)$ and can be embedded into $\mathbf{H} = \Bbbk[[x_1]] \times \Bbbk[[x_2]]$ as the subring of pairs (f,g) such that $f(0) = g(0)$: x maps to $(x_1, 0)$ and y to $(0, x_2)$. Evidently this embedding satisfies conditions of Definition 3.3.

2. The *dihedral algebra* $\mathbf{A} = \Bbbk\langle\langle\, x, y\,\rangle\rangle/(x^2, y^2)$ is another example of a nodal ring. In this case $\mathbf{H} = \mathbf{H}(\Bbbk[[t]], 2)$ and the embedding $\mathbf{A} \to \mathbf{H}$ is given by the rule

$$x \mapsto \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

3. The "Gelfand problem," arising from the study of Harish-Chandra modules over the Lie group $\mathrm{SL}\,(2,\mathbb{R})$, is that of classification of diagrams with relations

$$\begin{tikzcd} 2 \arrow[r, bend left, "x_+"] & 1 \arrow[l, bend left, "x_-"] \arrow[r, bend left, "y_-"'] & 3 \arrow[l, bend left, "y_+"'] \end{tikzcd} \qquad x_+x_- = y_+y_-.$$

If we consider the case when x_+x_- is nilpotent (the nontrivial part of the problem), such diagrams are just modules over the ring $\mathbf{A}$, which is the subring of $\mathrm{Mat}(3, \Bbbk[[t]])$ consisting of all matrices (a_{ij}) with $a_{12}(0) = a_{13}(0) = a_{23}(0) = a_{32}(0) = 0$. The arrows of the diagram correspond to the following matrices:

$$x_+ \mapsto te_{12}, \quad x_- \mapsto e_{21}, \quad y_+ \mapsto te_{13}, \quad y_- \mapsto e_{31},$$

where e_{ij} are the matrix units. It is also a nodal ring with $\mathbf{H}$ being the subring of $\mathrm{Mat}(3, \Bbbk[[t]])$ consisting of all matrices (a_{ij}) with $a_{12}(0) = a_{13}(0) = 0$ (it is Morita equivalent to $\mathbf{H}(\Bbbk[[t]], 2)$). More general cases, arising in representation theory of Lie groups $\mathrm{SO}\,(1,n)$, were considered in [41] (cf. also [11, Section 7], where the corresponding diagrams are treated as nodal rings).

4. The classification of *quadratic functors*, which play an important role in algebraic topology (cf. [5]), reduces to the study of modules over the ring $\mathbf{A}$, which is the subring of $\mathbb{Z}_2^2 \times \mathrm{Mat}(2, \mathbb{Z}_2)$ consisting of all triples

$$\left(a, b, \begin{pmatrix} c_1 & 2c_2 \\ c_3 & c_4 \end{pmatrix}\right) \quad \text{with } a \equiv c_1 \mod 2 \quad \text{and} \quad b \equiv c_4 \mod 2,$$

where $\mathbb{Z}_2$ is the ring of 2-adic integers [24]. It is again a (split) nodal ring: one can take for $\mathbf{H}$ the ring of all triples as above, but without congruence conditions; then $\mathbf{H} = \mathbb{Z}_2^2 \times \mathbf{H}(\mathbb{Z}_2, 2)$.

Certainly, we shall apply Theorem 3.2 to study the derived categories of modules over nodal rings. Moreover, in this case the resulting problem belongs to a well-known type, considered in [7, 8, 16] (for its generalization to the non-split case, see [19]). We denote by $U_1, U_2, \dots, U_s$ indecomposable non-isomorphic projective (left) modules over $\mathbf{A}$, by $V_1, V_2, \dots, V_r$ those over $\mathbf{H}$ and consider the decompositions of $\mathbf{H} \otimes_{\mathbf{A}} U_i$ into direct sums of V_j. Condition 2 from Definition 3.3 implies that there are three possibilities:

1. $\mathbf{H} \otimes_{\mathbf{A}} U_i \simeq V_j$ for some j and V_j does not occur as a direct summand in $\mathbf{H} \otimes_{\mathbf{A}} U_k$ for $k \neq i$;
2. $\mathbf{H} \otimes_{\mathbf{A}} U_i \simeq V_j \oplus V_{j'}$ $(j \neq j')$ and neither V_j nor $V_{j'}$ occur in $\mathbf{H} \otimes_{\mathbf{A}} U_k$ for $k \neq i$;
3. There are exactly two indices $i \neq i'$ such that $\mathbf{H} \otimes_{\mathbf{A}} U_i \simeq \mathbf{H} \otimes_{\mathbf{A}} U_{i'} \simeq V_j$ and V_j does not occur in $\mathbf{H} \otimes_{\mathbf{A}} U_k$ for $k \notin \{ i, i' \}$.

We denote by H_j the indecomposable projective $\mathbf{H}$-module such that $H_j / \mathbf{J} H_j \simeq V_j$. Since $\mathbf{H}$ is a semi-perfect *hereditary order*, any indecomposable complex from $\mathscr{D}^-(\mathbf{H}\text{-mod})$ is isomorphic either to $0 \to H_k \xrightarrow{\phi} H_j \to 0$ or to $0 \to H_j \to 0$ (it follows, for instance, from [22]). Moreover, the former complex is completely defined by either j or k and the length $l = \operatorname{length}_{\mathbf{H}}(\operatorname{Cok} \phi)$. We shall denote it both by $C(j, -l, n)$ and by $C(k, l, n+1)$, while the latter complex will be denoted by $C(j, \infty, n)$, where n is the number of the place of H_j in the complex (so the number of the place of H_k is $n+1$). We denote by $\widetilde{\mathbb{Z}}$ the set $(\mathbb{Z} \setminus \{ 0 \}) \cup \{ \infty \}$ and consider the ordering $\leq$ on $\widetilde{\mathbb{Z}}$, which coincides with the usual ordering separately on positive integers and on negative integers, but $l < \infty < -l$ for any positive l. Note that for each j the submodules of H_j form a chain with respect to inclusion. It immediately implies the following result.

Lemma 3.5. *There is a homomorphism $C(j, l, n) \to C(j, l', n)$, which is an isomorphism on the n-th components, if and only if $l \leq l'$ in $\widetilde{\mathbb{Z}}$. Otherwise the n-th component of any homomorphism $C(j, l, n) \to C(j, l', n)$ is zero modulo* $\mathbf{J}$.

We transfer the ordering from $\widetilde{\mathbb{Z}}$ to the set $\mathfrak{E}_{j,n} = \left\{ C(j, l, n) \,\middle|\, l \in \widetilde{\mathbb{Z}} \right\}$, so the latter becomes a chain with respect to this ordering. We also consider one element sets $\mathfrak{F}_{j,n} = \{ (j, n) \}$ and denote

$$\mathfrak{F}^*_{j,n} = \{ (i, j, n) \,|\, V_j \text{ is a direct summand of } \mathbf{H} \otimes_{\mathbf{A}} U_i \} \ .$$

If j is fixed, there can be at most two such values of i. It happens when case 3 from page 103 occurs: $\mathbf{H} \otimes_{\mathbf{A}} U_i \simeq \mathbf{H} \otimes_{\mathbf{A}} U_{i'} \simeq V_j$. Then we write $(j, n) \sim (j, n)$. We also write $C(j, -l, n) \sim C(k, l, n+1)$ if these symbols denote the same complex $0 \to H_k \xrightarrow{\phi} H_j \to 0$, and $(j, n) \sim (j', n)$ $(j \neq j')$ if case 2 from page 103 occurs: $\mathbf{H} \otimes_{\mathbf{A}} U_i \simeq V_j \oplus V_{j'}$ (if j is fixed, there can be only one j' with this property). Thus a triple $(A_\bullet, B_\bullet, \iota)$ from the category $\mathcal{T}(\mathbf{A})$ is given by homomorphisms $\phi^{ijn}_{jln} : d_{i,j,n}U_i \to r_{j,l,n}V_j$, where $C(j,l,n) \in \mathfrak{E}_{j,n}$ and $(i,j,n) \in \mathfrak{F}^*_{j,n}$. Here the left U_i comes from B_n and the right V_j comes from the direct summands $r_{j,l,n}C(j,l,n)$ of $A_\bullet$ after tensoring by $\mathbf{R}$. Note that if $C(j,-l,n) \sim C(k,l,n+1)$, we have $r_{j,-l,n} = r_{k,l,n+1}$, and if $(j,n) \simeq (j',n)$, we have $d_{i,j,n} = d_{i,j',n}$ for the unique possible value of i. We present ϕ^{ijn}_{jln} by its matrix $M^{ijn}_{jln} \in \mathrm{Mat}(r_{j,l,n} \times d_{i,j,n}, \Bbbk)$. Then Lemma 3.5 implies the following

Proposition 3.6. *Two sets of matrices $\{ M^{ijn}_{jln} \}$ and $\{ N^{ijn}_{jln} \}$ describe isomorphic triples if and only if one of them can be transformed to the other one by a sequence of the following "elementary transformations":*

1. *For any given values of i,n, simultaneously $M^{ijn}_{jln} \mapsto M^{ijn}_{jln}S$ for all j,l such that $(i,j,n) \in \mathfrak{F}^*_{j,n}$, where S is an invertible matrix of appropriate size.*
2. *For any given values of j,l,n, simultaneously $M^{ijn}_{jln} \mapsto S'M^{ijn}_{jln}$ for all $(i,j,n) \in \mathfrak{F}^*_{j,n}$ and $M^{i,k,n-\operatorname{sgn} l}_{k,-l,n-\operatorname{sgn} l} \mapsto S'M^{i,k,n-\operatorname{sgn} l}_{k,-l,n-\operatorname{sgn} l}$ for all $(i,k,n-\operatorname{sgn} l) \in \mathfrak{F}^*_{k,n-\operatorname{sgn} l}$, where S' is an invertible matrix of appropriate size and $C(j,l,n) \sim C(k,-l,n-\operatorname{sgn} l)$. If $l = \infty$, it just means $M^{ijn}_{j\infty n} \mapsto S'M^{ijn}_{j\infty n}$.*
3. *For any given values of j, $l' < l$, n, simultaneously $M^{ijn}_{jln} \mapsto M^{ijn}_{jln} + RM^{ijn}_{jl'n}$ for all $(i,j,n) \in \mathfrak{F}^*_{j,n}$, where R is an arbitrary matrix of appropriate size.* (Note that, unlike the preceding transformation, this one does not touch the matrices $M^{i,k,n-\operatorname{sgn} l}_{k,-l,n-\operatorname{sgn} l}$ such that $C(j,l,n) \sim C(k,-l,n-\operatorname{sgn} l)$.)

This sequence can be infinite, but must contain finitely many transformations for every fixed values of j and n.

Therefore, we obtain *representations of the bunch of chains* $\{ \mathfrak{E}_{j,n}, \mathfrak{F}_{j,n} \}$ considered in [7, 8],[2] so we can deduce from these papers a description of indecomposables in $\mathscr{D}^-(\mathbf{A}\text{-mod})$ (for infinite words, which correspond to infinite strings, see [12]). We arrange it in terms of *strings and bands* often used in representation theory.

Definition 3.7. 1. We define the *alphabet* $\mathfrak{X}$ as the set $\bigcup_{j,n}(\mathfrak{E}_{j,n} \cup \mathfrak{F}_{j,n})$. We define symmetric relations $\sim$ and $-$ on $\mathfrak{X}$ by the following exhaustive rules:

(a) $C(j,l,n) - (j,n)$ for all $l \in \mathbb{Z}$;
(b) $C(j,-l,n) \sim C(k,l,n+1)$ if these both symbols correspond to the same complex $0 \to H_k \xrightarrow{\phi} H_j \to 0$;

[2] Note that in [7, 8] they are called "bunches of semichained sets," but we prefer to say "bunches of chains," as in [29, 11].

(c) $(j,n) \sim (j',n)$ $(j' \neq j)$ if $V_j \oplus V_{j'} \simeq \mathbf{H} \otimes_{\mathbf{A}} U_i$ for some i;
(d) $(j,n) \sim (j,n)$ if $V_j \simeq \mathbf{H} \otimes_{\mathbf{A}} U_i \simeq \mathbf{H} \otimes_{\mathbf{A}} U_{i'}$ for some $i' \neq i$.

2. We define an $\mathfrak{X}$-*word* as a sequence $w = x_1 r_1 x_2 r_2 x_3 \dots r_{m-1} x_m$, where $x_k \in \mathfrak{X}$, $r_k \in \{-, \sim\}$ such that

(a) $x_k r_k x_{k+1}$ in $\mathfrak{X}$ for $1 \leq k < m$;
(b) $r_k \neq r_{k+1}$ for $1 \leq k < m-1$.

We call x_1 and x_m the *ends of the word* w.

3. We call an $\mathfrak{X}$-word w *full* if

(a) $r_1 = r_{m-1} = -$
(b) $x_1 \not\sim y$ for each $y \neq x_1$;
(c) $x_m \not\sim z$ for each $z \neq x_m$.

Condition (a) reflects the fact that ι^R must be an isomorphism, while conditions (b,c) come from generalities on bunches of chains [8, 11].

4. A word w is called *symmetric* if $w = w^*$, where $w^* = x_m r_{m-1} x_{m-1} \dots r_1 x_1$ (the *inverse word*), and *quasi-symmetric* if there is a shorter word v such that $w = v \sim v^* \sim \dots \sim v^* \sim v$.

5. We call the end x_1 (x_m) of a word w *special* if $x_1 \sim x_1$ and $r_1 = -$ (respectively, $x_m \sim x_m$ and $r_{m-1} = -$). We call a word w

(a) *usual* if it has no special ends;
(b) *special* if it has exactly one special end;
(c) *bispecial* if it has two special ends.

Note that a special word is never symmetric, a quasi-symmetric word is always bispecial, and a bispecial word is always full.

6. We define a *cycle* as a word w such that $r_1 = r_{m-1} = \sim$ and $x_m - x_1$. Such a cycle is called *non-periodic* if it cannot be presented in the form $v - v - \dots - v$ for a shorter cycle v. For a cycle w we set $r_m = -$, $x_{qm+k} = x_k$ and $r_{qm+k} = r_k$ for any $q, k \in \mathbb{Z}$.

7. A k-th *shift* of a cycle w, where k is an even integer, is the cycle $w^{[k]} = x_{k+1} r_{k+1} x_{k+2} \dots r_{k-1} x_k$. A cycle w is called *symmetric* if $w^{[k]} = w^*$ for some k.

8. We also consider *infinite words* of the sorts $w = x_1 r_1 x_2 r_2 \dots$ (with one end) and $w = \dots x_0 r_0 x_1 r_1 x_2 r_2 \dots$ (with no ends) with the following restrictions:

(a) every pair (j,n) occurs in this sequence only finitely many times;

(b) there is an n_0 such that no pair (j,n) with $n < n_0$ occurs.

We extend to such infinite words all above notions in the obvious manner.

Definition 3.8 (String and band data). 1. *String data* are defined as follows:

(a) a *usual string datum* is a full usual non-symmetric $\mathfrak{X}$-word w;

(b) a *special string datum* is a pair (w,δ), where w is a full special word and $\delta \in \{0,1\}$;

(c) a *bispecial string datum* is a quadruple (w,m,δ_1,δ_2), where w is a bispecial word that is neither symmetric nor quasi-symmetric, $m \in \mathbb{N}$ and $\delta_1,\delta_2 \in \{0,1\}$.

2. A *band datum* is a triple (w,m,λ), where w is a non-periodic cycle, $m \in \mathbb{N}$ and $\lambda \in \Bbbk^*$; if w is symmetric, we also suppose that $\lambda \neq 1$.

The results of [7, 8] (and [11] for infinite words) imply

Theorem 3.9. *Every string or band datum* $\mathbf{d}$ *defines an indecomposable object* $C_\bullet(\mathbf{d})$ *from* $\mathscr{D}^-(\mathbf{A}\text{-mod})$*, so that*

1. *Every indecomposable object from* $\mathscr{D}^-(\mathbf{A}\text{-mod})$ *is isomorphic to* $C_\bullet(\mathbf{d})$ *for some* $\mathbf{d}$.

2. *The only isomorphisms between these complexes are the following:*

   (a) $C(w) \simeq C(w^*)$ and $C(w,\delta) \simeq C(w^*,\delta)$;

   (b) $C(w,m,\delta_1,\delta_2) \simeq C(w^*,m,\delta_2,\delta_1)$;

   (c) $C(w,m,\lambda) \simeq C(w^{[k]},m,\lambda) \simeq C(w^{*[k]},m,1/\lambda)$ if $k \equiv 0 \mod 4$;

   (d) $C(w^*,m,\lambda) \simeq C(w^{[k]},m,1/\lambda) \simeq C(w^{*[k]},m,\lambda)$ if $k \equiv 2 \mod 4$.

3. *Every object from* $\mathscr{D}^-(\mathbf{A}\text{-mod})$ *uniquely decomposes into a direct sum of indecomposable objects.*

The construction of complexes $C_\bullet(\mathbf{d})$ is rather complicated, especially in the case, when there are pairs (j,n) with $(j,n) \sim (j,n)$ (e.g. special ends are involved). So we only show several examples arising from simple node, dihedral algebra and Gelfand problem.

3.3 Examples

3.3.1 Simple Node

In this case there is only one indecomposable projective $\mathbf{A}$-module ($\mathbf{A}$ itself) and two indecomposable projective $\mathbf{H}$-modules H_1, H_2 corresponding to the first and the second direct factors of the ring $\mathbf{H}$. We have $\mathbf{H} \otimes_{\mathbf{A}} \mathbf{A} \simeq \mathbf{H} \simeq H_1 \oplus H_2$. So the $\sim$-relation is given by:

1. $(1, n) \sim (2, n)$;
2. $C(j, l, n) \sim C(j, -l, n - \operatorname{sgn} l)$ for any $l \in \mathbb{Z} \setminus \{ 0 \}$.

Therefore, there are no special ends at all. Moreover, any end of a full string must be of the form $C(j, \infty, n)$. Note that the homomorphism in the complex corresponding to $C(j, -l, n)$ and $C(j, l, n+1)$ ($l \in \mathbb{N}$) is just multiplication by x_j^l. Consider several examples of strings and bands.

Example 3.10. 1. Let w be the cycle

$$\begin{aligned} C(2,1,1) &\sim C(2,-1,0) - (2,0) \sim (1,0) - C(1,-2,0) \sim C(1,2,1) - (1,1) \\ &\sim (2,1) - C(2,4,1) \sim C(2,-4,0) - (2,0) \sim (1,0) - C(1,-1,0) \\ &\sim C(1,1,1) - (1,1) \sim (2,1) - C(2,-3,1) \sim C(2,3,2) - (2,2) \\ &\sim (1,2) - C(1,2,2) \sim C(1,-2,1) - (1,1) \sim (2,1) \end{aligned}$$

Then the band complex $C_\bullet(w, 1, \lambda)$ is obtained from the complex of $\mathbf{H}$-modules

$$\begin{tikzcd}
& H_2 \arrow[r, "x_2"] \arrow[ddddd, dash, dashed, bend right=40, "\lambda"'] & H_2 \arrow[d, dash, dashed] \\
& H_1 \arrow[r, "x_1^2"] \arrow[d, dash, dashed] & H_1 \\
& H_2 \arrow[r, "x_2^4"] & H_2 \arrow[d, dash, dashed] \\
& H_1 \arrow[r, "x_1"] \arrow[d, dash, dashed] & H_1 \\
H_2 \arrow[r, "x_2^3"] \arrow[d, dash, dashed] & H_2 & \\
H_1 \arrow[r, "x_1^2"] & H_1 &
\end{tikzcd}$$

by gluing along the dashed lines (they present the $\sim$ relations $(1, n) \sim (2, n)$). All gluings are trivial, except the last one marked with 'λ'; the latter must be twisted by λ. It gives the $\mathbf{A}$-complex

$$\begin{tikzcd}
& \mathbf{A} \arrow[r, "y"] & \mathbf{A} \\
\mathbf{A} \arrow[ur, "\lambda x^2"] \arrow[dr, "y^3"'] & \mathbf{A} \arrow[ur, "x^2"] \arrow[r, "y^4"] & \mathbf{A} \\
& \mathbf{A} \arrow[ur, "x"'] &
\end{tikzcd} \tag{6}$$

Here each column presents direct summands of a non-zero component C_n (in our case $n = 2, 1, 0$) and the arrows show the non-zero components of the differential. According to the embedding $\mathbf{A} \to \mathbf{H}$, we have to replace x_1 by x and x_2 by y. Gathering all data, we can rewrite this complex as

$$\mathbf{A} \xrightarrow{\begin{pmatrix} \lambda x^2 \\ 0 \\ y^3 \end{pmatrix}} \mathbf{A} \oplus \mathbf{A} \oplus \mathbf{A} \xrightarrow{\begin{pmatrix} y & 0 \\ x^2 & y^4 \\ 0 & x \end{pmatrix}} \mathbf{A} \oplus \mathbf{A}\,,$$

though the form (6) seems more expressive, so we use it further. If $m > 1$, one only has to replace $\mathbf{A}$ by $m\mathbf{A}$, each element $a \in \mathbf{A}$ by aE, where E is the identity matrix, and λa by $aJ_m(\lambda)$, where $J_m(\lambda)$ is the Jordan $m \times m$ cell with eigenvalue λ. So we obtain the complex

$$\begin{tikzcd}
 & m\mathbf{A} \arrow[r, "yE"] & m\mathbf{A} \\
m\mathbf{A} \arrow[ur, "x^2J_m(\lambda)"] \arrow[dr, "y^3E"'] & m\mathbf{A} \arrow[ur, "x^2E"] \arrow[r, "y^4E"] & m\mathbf{A} \\
 & m\mathbf{A} \arrow[ur, "xE"'] &
\end{tikzcd}$$

or, the same,

$$m\mathbf{A} \xrightarrow{\begin{pmatrix} x^2J_m(\lambda) \\ 0 \\ y^3E \end{pmatrix}} m\mathbf{A} \oplus m\mathbf{A} \oplus m\mathbf{A} \xrightarrow{\begin{pmatrix} yE & 0 \\ x^2E & y^4E \\ 0 & xE \end{pmatrix}} m\mathbf{A} \oplus m\mathbf{A}\,.$$

2. Let w be the word

$$\begin{aligned}
&C(1,\infty,1) - (1,1) \sim (2,1) - C(2,2,1) \sim C(2,-2,0) - (2,0) \\
&\quad \sim (1,0) - C(1,-3,0) \sim C(1,3,1) - (1,1) \sim (2,1) - C(2,-1,1) \\
&\quad \sim C(2,1,2) - (2,2) \sim (1,2) - C(1,1,2) \sim C(1,-1,1) - (1,1) \\
&\quad \sim (2,1) - C(2,2,1) \sim C(2,-2,0) - (2,0) \sim (1,0) - C(1,\infty,0)\,.
\end{aligned}$$

Then the string complex $C_\bullet(w)$ is

$$\begin{tikzcd}
 & \mathbf{A} \arrow[r, "y^2"] & \mathbf{A} \\
\mathbf{A} \arrow[r, "y"] \arrow[dr, "x"'] & \mathbf{A} \arrow[ur, "x^3"'] & \\
 & \mathbf{A} \arrow[r, "y^2"] & \mathbf{A}
\end{tikzcd}$$

Note that for string complexes (which are always usual in this case) there are no multiplicities m and all gluings are trivial.

3. Set $a = x + y$. Then the factor $\mathbf{A}/a\mathbf{A}$ is represented by the complex $\mathbf{A} \xrightarrow{a} \mathbf{A}$, which is the band complex $C_\bullet(w, 1, 1)$, where

$$\begin{aligned} w = C(1,1,1) \ &\sim \ C(1,-1,0) - (1,0) \sim (2,0) - C(2,-1,0) \\ &\sim \ C(2,1,1) - (2,1) \sim (1,1). \end{aligned}$$

Consider the morphism of this complex to $\mathbf{A}[1]$ given on the 1-component by multiplication $\mathbf{A} \xrightarrow{x} \mathbf{A}$. It is non-zero in $\mathscr{D}^-(\mathbf{A}\text{-mod})$ (presenting a non-zero element from $\operatorname{Ext}^1(\mathbf{A}/a\mathbf{A}, \mathbf{A})$), but the corresponding morphism of triples is $(\Phi, 0)$, where Φ arises from the morphism of the complex $\mathbf{H} \xrightarrow{a} \mathbf{H}$ to $\mathbf{H}[1]$ given by multiplication with x_1. But Φ is homotopic to 0: $x_1 = e_1 a$, where $e_1 = (1, 0) \in \mathbf{H}$, thus $(\Phi, 0) = 0$ in the category of triples. So the functor $\mathbf{F}$ from Theorem 3.2 is not faithful in this case.

4. The string complex $C_\bullet(\mathbf{l}, 0)$, where w is the word

$$\begin{aligned} &C(1,\infty,0) - (1,0) \sim (2,0) - C(2,-1,0) \sim C(2,1,1) - (2,1) \\ &\qquad \sim \ (1,1) - C(1,-2,1) \sim C(1,1,2) - (1,2) \sim (2,2) - C(2,-1,2) \\ &\qquad \sim \ C(2,1,3) - (2,3) \sim (1,3) - C(1,-2,3) \sim C(1,2,4) - \dots, \end{aligned}$$

is

$$\dots \ \mathbf{A} \xrightarrow{x^2} \mathbf{A} \xrightarrow{y} \mathbf{A} \xrightarrow{x^2} \mathbf{A} \xrightarrow{y} \mathbf{A} \longrightarrow 0.$$

Its homologies are not left bounded, so it does not belong to $D^b(\mathbf{A}\text{-mod})$.

3.3.2 Dihedral Algebra

This case is very similar to the preceding one. Again there is only one indecomposable projective $\mathbf{A}$-module ($\mathbf{A}$ itself) and two indecomposable projective $\mathbf{H}$-modules H_1, H_2, corresponding to the first and the second columns of matrices from the ring $\mathbf{H}$, and we have $\mathbf{H} \otimes_{\mathbf{A}} \mathbf{A} \simeq \mathbf{H} \simeq H_1 \oplus H_2$. The main difference is that now the unique maximal submodule of H_j is isomorphic to H_k, where $k \neq j$. So the $\sim$-relation is given by:

1. $(1, n) \sim (2, n)$;
2. $C(j, l, n) \sim C(j, -l, n - \operatorname{sgn} l)$ if $l \in \mathbb{Z} \setminus \{0\}$ is even, and $C(j, l, n) \sim C(j', -l, n - \operatorname{sgn} l)$, where $j' \neq j$, if $l \in \mathbb{Z} \setminus \{0\}$ is odd.

Again there are no special ends. The embeddings $H_k \to H_j$ are given by right multiplications with the following elements from $\mathbf{H}$:

$$\begin{array}{llll} H_1 \to H_1 & - \ \text{by} & t^r e_{11} & (\text{colength } 2r), \\ H_1 \to H_2 & - \ \text{by} & t^r e_{12} & (\text{colength } 2r - 1), \\ H_2 \to H_1 & - \ \text{by} & t^r e_{21} & (\text{colength } 2r + 1), \\ H_2 \to H_2 & - \ \text{by} & t^r e_{22} & (\text{colength } 2r). \end{array}$$

When gluing **H**-complexes into **A**-complexes we have to replace them respectively

$$\begin{array}{llllll} t^r e_{11} & - \text{ by } & (xy)^r, & \qquad t^r e_{22} & - \text{ by } & (yx)^r, \\ t^r e_{12} & - \text{ by } & (xy)^{r-1}x, & \qquad t^r e_{21} & - \text{ by } & (yx)^r y. \end{array}$$

The gluings are quite analogous to those for simple node, so we only present the results, without further comments.

Example 3.11. 1. Consider the band datum $(w, 1, \lambda)$, where

$$\begin{aligned} w = C(1,-2,0) &\sim C(1,2,1) - (1,1) \sim (2,1) - C(2,-5,1) \\ &\sim \; C(1,5,2) - (1,2) \sim (2,2) - C(2,4,2) \sim C(2,-4,1) - (2,1) \\ &\sim \; (1,1) - C(1,3,1) \sim C(2,-3,0) - (2,0) \sim (1,0). \end{aligned}$$

The corresponding complex $C_\bullet(w,m,\lambda)$ is

$$\begin{tikzcd} & m\mathbf{A} \arrow[r, "xyE"] & m\mathbf{A} \\ m\mathbf{A} \arrow[ur, "(xy)^2xE"] \arrow[r, "(yx)^2E"'] & m\mathbf{A} \arrow[ur, "xyxJ_m(\lambda)"'] & \end{tikzcd}$$

2. Let w be the word

$$\begin{aligned} C(2,\infty,0) - (2,0) &\sim (1,0) - C(1,-1,0) \sim C(2,1,1) - (2,1) \\ &\sim \; (1,1) - C(1,3,1) \sim C(2,-3,0) - (2,0) \sim (1,0) - C(1,-3,0) \\ &\sim \; C(2,3,1) - (2,1) \sim (1,1) - C(1,\infty,1). \end{aligned}$$

Then the string complex $C_\bullet(w)$ is

$$\begin{tikzcd} \mathbf{A} \arrow[r, "e_{21}"] \arrow[dr, "t^2e_{12}"] & \mathbf{A} \\ \mathbf{A} \arrow[r, "te_{21}"'] & \mathbf{A} \end{tikzcd}$$

3. The factor $\mathbf{A}/\mathbf{J}$ is described by the infinite string complex $C_\bullet(w)$:

$$\begin{tikzcd} \cdots \arrow[r, "e_{21}"] & \mathbf{A} \arrow[r, "te_{12}"] & \mathbf{A} \arrow[r, "e_{21}"] & \mathbf{A}. \\ \cdots \arrow[r, "te_{12}"] & \mathbf{A} \arrow[r, "e_{21}"] & \mathbf{A} \arrow[ur, "te_{12}"'] & \end{tikzcd}$$

The corresponding word w is

$$\begin{aligned} \cdots - C(2,1,2) &\sim C(1,-1,1) - (1,1) \sim (2,1) - C(2,1,1) \\ &\sim \; C(1,-1,0) - (1,0) \sim (2,0) - C(2,-1,0) \sim C(1,1,1) - (1,1) \\ &\sim \; (2,1) - C(2,-1,1) \sim C(1,1,2) - \ldots \end{aligned}$$

3.3.3 Gelfand Problem

In this case there are 2 indecomposable projective **H**-modules H_1 (the first column) and H_2 (both the second and the third columns). There are 3 indecomposable **A**-projectives A_i ($i = 1, 2, 3$); A_i correspond to the i-th column of **A**. We have $\mathbf{H} \otimes_{\mathbf{A}} A_1 \simeq H_1$ and $\mathbf{H} \otimes_{\mathbf{A}} A_2 \simeq \mathbf{H} \otimes_{\mathbf{A}} A_3 \simeq H_2$. So the relation $\sim$ is given by:

1. $(2, n) \sim (2, n)$;
2. $C(j, l, n) \sim C(j, -l, n - \operatorname{sgn} l)$ if l is even;
3. $C(j, l, n) \sim C(j', -l, n - \operatorname{sgn} l)$ $(j' \neq j)$ if l is odd.

Hence a special end is always $(2, n)$.

Example 3.12. 1. Consider the special word w:

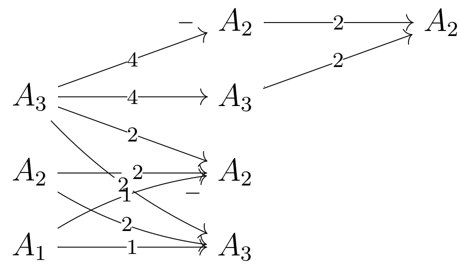
$$\begin{aligned} (2,0) - C(2,-2,0) \sim C(2,2,1) - (2,1) \sim (2,1) - C(2,-4,1) \\ \sim \quad C(2,4,2) - (2,2) \sim (2,2) - C(2,2,2) \sim C(2,-2,1) - (2,1) \\ \sim \quad (2,1) - C(2,-1,1) \sim C(1,1,2) - (1,2)\,. \end{aligned}$$

The complex $C_\bullet(w, 0)$ is obtained by gluing from the complex of **H**-modules

$$\begin{tikzcd}
 & H_2 \arrow[r, "2"] \arrow[d, dashed] & H_2 \\
H_2 \arrow[r, "4"] & H_2 & \\
H_2 \arrow[r, "2"] \arrow[u, dashed] & H_2 \arrow[d, dashed] & \\
H_1 \arrow[r, "1"] & H_2 &
\end{tikzcd}$$

Here the numbers inside arrows show the colengths of the corresponding images. We mark dashed lines defining gluings with arrows going from the bigger complex (with respect to the ordering in $\mathfrak{E}_{j,n}$) to the smaller one. When we construct the corresponding complex of **A**-modules, we replace each H_2 by A_2 and A_3 starting with A_2 (since $\delta = 0$; if $\delta = 1$ we start from A_3). Each next choice is arbitrary with the only requirement that every dashed line must touch both A_2 and A_3. (Different choices lead to isomorphic complexes: one can see it from the pictures below.) All horizontal mappings must be duplicated by slanting ones, carried along the dashed arrow from the starting point or opposite the dashed arrow with the opposite sign from the ending point (the latter procedure will be marked by '$-$' near the duplicated arrow).

So we get the **A**-complex

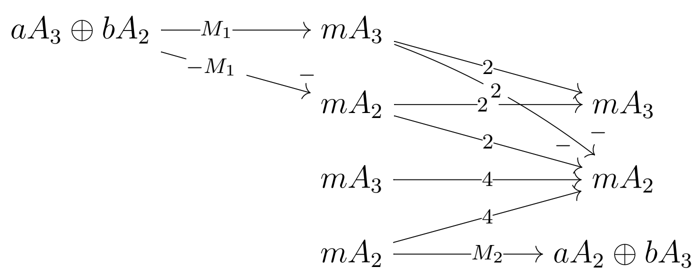


All mappings are uniquely defined by the colengths in the **H**-complex, so we just mark them with 'l.'

2. Let w be the bispecial word

$$\begin{aligned}(2,2)-C(2,2,2)\sim C(2,-2,1)-(2,1)\sim (2,1)-C(2,2,1)\\ \sim\quad C(2,-2,0)-(2,0)\sim(2,0)-C(2,-4,0)\sim C(2,4,1)-(2,1)\\ \sim\quad (2,1)-C(2,6,1)\sim C(2,-6,0)-(2,0)\end{aligned}$$

The complex $C_\bullet(w,m,1,0)$ is the following one:



where $a=[(m+1)/2]$, $b=[m/2]$, so $a+b=m$. (The change of δ_1,δ_2 transpose A_2 and A_3 at the ends.) All arrows are just $\alpha_l E$, where α_l is defined by the colength l, except of the "end" matrices M_i. To calculate the latter, write $\alpha_l E$ for one of them (say, M_1) and $\alpha_l J$ for another one (say, M_2), where J is the Jordan $m\times m$ cell with eigenvalue 1, then put the odd rows or columns into the first part of M_i and the even ones to its second part. In our example we get

$$M_1=\alpha_2\left(\begin{array}{ccc|cc}1&0&0&0&0\\0&0&0&1&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&1&0&0\end{array}\right),\qquad M_2=\alpha_6\left(\begin{array}{ccccc}1&1&0&0&0\\0&0&1&1&0\\0&0&0&0&1\\\hline 0&1&1&0&0\\0&0&0&1&1\end{array}\right).$$

(We use columns for M_1 and rows for M_2 since the left end is the source and the right end is the sink of the corresponding mapping.)

3. The band complex $C_\bullet(w,1,\lambda)$, where w is the cycle

$$\begin{aligned}(2,1) &\sim (2,1)-C(2,-2,1)\sim C(2,2,2)-(2,2)\sim(2,2)-C(2,4,2)\\ &\sim C(2,-4,1)-(2,1)\sim(2,1)-C(2,6,1)\sim C(2,-6,0)-(2,0)\\ &\sim (2,0)-C(2,-4,0)\sim C(2,4,1)\end{aligned}$$

is

$$\begin{tikzcd}
mA_2 \arrow[r, "2" description] \arrow[dddr, "2" description, "-"' near end] & mA_2 \arrow[ddr, bend left, "4^\lambda" description, "-" near end] \arrow[dddr, bend left=15, "4^\lambda" description] & \\
mA_3 \arrow[ur, "2" description] \arrow[r, "4" description] \arrow[dr, "4" description, "-"' near end] \arrow[ddr, bend right, "2" description, "-"' near end] & mA_2 \arrow[dr, "6" description] & \\
& mA_3 \arrow[r, "6" description] & mA_2 \\
& mA_3 \arrow[ur, "4^\lambda" description, "-"' near end] \arrow[r, "4^\lambda" description] & mA_3
\end{tikzcd}$$

Superscript 'λ' denotes that the corresponding mapping must be twisted by $J_m(\lambda)$.

4. The projective resolution of the simple $\mathbf{A}$-module U_1 is

$$\begin{tikzcd}
& A_2 \arrow[r, "1" description] & A_1 \\
A_1 \arrow[ur, "1" description, "-" near end] \arrow[r, "1" description] & A_3 \arrow[ur, "1" description] &
\end{tikzcd}$$

It coincides with the usual string complex $C_\bullet(w)$, where w is

$$\begin{aligned}(1,0)-C(1,-1,0) &\sim C(2,1,1)-(2,1)\sim(2,1)-C(2,-1,1)\\ &\sim C(1,1,2)-(1,2).\end{aligned}$$

The projective resolution of U_2 (U_3) is $A_1\to A_2$ (respectively $A_1\to A_3$), which is the special string complex $C_\bullet(w,0)$ (respectively $C_\bullet(w,1)$), where

$$w = \ (2,0)-C(2,-1,0)\sim C(1,1,1)-(1,1).$$

Note that $\operatorname{gl.dim}\mathbf{A}=2$. It is due to the fact that the case 1 from page 103 occur: $\mathbf{H}\otimes_{\mathbf{A}}A_1\simeq H_1$. One can prove the following consequence of the above calculations.

Corollary 3.13. *Let* $\mathbf{A}$ *be a nodal ring. Suppose that there is no simple* $\mathbf{A}$-*module* U *such that* $\mathbf{H}\otimes_{\mathbf{A}}U$ *is a simple* $\mathbf{H}$-*module. Then* $\operatorname{gl.dim}\mathbf{A}=\infty$*; moreover, the* finitistic dimension *(in the sense of* [3]*) of* $\mathbf{A}$ *equals* 1, *i.e. for every* $\mathbf{A}$-*module* M *either* $\operatorname{proj.dim}M\le 1$ *or* $\operatorname{proj.dim}M=\infty$.

4 Projective Curves

In this section we consider "global" analogues of the results of the preceding one, namely, the derived categories of the categories $\operatorname{Coh} X$ of coherent sheaves over some projective curves X. Again we first consider a general framework ("projective configurations," which are an analogue of Backström rings), when the calculations in $\operatorname{Coh} X$ can be reduced to some matrix problems. Then we apply this technique to those classes of projective configurations, where the resulting matrix problem is tame. Throughout this section we suppose that the field $\Bbbk$ is algebraically closed. Analogous results can also be deduced for non-closed fields using the technique of [19], though the picture becomes more complicated.

4.1 Projective Configurations

Definition 4.1. Let X be a projective curve over $\Bbbk$, which we suppose reduced but possibly reducible. We denote by $\pi : \tilde{X} \to X$ its *normalization*; then $\tilde{X}$ is a disjoint union of smooth curves. We call X a *projective configuration* if all components of $\tilde{X}$ are rational curves (i.e. of genus 0) and all singular points p of X are *ordinary*, i.e. the dimension of the tangent cone at p or, the same, the number of linear independent tangent directions at this point equals its multiplicity. Algebraically it means that, if $\pi^{-1}(p) = \{ y_1, y_2, \dots, y_m \}$, the image of $\mathcal{O}_{X,p}$ in $\prod_{i=1}^m \mathcal{O}_{\tilde{X},y_i}$ contains $\prod_{i=1}^m \mathfrak{m}_i$, where $\mathfrak{m}_i$ is the maximal ideal of $\mathcal{O}_{\tilde{X},y_i}$.

We denote by S the set of singular points of X, by $\tilde{S} = \pi^{-1}(S)$ its preimage in $\tilde{X}$ and consider S ($\tilde{S}$) as a closed subvariety of X (resp. $\tilde{X}$). Let $\varepsilon : S \to X$ and $\tilde{\varepsilon} : \tilde{S} \to \tilde{X}$ be their embeddings, and $\overline{\pi} : \tilde{S} \to S$ be the restriction of π onto $\tilde{S}$. We also put $\mathcal{O} = \mathcal{O}_X$, $\widetilde{\mathcal{O}} = \mathcal{O}_{\tilde{X}}$, $\mathcal{S} = \mathcal{O}_S$, $\mathcal{R} = \mathcal{O}_{\tilde{S}}$, and denote by $\mathcal{J}$ the conductor of $\widetilde{\mathcal{O}}$ in $\mathcal{O}$, i.e. the maximal sheaf of $\pi_*\widetilde{\mathcal{O}}$-ideals contained in $\mathcal{O}$. Note that $\mathcal{S}_p \simeq \mathcal{O}_p/\mathcal{J}_p$ and $\mathcal{R}_y \simeq \widetilde{\mathcal{O}}_y/(\pi_*\mathcal{J})_y$. Since S and $\tilde{S}$ are 0-dimensional, hence affine, the categories $\operatorname{Coh} S$ and $\operatorname{Coh} \tilde{S}$ can be identified with the categories of modules, respectively, $\mathbf{S}$-mod and $\mathbf{R}$-mod, where $\mathbf{S} = \prod_{p\in S} \mathcal{S}_p$ and $\mathbf{R} = \prod_{y\in\tilde{S}} \mathcal{R}_y$. If X is a projective configuration, these algebras are semisimple, namely $\mathcal{S}_p \simeq \Bbbk(p)$ and $\mathcal{R}_y \simeq \Bbbk(y)$. Moreover, one easily sees that $\mathcal{J} \simeq \pi_*\widetilde{\mathcal{O}}(-\tilde{S})$, where $\widetilde{\mathcal{O}}(-\tilde{S}) = \widetilde{\mathcal{O}}(-\sum_{y\in\tilde{S}} y)$.

Since X is a projective variety, Serre's theorem [40, Theorem III.5.17] shows that for every coherent sheaf $\mathcal{F} \in \operatorname{Coh} X$ there is an integer n_0 such that all sheaves $\mathcal{F}(n)$ for $n \geq n_0$ are generated by their global sections, or, the same, there are epimorphisms $m\mathcal{O} \to \mathcal{F}(n)$. It easily implies that the derived category $\mathscr{D}^-(\operatorname{Coh} X)$ can be identified with the category of fractions $\mathscr{H}^-(\mathscr{VB}\, X)[\mathcal{Q}^{-1}]$, where $\mathscr{VB}\, X$ is the category of locally free coherent sheaves (equivalently, the category of vector bundles [40, Exercise II.5.18]) over X and $\mathcal{Q}$ is the set of quasi-isomorphisms in $\mathscr{H}^-(\mathscr{VB}\,X)$. So we always present objects from $\mathscr{D}^-(\operatorname{Coh} X)$ and from $\mathscr{D}^-(\operatorname{Coh}\tilde{X})$ as complexes of vector bundles. We denote by $T : \mathscr{D}^-(\operatorname{Coh} X) \to \mathscr{D}^-(\operatorname{Coh}\tilde{X})$ the left derived functor $L\pi^*$. Again if $\mathcal{C}_\bullet$ is a complex of vector bundles, $T\mathcal{C}_\bullet$ coincides with $\pi^*\mathcal{C}_\bullet$.

Just as in Subsection 3.1, we define the *category of triples* $\mathcal{T} = \mathcal{T}(X)$: Objects of $\mathcal{T}$ are triples $(\mathcal{A}_\bullet, \mathcal{B}_\bullet, \iota)$, where

- $\mathcal{A}_\bullet \in \mathscr{D}^-(\operatorname{Coh}\tilde{X})$ (we always present it as a complex of vector bundles);
- $\mathcal{B}_\bullet \in \mathscr{D}^-(\operatorname{Coh}\mathcal{S})$ (we always present it as a complex with zero differential);
- ι is a morphism $\mathcal{B}_\bullet \to \overline{\pi}_*\tilde{\varepsilon}^*\mathcal{A}_\bullet$ from $\mathscr{D}^-(\operatorname{Coh}\mathcal{S})$ such that the induced morphism $\iota^R : \overline{\pi}^*\mathcal{B}_\bullet \to \tilde{\varepsilon}^*\mathcal{A}_\bullet$ is an isomorphism in $\mathscr{D}^-(\operatorname{Coh}\mathcal{R})$.

A morphism from a triple $(\mathcal{A}_\bullet, \mathcal{B}_\bullet, \iota)$ to a triple $(\mathcal{A}'_\bullet, \mathcal{B}'_\bullet, \iota')$ is a pair (Φ, ϕ), where

- $\Phi : \mathcal{A}_\bullet \to \mathcal{A}'_\bullet$ is a morphism from $\mathscr{D}^-(\operatorname{Coh}\tilde{X})$;
- $\phi : \mathcal{B}_\bullet \to \mathcal{B}'_\bullet$ is a morphism from $\mathscr{D}^-(\operatorname{Coh}\mathcal{S})$;
- the diagram
$$\begin{array}{ccc} \mathcal{B}_\bullet & \xrightarrow{\iota} & \overline{\pi}_*\tilde{\varepsilon}^*\mathcal{A}_\bullet \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\overline{\pi}_*\tilde{\varepsilon}^*\Phi} \\ \mathcal{B}'_\bullet & \xrightarrow{\iota'} & \overline{\pi}_*\tilde{\varepsilon}^*\mathcal{A}'_\bullet \end{array} \tag{7}$$
  commutes in $\mathscr{D}^-(\operatorname{Coh}\mathcal{S})$.

We define a functor $\mathbf{F} : \mathscr{D}^-(\operatorname{Coh} X) \to \mathcal{T}(X)$ setting $\mathbf{F}(\mathcal{C}_\bullet) = (\pi^*\mathcal{C}_\bullet, \varepsilon^*\mathcal{C}_\bullet, \iota)$, where $\iota : \varepsilon^*\mathcal{C}_\bullet \to \overline{\pi}_*\tilde{\varepsilon}^*(\pi^*\mathcal{C}_\bullet)$ is induced by the natural isomorphism $\overline{\pi}^*\varepsilon^*\mathcal{F}_\bullet \simeq \tilde{\varepsilon}^*\pi^*\mathcal{F}_\bullet$.. Just as in Section 1, the following theorem holds (with almost the same proof, see [12]).

Theorem 4.2. *The functor* $\mathbf{F}$ *is a representation equivalence, i.e. it is dense and conservative.*

Remark. We do not now whether it is *full*, though it seems very plausible.

Just as for Backström rings, most projective configurations are *vector bundle wild*. Namely, in [29] it was shown that the only projective curves, which are not vector bundle wild, are the following:

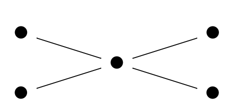
- Projective line $\mathbb{P}^1$.
- Elliptic curves, i.e. smooth projective curves of genus 1, or, the same, smooth plane cubics.

- Projective configurations of types A and $\tilde{A}$ (see the next subsection for their definitions).

Actually, projective line and projective configurations of type A are *vector bundle finite*, i.e. have only finitely many indecomposable vector bundles (up to isomorphism and natural twists), while elliptic curves and projective configurations of type $\tilde{A}$ are vector bundle tame. Since the derived category $\mathscr{D}^-(\operatorname{Coh} X)$ (even $\mathscr{D}^b(\operatorname{Coh} X)$) contains $\operatorname{Coh} X$ as a full subcategory, it can never be representation finite. We always have one-parameter family of skyscrapers, such as $\Bbbk(x)$ $(x \in X)$. If the curve X is smooth, the category $\operatorname{Coh} X$ is hereditary, thus its indecomposable objects are just shifts of sheaves. Moreover, every coherent sheaf is a direct sum of a vector bundle and several *skyscrapers*, i.e. sheaves supported in one point. The latter are just $\mathcal{O}/\mathfrak{m}_x^k$ for some $x \in X$ and some integer k, so they form one-parameter families. Hence, if a smooth curve is vector bundle tame, it is derived tame as well. It happens, just as in the case of pure noetherian rings, that *all* vector bundle tame projective curves are also derived tame, though for projective configurations of types A and $\tilde{A}$ the structure of skyscrapers is more complicated (it involve modules over local rings, which are nodal) and, moreover, there are "mixed" sheaves, which are neither vector bundles (even not torsion free) nor skyscrapers.

4.2 Configurations of Types A and $\tilde{A}$

Now we suppose that X is a projective configurations and all singular points of X are *nodes* (or double points). To such a curve one associates a graph $\Delta(X)$ called its *intersection graph* or *dual graph*. The vertices of $\Delta(X)$ are the irreducible components of X and the edges of $\Delta(X)$ are the singular points of X. The ends of an edge p are the components containing this point. In particular, if p only belongs to one component, it is a loop in $\Delta(X)$. Note that the graph $\Delta(X)$ does not completely define X. For instance, consider the case, when $\Delta(X)$ is the graph of type $\tilde{D}_4$, i.e.



The component corresponding to the central point contains 4 singular points. Therefore, their *harmonic ratio* is invariant under isomorphisms of $\mathbb{P}^1$ and can be an arbitrary scalar $\lambda \in \Bbbk \setminus \{ 0, 1 \}$ (these points can always be chosen as $0, 1, \lambda, \infty$). Thus the configurations with this dual graph but different values of λ are not isomorphic.

We say that a projective configuration X is

- *of type* A if its intersection graph is a chain:

$$1 \;\text{———}\; 2 \;\text{———}\; \cdots \;\text{———}\; s$$

- *of type* $\tilde{A}$ if its intersection graph is a cycle:

$$\begin{tikzcd} 1 \arrow[r, dash] \arrow[rrr, dash, bend right] & 2 \arrow[r, dash] & \cdots \arrow[r, dash] & s \end{tikzcd}$$

  (If $s = 1$, the projective configuration of type A is just a projective line, while the projective configuration of type $\tilde{A}$ is a nodal cubic.)

In other words, in the A-case irreducible components $X_1, X_2, \dots, X_s$ and singular points $p_1, p_2, \dots, p_{s-1}$ can be arranged so that $p_i \in X_i \cap X_{i+1}$, while in the $\tilde{A}$-case the components $X_1, X_2, \dots, X_s$ and the singular points $p_1, p_2, \dots, p_s$ can be so arranged that $p_i \in X_i \cap X_{i+1}$ for $i < s$ and $p_s \in X_s \cap X_1$. Note that in the A-case $s > 1$, while in the $\tilde{A}$-case $s = 1$ is possible: then there is one component with one ordinary double point (a nodal plane cubic). These projective configurations are global analogues of nodal rings, and the calculations according Theorem 4.2 are quite similar to those of Section 3. We present here the calculations for the $\tilde{A}$-case and add remarks explaining what changes should be made for the A-case.

If $s > 1$, the normalization of X is just a disjoint union $\bigsqcup_{i=1}^{s} X_i$; for uniformity, we write $X_1 = \tilde{X}$ if $s = 1$. We also denote $X_{qs+i} = X_i$. Certainly, $X_i \simeq \mathbb{P}_1$ for all i. Every singular point p_i has two preimages p_i', p_i'' in $\tilde{X}$; we suppose that $p_i' \in X_i$ corresponds to the point $\infty \in \mathbb{P}^1$ and $p_i'' \in X_{i+1}$ corresponds to the point $0 \in \mathbb{P}^1$. Recall that any indecomposable vector bundle over $\mathbb{P}^1$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(d)$ for some $d \in \mathbb{Z}$. So every indecomposable complex from $\mathscr{D}^-(\operatorname{Coh} \tilde{X})$ is isomorphic either to $0 \to \mathcal{O}_i(d) \to 0$ or to $0 \to \mathcal{O}_i(-lx) \to \mathcal{O}_i \to 0$, where $\mathcal{O}_i = \mathcal{O}_{X_i}$, $d \in \mathbb{Z}$, $l \in \mathbb{N}$ and $x \in X_i$. The latter complex corresponds to the indecomposable sky-scraper sheaf of length l and support $\{\, x \,\}$. (It is isomorphic in the derived category to any complex $0 \to \mathcal{O}_i((k-l)x) \to \mathcal{O}_i(kx) \to 0$ with arbitrary $k \in \mathbb{Z}$.) We denote this complex by $C(x, -l, n)$ and by $C(x, l, n+1)$. The complex $0 \to \mathcal{O}_i(d) \to$ is denoted by $C(p_i', d\omega, n)$ and by $C(p_{i-1}'', d\omega, n)$. As before, n is the unique place, where the complex has non-zero homologies. We define the symmetric relation $\sim$ for these symbols setting $C(x, -l, n) \sim C(x, l, n+1)$ and $C(p_i', d\omega, n) \sim C(p_{i-1}'', d\omega, n)$.

Let $\mathbb{Z}^\omega = (\mathbb{Z} \oplus \{\, 0 \,\}) \cup \mathbb{Z}\omega$, where $\mathbb{Z}\omega = \{\, d\omega \mid d \in \mathbb{Z} \,\}$. We introduce an ordering on $\mathbb{Z}^\omega$, which is natural on $\mathbb{N}$, on $-\mathbb{N}$ and on $\mathbb{Z}\omega$, but $l < d\omega < -l$ for each $l \in \mathbb{N}$, $d \in \mathbb{Z}$. Recall that $\operatorname{Hom}(\mathcal{O}_i(d), \mathcal{O}_i(d'))$ can be considered as the space of homogeneous polynomial of degree $d' - d$ in homogeneous coordinates on $\mathbb{P}^1$ if $d' \geq d$; otherwise it is zero. Note also that $\mathcal{C}_n(x) \simeq \Bbbk$ if $\mathcal{C} = C(x, l, n)$ for some $l \in \mathbb{Z}^\omega$. It easily implies the following analogue of Lemma 3.5.

Lemma 4.3. *There is a morphism of complexes $\mathcal{C}_\bullet = C(x,z,n) \to \mathcal{C}'_\bullet = C(x,z',n)$ such that its nth component induces a non-zero mapping (actually an isomorphism) $\mathcal{C}_n(x) \to \mathcal{C}'_n(x)$ if and only if $z \le z'$ in $\mathbb{Z}^\omega$. Moreover, if $z = d\omega$, $z' = d'\omega$, $d' > d$ and $x \in S$, hence also $\mathcal{C}_\bullet = C(x',z,n)$ and $\mathcal{C}'_\bullet = C(x',z',n)$ for another singular point x', there is a morphism $\phi : \mathcal{C}_\bullet \to \mathcal{C}'_\bullet$ such that $\phi(x) \neq 0$, but $\phi(x') = 0$.*

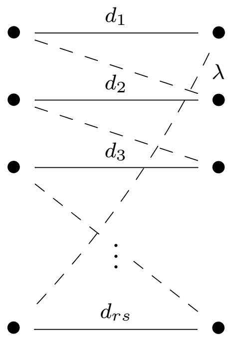
We introduce the ordered sets $\mathfrak{E}_{x,n} = \{\, C(x,z,n) \mid z \in \mathbb{Z}^\omega \,\}$ with the ordering inherited from $\mathbb{Z}^\omega$, We also put $\mathfrak{F}_{x,n} = \{\,(x,n)\,\}$ and $(p'_i,n) \sim (p''_{i-1},n)$ for all i,n. Lemma 4.3 shows that the category of triples $\mathcal{T}(X)$ can be again described in terms of the bunch of chains $\{\, \mathfrak{E}_{x,n},\, \mathfrak{F}_{x,n} \,\}$. Thus we can describe indecomposable objects in terms of strings and bands just as for nodal rings. We leave the corresponding definitions to the reader; they are quite analogous to those from Section 3. If we consider a configuration of type A, we have to exclude the points p'_s, p''_s and the corresponding symbols $C(p'_s,z,n)$, $C(p''_s,z,n)$, (p'_s,n), (p''_s,n). Thus in this case $C(p''_{s-1}, d\omega, n)$ and $C(p'_1, d\omega, n)$ are not in $\sim$ relation with any symbol. It makes possible finite or one-side infinite full strings, while in the $\tilde{A}$-case only two-side infinite strings are full. Note that an infinite word must contain a finite set of symbols (x,n) with any fixed n; moreover there must be n_0 such that $n \ge n_0$ for all entries (x,n) that occur in this word.

If $x \notin \tilde{S}$ (thus $z \notin \mathbb{Z}\omega$), the complex $C(x,z,n)$ vanishes under $\tilde{\varepsilon}^*$, so gives no essential input into the category of triples. It gives rise to the n-th shift of a sky-scraper sheaf with support at the regular point $\pi(x)$. In the language of bunches of chains it follows from the fact that $(x,n) \not\sim (x',n)$ for any $x' \neq x$, hence the only full words containing (x,n) are $(x,n) - C(x,l,n)$ for some $l \in \mathbb{Z} \setminus \{\,0\,\}$. Therefore, in the following examples we only consider complexes $C(x,z,n)$ with $x \in \tilde{S}$. Moreover, we confine most examples to the case $s = 1$ (so X is a nodal cubic). If $s > 1$, one must distribute vector bundles in the pictures below among the components of $\tilde{X}$.

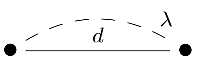
Example 4.4. 1. First of all, even a classification of vector bundles is nontrivial in $\tilde{A}$ case. They correspond to the bands concentrated at 0 place, i.e. such that the underlying cycle w is of the form

$$\begin{aligned}(p'_s,0) &\sim (p''_s,0) - C(p''_s, d_1\omega, 0) \sim C(p'_1, d_1\omega, 0) - (p'_1, 0)\\ &\sim (p''_1,0) - C(p''_1, d_2\omega, 0) \sim C(p'_2, d_2\omega, 0) - (p'_2, 0)\\ &\sim (p''_2,0) - C(p''_2, d_3\omega, 0) \sim \cdots \sim C(p'_s, d_{rs}\omega, 0)\end{aligned}$$

(obviously, its length must be a multiple of s, and we can start from any place p'_k, p''_k). Then $\mathcal{C}_\bullet(w,m,\lambda)$ is actually a vector bundle, which can be schematically described as the following gluing of vector bundles over $\tilde{X}$.



Here horizontal lines symbolize line bundles over X_i of the superscripted degrees, their left (right) ends are basic elements of these bundles at the point ∞ (respectively 0), and the dashed lines show which of them must be glued. One must take m copies of each vector bundle from this picture and make all gluings trivial, except one going from the uppermost right point to the lowermost left one (marked by 'λ'), where the gluing must be performed using the Jordan $m \times m$ cell with eigenvalue λ. In other words, if $e_1, e_2, \dots, e_m$ and $f_1, f_2, \dots, f_m$ are bases of the corresponding spaces, one has to identify f_1 with λe_1 and f_k with $\lambda e_k + e_{k-1}$ for $k > 1$. We denote this vector bundle over X by $\mathcal{V}(\mathbf{d}, m, \lambda)$, where $\mathbf{d} = (d_1, d_2, \dots, d_{rs})$; it is of rank mr and of degree $m \sum_{i=1}^{r} d_i$. If $r = s = 1$, this picture becomes



If $r = m = 1$, we obtain all line bundles: they are $\mathcal{V}((d_1, d_2, \dots, d_s), 1, \lambda)$ (of degree $\sum_{i=1}^{s} d_i$). Thus the Picard group is $\mathbb{Z}^s \times \mathbb{k}^*$.

In the A-case, there are no bands concentrated at 0 place, but there are finite strings of this sort:

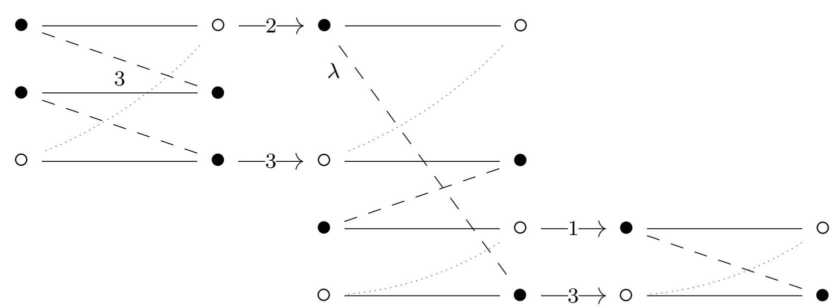
$$\begin{gathered} C(p_1'', d_1\omega, 0) - (p_1', 0) \sim (p_1'', 0) - C(p_1'', d_2\omega, 0) \sim \\ \sim C(p_2', d_2, 0) - (p_2', 0) \sim (p_2'', 0) - C(p_2'', d_3, 0) \sim \\ \cdots \sim C(p_{s-1}', d_{s-1}\omega, 0) - (p_{s-1}', 0) \sim (p_{s-1}'', 0) - C(p_{s-1}'', d_s\omega, 0) \end{gathered}$$

So vector bundles over such configurations are in one-to-one correspondence with integral vectors $(d_1, d_2, \dots, d_s)$; in particular, all of them are line bundles and the Picard group is $\mathbb{Z}^s$. In the picture above one has to set $r = 1$ and to omit the last gluing (marked with 'λ').

2. From now on $s=1$, so we write p instead of p_1. Let w be the cycle

$$\begin{aligned}(p'',1) \sim (p',1) - C(p',-2,1) \sim C(p',2,2) - (p',2) \sim (p'',2)-\\ - C(p'',3\omega,2) \sim C(p',3\omega,2) - (p',2) \sim (p'',2) - C(p'',3,2) \sim\\ \sim C(p'',-3,1) - (p'',1) \sim (p',1) - C(p',1,1) \sim C(p',-1,0)-\\ - (p',0) \sim (p'',0) - C(p'',-2,0) \sim C(p'',2,1).\end{aligned}$$

Then the band complex $\mathcal{C}_\bullet(w,m,\lambda)$ can be pictured as follows:

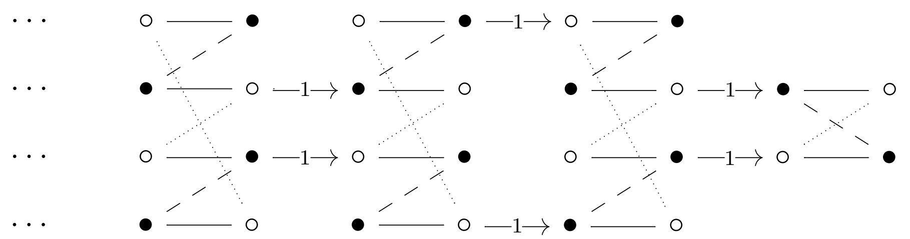


Again horizontal lines describe vector bundles over $\tilde{X}$. Bullets and circles correspond to the points ∞ and 0; circles show those points, where the corresponding complex gives no input into $\overline{\pi}_*\tilde{\varepsilon}^*\mathcal{A}_\bullet$. Horizontal arrows show morphisms in $\mathcal{A}_\bullet$; the numbers l inside give the lengths of factors. For instance, the first row in this picture describes the complex $C(p',-2,1)$, the second one is $C(p',3\omega,2)$ (or, the same, $C(p'',3\omega,2)$) and the last one is $C(p'',-3,0)$. Dashed and dotted lines describe gluings. Dashed lines (between bullets) correspond to mandatory gluings arising from relations $(p',n) \sim (p'',n)$ in the word w, while dotted lines (between circles) can be drawn arbitrarily; the only conditions are that each circle must be an end of a dotted line and the dotted lines between circles sitting at the same level must be parallel (in our picture they are between the 1st and 3rd levels and between the 4th and 5th levels). The degrees of line bundles in complexes $C(x,z,n)$ with $z \in \mathbb{N}\cup(-\mathbb{N})$ (they are described by the levels containing 2 lines) can be chosen as $d-l$ and d with arbitrary d, otherwise (in the second row) they are superscripted over the line. We set $d=1$ in the last row and $d=0$ elsewhere. Thus the resulting complex is

$$\mathcal{V}((-2,3,-3),m,1) \longrightarrow \mathcal{V}((0,0,-1,-2),m,\lambda) \longrightarrow \mathcal{V}((0,1),m,1)$$

(we do not precise mappings, but they can be easily restored). Note that our choice of d's enables to consider the components of this complex as the "standard" vector bundles $\mathcal{V}(\mathbf{d},m,\lambda)$ from the preceding example.

3. If $s = 1$, the sky-scraper sheaf $\Bbbk(p)$ is described by the complex



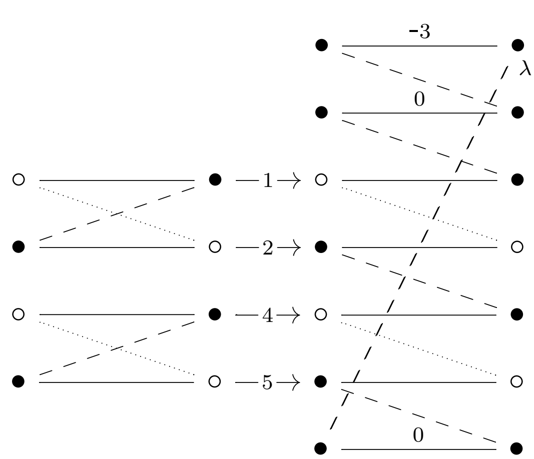
which is the string complex corresponding to the word

$$\begin{aligned}\dots\ C(p', -1, 2) - (p', 2) \sim (p'', 2) - C(p'', 1, 2) \sim C(p'', -1, 1) - \\ - (p'', 1) \sim (p', 1) - C(p', 1, 1) \sim C(p', -1, 0) - (p', 0) \sim \\ \sim (p'', 0) - C(p'', -1, 0) \sim C(p'', 1, 1) - (p'', 1) \sim (p', 1) - \\ - C(p', -1, 1) \sim C(p', 1, 2) - (p', 2) \sim (p'', 2) - C(p'', -1, 2)\ \dots\end{aligned}$$

4. The band complex $\mathcal{C}(w, m, \lambda)$, where w is the cycle

$$\begin{aligned}(p', 0) \sim (p'', 0) - C(p'', -3\omega, 0) \sim C(p', -3\omega, 0) - \\ - (p', 0) \sim (p'', 0) - C(p'', 0\omega, 0) \sim C(p', 0\omega, 0) - (p', 0) \sim \\ \sim (p'', 0) - C(p'', -1, 0) \sim C(p'', 1, 1) - (p'', 1) \sim (p', 1) - \\ - C(p'', 2, 1) \sim C(p', -2, 0) - (p', 0) \sim (p'', 0) - C(p'', -4, 0) \sim \\ \sim C(p'', 4, 1) - (p'', 1) \sim (p', 1) - C(p', 5, 1) \sim C(p', -5, 0) - \\ - (p', 0) \sim (p'', 0) - C(p'', 0\omega, 0) \sim C(p', 0\omega, 0)\end{aligned}$$

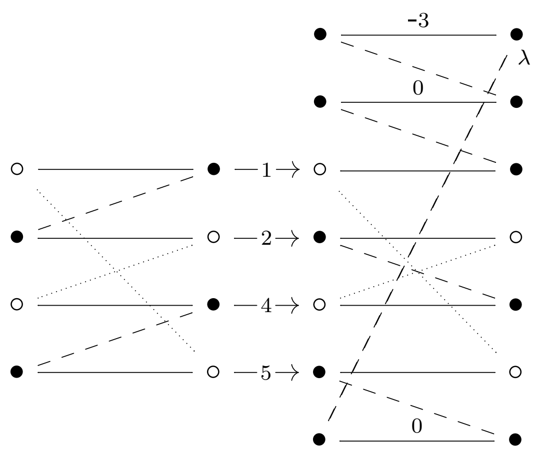
describes the complex



or

$$\mathcal{V}((0,0), m, 1) \oplus \mathcal{V}((0,0), m, 1) \longrightarrow \mathcal{V}((-3, 0, 1, 2, 4, 5, 0), m, \lambda).$$

Its homologies are zero except the place 0, so it corresponds to a coherent sheaf. One can see that this sheaf is a “mixed” one (neither torsion free nor sky-scraper). Note that this time we could trace dotted lines another way, joining the first free end with the last one and the second with the third:



It gives an isomorphic object in $\mathscr{D}(\operatorname{Coh} X)$:

$$\mathcal{V}((0,0,0,0),m,1) \longrightarrow \mathcal{V}((\text{-}3,0,1,5,0),m,\lambda) \oplus \mathcal{V}((2,4),m,1).$$

Remark 4.5. In [12] we used another encoding of strings and bands for projective configurations, which is equivalent but uses more specifics of the situation. In this paper we prefer to use a uniform encoding, which is the same both for nodal rings and for projective configurations.

4.3 Application to Cohen–Macaulay Modules

The description of vector bundles has an important application in the theory of Cohen–Macaulay modules over *surface singularities*.

Definition 4.6. 1. By a *normal surface singularity* over the field $\Bbbk$, which we suppose algebraically closed, we mean a complete noetherian $\Bbbk$-algebra $\mathbf{A}$ such that:

- $\operatorname{Kr.dim} \mathbf{A} = 2$;
- $\mathbf{A}/\mathfrak{m} \simeq \Bbbk$, where $\mathfrak{m}$ is the maximal ideal of $\mathbf{A}$;
- $\mathbf{A}$ has no zero divisors and is *normal*, i.e. integrally closed in its field of fractions;
- $\mathbf{A}$ is not regular, i.e. $\operatorname{gl.dim} \mathbf{A} = \infty$.

We denote by X the scheme $\operatorname{Spec} \mathbf{A}$, by $p \in X$ the point corresponding to the maximal ideal $\mathfrak{m}$ (the unique closed point of X) and by $\check{X}$ the open subscheme $X \setminus \{p\}$.

2. A *resolution* of such a singularity is a morphism of $\Bbbk$-schemes $\pi : \tilde{X} \to X$ such that:

- $\tilde{X}$ is smooth;
- π is projective (hence closed) and birational;
- the restriction of π onto $\tilde{X} \setminus E$, where $E = \pi^{-1}(p)_{\mathrm{red}}$, is an isomorphism $\tilde{X} \setminus E \to \check{X}$; we shall identify $\tilde{X} \setminus E$ with $\check{X}$ using this isomorphism.

We call E the *exceptional curve* of the resolution π (it is indeed a projective curve) and denote by $E_1, E_2, \dots, E_s$ its irreducible components.

3. A resolution $\pi : \tilde{X} \to X$ is called *minimal*, if it cannot be decomposed as $\tilde{X} \to X' \to X$, where X' is also smooth.

Recall that such a resolution, as well as a minimal resolution, always exists (cf. e.g. [47]).

In [43] Kahn established a one-to-one correspondence between Cohen–Macaulay modules over a normal surface singularity $\mathbf{A}$ and a class of vector bundles over a *reduction cycle* $Z \subseteq \tilde{X}$, which is given by a specially chosen effective divisor $\sum_{i=1}^{s} m_i E_i$ $(m_i > 0)$. His result becomes especially convenient if this singularity is *minimally elliptic* in the sense of [46]. It means that $\mathbf{A}$ is Gorenstein (i.e. $\operatorname{inj.dim} \mathbf{A} = 2$) and $\dim \mathrm{H}^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 1$. Let $\pi : \tilde{X} \to X$ be the minimal resolution of a minimally elliptic singularity, Z be its *fundamental cycle*, i.e. the smallest effective cycle such that all intersection numbers $(Z.E_i)$ are non-positive. Then Z is a reduction cycle in the sense of Kahn, and the following result holds.

Theorem 4.7 (Kahn). *There is one-to-one correspondence between Cohen–Macaulay modules over* $\mathbf{A}$ *and vector bundles* $\mathcal{F}$ *over* Z *such that* $\mathcal{F} \simeq \mathcal{G} \oplus n\mathcal{O}_Z$*, where*

(i) $\mathcal{G}$ *is* generically spanned*, i.e. global sections from* $\Gamma(E, \mathcal{G})$ *generate* $\mathcal{G}$ *everywhere, except maybe finitely many closed points;*

(ii) $\mathrm{H}^1(E, \mathcal{G}) = 0$*;*

(iii) $n \geq \dim_{\Bbbk} \mathrm{H}^0(E, \mathcal{G}(Z))$.

Especially, indecomposable Cohen–Macaulay $\mathbf{A}$*-modules correspond to vector bundles* $\mathcal{F} \simeq \mathcal{G} \oplus n\mathcal{O}_Z$*, where either* $\mathcal{G} = 0$, $n = 1$ *or* $\mathcal{G}$ *is indecomposable, satisfies the above conditions* (i,ii) *and* $n = \dim_{\Bbbk} \mathrm{H}^0(E, \mathcal{G}(Z))$. *(The vector bundle* $\mathcal{O}_Z$ *corresponds to the regular* $\mathbf{A}$*-module, i.e.* $\mathbf{A}$ *itself.)*

Kahn himself deduced from this theorem and the results of Atiyah [1] a description of Cohen–Macaulay modules over *simple elliptic* singularities, i.e. such that E is an elliptic curve (smooth curve of genus 1). Using the results of subsection 4.2, one can obtain an analogous description for *cusp singularities*, i.e. such that E is a projective configuration of type $\tilde{A}$. Briefly, one gets the following theorem (for more details see [30]).

Theorem 4.8. *There is a one-to-one correspondence between indecomposable Cohen–Macaulay modules over a cusp singularity* $\mathbf{A}$*, except the regular module* $\mathbf{A}$*, and vector bundles* $\mathcal{V}(\mathbf{d}, m, \lambda)$*, where* $\mathbf{d} = (d_1, d_2, \dots, d_{rs})$ *satisfies the following conditions:*

- $\mathbf{d} > \mathbf{0}$*, i.e.* $d_i \geq 0$ *for all* i *and* $\mathbf{d} \neq (0, 0, \dots, 0)$*;*
- *no shift of* $\mathbf{d}$*, i.e. a sequence* $(d_{k+1}, \dots, d_{rs}, d_1, \dots, d_k)$*, contains a subsequence* $(0, 1, 1, \dots, 1, 0)$*, in particular* $(0, 0)$*;*
- *no shift of* $\mathbf{d}$ *is of the form* $(0, 1, 1, \dots, 1)$*.*

Moreover, from Theorem 4.7 and the results of [29] one gets the following corollary [30]:

Theorem 4.9. *If a minimally elliptic singularity* $\mathbf{A}$ *is neither simple elliptic nor cusp, it is Cohen–Macaulay wild, i.e. the classification of Cohen–Macaulay* $\mathbf{A}$*-modules includes the classification of representations of all finitely generated* $\Bbbk$*-algebras.*

An important example of Cohen–Macaulay tame minimally elliptic singularities are the surface singularities of type T_{pqr}, i.e. factor rings

$$\Bbbk[[x, y, z]]/(x^p + y^q + z^r + \lambda xyz) \quad (1/p + 1/q + 1/r \leq 1).$$

They are simple elliptic if $1/p + 1/q + 1/r = 1$ and cusp otherwise [49].

As a consequence of Theorem 4.8 and the Knörrer periodicity theorem [44, 50], one also obtains a description of Cohen–Macaulay modules over *hypersurface singularities* of type T_{pqr}, i.e. factor rings

$$\Bbbk[[x_1, x_2, \dots, x_n]]/(x_1^p + x_2^q + x_3^r + \lambda x_1 x_2 x_3 + Q) \quad (1/p + 1/q + 1/r \leq 1),$$

where Q is a non-degenerate quadratic form of $x_4, \dots, x_n$, and over *curve singularities* of type T_{pq}, i.e. factor rings

$$\Bbbk[[x, y]]/(x^p + y^q + \lambda x^2 y^2) \quad (1/p + 1/q \leq 1/2).$$

The latter fills up a flaw in the result of [27], where one has only proved that the curve singularities of type T_{pq} are Cohen–Macaulay tame, but got no explicit description of modules.

Suppose that $\operatorname{char} \Bbbk = 0$. Then it is known [2, 32] that a normal surface singularity $\mathbf{A}$ is Cohen–Macaulay finite, i.e. has only a finite number of non-isomorphic indecomposable Cohen–Macaulay modules, if and only if it is a *quotient singularity*, i.e. $\mathbf{A} \simeq \Bbbk[[x,y]]^G$, where G is a finite group of automorphisms. (I do not know a criterion of finiteness if $\operatorname{char} \Bbbk > 0$). Just in the same way one can show that all singularities of the form $\mathbf{A} = \mathbf{B}^G$, where $\mathbf{B}$ is either simple elliptic or cusp, are Cohen–Macaulay tame, and obtain a description of Cohen–Macaulay modules in this case. Actually such singularities coincide with the so called *log-canonical* singularities [45]. There is an evidence that all other singularities are Cohen–Macaulay wild, so Table 1 completely describes Cohen–Macaulay types of isolated singularities (for the curve case see [27]; we mark by '?' the places, where the result is still a conjecture).

Table 1.

Cohen–Macaulay types of singularities

CM type	curves	surfaces	hypersurfaces
finite	dominate A-D-E	quotient	simple (A-D-E)
tame	dominate T_{pq}	log-canonical (only ?)	T_{pqr} (only ?)
wild	all other	all other ?	all other ?

Acknowledgements

I am greatful first of all to Gert-Martin Greuel, who has been my co-author since 1990. In particular, our joint work inspired me on this research, so I am highly indebted to him for the results presented here. Thus it was especially pleasant for me to present them during his anniversary conference in Kaiserslautern. I am also thankful to the University of Kaiserslautern for the support of our collaboration.

I am also greatful to Birge Huisgen-Zimmermann who explained me some details of her joint work with Manolo Saorín, which led me to the idea that the techniques of boxes and "almost versal families" can be successfully applied to derived categories. And, of course, I very much obliged to my co-authors in this research Viktor Bekkert and Igor Burban.

The final version of this paper has been prepared during my visit to Max-Plank-Institut für Mathematik in Bonn. I am very thankful to the Institute

for its hospitality and the perfect working conditions.

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Monodromy

Wolfgang Ebeling

Dedicated to Gert-Martin Greuel on the occasion of his 60th birthday

Abstract

Let (X, x) be an isolated complete intersection singularity and let $f : (X, x) \to (\mathbb{C}, 0)$ be the germ of an analytic function with an isolated singularity at x. An important topological invariant in this situation is the Picard-Lefschetz monodromy operator associated to f. We give a survey on what is known about this operator. In particular, we review methods of computation of the monodromy and its eigenvalues (zeta function), results on the Jordan normal form of it, definition and properties of the spectrum, and the relation between the monodromy and the topology of the singularity.

Introduction

The word 'monodromy' comes from the greek word $\mu o \nu o - \delta \rho o \mu \psi$ and means something like 'uniformly running' or 'uniquely running'. According to [99, 3.4.4], it was first used by B. Riemann [135]. It arose in keeping track of the solutions of the hypergeometric differential equation going once around a singular point on a closed path (cf. [30]). The group of linear substitutions which the solutions are subject to after this process is called the *monodromy group*.

Since then, monodromy groups have played a substantial rôle in many areas of mathematics. As is indicated on the webside 'www.monodromy.com' of N.M. Katz, there are several incarnations, classical and l-adic, local and global, arithmetic and geometric. Here we concentrate on the classical local geometric monodromy in singularity theory. More precisely we focus on the monodromy operator of an isolated hypersurface or complete intersection singularity. The investigation of this operator started in 1967 with the proof

1991 *Mathematics Subject Classification.* 14D05, 32S40

Key words. Monodromy, zeta function, spectrum, isolated singularity

of the famous monodromy theorem (see §1). This theorem can be proved using the theory of the Gauß-Manin connection which was introduced by E. Brieskorn for isolated hypersurface singularities [28]. The study of this connection for isolated complete intersection singularities was started by G.-M. Greuel in his thesis [76].

We try to review the results of 37 years of investigation of the monodromy operator. The results include results on the zeta function of the monodromy and the spectrum of a singularity. The monodromy contains a lot of information about the topology of the singularity. This was one motivation to study the monodromy. We review the known facts in the last section.

For a basic introduction to the subject for non-specialists see [55].

Aspects which are not mentioned or only touched in this survey are

- Monodromy of a polynomial function. For a survey on this topic see [40] and [87].
- Generalizations to non-isolated singularities. Here we refer to the survey of D. Siersma [151].
- Monodromy groups. We do not talk about monodromy groups of isolated complete intersection singularities. We refer to our book [51] for this topic.
- Braid monodromy. Recently, M. Lönne introduced a notion of braid monodromy of singularities. He computed the braid monodromy and the fundamental group of the complement of the discriminant of a Brieskorn-Pham singularity [116], making a substantial contribution to the last problem [29, Problème 20] of Brieskorn's list of problems on monodromy.

1 The Monodromy Operator

Let $(Y,0) \subset (\mathbb{C}^N,0)$ be an isolated complete intersection singularity (abbreviated ICIS in the sequel) of dimension $n+1$, i.e. $(Y,0)$ is the germ of an analytic variety of pure dimension $n+1$ with an isolated singularity at the origin given by $Y = F^{-1}(0)$, where $F = (f_1,\dots,f_{N-n-1}) : (\mathbb{C}^N,0) \to (\mathbb{C}^{N-n-1},0)$ is the germ of an analytic mapping. Let $f : \mathbb{C}^N \to \mathbb{C}$ be an analytic function such that the restriction $f : Y \to \mathbb{C}$ which we denote by the same symbol has an isolated singularity at the origin. We assume that $f(0) = 0$. Let $\varepsilon > 0$ be small enough such that the closed ball $B_\varepsilon \subset \mathbb{C}^N$ of radius ε around the origin in $\mathbb{C}^N$ intersects the fibre $f^{-1}(0)$ transversely. Let $0 < \delta \ll \varepsilon$ be such that for t in the disc $D_\delta \subset \mathbb{C}$ around the origin, the fibre $f^{-1}(t) \cap Y$ intersects the ball B_ε transversely. Let $X_t := f^{-1}(t) \cap B_\varepsilon \cap Y$ for $t \in D_\delta$,

$$X := f^{-1}(D_\delta) \cap B_\varepsilon \cap Y\,, \quad X' := X \setminus X_0\,, \quad D' := D \setminus \{0\}\,.$$

Then $(X_0, 0)$ is an ICIS of dimension n. In the important special case when Y is smooth, $(Y, 0) = (\mathbb{C}^{n+1}, 0)$, then $(X_0, 0)$ is an isolated hypersurface singularity. By a result of J. Milnor [125] in the case when Y is smooth and H. Hamm [89] in the general case, the mapping $f|_{X'} : X' \to D'$ is the projection of a locally trivial C^∞ fibre bundle. A fibre X_t of this bundle is called *Milnor fibre*. It has the homotopy type of a bouquet of μ n-spheres where μ is the Milnor number. Therefore its only interesting homology group is the group $H_n(X_t, \mathbb{Z})$. It is of rank μ. Parallel translation along the path

$$\gamma : [0, 1] \to D_\delta, \quad t \mapsto \delta e^{2\pi i t},$$

yields a diffeomorphism $h : X_\delta \to X_\delta$ called the *geometric monodromy* of the singularity. It is determined up to isotopy.

Definition 1.1. The induced homomorphism $h_*^{\mathbb{C}} : H_n(X_\delta, \mathbb{C}) \to H_n(X_\delta, \mathbb{C})$ (resp. $h_*^{\mathbb{Z}} : H_n(X_\delta, \mathbb{Z}) \to H_n(X_\delta, \mathbb{Z})$) is called the *complex* (resp. *integral*) *monodromy (operator)* of the singularity.

This operator is also sometimes called the *Picard-Lefschetz monodromy operator* since the consideration of this operator goes back to E. Picard [133] and S. Lefschetz [107] (see also [98], [132]).

Theorem 1.2 (Monodromy theorem).

(a) *The eigenvalues of h_* are roots of unity.*

(b) *The size of the blocks in the Jordan normal form of h_* is at most $(n + 1) \times (n + 1)$.*

(c) *If $(Y, 0)$ is smooth, then the size of the Jordan blocks for the eigenvalue 1 is at most $n \times n$.*

There are many different proofs of this theorem: by A. Borel (unpublished), E. Brieskorn [28] (for $(Y, 0)$ smooth, generalized by G.-M. Greuel [76]), C. H. Clemens [33], P. Deligne [35, 36], P. A. Griffiths [79], A. Grothendieck [80], N. M. Katz [95], A. Landman [100, 101], Lê Dũng Tráng [106] (of (a), for (b) see the book of E. Looijenga [117]), B. Malgrange [120], and W. Schmid [146] (see also the survey [79]).

Examples of B. Malgrange [118] show that the bounds on the size of the Jordan blocks are sharp.

For weighted homogeneous singularities with $(Y, 0)$ smooth, Milnor [125] has shown that the complex monodromy $h_*^{\mathbb{C}}$ is diagonalizable. For weighted homogeneous ICIS this was shown by A. Dimca [39]. For irreducible plane curve singularities, Lê [104] has shown that the monodromy is of finite order. N. A'Campo [2] has shown that for isolated plane curve singularities with more than one branch the monodromy is in general not of finite order. A. H. Durfee [47] has given a necessary and sufficient condition for the monodromy of a degenerating family of curves to be of finite order.

We now mention several results which are only valid in the case when $(Y, 0) = (\mathbb{C}^{n+1}, 0)$. J. Scherk [144] has shown that if f^{r+1} belongs to the ideal $(\partial f/\partial x_0, \ldots, \partial f/\partial x_n)$ of the ring $\mathcal{O}_{n+1}$ of germs of holomorphic functions on $\mathbb{C}^{n+1}$, then the size of the Jordan blocks of $h_*^{\mathbb{C}}$ is at most $(r+1) \times (r+1)$. By a theorem of J. Briançon and H. Skoda [27], $f^{n+1} \in (\partial f/\partial x_0, \ldots, \partial f/\partial x_n)$. Therefore Scherk's theorem implies the Monodromy theorem. Generalizations of Scherk's theorem can be found in [145].

M. G. M. van Doorn and J. H. M. Steenbrink [41] have proved the following supplement to the Monodromy theorem: If there exists a Jordan block of size $(n+1) \times (n+1)$, then there exists a Jordan block of size $n \times n$ for the eigenvalue 1. Since a plane curve singularity is reducible if and only if $h_*^{\mathbb{C}}$ has an eigenvalue 1, this implies Lê's theorem.

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ be two germs of analytic functions with an isolated singularity at 0. Denote by c_f and c_g the complex monodromy operators of f and g respectively. Denote by c_{f+g} the complex monodromy operator of the germ $f+g$. The famous theorem of M. Sébastiani and R. Thom [150] states that

$$c_{f+g} = c_f \otimes c_g.$$

The author and Steenbrink [64] have proved a generalization of this theorem for a suspension of an ICIS.

In the case when $Y = \mathbb{C}^{n+1}$ one can associate to $f \in \mathbb{C}\{x_0, \ldots, x_n\}$ its *Bernstein-Sato polynomial*. This is defined as follows. Let s be a new variable. Then there exists a differential operator $P = P(x, s, \partial/\partial x)$ whose coefficients are convergent power series in s and $x_0, \ldots, x_n$ and a nonzero polynomial $b(s) \in \mathbb{C}[s]$ satisfying the formal identity

$$Pf^s = b(s)f^{s-1}.$$

The set of all polynomials $b(s) \in \mathbb{C}[s]$ for which such an identity holds (for some operator P) forms an ideal, and the unique monic generator for this ideal is called the *Bernstein-Sato polynomial* of f. It is denoted by $b_f(s)$. According to Malgrange [119, 121] (see also [22]) there is the following relation to the monodromy of f: The zeros $s_1, s_2, \ldots$ of $\widetilde{b}_f(s) := b_f(s)/s$ are rational and less than 1, the minimal polynomial of the monodromy divides the polynomial $p(t) := \prod(t - \exp(-2\pi i s_j))$, and on the other hand, $p(t)$ divides the characteristic polynomial of the monodromy. D. Barlet [18] has shown that if there exists a $k \times k$ Jordan block for the eigenvalue $\exp(-2\pi i u)$ of the monodromy, then there exist at least k (counted with multiplicity) roots of

$b_f(s)$ of the form $-q-u$ with $q \in [0,n]$ $(0 \le u < 1)$. A generalization of the Bernstein-Sato polynomial for the case that $(Y,0)$ is an ICIS was studied by T. Torrelli [157, 158].

2 Computation of Monodromy

We now review methods to compute the monodromy operator.

We first consider the complex monodromy operator. In the case when $(Y,0)$ is smooth, Brieskorn [28] has indicated a method to compute the complex monodromy operator. This method has been implemented by M. Schulze to Singular [147] (see also [148]).

For plane curve singularities there is an algorithm in the book of D. Eisenbud and W. Neumann [68] to compute the Jordan normal form of the complex monodromy operator from a splicing diagram of the singularity.

For superisolated surface singularities (see the article of E. Artal-Bartolo, I. Luengo and A. Melle-Hernández in this volume [17]), Artal-Bartolo [12, 14] has determined the Jordan normal form of the complex monodromy operator.

We now consider the integral monodromy operator. Let $Y_\eta := F^{-1}(\eta) \cap B_\varepsilon$ where η is a regular value of F sufficiently close to 0. Let $c := h_*^{\mathbb{Z}}$ be the integral monodromy operator. The above path γ also induces a map $\widehat{c} : H_{n+1}(Y_\eta, X_\delta) \to H_{n+1}(Y_\eta, X_\delta)$ on the relative homology groups. (Unless otherwise stated, we consider homology with integral coefficients). We have the following diagram with exact rows and commutative squares (cf. [51]):

$$\begin{tikzcd}
0 \arrow[r] & H_{n+1}(Y_\eta) \arrow[r] \arrow[d, "\mathrm{id}"'] & H_{n+1}(Y_\eta, X_\delta) \arrow[r] \arrow[d, "\widehat{c}"'] & H_n(X_\delta) \arrow[r] \arrow[d, "c"'] & 0 \\
0 \arrow[r] & H_{n+1}(Y_\eta) \arrow[r] & H_{n+1}(Y_\eta, X_\delta) \arrow[r] & H_n(X_\delta) \arrow[r] & 0
\end{tikzcd}$$

Let A be the intersection matrix on $H_{n+1}(Y_\eta, X_\delta)$ with respect to a distinguished basis of thimbles (cf. [51]). It is a $\nu \times \nu$-matrix where ν is the number of thimbles in a distinguished basis of thimbles. One has $\nu = \mu + \mu'$ where μ' is the Milnor number of the singularity $(Y,0)$. The matrix A is encoded in the *Coxeter-Dynkin diagram* of the singularity. This matrix is of the form $A = V + (-1)^n V^t$ for some upper triangular matrix

$$V = \begin{pmatrix} (-1)^{\frac{n(n+1)}{2}} & * & \cdots & * & * \\ 0 & (-1)^{\frac{n(n+1)}{2}} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & (-1)^{\frac{n(n+1)}{2}} & * \\ 0 & \cdots & \cdots & 0 & (-1)^{\frac{n(n+1)}{2}} \end{pmatrix}.$$

If $Y = \mathbb{C}^{n+1}$ then V is the matrix of the *(integral) Seifert form* or of the inverse of the *variation operator* of the singularity (see [11, 45, 98]). For general $(Y, 0)$, the author and S. M. Gusein-Zade defined in [61] a variation operator the inverse of which has the matrix V. The operator $\widehat{c}$ is the product of the Picard-Lefschetz transformations corresponding to the elements of a distinguished basis of thimbles (cf. [51]). In the case when $Y = \mathbb{C}^3$ and f defines a simple singularity, the Coxeter-Dynkin diagram is the classical Coxeter-Dynkin diagram of a root system of type A_μ, D_μ, E_6, E_7, or E_8 and $\widehat{c} = c$ is the corresponding *Coxeter element*. It follows from [23, Chap. V, §6, Exercice 3] that the matrix $\widehat{C}$ of the operator $\widehat{c}$ is given by $\widehat{C} = (-1)^{n+1} V^{-1} V^t$ (see also [108]).

F. Lazzeri [102] (see also [103]) and independently A.M. Gabrielov [71] showed that in the case when $Y = \mathbb{C}^{n+1}$ the Coxeter-Dynkin diagram is connected thus extending an earlier result of C.H. Bey [20, 21] for curves. A. Hefez and Lazzeri [91] computed the intersection matrix of Brieskorn-Pham singularities solving in this way an open problem stated by Brieskorn [29] and F. Pham [132].

Gabrielov [70, 72] has given methods to compute the intersection matrix A for some special singularities. A'Campo [4, 5] and Gusein-Zade [81, 82] have found a beautiful method to compute the intersection matrix for isolated plane curve singularities using real morsifications. This method was generalized in [59, 60] to suspensions of fat points.

A rather general method to compute an intersection matrix for isolated hypersurface singularities using polar curves was found by Gabrielov [73]. This method was generalized to ICIS in [51]. The author [48, 49] has computed the characteristic polynomial of the monodromy for the uni- and bimodal hypersurface singularities in Arnold's classification [8] and the intersection matrix A for the elliptic hypersurface singularities [50]. Gusein-Zade [83] gave a recursive formula for the characteristic polynomials of the monodromy for the singularities of Arnold's series of singularities [8].

In [51] the matrices A were computed for the simple space curve singularities classified by M. Giusti [74] except Z_9 and Z_{10} and many of the $\mathcal{K}$-unimodal isolated singularities of complete intersection surfaces classified by C. T. C. Wall [170]. The missing cases Z_9 and Z_{10} were studied in [58] and the case $I_{1,0}$ was treated in [53].

P. Orlik and R. Randell [130] computed the integral monodromy for some classes of weighted homogeneous singularities.

If $Y = \mathbb{C}^{n+1}$ and f is a real analytic function, i.e. takes real values on $\mathbb{R}^{n+1} \subset \mathbb{C}^{n+1}$, then Gusein-Zade [84] showed that the integral monodromy is the product of two involutions (see also [6]).

3 Zeta Function

The *zeta function* of the monodromy is defined to be

$$\zeta(t) := \prod_{q \geq 0} \{\det(\mathrm{id}_* - t h_*; H_q(X_\delta, \mathbb{C}))\}^{(-1)^{q+1}}.$$

The relation with the characteristic polynomial $\Delta(t)$ of the monodromy is

$$\Delta(t) := \det(t\mathrm{id}_* - h_*) = t^\mu \left[\frac{t-1}{t}\zeta\left(\frac{1}{t}\right)\right]^{(-1)^{n+1}}.$$

The *Lefschetz numbers* are defined by

$$\Lambda_k := \Lambda(h_*^k) = \sum_{q \geq 0} (-1)^q \mathrm{Tr}[h_*^k; H_q(X_\delta, \mathbb{C})].$$

We define rational numbers χ_m by

$$\Lambda_k = \sum_{m|k} m\chi_m.$$

Explicitely, these numbers can be defined by Möbius inversion

$$\chi_m = \frac{1}{m}\left(\sum_{k|m} \mu\left(\frac{m}{k}\right)\Lambda_k\right),$$

where $\mu(\ )$ denotes the Möbius function. By A. Weil (cf. [125]) we have

$$\zeta(t) = \prod_{m \geq 1} (1 - t^m)^{-\chi_m}.$$

The following statements were explained to me by D. Zagier.

Proposition 3.1 (Zagier). (i) *The numbers χ_m are integers.*

(ii) *The following statements are equivalent:*

(a) *$\Delta(t)$ is a product of cyclotomic polynomials.*
(b) *$\chi_m \neq 0$ for only finitely many m.*
(c) *The sequence (Λ_k) is periodic.*

Proof. (i) is proved by induction: Assume that $\chi_m \in \mathbb{Z}$ for $m \leq \ell$. Then

$$\prod_{m=1}^{\ell} (1 - t^m)^{-\chi_m}$$

is a formal power series with integer coefficients which starts with 1. Since $\zeta(t)$ is also a power series with integral coefficients, the same is true for

$$\frac{\zeta(t)}{\prod_{m=1}^{\ell}(1-t^m)^{-\chi_m}} = \prod_{m=\ell+1}^{\infty}(1-t^m)^{-\chi_m}.$$

But this power series starts with $1+\chi_{\ell+1}t^{\ell+1}$.

The proof of (ii) is done in several steps: The implication (a) $\Rightarrow$ (b) is clear.

(b) $\Rightarrow$ (c): Let $\chi_m = 0$ for all m which do not divide a number Q. Then for positive integers d, r with $0 < r \leq Q$ we have

$$\Lambda_{dQ+r} = \sum_{m|dQ+r} m\chi_m = \sum_{m|r} m\chi_m = \Lambda_r.$$

(c) $\Rightarrow$ (a): Let the sequence (Λ_k) be periodic of period Q. Let $s_k := \operatorname{Tr} h_*^k$ and let $p(t) := \det(\mathrm{id}_* - th_*)$. Then

$$p(t) = \exp(\operatorname{Tr}(\log(\mathrm{id}_* - th_*))) = \exp\left(-\sum_{k=1}^{\infty} s_k \frac{t^k}{k}\right).$$

For the logarithmic derivative of $p(t)$ we get from this

$$\frac{p'(t)}{p(t)} = -\sum_{k=1}^{\infty} s_k t^{k-1}.$$

Since s_k is periodic of period Q, we get

$$\begin{aligned}\frac{p'(t)}{p(t)} &= -\sum_{k=1}^{Q} s_k t^{k-1}(1+t^Q+t^{2Q}+\ldots)\\ &= -\sum_{k=1}^{Q} s_k \frac{t^{k-1}}{1-t^Q} = \frac{q(t)}{1-t^Q}\end{aligned}$$

for some polynomial $q(t)$. But each zero of $p(t)$ must be a simple pole of the logarithmic derivative and hence a zero of $1-t^Q$. $\square$

If $(Y,0)$ is smooth and f is singular, then by [1] $\Lambda_1 = 0$. Lê [105] proved that in this case there exists a characteristic diffeomorphism h without fixed points. In a letter to A'Campo, P. Deligne showed that more generally in the case when $(Y,0)$ is smooth, $\Lambda_k = 0$ for $0 < k < \mathrm{mult}(f)$ where $\mathrm{mult}(f)$ is the multiplicity of f. G. G. Il'yuta [94] gave formulae expressing the Lefschetz numbers in terms of cycles of the Coxeter-Dynkin diagram. In [52] the author showed that $\operatorname{Tr} C^2 = (-1)^r$ where r is the corank of f.

Let $\pi : \widetilde{Y} \to Y$ be a resolution of Y and let $\widetilde{f} := f \circ \pi$ be the composition. We denote by $\widetilde{X}_0$ the proper transform of $X_0 = f^{-1}(0)$. Let $\widetilde{f}^{-1}(0) = \widetilde{X}_0 \cup E_1 \cup \cdots \cup E_s$ where E_i is irreducible. We assume that the following conditions are satisfied:

(1) $\pi : \widetilde{X}_0 \to X_0$ is a resolution of X_0.

(2) Each exceptional divisor E_i is smooth and $\widetilde{f}^{-1}(0)$ has only normal crossings.

Let m_i be the order of the function $\widetilde{f}$ along the divisor E_i and let

$$E'_i := E_i \setminus \left(\bigcup_{j \neq i} E_j \right) \cup \widetilde{X}_0.$$

Then we have the following famous theorem of A'Campo [3]:

Theorem 3.2 (A'Campo). *Under the above assumptions, we have*

$$\zeta(t) = \prod_{i=1}^{s} (1 - t^{m_i})^{-\chi(E'_i)}$$

where $\chi(E'_i)$ is the topological Euler characteristic of E'_i.

The theorem was formulated by A'Campo only for the case $Y = \mathbb{C}^{n+1}$ but it can be easily generalized to this more general situation (see e.g. [128]). From Proposition 3.1 we see that A'Campo's theorem implies Part (a) of the Monodromy theorem. In [11] it is shown that Part (b) can also be derived from that theorem.

A generalization of A'Campo's theorem using partial resolutions was given by Gusein-Zade, Luengo and Melle-Hernández [86].

If $Y = \mathbb{C}^{n+1}$ and f is non-degenerate with respect to the Newton diagram, then A. Varchenko [160] (and also independently F. Ehlers [66]) have given a formula to compute $\zeta(t)$ from the Newton diagram. This formula was generalized to the general case by M. Oka [128].

A. Campillo, F. Delgado and Gusein-Zade [85] have shown that for an irreducible curve singularity, the zeta function $\zeta(t)$ coincides with the Poincaré series $P(t)$ of the natural filtration on the ring of functions of such a singularity given by the order with respect to a uniformization.

Now suppose $Y = \mathbb{C}^{n+1}$ and f is weighted homogeneous of weights $q_0, \ldots, q_n$ and degree d. Here $q_0, \ldots, q_n$ are assumed to be coprime. Then the geometric monodromy $h : X_1 \to X_1$ can be described as follows [125]:

$$h(z_0, \ldots, z_n) = (e^{2\pi i/q_0} z_0, \ldots, e^{2\pi i/q_n} z_n).$$

The monodromy operator h_* is of order d. Milnor and P. Orlik [126] have shown how to compute $\zeta(t)$ from the weights and the degree of f. Greuel and H. Hamm [77] have given a more general formula for a weighted homogeneous ICIS $(Y,0)$.

K. Saito [137] has shown that if $Y = \mathbb{C}^3$ then all the primitive d-th roots of unity are eigenvalues of h_*.

Saito [138, 139] also introduced a duality between rational functions of the form of the zeta function. If $\phi(t)$ is a rational function of the form

$$\phi(t) = \prod_{m|d} (1 - t^m)^{\chi_m} \quad \text{for } \chi_m \in \mathbb{Z} \text{ and some } d \in \mathbb{N},$$

then he defines

$$\phi^*(t) = \prod_{k|d} (1 - t^k)^{-\chi_{d/k}}.$$

Let f be a function defining one of the 14 exceptional unimodal hypersurface singularities in the sense of V. I. Arnold [9]. Arnold has observed a strange duality between these singularities [8]. Saito has observed the following fact: If $\Delta(t)$ is the characteristic polynomial of the monodromy of f then $\Delta^*(t)$ is the characteristic polynomial of the monodromy of the dual singularity. The author and C. T. C.Wall [65] have found an extension of Arnold's strange duality embracing also ICIS. The author [53] has shown that Saito's duality also holds for this extension and he has related it to polar duality and to a duality of weight systems found by M. Kobayashi [54, 57].

If $n = 2$ and $Y = \mathbb{C}^3$ or $(Y,0)$ is a certain special ICIS, then it was shown [56] that the Saito dual $\Delta^*(t)$ of the characteristic polynomial of the monodromy is equal to the product of the Poincaré series $P(t)$ of the coordinate algebra and some rational function $\mathrm{Or}(t)$ depending only on the orbit invariants of the natural $\mathbb{C}^*$-action on the singularity. In the case when $Y = \mathbb{C}^{n+1}$ and f is a Newton non-degenerate function, the author and Gusein-Zade [62] showed that the same holds for the Saito dual of the inverse of the reduced zeta function $\widetilde{\zeta}(t)$ (reduced means considering reduced homology). Finally, in [63] this was generalized to the case when Y is a complete intersection given as the zero set of functions $f_1, \ldots, f_{k-1}$ and $f = f_k$ to the product of the Saito duals of the inverse reduced zeta functions $\widetilde{\zeta}_j(t)$ of the monodromy operators of f_j on $f_1 = \ldots = f_{j-1} = 0$ for $j = 1, \ldots, k$. J. Stevens [156] proved that this result implies the theorem of Campillo, Delgado and Gusein-Zade [85].

Now let $Y = \mathbb{C}^{n+1}$ and assume that $f \in \mathbb{Z}[x_0, \ldots, x_n]$. For a prime number p denote by $\mathbb{Z}_p$ the p-adic integers. Consider the p-adic integral

$$Z_p(s) := \int_{\mathbb{Z}_p^{n+1}} |f(x)|_p^s |dx|$$

for $s \in \mathbb{C}$, $\mathrm{Re}(s) > 0$, where $|dx|$ denotes the Haar measure on $\mathbb{Q}_p^{n+1}$ normalized in such a way that $\mathbb{Z}_p^{n+1}$ is of volume 1. This function is called the *p-adic Igusa zeta function*. Now there is the following famous conjecture [93] (see also [37]):

Conjecture 3.3 (Igusa's monodromy conjecture). *For almost all prime numbers p, if s_0 is a pole of $Z_p(s)$ then $e^{2\pi i \mathrm{Re}(s_0)}$ is an eigenvalue of the monodromy operator h_* at some point of $\{f = 0\}$.*

J. Denef and F. Loeser [38] defined a topological zeta function $Z_{\mathrm{top}}(t)$ generalizing Igusa's zeta function. The analogous conjecture is stated for this function. Loeser [113, 114], W. Veys [167], Artal-Bartolo, P. Cassou-Noguès, Luengo and Melle-Hernández [15, 16] and B. Rodrigues and Veys [136] proved various special cases of this conjecture. See [168] for an excellent survey on this topic and the article of Artal-Bartolo, Luengo and Melle-Hernández in this volume [17].

In [88] a motivic version of the zeta function of the monodromy is discussed and compared with the motivic zeta function of Denef and Loeser.

4 Spectrum

In the case $(Y, 0)$ smooth, Steenbrink [153] showed that there exists a mixed Hodge structure on the Milnor fibre. Let $H = H^n(X_\delta, \mathbb{Z})$. Such a mixed Hodge structure consists of an increasing *weight* filtration

$$0 = W_{-1} \subset W_0 \subset \cdots \subset W_{2n} = H \otimes \mathbb{Q}$$

of $H \otimes \mathbb{Q}$ and a decreasing *Hodge* filtration

$$H \otimes \mathbb{C} = F^0 \supset F^1 \supset \cdots \supset F^n \subset F^{n+1} = 0.$$

It follows from a result of M. Saito [141] that in the general case, the analogue in cohomology of the short exact sequence in §2 can be considered as a sequence of mixed Hodge structures (see [64]).

The mixed Hodge structure is used to define the spectrum of a singularity. The spectrum was defined by Steenbrink [153] and Arnold [10] in the case when $(Y, 0)$ is smooth and in [64] in the general case.

Definition 4.1. The *spectrum* $\mathrm{Sp}(f)$ of f is defined as follows. Let $p \in \mathbb{Z}$, $0 \leq p \leq n$. A rational number $\alpha \in \mathbb{Q}$ with $n-p-1 < \alpha \leq n-p$ is in $\mathrm{Sp}(f)$ if and only if $e^{2\pi i\alpha}$ is an eigenvalue of the semisimple part of h^* on $F^pH/F^{p+1}H$. Here $H = H^n(X_\delta, \mathbb{C})$ if $Y = \mathbb{C}^{n+1}$ and $H = H^{n+1}(Y_\eta, X_\delta, \mathbb{C})$ in the general case. The multiplicity of α is the dimension of the corresponding eigenspace.

The spectrum is an unordered tuple of ν rational numbers $\alpha_1, \ldots, \alpha_\nu$ which lie between -1 and n. We order these numbers as follows:

$$-1 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_\nu < n.$$

There is a symmetry property

$$\alpha_i + \alpha_{\nu+1-i} = n - 1.$$

V.V. Goryunov [75] computed the spectra of the simple, uni- and bimodal hypersurface singularities. Steenbrink [155] compiled tables of the spectra for all $\mathcal{K}$-unimodal ICIS. If $Y = \mathbb{C}^{n+1}$ and f is Newton non-degenerate then the spectrum can be computed from the Newton diagram, see [140, 165]. Other methods to compute the spectrum in the case when $(Y, 0)$ is smooth have been given by Schulze and Steenbrink [149].

The most famous property of the spectrum is the semicontinuity conjectured by Arnold [10] and proved by Steenbrink [154] for the case when $(Y, 0)$ smooth and the author and Steenbrink in the general case [64]. A weaker form of this theorem has already been proved by Varchenko [164].

Theorem 4.2 (Semicontinuity theorem). *The spectrum behaves semicontinuously under deformation of the singularity in the following sense: If f' (with $\nu' < \nu$) appears in the semi-universal deformation of f, then*

$$\alpha_i \leq \alpha'_i.$$

The variance of the spectrum measures the distribution of the spectral numbers with respect to the central point and is defined by

$$V = \frac{1}{\nu} \sum_{i=1}^{\nu} \left(\alpha_i - \frac{n-1}{2} \right)^2.$$

C. Hertling [92] proposed the following conjecture

Conjecture 4.3 (Hertling). *If $(Y, 0)$ is smooth (so $\nu = \mu$), then*

$$V \leq \frac{\alpha_\mu - \alpha_1}{12}.$$

One has equality if f is weighted homogeneous, as shown by A. Dimca [40] and Hertling [92]. M. Saito [143] showed that Hertling's conjecture holds for all irreducible plane curve singularities. Th. Brélivet [24, 25] showed that the conjecture holds for all curve singularities. Brélivet and Hertling have stated more general conjectures involving higher moments [26].

Let $Y = \mathbb{C}^{n+1}$. We shall now give several different interpretations of the smallest exponent α_1.

Let ω be a holomorphic $(n+1)$-form on $\mathbb{C}^{n+1}$. For $0 < |t| < \delta$ let $\eta(t)$ be a continuously varying homology class of dimension n on X_t and consider the function

$$I(t) = \int_{\eta(t)} \frac{\omega}{df}.$$

This function admits an asymptotic expansion as t tends to zero:

$$I(t) = \sum_{\alpha,q} \frac{1}{q!} C_{\alpha,q}^{\omega,\eta} t^{\alpha} (\log t)^q,$$

such that $q \in \mathbb{Z}$, $0 \leq q \leq n$, $\alpha \in \mathbb{Q}$, $\alpha > -1$ and $e^{2\pi i \alpha}$ is an eigenvalue of the semisimple part of the monodromy operator. By [162] we have

$$\alpha_1 = \beta_{\mathbb{C}} - 1 := \min\{\alpha \mid \exists \omega, \eta, q \ C_{\alpha,q}^{\omega,\eta} \neq 0\}.$$

The number $\beta_{\mathbb{C}}$ is the *complex singularity index* (cf. [7], where in fact the number $\frac{n+1}{2} - \beta_{\mathbb{C}}$ is called the complex singularity index). For a simple singularity in $\mathbb{C}^3$, one has $\beta_{\mathbb{C}} = 1 + \frac{1}{N}$ where N is the Coxeter number of the singularity (cf. [7]).

With the notations of §3, let k_i be the multiplicity of $\pi^*(dx_0 \wedge \ldots \wedge dx_n)$ along the divisor E_i, $i = 1, \ldots, s$. Let m_0 and k_0 be the order of $\widetilde{f}$ and the multiplicity of $\pi^*(dx_0 \wedge \ldots \wedge dx_n)$ respectively along the divisor $\widetilde{X}_0$. So $m_0 = 1$ and $k_0 = 0$. Let

$$\lambda := \min \left\{ \frac{k_i + 1}{m_i} \,\middle|\, i = 0, \ldots, s \right\}.$$

K.-Ch. Lo [112] has shown that

$$\beta_{\mathbb{C}} \geq \lambda.$$

T. Yano [172] and B. Lichtin [111] have shown that if $\lambda < 1$ then

$$\beta_{\mathbb{C}} = \lambda.$$

Varchenko [161] (see also [67, 166]) has shown that for a Newton non-degenerate function, $\beta_{\mathbb{C}} = 1/t_0$ where $(t_0, \ldots, t_0)$ is the intersection point of the diagonal $t \mapsto (t, \ldots, t)$ with the Newton diagram of f.

J. Kollar [97] has shown that λ is equal to the *log canonical threshold*.

Moreover, we have the following relations which were recently brought back into attention in the framework of multiplier ideals (see e.g. [69]).

For a rational number α we define the following ideal $\mathcal{A}_\alpha$ in the ring $\mathcal{O}_{Y,0}$ of analytic functions on $(Y, 0)$:

$$\mathcal{A}_\alpha := \left\{ \phi \in \mathcal{O}_{Y,0} \,\middle|\, \inf_{1 \leq i \leq s} \left(\frac{1 + k_i + \nu_i(\phi)}{m_i} - 1 \right) > \alpha \right\}$$

where $\nu_i(\phi)$ denotes the order of ϕ along the divisor E_i. This is a *multiplier ideal* in the sense of Y.-T. Siu and A. Nadel (see [69]).

Definition 4.4. We define a sequence of numbers

$$\xi_0 = 0 < \xi_1 < \xi_2 < \dots$$

as follows: $\mathcal{A}_\alpha = \mathcal{A}_{\xi_i}$ for $\alpha \in [\xi_i, \xi_{i+1})$ and $\mathcal{A}_{\xi_{i+1}} \neq \mathcal{A}_{\xi_i}$ for $i = 0, 1, \dots$. These numbers are called *jumping numbers*.

These numbers first appeared implicitly in a paper of A. Libgober [110]. The above definition is due to Loeser and M. Vaquié [115, 159].

Varchenko [163] (see also [31]) proved the following statement: If $\alpha \in (-1, 0)$, then

$$\alpha \in \mathrm{Sp}(f) \Leftrightarrow \alpha + 1 = \xi_i \text{ for some } i.$$

M. Saito [142] showed that the Bernstein-Sato polynomial $b_f(s)$ (see §1) has roots in $[0, 1)$ which do not come from the spectrum of f.

5 Monodromy and the Topology of the Singularity

Let B_ε be a closed ball as in §1 and let $K := f^{-1}(0) \cap \partial B_\varepsilon \cap Y$ be the link of the singularity $(X_0, 0)$.

First assume that $Y = \mathbb{C}^{n+1}$. Milnor [125] has shown that the manifold K is a homology sphere (and when $n \neq 2$ actually a topological sphere) if and only if the integer

$$\Delta(1) = \det(\mathrm{id}_* - h_*)$$

is equal to ± 1. Let $n \neq 2$ and assume that K is a topological sphere. The differentiable structure of K is completely determined by the Kervaire invariant $c(X_\delta) \in \mathbb{Z}_2$ if n is odd, or by the signature of the intersection matrix A if n is even (cf. [125]). If n is odd, then by a theorem of J. Levine [108] the Kervaire invariant is given by

$$c(X_\delta) = \begin{cases} 0 & \text{if } \Delta(-1) \equiv \pm 1 \pmod 8, \\ 1 & \text{if } \Delta(-1) \equiv \pm 3 \pmod 8. \end{cases}$$

If n is even and f is weighted homogeneous, then the signature of the intersection matrix A is determined by the eigenvalues of the monodromy, see [152]. Hence in many cases the complex monodromy operator determines the differentiable structure of K.

In fact it is shown in [125] that K is $(n-2)$-connected, that the rank of $H_{n-1}(K)$ is equal to the dimension of the eigenspace of h_* corresponding to the eigenvalue 1, and that, if the rank is equal to zero, the order of $H_{n-1}(K)$

is equal to $|\Delta(1)|$. Here we use reduced homology if $n = 1$. If f is weighted homogeneous, a formula for the rank of $H_{n-1}(K)$ and for $\Delta(1)$ in terms of weights and degree is given in [126].

If $n = 1$, Durfee [46] relates the topology of a branched cyclic cover of the link K to the characteristic polynomial of the monodromy. B. G. Cooper [34] has calculated the homology of the link K for some special weighted homogeneous polynomials f.

Let C be the matrix of h_* with respect to a basis of $H_n(X_\delta, \mathbb{C})$ and let I be the $\mu \times \mu$ identity matrix. In the case when f is weighted homogeneous, Orlik [129] has stated the following conjecture:

Conjecture 5.1 (Orlik). *The matrix C can be diagonalized over the integers, i.e. there exist unimodular matrices $U(t)$ and $V(t)$ with entries in the ring $\mathbb{Z}[t]$ so that*

$$U(t)(tI - C)V(t) = \mathrm{diag}(m_1(t), \ldots, m_\mu(t))$$

where $m_i(t)$ divides $m_{i+1}(t)$ for $i = 1, \ldots, \mu - 1$.

Since the ring $\mathbb{C}[t]$ is a principal ideal domain, such matrices exist over $\mathbb{C}[t]$. The conjecture implies that

$$H_{n-1}(K) = \mathbb{Z}_{m_1(1)} \oplus \ldots \oplus \mathbb{Z}_{m_\mu(1)}$$

where $\mathbb{Z}_1$ is the trivial group and $\mathbb{Z}_0$ is the infinite cyclic group.

The conjecture holds for f weighted homogeneous and $n = 2$ as follows from [131].

Sometimes the conjecture is extended to germs f with finite monodromy. Then the following is known about the more general conjecture. From [2] one can derive that Orlik's conjecture is true for irreducible plane curve singularities. F. Michel and C. Weber [123, 124] have shown that Orlik's conjecture is false for plane curve singularities with more than one branch.

Let $n = 2$ and f be weighted homogeneous with weights q_1, q_2, q_3 and degree d. Then Y. Xu and S.-T. Yau [171] have shown that the characteristic polynomial $\Delta(t)$ of the monodromy and the fundamental group $\pi_1(K)$ of the link determine the embedded topological type of $(X_0, 0)$. Let K be in addition a rational homology sphere. Then R. Mendris and A. Némethi [122] have observed that it follows from [56] that $\Delta(t)$ is already determined by $\pi_1(K)$. Define

$$R := d - q_1 - q_2 - q_3.$$

Némethi and L. I. Nicolaescu [127] have derived from [56] that

$$\frac{\Delta(t)}{\Delta(1)} = 1 + \frac{\mu}{2}(t-1) + \ldots \quad \text{and} \quad \frac{\Delta^*(t)}{\Delta^*(1)} = 1 + \frac{R}{2}(1 - e_{\mathrm{st}})(t-1) + \ldots$$

where e_{st} is Batyrev's stringy Euler characteristic of $(X_0, 0)$ (cf. [19]) as generalized by Veys in [169].

Now let $Y = \mathbb{C}^{n+1}$ and let f be again general. The matrix V of §2 is the matrix of the (integral) Seifert form of the singularity $(X_0, 0)$. If $n \geq 3$ then results of M. Kervaire [96] and J. Levine [109] show that the Seifert form determines the (embedded) topological type of the singularity, see also [45]. If $n = 1$ and f defines an irreducible curve singularity, then it follows from [32] and [173] that the integral monodromy and even the rational monodromy determines the topology of the singularity. M.-C. Grima [78] has given examples of plane curve singularities with two branches of different topological types with the same rational monodromy, but different integral monodromy. Ph. Du Bois and Michel [42, 44] have shown that the integral Seifert form does not always determine the topology of the singularity in the case $n = 1$. Using suspensions of the examples of Du Bois and Michel, Artal-Bartolo [13] has shown that the same applies to the case $n = 2$.

Let $(Y, 0)$ be a weighted homogeneous ICIS and let f be weighted homogeneous. Let $L := \partial B_\varepsilon \cap Y$. Dimca [39] has shown that for $n \geq 2$ one has the following formula for the Betti numbers of L and K:

$$b_{n+1}(L) + b_n(K) = \dim \ker(\mathrm{Id}_* - h_*).$$

Hamm [90] computed the characteristic polynomial of the monodromy for some ICIS which are generalizations of the Brieskorn-Pham singularities, the Brieskorn-Hamm-Pham singularities. The homology torsion of the link of a Brieskorn-Hamm-Pham singularity was computed by Randell [134].

Acknowledgements

This research was partially supported by the DFG-programme "Global methods in complex geometry" (Eb 102/4–3). The author is grateful to Th. Eckl for drawing his attention to jumping numbers of multiplier ideals and the connection with the spectrum (see §4). He would like to thank N. Budur, S. Gusein-Zade, A. Melle-Hernández, T. Torrelli, and W. Veys for their useful comments.

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Algorithmic Resolution of Singularities

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Dedicated to Gert-Martin Greuel on the occasion of his 60th birthday

Abstract

Although the problem of the existence of a resolution of singularities in characteristic zero was already proved by Hironaka in the 1960s and although algorithmic proofs of it have been given independently by the groups of Bierstone and Milman and of Encinas and Villamayor in the early 1990s, the explicit construction of a resolution of singularities of a given variety is still a very complicated computational task. In this article, we would like to outline the algorithmic approach of Encinas and Villamayor and simultaneously discuss the practical problems connected to the task of implementing the algorithm.

Introduction

The problem of existence and construction of a resolution of singularities is one of the central tasks in algebraic geometry. In its shortest formulation it can be stated as: Given a variety X over a field K, a resolution of singularities of X is a proper birational morphism $\pi : Y \longrightarrow X$ such that Y is a non-singular variety.

Historically, a question of this type has first been considered in the second half of the 19th century – in the context of curves over the field of complex numbers. It was already a very active area of research at that time with a large number of contributions (of varying extent of rigor) and eventually lead to a proof of existence of resolution of singularities in this special situation at the end of the century. Although it does not seem feasable to even give a nearly complete list of important contributions to the field during that

1991 *Mathematics Subject Classification.* 14E15, 14Q15, 14B05, 32S45
Key words. Algorithmic resolution of singularities, blowing up, center

period, there are certain names which have to be mentioned in this context, e.g. L. Kronecker, M. Noether and A. Brill. After the case of curves over the complex numbers had been proved, the task was generalized slightly by passing from curves to surfaces. To this extended task, many contributions were made by the italian school, among others by O. Chisini, G. Albanese and F. Severi; but many of these treatments lacked the necessary rigor in the proofs. Thus the first mathematically rigorous proof of the existence of resolution of singularities for surfaces over the field $\mathbb{C}$ was presented by R.J. Walker in 1935 ([13]) by patching the local arguments of the article of H.W. Jung dating from 1908 ([11]) in a suitable way; this article of Jung had locally studied surfaces in 3–dimensional space by means of a projection to the plane, proving that a resolution of singularities exists locally in the given setting.

All of these early contributions to the task of resolving singularities relied on analytic arguments. It was not until the early 1930s that a more algebraic approach to dealing with geometric problems became established which allowed a more systematic treatment. This change of point of view and methods manifests itself in the groundbreaking work of O. Zariski, e.g. in his 1939 proof of the existence of resolution of singularities of surfaces over an algebraically closed field of characteristic zero ([14]) and the proof for the three-dimensional case in 1944 ([15]). It also lead the way to considering the problem in full generality, i.e. without restriction to the dimension of X or in arbitrary characteristic. In positive characteristic, only partial results are known; the general case is still open ([1], [5]). In characteristic zero, however, the existence of resolution of singularities in the general case has been proved by H. Hironaka in his monumental article in 1964 ([10]). In fact, he was the first one to consider the non-hypersurface case and introduced the concept of a standard basis and a generalization of the order of a hypersurface at a point as tool for achieving his goal of proving the general case. But his proof is highly non-constructive, which led to an intensive interest in the search for a constructive approach, whether it is the quest for fast algorithms in special cases like the toric one or the task of finding an algorithm for the general case. To the latter problem, important contributions have been made independently by the groups of E. Bierstone and P. Milman and of O. Villamayor and S. Encinas since the late 1980s, which eventually gave rise to implementations in recent years.

In this article, we would like to give a brief overview of the constructive approach of Villamayor and Encinas and of the computational tasks arising when implementing the algorithm. To this end, it is important to understand that this algorithm actually considers the more general set-up, not only constructing a resolution, but a strong factorizing desingulariztion, from which principalization of ideals and embedded resolution of singularities can be obtained as corollaries – as well as resolution of singularities without reference to an embedding. Since the problem we shall be considering in this article is embedded resolution of singularities, we need to state this task in a more detailed way: Given a subscheme X of a smooth algebraic scheme W, the task is to construct a sequence of blowing ups of W at smooth centers such that

- the exceptional divisors in each step are normal crossing,
- the respective centers are normal crossing with them,
- the strict transform of X under the sequence of blowing ups is eventually smooth and normal crossing with the exceptional divisors and
- the blowing ups have only altered W in the points of $\mathrm{Sing}(X)$.

As the resolution process consists of a sequence of blowing ups, the first issue which we consider is the blowing up (in section 1), which is a very well known type of a birational map in algebraic geometry. Therefore the main purpose of this section is to fix notation and explain implementational aspects.
In the following section, the notions of the b-singular locus and of basic objects are introduced. These are Encinas' and Villamayor's way of describing the collection of data that is used to describe the current situation at each step of the resolution process, including appropriate information on the history of the process.
After treating the special case of monomial basic objects separately in section 3, the algorithm for finding appropriate centers is then described and illustrated by a detailed example in section4.
The final section 5 then briefly outlines how to use the implemented algorithm for some applications focusing on the problem of how to represent the final result of the resolution algorithm and how to extract information from it. This is again illustrated by means of the example of the preceding section.
The authors are supported in part by the DFG-Schwerpunkt "Globale Methoden in der komplexen Geometrie".

1 Blowing Up

As the main goal of this article is to explain how to construct a resolution of singularities algorithmically and how to compute this in practice, we start by explaining the main ingredients to the algorithms. The first one to consider is the blowing up at a given center. After briefly recalling the notion of the blowing up of a variety and listing some properties, we explain how to compute it by means of Gröbner bases techniques. We refer to [8] for more details on blowing ups and to [6] for the computational details.

Definition 1.1. Let W be an algebraic variety, $C \subseteq W$ a closed sub–variety defined by the ideal sheaf $\mathcal{K} \subseteq \mathcal{O}_W$. The blowing up of W with center C is

$$\pi : \widetilde{W} := \mathrm{Proj}\big(\textstyle\bigoplus_{n\geq 0}\mathcal{K}^n\big) \to W.$$

Theorem 1.2. *(Universal property of blowing up): Let $f : Y \to W$ be a morphism such that $\mathcal{K}\mathcal{O}_Y$ is locally principal. Then there is a unique morphism $g : Y \to \widetilde{W}$ such that $f = \pi \circ g$.*

Remark 1.3. The blowing up with center C has the following properties:

(1) $\widetilde{W}$ is an algebraic variety.
(2) π is proper.
(3) π induces an isomorphism over $W \setminus C$
(4) $\mathcal{K}\mathcal{O}_{\widetilde{W}}$ is a locally principal ideal sheaf.
(5) If W is projective then $\widetilde{W}$ is projective

In the context of this article, we only consider blowing ups at non-singular centers, as these are the only ones appearing in the resolution process; more precisely, we will later impose further conditions on the choice of the center leading to the notion of a permissible center:

Definition 1.4. Let W be an algebraic variety.

(1) A subscheme $E \subset W$ is called normal crossing at a point $p \in E$ if p is a regular point of W and there is a regular system of parameters $f_1, \dots, f_k$ for $p \in W$ such that E is given by the equation $f_1 \cdot \dots \cdot f_\ell = 0$ for some $1 \leq \ell \leq k$ on a Zariski neighbourhood of p.

(2) Let a subscheme $E \subset W$ be normal crossing at all points of E. Then a regular closed subscheme $Y \subset W$ is called a permissible center w.r.t. W and E, if Y has normal crossings with E.

For dealing with explicit examples, it is more convenient to pass to a covering by affine charts. In particular, the calculation of the blowing up can easily be formulated in the affine case in a way which is also accessible to direct implementation:

Remark 1.5. Let $U \subset W$ be an affine open subset and and denote $\Gamma(U, \mathcal{O}_W)$ by A and $\Gamma(U, \mathcal{K}) = \langle f_1, \dots, f_m \rangle \subseteq A$ by K. Then the blowing up of U at the center $C \cap U$ is

$$\pi^{-1}(U) = \mathrm{Proj}\left(\bigoplus_{n \geq 0} K^n\right) = \bigcup_{i=1}^{m} \mathrm{Spec}\ A\left[\frac{f_1}{f_i}, \dots, \frac{f_m}{f_i}\right].$$

$\mathrm{Spec}\ A\left[\frac{f_1}{f_i}, \dots, \frac{f_m}{f_i}\right]$ is called the *i–th affine chart* of the blowing up.

To compute the blowing up explicitly, we consider the graded A-algebra homorphism $\Phi : A[y_1, \dots, y_m] \to \bigoplus_{n \geq 0} K^n t^n \subseteq A[t]$ defined by $\Phi(y_i) = tf_i$.

Then $\bigoplus_{n\geq 0} K^n$ is obviously isomorphic to $A[y_1,\ldots,y_m]/\mathrm{Ker}(\Phi)$ and we can describe the situation by means of the embedding

$$\pi^{-1}(U) \cong V(\mathrm{Ker}(\Phi)) \subseteq \mathrm{Spec}(A) \times \mathbb{P}^{m-1}\,.$$

Now let $X = V(J) \subseteq W$ be a subvariety defined by the ideal sheaf[1] $J \subseteq \mathcal{O}_W$. For describing how X is transformed under the blow-up, the following notions are used:

Definition 1.6. (1) The *total transform* of X, $\pi^*(X)$, is the subvariety of $\widetilde{W}$ defined by $\pi^*(J) = J\mathcal{O}_{\widetilde{W}}$.

(2) The *strict transform* of X, $\widetilde{X}$ is the Zariski closure of $\pi^{-1}(X \smallsetminus V(\mathcal{K}))$ in $\widetilde{W}$. Its ideal sheaf is $\widetilde{J} := J\mathcal{O}_{\widetilde{W}} : \mathcal{K}\mathcal{O}_{\widetilde{W}}^{\infty}$.

(3) The *exceptional hypersurface* E is the reduced subvariety of $\widetilde{W}$ defined by $\mathcal{K}\mathcal{O}_{\widetilde{W}}$; the corresponding ideal sheaf is denoted by $I(E)$

(4) The *weak transform* of X, $\overline{X}$, is defined by the ideal sheaf $\overline{J}$ satisfying the properties $J\mathcal{O}_{\widetilde{W}} = I(E)^c\overline{J}$ and $I(E) \nmid \overline{J}$

Remark 1.7. The strict transform $\widetilde{X}$ is the blowing up of X in the subvariety defined by $\mathcal{K}\mathcal{O}_X$,

$$\widetilde{X} = \mathrm{Proj}\big(\textstyle\bigoplus_{n\geq 0}(\mathcal{K}\mathcal{O}_X)^n\big).$$

Example 1.8. To illustrate the difference between the weak and the strict transform under a blow-up, we will now consider the blow-up of the affine variety defined by the ideal $J = \langle xy, x^3 + y^3 + z^3\rangle \subset \mathbb{C}[x,y,z] = \mathcal{O}_W$ at the origin, that is at the center C defined by $K = \langle x, y, z\rangle$.

Then $\widetilde{W} \subset \mathbb{A}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ is the set $\{(x,y,z;u:v:w) \in \mathbb{A}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}} \mid uy - xv = uz - xw = vz - yw = 0\}$ which can be covered by the three affine charts corresponding to the open sets $D(u), D(v)$ and $D(w)$. In each of these, the above equations imply that the affine chart looks again like an $\mathbb{A}^3_{\mathbb{C}}$.

Chart 1: $u \neq 0$.

$$\begin{array}{lcl} \mathcal{I}(H) & = & \langle x\rangle\,, \\ \pi^*(J) & = & \langle x^2v, x^3 + x^3v^3 + x^3w^3\rangle \subset \mathbb{C}\,[x,v,w]\,, \\ \widetilde{J} & = & \langle v, 1 + w^3\rangle\,, \\ \overline{J} & = & \langle v, x + xv^3 + xw^3\rangle = \langle v, 1 + w^3\rangle \cap \langle v, x\rangle\,. \end{array}$$

Chart 2: $v \neq 0$.

[1] Here $V(J)$ is considered with the structure sheaf defined by $\mathcal{O}_W/J$, i.e. not necessarily reduced.

By symmetry in x and y in the generators of the original ideal, the equations here are the same as in chart 1 after exchanging x by y and v by u.

Chart 3: $w \neq 0$.

$$\begin{array}{lcl} \mathcal{I}(H) & = & \langle z \rangle \,, \\ \pi^*(J) & = & \langle uvz^2, u^3z^3 + v^3z^3 + z^3 \rangle \subset \mathbb{C}\,[u, v, z]\,, \\ \widetilde{J} & = & \langle uv, u^3 + v^3 + 1 \rangle = \langle u, 1 + v^3 \rangle \cap \langle v, 1 + u^3 \rangle\,, \\ \overline{J} & = & \langle uv, u^3z + v^3z + z \rangle = \langle u, 1 + v^3 \rangle \cap \langle u, z \rangle \cap \langle v, z \rangle\,. \end{array}$$

In particular, we see that the strict transform does not contain any components which are contained in the hypersurface H, whereas in the weak transform all of those components are present except the hypersurface H itself.

Computational Remark 1.9. In example 1.8, the explicit calculation of the blowing up was rather straight forward due to the simplicity of the generators of the ideal K. In general, however, generators for the ideal of the center (in an affine chart) are not of such a simple structure and hence the preimage computation of remark 1.5 for the blowing up, which is a Gröbner basis calculation involving the original variables and additionally the new variable t and one new variable for each generator of K, can become quite expensive. In practice, this problem can be tackled in two ways. First of all, blowing up at a center consisting of several disjoint components may be implemented as a single blowing up or alternatively as a sequence of blowing ups, each of which involving just one of the components, because outside the center the blowing up is an isomorphism as we already mentioned; the latter leads to fewer and simpler generators for the centers and turns out to be faster than the other variant, although it produces more affine charts. The other improvement, that can be implemented, is that instead of using the given generators of the center, it is possible to drop redundant generators before continuing.

After computing the blowing up of the ambient space W, determining the total, weak and strict transform of subvarieties does not pose any additional difficulties. The only further issue that has to be considered is that the calculations of the weak and strict transform are carried out by means of iterated ideal quotients which are in turn Gröbner basis calculations. Therefore it is again vital for the efficiency of an implementation that the number of variables is kept as small as possible.

2 The *b*-Singular Locus and Basic Objects

After outlining in the previous section that the blowing up itself is not too difficult to handle algorithmically, we now turn our focus to the heart of the algorithm, the choice of the center. Before we can describe how it is constructed in section 4, we need some preparations including the notions of the b-singular locus and of basic objects in this section and the separate treatment of a special case in the following section.

In the simplest case, the case of resolving singularities of plane curves, it is a well known fact that the centers are always finite sets of points and the choice of the respective points is governed by an invariant whose main ingredient is the order of the power series locally generating the ideal of the curve. In the general case, the first important ingredient to the governing invariant is a generalization of this, the order of an ideal at a point:

Definition 2.1. Let W be a non-singular algebraic variety, $J \subseteq \mathcal{O}_W$ an ideal sheaf and $w \in W$. The order at w with respect to J is defined as

$$v_J(w) = \sup\{m \mid J_w \subseteq \mathfrak{m}_{W,w}^m\}.$$

The function $v_J : W \to \mathbb{N}$ is upper semi–continuous[2]

If $X \subseteq W$ is the subvariety defined by J, often the notation v_X is used instead of v_J.

Computing the order $v_J(w)$ at a point $w \in W$ can be done using the following construction: Let $A = k[[x_1, \ldots, x_n]]$ be the power series ring and let $J = \langle f_1, \ldots, f_r \rangle \subseteq A$ be an ideal. We set

$$\widehat{\Delta}(J) := \left\langle f_1, \ldots, f_r, \left\{ \frac{\partial f_i}{\partial x_j} \right\}_{i,j} \right\rangle.$$

It is not difficult to prove that the definition of $\widehat{\Delta}(J)$ neither depends on the choice of generators of J nor on the choice of regular parameters of A. Its properties and its relation to the order at a point are outlined by the following propositions:

Proposition 2.2. *Let W be a non-singular algebraic variety, $J \subseteq \mathcal{O}_W$ an ideal sheaf. Then there exists an ideal sheaf $\Delta(J) \subseteq \mathcal{O}_W$ such that $\Delta(J)\widehat{\mathcal{O}}_{W,w} = \widehat{\Delta}(J\widehat{\mathcal{O}}_{W,w})$ for all $w \in W$.*

See [4] for a proof. Defining inductively $\Delta^0(J) = J$ and $\Delta^r(J) = \Delta(\Delta^{r-1}(J))$, it is then easy to see that

Lemma 2.3. *(1) $v_J(w) = b > 0$ if and only if $v_{\Delta(J)}(w) = b - 1$.*

(2) $v_J(w) \geq b > 0$ if and only if $w \in V(\Delta^{b-1}(J))$.

[2]To be more precise, it is Zariski upper-semicontinuous and infinitessimally upper-semincontinuous; in particular, it can only stay constant or drop upon blowing up, but never increase.

On the basis of these observations, we can now describe the b-singular locus of J, i.e. the set of points where the order is at least b:

$$\mathrm{Sing}_b(J) := \{w \in W \mid v_J(w) \geq b\} = V(\Delta^{b-1}(J)).$$

Computational Remark 2.4. In practice, it is, of course, not feasible to compute $\widehat{\Delta}(J)$ at each point $w \in W$ separately. Passing to an affine covering, we can, however, obtain the desired result as follows: Let $W = V(g_1, \ldots, g_r)$ be a smooth affine algebraic variety and $J = \langle f_1, \ldots, f_s \rangle \subseteq \mathcal{O}_W$ an ideal.

If $r = 0$, i.e. W is an affine space, the calculation can be performed by directly applying the definition:

$$\Delta(J) = \left\langle f_1, \ldots, f_r \ , \ \left\{ \frac{\partial f_i}{\partial x_j} \right\}_{i,j} \right\rangle .$$

In the general case, the difficulty arises that we have to determine a system which induces a local system of parameters at each point w. To this end, we use the fact that W is smooth; more precisely, let $m := n - \dim(W)$ and L be the set of $m \times m$ submatrices of the Jacobian matrix of $(g_1, \ldots, g_r)$ with non-vanishing determinant. For $M \in L$ let $r(M)$ (resp. $c(M)$) be the set of row indices (resp. column indices) of the Jacobian matrix defining M. Let $A(M) = (A_{ij}(M))$ be defined by $A(M) \cdot M = \det(M) \cdot E_m$. On the open set defined by $\det(M) \neq 0$ in W we can use $\{x_i\}_{i \notin r(M)}$ as a regular system of parameters. There we can hence compute the respective partial derivatives and define the ideal $\Delta(J, M)$

$$\Delta(J,M) := \left(J + \left\langle \left\{ \det(M) \frac{\partial f_i}{\partial x_j} - \sum_{\substack{k \in r(M) \\ l \in c(M)}} \frac{\partial g_l}{\partial x_j} A_{lk}(M) \frac{\partial f_i}{\partial x_k} \right\}_{\substack{i \leq j \leq s \\ j \notin r(M)}} \right\rangle \right) : \det(M)^{\infty}.$$

Outside of $V(\det(M))$ this ideal coincides with $\Delta(J)$ as can easily be checked by direct computation; the saturation, on the other hand, enables us to remove all components which are contained in $V(\det(M))$. Hence $\Delta(J)$ can be obtained by computing the intersection of all $\Delta(J, M)$ where $M \in L$:

$$\Delta(J) = \bigcap_{M \in L} \Delta(J, M).$$

Remark 2.5. The notion of the b-singular locus, as it is defined above, apprears in the algorithmic approach of Villamayor and Encinas, which we follow in this article, but not in the approach of Bierstone and Milman. In the latter algorithm, the Hilbert-Samuel function and a slightly different notion of an order are used in its place. More precisely, they use a particular choice of the local system of parameters, imposing the condition that it contains the local generators of all exceptional hypersurfaces meeting this point; instead of considering the ideal generated by the generators of the ideal and their partial derivatives w.r.t. each of the elements of the local system of parameters, they then use $x_i \cdot \frac{\partial f_j}{\partial x_i}$ instead of $\frac{\partial f_j}{\partial x_i}$ whenever x_i corresponds to one of the exceptional hypersurfaces.

On the other hand, the b-singular locus of a variety X is not the only piece of data that has influence on the resolution process. If exceptional divisors are present, these need to be taken into account in a suitable way as well. To this end, all necessary data is collected into the notion of a basic object:

Definition 2.6. Let b be a positive integer, W a pure-dimensional smooth algebraic variety of dimension d, $X \subseteq W$ a subvariety. Let $E = \{H_1, \ldots, H_k\}$ be an ordered set of normal crossing hypersurfaces in W, $E_{bad} \subseteq E$ a subset.[3] The tuple $B = (W, X, b, E, E_{bad})$ is called a (d–dimensional) basic object. B is called monomial, if $I(X) = \prod_{H \in E} I(H)^{a(H)}$.

Consequently, the task of resolution of singularities needs to be reformulated in terms of resolution of basic objects. To this end, we first need to define how a basic object is transformed under a permissible blow-up:

Definition 2.7. Let $B = (W, X, b, E, E_{bad})$ be a basic object, $J = I(X)$, and $E = \{H_1, \ldots, H_r\}$. Let $Y \subseteq W$ be a smooth closed subvariety which is permissible w.r.t. W and E and which satisfies $Y \subseteq \mathrm{Sing}_b(B) := \mathrm{Sing}_b(X)$. Then the blowing up $\pi : \widetilde{B} \to B$ of the basic object B at the center Y is induced by the blowing up $\pi : \widetilde{W} \to W$ of W at Y (which gives rise to a new exceptional divisor H) in the following way:

Because $Y \subseteq \mathrm{Sing}_b(X)$, we may now consider $J\mathcal{O}_{\widetilde{W}} = I(H)^b \widetilde{J}$ for a suitable ideal $\widetilde{J} \subset \mathcal{O}_{\widetilde{W}}$, the weak transform of J under the blowing up. Denoting by $\{\widetilde{H}_1, \ldots, \widetilde{H}_r\}$ the set of strict transforms of the H_i, we define $\widetilde{E} = \{\widetilde{H}_1, \ldots, \widetilde{H}_r, H\}$. Then $\widetilde{B} = (\widetilde{W}, \widetilde{X}, b, \widetilde{E}, \widetilde{E}_{bad})$, where $\widetilde{X}$ is the algebraic variety defined by $\widetilde{J}$. The last piece of data which still needs to be specified is $\widetilde{E}_{bad}$; we postpone this to the following sections 3 and 4, as it is rather technical and will not be needed any earlier.

[3]The role of the subset E_{bad} in the resolution process is rather technical. As the construction of the center involves a induction on the dimension of the ambient space and hence the construction of lower dimensional auxilliary objects, E_{bad} is used to indicate those among the execeptional divisors which need to be taken into account before the subsequent descent in dimension.

Definition 2.8. A resolution of the singularities of a basic object B is a sequence of blowing ups

$$B(n) \xrightarrow{\pi_n} B(n-1) \to \cdots \to B(1) \xrightarrow{\pi_1} B,$$

where the $B(i) = (W(i), X(i), b, E(i), E_{bad}(i))$ are basic objects and

$$\pi_i : W(i) \to W(i-1)$$

blowing ups at permissible centers $Y_i \subseteq \mathrm{Sing}_b(X(i-1))$, such that

(a) $\mathrm{Sing}_b(X(n), b) = \emptyset$

(b) $W(n) \setminus \bigcup_{H \in E(n)} H \cong W \setminus \mathrm{Sing}_b(X)$

(c) $X(n)$ has normal crossings with $E(n)$.

Example 2.9. Consider the basic object $B = (\mathbb{C}^2, V(x^3 - y^5), 3, \emptyset, \emptyset)$ and let $\pi : W(1) \to \mathbb{C}^2 = W$ be the blowing up of $\mathbb{C}^2$ at 0. Denoting by $X(1)$ the strict transform of $X = V(x^3 - y^5)$ and by H the exceptional divisor, we obtain the following resolution of singularities of B:

$$B(1) = (W(1), X(1), 3, \{H\}, \{H\}) \to B.$$

Here it is important to observe that $\mathrm{Sing}_3(B(1)) = \emptyset$ and $B(1)$ is resolved, although $X(1)$ still has a singularity at the origin which it is a cusp, i.e. of order 2.

The key point to the whole resolution process, the choice of the suitable centers, is governed by an invariant $f_{(W,X,b,E,E_{bad})} : X \to \mathcal{I}$ assigning to each point of X a value in a totally ordered set $\mathcal{I}$. As the maximal locus of this invariant is determining the upcoming center and as the decrease of its maximal value under blowing up is the measure for the progress in the resolution process, the invariant has to be Zariski upper semicontinuous as well as infinitessimally upper semicontinuous. In the general case, the construction of this invariant it rather complicated and involves an iterated descent in dimension each giving rise to a new auxilliary basic object. The details of this construction are outlined in section 4, following the algorithmic approach of Villamayor and Encinas. One special case, however, has to be treated separately, the monomial case, because in this situation the general invariant does not provide a suitable approach.

Remark 2.10. In the algorithmic approach of Bierstone and Milman resp. the one of Wlodarczyk a similar collection of data is used, which Wlodarczyk calls a marked ideal; it contains information on the ambient space, the ideal itself, the exceptional divisors and an integer used in a way similar to the b in a

basic object. Although the Bierstone-Milman approach uses the strict instead of the weak transform, the way in which the invariant is constructed (including the descent in dimension) also follows the same type of main ideas which will be outlined for Villamayor's construction in the subsequent sections. This reflects the fact that all of these approaches have their roots in the non-constructive proof of Hironaka; the subtle but important differences arise from the different approaches to filling in the constructive details.
As the main goal of this article is to explain a construction of resolution of singularities and the practical problems arising in its implementation, the simultaneous treatment of the approaches of Villamayor and Encinas and of Bierstone and Milman is beyond the scope of this article and we hence focus on one of the two, the one of Villamayor and Encinas.

3 The Monomial Case

Before turning to the general case, we still need to deal with one special situation separately, the case of a monomial basic object:

Let $B = (W, X, b, E, E_{bad})$ be a monomial basic object where $E = \{H_1, \dots, H_r\}, I(X) = \prod_{i=1}^r I(H_i)^{a_i}$. We define

$$\begin{array}{rcl} f_B : X & \longrightarrow & \mathbb{Z}^{\#E+2} \\ x & \longmapsto & (-p(x), w(x), \mathfrak{i}(x)) \end{array}$$

where $p(x)$, $w(x)$ and $\mathfrak{i}(x)$ are defined by

$$\begin{aligned} p(x) &= \min\left\{ q \;\middle|\; x \in H_{i_1} \cap \ldots \cap H_{i_q} \text{ and } \sum_{j=1}^{q} a_{i_j} \geq b \right\} \\ w(x) &= \max\left\{ \sum_{j=1}^{p(x)} a_{i_j} \;\middle|\; x \in H_{i_1} \cap \ldots \cap H_{i_{p(x)}} \;,\; \sum_{j=1}^{p(x)} a_{i_j} \geq b \right\} \\ \mathfrak{i}(x) &= \max\left\{ (i_1, \dots, i_{p(x)}) \;\middle|\; \begin{array}{c} i_1 < \ldots < i_{p(x)}, \\ x \in H_{i_1} \cap \ldots \cap H_{i_{p(x)}} \end{array} \;,\; \sum_{j=1}^{p(x)} a_{i_j} \geq b \right\} \end{aligned}$$

Remark 3.1. At this point, it is important to observe that in the monomial case the subset E_{bad} of E is not considered in any way.

Example 3.2. As an example for the monomial case, let us consider the problem of resolving the basic object

$$B = (\mathbb{C}^2, V(x^2y^2), 2, \{V(x), V(y)\}, \emptyset).$$

By direct calculation, we can check that along the exceptional divisors the value of p is -1 and the one of w is 2 at all points. Therefore the choice of the upcoming center has to be made on the basis of the last entry $\mathfrak{i}$ which has the value (2) for points on $V(y)$ and (1) on $V(x)$. This leads to the center $V(y)$ with invariant $(-1, 2, (2))$.

After one blowing up at this center, the transformed object is

$$\widetilde{B} = (\mathbb{C}^2, V(x^2), 2, \{V(x), \emptyset, V(y)\}, \emptyset).$$

By the same direct calculation as before, the subsequent center is now $V(x)$ with invariant value $(-1, 2, (1))$. After this second blowing up, the object is clearly resolved.

4 The Tower of a Basic Object

In the general case, the governing invariant of the resolution algorithm is constructed inductively by means of a descent in dimension. Therefore, we will first define the respective 'fragment' of the invariant corresponding to a basic object and then explain how an auxilliary basic object is constructed whose ambient space is of smaller dimension. Iterating this process, we obtain a tower of basic objects and then construct the invariant by concatenating the 'fragments' of the invariant corresponding to the objects in the tower.

Given such a tower of basic objects, we then need to define how the tower is transformed under a blow-up and how the invariant is constructed for the transformed tower.

Construction 4.1 (Building the Tower). To define the 'fragments' of the invariant, let $B = (W, X, b, E, E_{bad})$ be a basic object and define

$$f_B : X \to \mathbb{Z}^2 \quad \text{by } f_B(x) = (v_X(x), n_x(E))$$

where $n_x(E) = \#\{H \in E_{bad} \mid x \in H\}$.

For a given basic object $B = (W, X, b, E, E_{bad})$, which we want to resolve, we now construct locally in the neighbourhood of every point $w \in W$ a tower of lower dimensional basic objects. If the dimension of W is one or if B is monomial then the tower is $T_0(B) = \{B\}$.

Otherwise, let $Y = \{x \in X \mid (v_X(x), n_x(E)) \text{ maximal }\}$ and consider two cases separately: If $\dim(Y) = \dim(W) - 1$, denote the reduced variety associated to the top-dimensional part of Y by Y_{eq} and define the tower as $T_0(B) = \{B, B_{aux}\}$ where B_{aux} is the auxilliary basic object of the form

$(Y_{eq}, Y_{eq}, 1, \emptyset, \emptyset)$.[4] If this is not the case, set $E_{bad} = \{H_1, \ldots, H_s\}$[5] and define $X' \subseteq W$ by

$$\begin{array}{rcl} b' & = & \max\{v_X(x) \mid x \in X\} \\ m & = & \max\{n_x(E) \mid x \in X, v_X(x) = b'\} \\ I(X') & = & I(X) + (\prod_{i_1 < \cdots < i_m} \sum_{j=1}^{m} I(H_{i_j}))^{b'} \\ E' & = & E \setminus E_{bad} \end{array}$$

Choose $U \subseteq W$ open and a smooth hypersurface $Z \subseteq U$ (a hypersurface of maximal contact) such that

- $Z \supseteq U \cap \{x \in X \mid (v_X(x), n_x(E) = (b', m)\}$
- Z intersects all $H \in E'$ transversally.
- $E' \cap Z := \{H \cap Z \mid H \in E'\}$ have normal crossings.

For every $w \in W$, such an open subset U and a hypersurface Z satisfying the above conditions exists (see [4]). To simplify the following notations, we assume $U = W$.[6] The coefficient ideal of X is defined (locally) as

$$\mathrm{Coeff}_Z(I(X')) = \sum_{i=0}^{b'-1} (\Delta^i(I(X'))\mathcal{O}_Z)^{\frac{b'!}{b'-i}}.$$

Let $C \subseteq Z$ be defined by $\mathrm{Coeff}_Z(I(X')) \subseteq \mathcal{O}_Z$, then the first auxiliary object is $B_Z := (Z, C, b'!, E, \emptyset)$. Note that $\mathrm{Sing}_{b'}(X') = \mathrm{Sing}_{b'!}(C)$.

At the beginning of the resolution process, we define the tower of the basic object B inductively by

$$T_0(B) = \{B\} \cup T_0(B_Z),$$

[4]Here, it is important to note that this auxilliary basic object is only introduced to also include this special case into the general way of expressing the construction of the center. The top-dimensional components of the locus of maximal invariant are hypersurfaces in W in this case; it can be checked directly (see [4]) that Y_{eq} consists of smooth hypersurfaces which do not intersect. These form the upcoming center which is needed in the algorithm to allow it to proceed in the usual way afterwards.

[5]If the tower is being constructed for the very first time, the set E_{bad} needs to be initialized and is set to contain all elements of E. If a tower has to be computed during the resolution process, the appropriate changes to the set E_{bad} are explained at respective steps in the algorithm.

[6]Actually, the fact that these constructions need to be performed in an open subset leads to the notion of a general basic object, a generalization of basic objects. But this is a rather technical concept which we would lead beyond the scope of this short article. In particular, it is not contributing to the general ideas behind the resolution process.

where the lower index 0 indicates that we have not performed any blowing up yet. Let $T_0(B) = \{B^{[0]}, \dots, B^{[e]}\}$ be the tower in the neighbourhood of a point $w \in W$. The invariant vector at w is then constructed as

$$\mathrm{inv}^{(0)}(w) = (f_{B^{[0]}}(w), \dots, f_{B^{[e]}}(w)).$$

Computational Remark 4.2. In practice, it is clearly not feasable to calculate the value of the invariant at each point $w \in W$ separately by a local construction. But this is not necessary for determining the center anyway, since the center is given by the locus of maximal value. It is therefore sufficient to calculate only the maximal value and its locus – of course iteratively passing from left to right through the invariant as the comparison is done lexicographically. The calculation of the b'-singular locus follows along the lines of construction 2.4; the calculation of the locus of maximal $n_x(E)$ (inside the b'-singular locus) is only a task of combinatorial nature, which can easily be implemented.

The crucial point in the construction is the descent in dimension, which forces the calculation to pass from the affine charts to open covers thereof in a similar way as in the calculation of the $\Delta(I(X))$. More precisely, b' denotes the maximal order and, hence, we know that the ideal $\Delta^{b'-1}(I(X))$ describing $\mathrm{Sing}_{b'}(X)$ is itself of order at most 1 at all points of W; on an affine chart, we may now choose a system of generators for $\Delta^{b'-1}(I(X))$ such that for a subset $f_1, \dots, f_s$ thereof the intersection of the singular loci of the corresponding hypersurfaces is empty and the conditions on the intersection properties of each of these hypersurfaces $V(f_i)$ with the exceptional divisors are satisfied outside $\mathrm{Sing}(V(f_i))$. As the open cover, we then choose the complements of the singular loci or, in practice, the complements of hypersurfaces generated by (an appropriate subset of the) partial derivatives of the f_i; the hypersurfaces Z on each of these open sets are chosen to be the respective $V(f_i)$.

The remaining part of the calculation of the Coeff-ideal does not cause any further difficulties. But we cannot recombine the local results to obtain an auxilliary basic object on W or on the affine chart, because the choice of the hypersurface Z clearly affects the auxilliary basic object. Nevertheless, the value of the invariant and hence the maximal locus are independent of the choice of Z and we can hence recombine the pieces of the maximal locus to obtain the center.

Construction 4.3 (Transformation of the Tower). After describing how a tower of basic objects is created from a given basic object, the next task is to consider how such a tower is transformed under a blowing up. (At this point it is important to note that the algorithm of Villamayor does not recompute the tower of basic objects from the transformed object, but transforms the whole tower instead – obtaining the new value of the invariant from the transformed tower.)

To this end, let $B = (W, X, b, E, E_{bad})$ be a basic object, say at the i-th step of the resolution process, $T_i(B) = \{B^{[0]}, B^{[1]}, \ldots, B^{[e]}\}$ the corresponding tower of basic objects, where $B^{[0]} = B$, and

$$Y = \{x \in X \mid \operatorname{inv}^{(i)}(x) \text{ is maximal } \}$$

the center computed by means of this tower, which is shown to be permissible for B in [4]. In particular, it consists of non-singular components which do not intersect.

Let $\widetilde{B}^{[j]}$ be the Blowing up of $B^{[j]}$ at the center Y and denote the collection of the transformed basic objects by $T'(B) = \{\widetilde{B}^{[0]}, \ldots, \widetilde{B}^{[e]}\}$. Let $\operatorname{inv}^{(i)}_{[j]} = \max\{f_{B^{[j]}}(x)\}$ be the maximal value of the j-th fragment of the invariant corresponding to the j-th auxilliary object before blowing up. Now, let k be minimal such that the locus of the fixed invariant value $\operatorname{inv}^{(i)}_{[k]}$ is empty[7] for $\widetilde{B}^{[k]}$. This can occur in four situations:

(a) the ideal of the total transform of the second entry of the basic object is a product of exceptional divisors, i.e. the new object is monomial

(b) the b-singular locus for the corresponding auxilliary object is empty

(c) the b'-singular locus for the corresponding auxilliary object is empty, but the b-singular locus is non-empty

(d) the b'-singular locus is non-empty, but the maximal number of exceptional divisors from E_{bad} simultaneously meeting a point thereof has dropped.

In the first case, the object in question is monomial. In case (b), the respective object is resolved and the maximal locus of the invariant truncated before $\operatorname{inv}^{(i)}_{[k]}$ is a permissible center. In case (c), we add all exceptional divisors of $\widetilde{B}^{[k]}$ to the corresponding set E_{bad} and define a new tower[8] by

$$T_{i+1}(B) := \{\widetilde{B}^{[0]}, \ldots, \widetilde{B}^{[k-1]}\} \cup T_0(\widetilde{B}^{[k]}).$$

In the last case, the set E_{bad} of $\widetilde{B}^{[k]}$ remains unchanged, we compute a descent in dimension with the new (lower) m to obtain an auxilliary object $B^{[k+1]}_{(i+1)}$

[7] As the invariant drops at each blowing up (by construction) and as the lowest dimension of an ambient space in the tower is in general 1, there is always such a k for which the locus of value $\operatorname{inv}^{(i)}_{[k]}$ is empty. In the special case that the tower does not reach 1 as the lowest dimension of the ambient space, the previous auxilliary object of lowest dimension is either monomial or is of the type B_{aux}. In the first case, the transformed auxilliary object is again monomial; in the other case, B_{aux} becomes empty after one blowing up by construction.

[8] By construction, all new auxilliary objects below $\widetilde{B}^{[k]}$ in the tower $T_0(\widetilde{B}^{[k]})$ are, hence, created with $E = \emptyset$.

whose set E_{bad} coincides with the set E' of the descent. Hence the new tower in this case is defined by

$$T_{i+1}(B) := \{\widetilde{B}^{[0]}, \ldots, \widetilde{B}^{[k]}\} \cup T_0(B^{[k+1]}_{(i+1)}).$$

Remark 4.4. The construction of the center ensures that all exceptional divisors which have been born after the corresponding fragment of the invariant last dropped are normal crossing with the centers arising from the maximal locus of the invariant. As soon as this fragment of the invariant drops, the subsequent auxilliary objects are recomputed and hence there is no way to predict the intersection properties of the exceptional divisors with the center arising from the new tower. Consequently, the subset $E_{bad} \subseteq E$ is used to mark those exceptional divisors which might cause problems concerning the normal crossing condition for the center; the value of E_{bad} for a given object in the tower is altered precisely at the moments when the subsequent parts of the tower need to be recomputed.

Example 4.5. To illustrate the rather technical construction, we consider a very simple example for which all calculations can still be done by hand:

$$X = V(z^2 - x^2y^2) \subseteq \mathbb{C}^3 = W$$

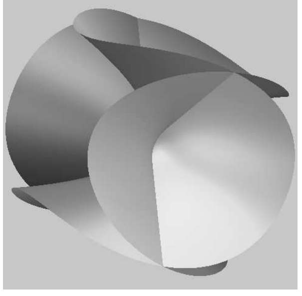

Figure 1: An illustration of the original surface $V(z^2 - x^2y^2)$.

(0) **Initialization step:** Construction of center of first blowing up.

Construction of the basic object B:

- $B = (\mathbb{C}^3, V(z^2 - x^2y^2), 2, \emptyset, \emptyset) =: B^{[0]}$

Maximal locus of $inv^{(0)}_{[0]}$:

- Computation of the maximal order:
$\Delta(z^2 - x^2y^2) = \langle z, xy^2, x^2y\rangle$

$\Delta^2(z^2 - x^2y^2) = \langle 1 \rangle$
Hence the maximal order b' is 2 and the b'-singular locus is $\mathrm{Sing}_2(B) = V(z, xy)$.

- Computation of maximal $n_x(E)$ unnecessary, because $E = \emptyset$

First descent in dimension:

- Choice of hypersurface of maximal contact:
  We choose $Z_0 = V(z)$ which clearly satisfies $\langle z \rangle \subseteq \Delta(z^2 - x^2y^2)$. As the set of exceptional divisors is empty, there are no further conditions to be checked.
- Construction of the first auxilliary object:
  $$\begin{aligned}\mathrm{Coeff}_{Z_0}(z^2 - x^2y^2) &= \langle x^2y^2, (xy^2)^2, (x^2y)^2 \rangle \\ &= \langle x^2y^2 \rangle\end{aligned}$$
  $B_{Z_0} = (\mathbb{C}^2, V(x^2y^2), 2, \emptyset, \emptyset) =: B^{[1]}$

Maximal locus of $inv^{(0)}_{[1]}$*:*

- Computation of the maximal order:
  $\Delta(x^2y^2) = \langle x^2y, xy^2 \rangle$
  $\Delta^2(x^2y^2) = \langle x^2, xy, y^2 \rangle$
  $\Delta^3(x^2y^2) = \langle x, y \rangle$
  $\Delta^4(x^2y^2) = \langle 1 \rangle$
  Hence the maximal order b' is 4 and the b'-singular locus is $\mathrm{Sing}_4(B^{[1]}) = V(x, y)$.
- Computation of maximal $n_x(E)$ unnecessary, because $E = \emptyset$

Second descent in dimension:

- Choice of hypersurface of maximal contact:
  We choose $Z_1 = V(x)$ which satisfies $\langle x \rangle \subseteq \Delta^3(x^2y^2)$. The other conditions hold trivially.
- Construction of the second auxilliary object:
  $$\begin{aligned}\mathrm{Coeff}_{Z_1}(x^2y^2) &= \langle (y^2)^{\frac{4!}{4-2}}, y^{\frac{4!}{4-1}} \rangle \\ &= \langle y^{24} \rangle\end{aligned}$$
  $B^{[1]}_{Z_1} = (\mathbb{C}, V(y^{24}), 24, \emptyset, \emptyset) =: B^{[2]}$

Maximal locus of $inv^{(0)}_{[2]}$*:*

- Maximal order: 24
- maximal $n_x(E)$: 0

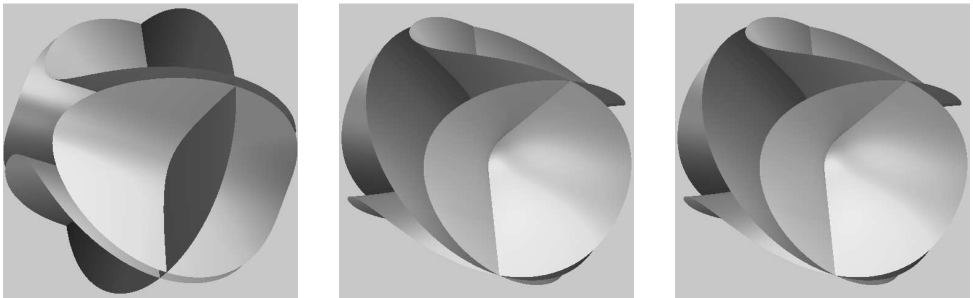

Figure 2: *These three images illustrate the three charts arising from the first blowing up, which introduced new variables u, v, w for the $\mathbb{P}^2$. Fixing u, v and w as names for the new variables introduced in this blowing up, the left image corresponds to the chart $w \neq 0$, that is $x = uz$ and $y = vz$, the ideal of the transformed surface (lighter grey) is generated by $1 - u^2v^2$ in this chart, the one of the exceptional divisor (darker grey to black) by z. The image in the center illustrates the chart $v \neq 0$, i.e. $x = uy$, $z = wy$; the ideal of the transformed surface is generated by $z^2 - x^2y^2$, the one of the exceptional divisor by y. The last image illustrates the third chart, which basically coincides with the second one up to exchange of the roles of x and y and of u and v.*

Hence the tower of the original basic object is $T_0(B) = \{B^{[0]}, B^{[1]}, B^{[2]}\}$, leading to the invariant values

$$\mathrm{inv}^{(0)}(w) = \begin{cases} (2,0;4,0;24,0) & w \;=\; (0,0,0) \\ (2,0;2,0;0,0) & w \;=\; (0,y,0), y \neq 0 \end{cases}$$

which in turn imply that the first center is $(0,0,0)$.

(1) **First Blowing Up** and Transformation of the Tower:

By blowing up this center, we obtain three charts (see Figure 2). It can easily be checked by direct computation that the first one is already resolved and that the two remaining ones are showing the same objects up to renaming of variables due to the symmetry of the original situation. Hence, we only consider one of the latter two in detail: the chart defined by $x = uy, z = wy$. As transformed basic objects, we obtain:

$$\begin{aligned} \widetilde{B}^{[0]} &= (\mathbb{C}^3, V(w^2 - u^2y^2), 2, \{V(y)\}, \emptyset) \\ \widetilde{B}^{[1]} &= (\mathbb{C}^2, V(u^2), 2, \{V(y)\}, \emptyset) \\ \widetilde{B}^{[2]} &= (\mathbb{C}^2, \emptyset, 24, \{V(y)\}, \emptyset). \end{aligned}$$

Obviously, $\mathrm{Sing}_2(\widetilde{B}^{[0]}) = \mathrm{Sing}_2(w^2 - u^2y^2) \neq \emptyset$, but $\mathrm{Sing}_{b'}(\widetilde{B}^{[1]}) =$

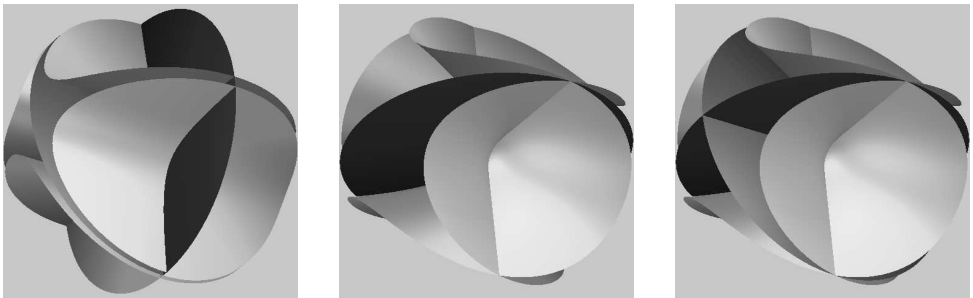

Figure 3: *These three images illustrate the three charts after the second blowing up. Again the transformed surface is drawn in lighter grey, the exceptional divisors in darker grey and black. The one on the left corresponds to the resolved object, the one in the center to case (2a) and the one on the left to case (2b).*

$\mathrm{Sing}_4(u^2) = \emptyset$.[9] Thus we can set $B_1^{[0]} = \widetilde{B}^{[0]}$, but we have to recompute $B_1^{[1]}$. As $\mathrm{Sing}_2(u^2) \neq \emptyset$, we only need to correct E_{bad} in $\widetilde{B}^{[1]}$ setting it to $E_{bad} = \{V(y)\}$ before assigning this basic object to $B_1^{[1]}$.

Additionally, we have to recompute the tower starting at $B_1^{[2]}$: According to the formulae, we obtain $b' = 2$, $m = 1$ and $I(X') = \langle u^2, y^2 \rangle$ from the basic object $B_1^{[1]}$ implying that $B_2^{[2]}$ is assigned the value $(\mathbb{C}, V(u^2), 2, \emptyset, \emptyset)$.

Therefore the new tower is $T_1(B) = \{B_1^{[0]}, B_1^{[1]}, B_1^{[2]}\}$, leading to the invariant values

$$\mathrm{inv}^{(1)}(w) = \begin{cases} (2,0;2,1;2,0) & w = (0,0,0) \\ (2,0;2,0;0,0,) & w = (0,y,0), y \neq 0 \end{cases}$$

which in turn imply that the second center is $(0,0,0)$.[10]

(2) **Second Blowing Up** and corresponding transformations of the towers.

Again, the first chart can easily be checked to be resolved by direct computation. The other two need to be considered separately:

(2a) **Chart** $u = yr$**,** $w = yt$.

[9] Testing whether the n_X have dropped is meaningless here, because they are nonnegative integers and had value zero in the previous step.

[10] Although we obtain a center which is just the coordinate origin in the other chart as well due to the symmetry in x and y of the original equation, these two points do not coincide. More precisely, one of the two is $((0,0,0);(1:0:0)) \subset \mathbb{A}^3 \times \mathbb{P}^2$, the other one is $((0,0,0);(0:1:0)) \subset \mathbb{A}^3 \times \mathbb{P}^2$.

As transformed basic objects, we obtain

$$\begin{array}{rcl}\widetilde{B}_1^{[0]} & = & (\mathbb{C}^3, V(t^2 - r^2y^2, 2, \{\emptyset, V(y)\}, \emptyset)\,, \\ \widetilde{B}_1^{[1]} & = & (\mathbb{C}^2, V(r^2), 2, \{\emptyset, V(y)\}, \{\emptyset\})\,, \\ \widetilde{B}_1^{[2]} & = & (\mathbb{C}^1, V(r^2), 2, \{V(y)\}, \emptyset)\,.\end{array}$$

Clearly, $\mathrm{Sing}_2(t^2 - r^2y^2) \neq \emptyset$ and $\mathrm{Sing}_2(r^2) \neq \emptyset$, but the locus of invariant value $\mathrm{inv}^{(1)}_{[1]}$ is empty. Therefore, we can use $\widetilde{B}_1^{[0]}$ and $\widetilde{B}_1^{[1]}$ as $B_2^{[0]}$ and $B_2^{[1]}$ in the new tower (without adding any further exceptional divisors to E_{bad} of $B_2^{[1]}$) and we have to recompute $B_2^{[2]}$ obtaining the auxilliary object $(\mathbb{C}^1, V(0), 2, \{V(y)\}, \{V(y)\})$ which is already resolved. Hence the upcoming center is determined by the maximal locus of $(\mathrm{inv}^{(2)}_{[0]}; \mathrm{inv}^{(2)}_{[1]})$ which is $V(t, r)$.
As the subsequent calculations in this branch are very similar to this one respectively those in the branch (2b), we do not discuss this branch of the resolution any further in this example.

(2b) **Chart** $y = us$**,** $w = ut$:

As transformed basic objects, we obtain

$$\begin{array}{rcl}\widetilde{B}_1^{[0]} & = & (\mathbb{C}^3, V(t^2 - u^2s^2), 2, \{V(s), V(u)\}, \emptyset)\,, \\ \widetilde{B}_1^{[1]} & = & (\mathbb{C}^2, \emptyset, 2, \{V(s), V(u)\}, \{V(s)\})\,.\end{array}$$

(Note that $\widetilde{B}_1^{[2]}$ is irrelevant due to the structure of $\widetilde{B}_1^{[1]}$.)

Because the second entry of $\widetilde{B}_1^{[1]}$ is the empty set, we know that the ideal of the total transform of the second entry of $B_1^{[1]}$ is a product of exceptional hypersurfaces and that we are in the monomial case. Using $\widetilde{B}_1^{[0]}$ as $B_2^{[0]}$, we recompute the auxilliary basic object for entering the algorithm of the monomial case: here we still have $b' = 2$, $m = 0$, $I(X') = \langle t^2 - u^2s^2 \rangle$ and $E' = \{V(s), V(u)\}$, which leads to the auxiliary object

$$B_2^{[1]} = (\mathbb{C}^2, V(u^2s^2), 2, \{V(s), V(u)\})$$

which has already been dealt with in Example 3.2[11], and hence to the tower

$$T_2(B) = \{B_2^{[0]}, B_2^{[1]}\}.$$

From the calculations of Example 3.2, we obtain the center $V(u)$ for the auxilliary basic object $B_2^{[1]}$ and therefore the new center $V(u, t)$ corresponding to the invariant value $(2, 0; -1, 2, (2))$.

[11] As E_{bad} is irrelevant for the calculations in the monomial case, we have omitted it here.

(3) **Third Blowing Up** in the (2b) branch: To simplify notation, we rename the variables to be x, y, z again. Then the tower before the blowing up is $T_2(B) = \{B_2^{[0]}, B_2^{[1]}\}$ where

$$\begin{aligned} B_2^{[0]} &= (\mathbb{C}^3, V(z^2 - x^2y^2), 2, \{V(y), V(x)\}, \emptyset)\,, \\ B_2^{[1]} &= (\mathbb{C}^2, V(x^2y^2), 2, \{V(y), V(x)\}, \emptyset)\,. \end{aligned}$$

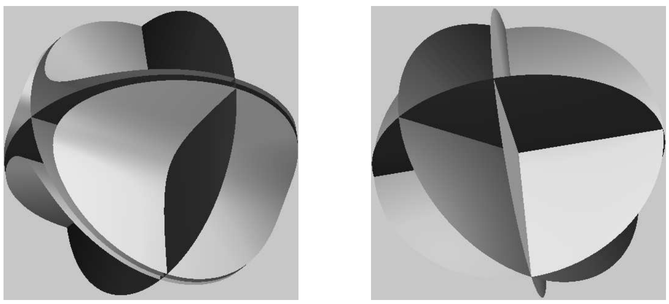

Figure 4: *Illustration of the two charts arising from the blowing up at the center determined in case (2b).*

Only the object in the second chart, defined by $z = ux$ is not resolved yet. For this one the transformed objects are

$$\begin{aligned} \widetilde{B}_2^{[0]} &= (\mathbb{C}^3, V(u^2 - y^2), 2, \{V(y), V(1), V(x)\}, \emptyset)\,, \\ \widetilde{B}_2^{[1]} &= (\mathbb{C}^2, V(y^2), 2, \{V(y), V(1), V(x)\}\{V(x)\})\,. \end{aligned}$$

The 2-singular locus of the first one is still non-empty and the second one is still monomial. Hence the transformed tower is now

$$T_3(B) = \{\widetilde{B}_2^{[0]}, \widetilde{B}_2^{[1]}\}.$$

As we are still in the monomial case for the first auxilliary object, we obtain the upcoming center from example 3.2: $V(z, y)$ with corresponding invariant $(2, 0; -1, 2, (1))$

(4) **Fourth Blowing Up**

The objects in both charts are resolved. For simplicity, we only consider the chart defined by $u = ty$. Here the transformed objects after the blowing up are:

$$\begin{aligned} \widetilde{B}_3^{[0]} &= (\mathbb{C}^3, V(t^2 - 1), 2, \{V(1), V(1), V(x), V(y)\}, \{V(y)\})\,, \\ \widetilde{B}_3^{[1]} &= (\mathbb{C}^2, \emptyset, 2, \{V(1), V(1), V(x), V(y)\}, \{V(y)\})\,, \end{aligned}$$

and $\mathrm{Sing}_2(\widetilde{B}_3^{[0]}) = \emptyset$ which is exactly what we wanted to achieve.

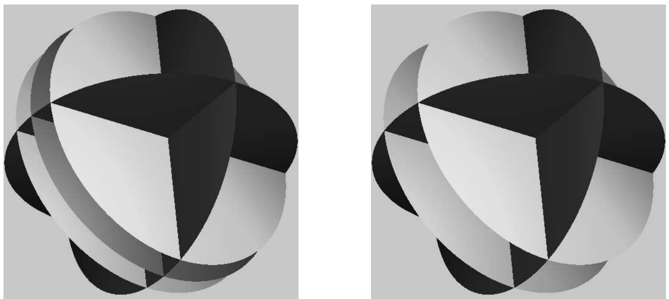

Figure 5: *Illustration of the two charts arising from the blowing up at the center determined after the third blowing up.*

Even in such a small example as this one, it is sometimes difficult to keep track of the parent-child relationships of the various charts. Therefore it is often useful to illustrate these in terms of a tree of charts:

Computational Remark 4.6. From the implementational point of view, there are three aspects of the resolution process which greatly affect the efficiency of the resulting program: first of all the construction of the tower which was discussed in 4.1, secondly the transformation of the towers, i.e. the blowing up (see 1.9) and last but not least to combinatorial complexity due to the use of charts – arising from blowing up and from passing to open covers – which, of course, have non-empty intersections in general. This last issue turns out to be the crucial point in the overall performance of the algorithm: It is simply the number of redundant blowing ups that often makes the algorithm of S. Encinas and O. Villamayor painfully slow in examples of practical relevance. To tackle this problem, we need to analyze the overall strategy of the choice of the centers. Clearly, the fundamental idea behind the construction of the centers is the need to choose them as large as possible while they still have to be subject to the conditions of permissibility.[12]

An obvious way to avoid unnecessary calculations is the use of symmetries in the system of equations – as we did in the above example when determining the center for the second blowing up. To achieve an even better improvement following this idea, it is usually helpful to try to detect and preserve these symmetries throughout the process as far as possible.

[12]In the algorithm of E. Bierstone and P. Milman, the choice of centers follows the same fundamental idea, but as the construction of the invariant differs from the other algorithm (e.g. using the Hilbert-Samuel function) and the strict transform is used instead of the weak one, the number of blowing ups in this algorithm tends to be lower. However, the computation of the invariant poses problems of a different kind, like the computation of the maximal locus of the Hilbert-Samuel function. Moreover, as it is still necessary to pass to charts, the problem of redundant blowing ups is also present here.

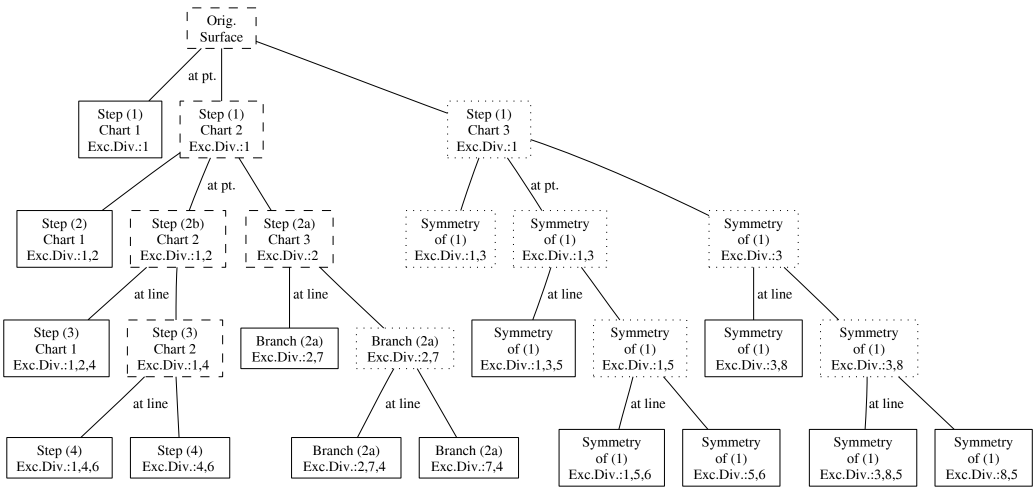

Figure 6: *Tree of Charts for the example $V(z^2 - x^2y^2)$: Boxes with solid border represent final charts; boxes with dashed border are charts, which have been discussed explicitly, whereas boxes with dotted border represent charts which have not been discussed in detail. In the whole resolution process 8 different exceptional divisors appeared.*

Additionally, there is a special situation in which it is often worth applying a simple heuristic: If the singular locus of the original object happens to be a permissible center, this can be used as the very first center. In the case of the Whitney umbrella, for instance, this make the difference between a tree leading to twelve final charts and one consisting of just one blowing up giving rise to 2 final charts.

Apart from the previously mentioned changes which do not alter the course of the algorithm, it is also possible to carefully change the choice of centers by means of a backtracking approach. More precisely, the algorithm of Encinas-Villamayor uses weak transforms, but for the final result of embedded resolution of singularities we are only interested in strict transforms. Thus a natural idea would be to pass from weak to strict transforms after each blowing up and compute a new tower corresponding to the new main basic object, keeping only the partition of the set of exceptional divisors into the various sets E_{bad} unless the order of the respective auxilliary object drops (compared to the one appearing before passing to the strict transform). Unfortunately, passing from weak to strict transform the invariant of the first auxilliary object can even go up. Therefore it is not always possible to pass from weak to strict transform, but it can be applied as a kind of heuristic, falling back to the original algorithm of Encinas-Villamayor whenever a step is detected where the invariant increased. In this case, of course, the algorithm has to follow the steps of the unchanged one until the maximal value of a fragment of the invariant corresponding to a higher dimensional object in the tower drops; only then the heuristic can be applied again. This backtracking approach can also be refined by slightly changing the construction of certain auxilliary objects, but this is beyond the scope of this article.

5 Some Remarks on Applications

When considering practical applications of the resolution of singularities, e.g. the calculation of an invariant like the topological zeta function, an obstacle, which is common to all these tasks, is the fact that the final result is represented by means of charts. This makes it possible – even highly probable – that the same point or subvariety may be present in several charts which, in turn, implies that rather simple tasks, like e.g. counting intersection points of two curves, cannot be performed in a direct way. Instead we need to a way to identify the same point in different charts by moving through the whole tree of charts in an appropriate way.

As blowing ups are isomorphisms away from the center, the process of sucessively blowing down and then blowing up again does not cause any problems for points which do not lie on an exceptional divisor at all or only lie on exceptional divisors, which already exist in the chart at which the history of the considered charts branched. If, however, the point lies on an exceptional divisor which arises later, then blowing down beyond the moment of birth of this divisor will inevitably lead to incorrect results, because this blow up map is not an isomorphism. To avoid this problem, we need to represent the point on the exceptional divisor as the locus of intersection of the exceptional divisor with an auxilliary variety which is not contained in the exceptional divisor. More formally speaking, we use the following simple fact from commutative algebra:

Let $I \subset K[x_1, \ldots, x_n]$ be a prime ideal, $J \subset K[x_1, \ldots, x_n]$ another ideal such that $I + J$ is equidimensional and $ht(I) = ht(I + J) - r$ for some integer $0 < r < n$. Then there exist polynomials $p_1, \ldots, p_r \in I + J$ and a polynomial $f \in K[x_1, \ldots, x_n]$ such that

$$\sqrt{I + J} = \sqrt{(I + (p_1, \ldots, p_r)) : f}.$$

In our situation, the ideal I is, of course, the ideal of the intersection of the exceptional divisors in which the point or subvariety $V(J)$ is contained. As any sufficiently general set of polynomials $p_1, \ldots, p_r \in J \setminus (I \cap J)$ leading to the correct height of $I + (p_1, \ldots, p_r)$ will do and as the only truely restricting condition on f is that it has to exclude all extra components of $I + (p_1, \ldots, p_r)$, we also have enough freedom of choice of the $p_1, \ldots, p_r, f$ to achieve that none of them is contained in any further exceptional divisor that might be in our way when blowing down.

Having solved the problem of identifying points which exist in more than one chart, we can now consider a very simple application: we determine which exceptional divisor in one chart coincides with which one in another chart. The method to do this is quite simple: we simply compare the centers leading to these exceptional divisors. To this end, we start at the root of the tree of charts of the resolution and work our way up to the final charts. The criteria for identifying the centers are quite simple: first of all, the centers can not be the same, if the corresponding values of the governing function do not agree, secondly, the centers cannot be the same if the exceptional divisors in which they are contained are not the same and, in the last step, the remaining candidates are compared explicitly by mapping them through the resolution tree as described above. In the example of the last section, this corresponding calculations are the following:

Example 5.1. In Example 4.5, identifying the exceptional divisor which arose from the first blowing up can be done by simply considering the tree. As we previously mentioned, the subsequent 0–dimensional centers in the charts 2 and 3 do not coincide and hence the identification of the exceptional divisors E_2 and E_3 does not involve any further calculations. For the remaining exceptional divisors, however, we cannot avoid passing through the tree. As all of these calculations are rather similar, we restrict our considerations to the comparison of the exceptional divisors which arise from the blowing ups (2a) and (2b) and the respective subsequent blowing ups to illustrate the main ideas of the identification process:

(a) Comparison of Centers in (2a) and (2b):
The ideal of the center (2a) in the respective chart is $\langle u, t\rangle$ as we had previously computed; the one of (2b) is $\langle t, r\rangle$ in another chart. Both of these charts arose from the same blowing up. Therefore we can look at the respective centers as subsets of $\mathbb{A}^3 \times \mathbb{P}^2$: the first one is $V(u, w, r, t)$, whereas the second one is $V(u, y, w, t)$. These are two different lines meeting in the point $V(u, y, w, r, t)$. Hence the two exceptional divisors arising from the blowing ups at these centers cannot be the same.

(b) Comparison of Center in (2b) and subsequent Center in Branch (2a):
The ideal of the subsequent (and last) center in the branch (2a) is $V(t, y, a) \subset \mathbb{A}^3 \times \mathbb{P}^1$, using variable names t, r, y for $\mathbb{A}^3$ and a, b for $\mathbb{P}^1$. Luckily, this is not contained in the newborn exceptional divisor $V(r)$, which allows us to blow it down directly to obtain $V(u, y, w, t)$ in the parent, coinciding with the center (2b).

(c) Comparison of Center in (2a) and subsequent Center in Branch (2b):
In this case, the subsequent (and last) center in the branch (2b) is $V(t,s,c) \subset \mathbb{A}^3 \times \mathbb{P}^1$, using variable names t,u,s for $\mathbb{A}^3$ and c,d for $\mathbb{P}^1$. Again this is not contained in the newborn exceptional divisor $V(u)$. Hence blowing down yields $V(y,w,s,t)$ which clearly does not coincide with the center of (2a).
Here we could also have proceeded by the argument that the newborn exceptional divisor in the (2b) branch also appears later on in the (2a) branch and hence any later divisor in the (2b) branch cannot coincide with the earliest exceptional divisor arising in the (2a) branch.

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Newton Polyhedra of Discriminants: A Computation

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Abstract

We compute the Newton polyhedron in the natural coordinates of the discriminant of a germ of complex analytic mapping $(\mathbf{C}^3 \times \mathbf{C}, 0) \to (\mathbf{C}^3 \times \mathbf{C}, 0)$ associated by the polar hypersurface construction to the degeneration of a plane analytic branch with two characteristic pairs to the monomial curve with the same semigroup. The result shows that the jacobian Newton polyhedron is not in general constant in an equisingular family of complete intersection branches (whereas it is constant in an equisingular family of plane branches). However, in this case the information that it contains, namely the semigroup, is constant and only the encoding changes.

Introduction

To any germ of an isolated complex analytic hypersurface singularity defined by a convergent power series equation $f(u_0, \ldots, u_n) = 0$, one can associate its *jacobian Newton polygon*, which is the Newton polygon *in the coordinates* (t_0, t_1) of the discriminant of the map

$$(\ell, f)\colon (\mathbf{C}^{n+1}, 0) \longrightarrow (\mathbf{C}^2, 0)$$

given by $t_0 = \ell(u_0, \ldots, u_n), t_1 = f(u_0, \ldots, u_n)$, where ℓ is a sufficiently general linear form. We say that a family of hypersurfaces with isolated singularities is equisingular if the singular locus of the total space of the family is a stratum of the minimal Whitney stratification of that total space. For a family of germs of plane complex analytic curves, this is equivalent to the usual definitions of equisingularity, and in particular to the constancy of the local embedded topological type.

1991 *Mathematics Subject Classification.* 32S55, 14H20
Key words. Newton polyhedron, discriminant, monomial curve

The discriminants associated in the way just described to the members $f_v = 0$ of an equisingular family of equations for germs do not in general form an equisingular family; the numbers of their branches may vary. It is therefore remarkable that their Newton polygons in the coordinates (t_0, t_1), which are the jacobian Newton polygons, are constant (see [6]).

Thanks to a result of Merle ([5]), it is even true that the jacobian Newton polygon of a plane branch is a complete invariant of its equisingularity type; it determines and is determined by the Puiseux characteristic (see §2). In particular the jacobian Newton polygon has g compact edges, where $g + 1$ is the number of Puiseux characteristic exponents, and they can be computed from the Puiseux exponents; we call this the *decomposition theorem*.

In the case of a plane curve C defined by $f(u_0, u_1) = 0$, the information contained in the jacobian Newton polygon concerns the possible contacts with C at 0 of the germs of analytically irreducible components (the branches) of the relative polar curve $\frac{\partial f}{\partial u_1} + \tau \frac{\partial f}{\partial u_0} = 0$ for a general value of τ. The invariants extracted from the jacobian Newton polygon appear in many different types of objects related to the singularity. For example in the JSJ decomposition of the complement in the sphere $\mathbf{S}^3_\epsilon$ (of radius ϵ centered at 0) of a small tubular neighborhood of the knot $\mathbf{S}^3_\epsilon \cap C$ for small enough ϵ. In fact, alternative proofs of the topological invariance of the inclinations of the edges of the jacobian Newton polyhedron mentioned above have been given using this fact (see [4]). They also appear in the description of the asymptotic behaviour of the Lipschitz-Killing curvature (as a real surface) of the Milnor fiber $\mathbf{B}^4_\epsilon \cap f^{-1}(t) \subset \mathbf{B}^4_\epsilon$ for $0 < |t| \ll \epsilon \ll 1$ (see [3]), in the Łojasiewicz exponent at 0 of $f(u_0, u_1)$ and so on. The constancy of the jacobian Newton polygon then appears as a tool to understand how the local topology determines geometric structures such as the JSJ decomposition, or even metric information.

It seems therefore interesting to examine whether this phenomenon of constancy of the jacobian Newton polygon in an equisingular family extends to other equisingular families of curves, for example those which are local complete intersections.

There is a particularly interesting such family, which is the specialization of a given plane branch to the monomial curve with the same semigroup (see [7]). The general fiber of this family is the plane branch suitably reimbedded in affine $(g + 1)$-dimensional space, where g is the number of its Puiseux exponents.

In this paper, we compute the jacobian Newton polyhedra of the jacobian discriminants of the fibers of such a family of complete intersections in the case of a branch with two characteristic pairs, and we obtain the following information:

- The jacobian Newton polyhedron is *not* constant, although the family is Whitney-equisingular.

- However, the information contained in the jacobian Newton polyhedra of the special and general fibers is the same, and is equivalent to the topology of the branch.

It is interesting to verify on this example that although the equations of the discriminants which we consider are, as usual, rather complicated, the method of computation by Fitting ideals makes it possible to determine at least their Newton polyhedron.

The interested reader will also note that the system of equations which we study is degenerate with respect to its Newton polyhedron in the usual sense for $v \neq 0$, so that the generic methods of computation of Newton polyhedra of discriminants à la Gel'fand-Kapranov-Zelevinski (see [2]) do not apply. This system of equations becomes non degenerate for $v = 0$, of course with respect to a different Newton polyhedron.

This work has a strongly computational flavour, and computer algebra tools did play a role in computing the first examples which led to conjecture the general shape of the result. Although SINGULAR was not used, we are happy to dedicate it to Gert-Martin Greuel, who did so much to develop computer algebra tools for singularists.

1 Plane Branches, Semigroups and Monomial Curves

(A reminder)

For us, a branch is an irreducible germ of a complex analytic curve. A plane branch is given by a convergent power series $f(u_0, u_1) \in \mathbf{C}\{u_0, u_1\}$ which is not a unit and is irreducible in that ring. The branch is the germ at 0 of the set of solutions of $f(u_0, u_1) = 0$. By the theorem of Newton, after possibly a change of coordinates to achieve that $u_0 = 0$ is transversal to it at 0, the branch C can be parametrized near 0 as follows

$$\begin{aligned} u_0(t) &= t^n \\ u_1(t) &= a_m t^m + a_{m+1} t^{m+1} + \cdots + a_j t^j + \cdots \qquad \text{with } m \geq n. \end{aligned}$$

Let us now consider the following grouping of the terms of the series $u_1(t)$: set $\beta_0 = n$ and let β_1 be the smallest exponent appearing in $u_1(t)$ which is not divisible by β_0. If no such exponent exists, it means that u_1 is a power series in u_0, so that our branch is analytically isomorphic to $\mathbf{C}$, hence non singular. Let us suppose that this is not the case, and set $e_1 = (n, \beta_1)$, the greatest common divisor of these two integers. Now define β_2 as the smallest exponent appearing in $u_1(t)$ which is not divisible by e_1. Define $e_2 = (e_1, \beta_2)$; we have $e_2 < e_1$, and we continue in this manner. Having defined $e_i = (e_{i-1}, \beta_i)$, we define β_{i+1} as the smallest exponent appearing in $u_1(t)$ which is not divisible by e_i. Since the sequence of integers

$$n > e_1 > e_2 > \cdots > e_i > \cdots$$

is strictly decreasing, there is an integer g such that $e_g = 1$. At this point, we have structured our parametric representation as follows:

$$\begin{aligned} u_0(t) &= t^n \\ u_1(t) &= a_n t^n + a_{2n} t^{2n} + \ldots + a_{kn} t^{kn} + a_{\beta_1} t^{\beta_1} + a_{\beta_1+e_1} t^{\beta_1+e_1} + \ldots + a_{\beta_1+k_1e_1} t^{\beta_1+k_1e_1} \\ &\qquad + a_{\beta_2} t^{\beta_2} + a_{\beta_2+e_2} t^{\beta_2+e_2} + \ldots + a_{\beta_q} t^{\beta_q} + a_{\beta_q+e_q} t^{\beta_q+e_q} + \ldots \\ &\qquad + a_{\beta_g} t^{\beta_g} + a_{\beta_g+1} t^{\beta_g+1} + \ldots \end{aligned}$$

where, by construction, the coefficients of the t^{β_i} for $i \geq 1$ are not zero.

The integers $(n = \beta_0, \beta_1, \ldots, \beta_g)$ are called the Puiseux characteristic exponents of the branch.

Let $\mathbf{C}\{u_0, u_1\}/(f(u_0, u_1)) = \mathcal{O}$ be the analytic algebra of a germ of analytically irreducible curve C, and let $\overline{\mathcal{O}}$ be its normalization; we have an injection $\mathcal{O} \hookrightarrow \overline{\mathcal{O}}$, in fact given by $u_0 \mapsto t^n$, $u_1 \mapsto u_1(t)$, which makes $\overline{\mathcal{O}}$ an $\mathcal{O}$-module of finite type, and $\overline{\mathcal{O}}$ is a subalgebra of the fraction field of $\mathcal{O}$. Since $\overline{\mathcal{O}}$ is isomorphic to $\mathbf{C}\{t\}$, the order in t of the series defines a mapping $\nu\colon \mathbf{C}\{t\} \setminus 0 \to \mathbf{N}$ which satisfies

i) $\nu(a(t)b(t)) = \nu(a(t)) + \nu(b(t))$ and

ii) $\nu(a(t) + b(t)) \geq \min(\nu(a(t)), \nu(b(t)))$ with equality if $\nu(a(t)) \neq \nu(b(t))$;

in other words, ν is a *valuation* of the ring $\mathbf{C}\{t\}$.

We consider the valuations of the elements of the subring $\mathcal{O}$, i.e., the image Γ of $\mathcal{O} \setminus \{0\}$ by ν; in view of i), it is a semigroup contained in $\mathbf{N}$. The fact that $\overline{\mathcal{O}}$ is a finite $\mathcal{O}$-module implies that $\mathbf{N} \setminus \Gamma$ is finite.

Now, we seek a minimal set of generators of Γ as a semigroup: Let $\overline{\beta_0}$ be the smallest nonzero element in Γ, let $\overline{\beta_1}$ be the smallest element of Γ which is not a multiple of $\overline{\beta_0}$, let $\overline{\beta_2}$ be the smallest element of Γ which is not a combination with non negative integral coefficients of $\overline{\beta_0}$ and $\overline{\beta_1}$, i.e., is not in the semigroup $\langle \overline{\beta_0}, \overline{\beta_1} \rangle$, and so on. Finally, since $\mathbf{N} \setminus \Gamma$ is finite, we find in this way a minimal set of generators:

$$\Gamma = \langle \overline{\beta_0}, \overline{\beta_1}, \ldots, \overline{\beta_g} \rangle.$$

This set of generators is uniquely determined by the semigroup Γ, and of course determines it.

Let us take the notations introduced for the Puiseux exponents; it is easy to check that if we set $\beta_0 = n$, the multiplicity, then $\overline{\beta_0} = \beta_0 = n, \overline{\beta_1} = \beta_1$.

After that is becomes more complicated. Zariski ([9], Th. 3.9) proved the following recursive formula: $\overline{\beta}_0 = \beta_0 = n$, $\overline{\beta}_1 = \beta_1$ and for $q \geq 2$,

$$\overline{\beta_q} = n_{q-1}\overline{\beta_{q-1}} - \beta_{q-1} + \beta_q,$$

where the integers n_i are defined inductively by $e_0 = n$ and $e_{i-1} = n_i e_i$, where the e_i are the successive greatest common divisors introduced at the beginning of the section, so that we have

$$n = \beta_0 = \overline{\beta}_0 = n_1 \dots n_g.$$

Thus, the datum of these generators, or of the semigroup, is equivalent to the datum of the Puiseux characteristic of $(X, 0)$, or of its topological type. The proof relies on a formula of Max Noether which computes the *contact exponent* $\frac{(C,D)_0}{m_0(D)}$ of two analytic branches at the origin in terms of the coincidence of their Puiseux expansions in fractional powers of x.
The semigroups coming from plane branches are characterized among all semigroups of analytically irreducible germs of curves by the following two properties:

1) $n_i\overline{\beta_i} \in \langle \overline{\beta_0}, \dots, \overline{\beta_{i-1}} \rangle$
2) $n_i\overline{\beta_i} < \overline{\beta}_{i+1}$.

That the semigroups of plane branches have these properties follows from the induction formula and the inequalities $\beta_i < \beta_{i+1}$. The converse can be proved by the construction outlined below (see [7]).

Conversely, given a semigroup Γ in $\mathbf{N}$ with finite complement, we can associate to it an analytic (in fact algebraic) curve, called the *monomial curve* associated to Γ. If $\Gamma = \langle \overline{\beta_0}, \overline{\beta_1}, \dots, \overline{\beta_g} \rangle$, the monomial curve C^Γ is described parametrically by

$$u_0 = t^{\overline{\beta_0}}, \quad u_1 = t^{\overline{\beta_1}}, \quad \dots, \quad u_g = t^{\overline{\beta_g}}.$$

On the other hand, the relations 1) above mean that there exist natural numbers $\ell_i^{(j)}$ satisfying

$$1') \qquad \begin{aligned} n_1\overline{\beta_1} &= \ell_0^{(1)}\overline{\beta_0}, \\ n_2\overline{\beta_2} &= \ell_0^{(2)}\overline{\beta_0} + \ell_1^{(2)}\overline{\beta_1}, \\ &\vdots \\ n_j\overline{\beta_j} &= \ell_0^{(j)}\overline{\beta_0} + \dots + \ell_{j-1}^{(j)}\overline{\beta_{j-1}} \\ &\vdots \\ n_g\overline{\beta_g} &= \ell_0^{(g)}\overline{\beta_0} + \dots + \ell_{g-1}^{(g)}\overline{\beta_{g-1}}. \end{aligned}$$

These relations translate into equations for the curve $C^\Gamma \subset \mathbf{C}^{g+1}$; since $u_i = t^{\overline{\beta_i}}$, our curve satisfies the g equations

$$f_j \;=\; u_j^{n_j} - u_0^{\ell_0^{(j)}} u_1^{\ell_1^{(j)}} \dots u_{j-1}^{\ell_{j-1}^{(j)}} = 0, \quad 1 \le j \le g,$$

and it can be shown that they actually define $C^\Gamma \subset \mathbf{C}^{g+1}$, so that if Γ is the semigroup of a plane branch, C^Γ is a complete intersection.

The relations $1')$ are not uniquely determined, but there is a canonical choice: dividing each $\ell_k^{(j)}$ by n_k we can request that for every $k \ge 1$ we have $\ell_k^{(j)} < n_k$; it is the choice we shall make in the sequel.

Remark that if we give to u_i the weight $\overline{\beta_i}$, the i-th equation is homogeneous of degree $n_i\overline{\beta_i}$.

The connection between a plane curve C having semigroup Γ and the monomial curve is much more precise and interesting than the formal relation we have just seen; by small deformations of the monomial curve one obtains all the branches with the same semigroup. In fact, the best way to understand all branches with semigroup Γ is to consider the not necessarily plane curve C^Γ (C^Γ is plane if and only if C has only one characteristic exponent).

By definition of Γ, there are elements $\xi_q \in \mathcal{O}$ with $\nu(\xi_q) = \overline{\beta_q}$. We can write these elements in $\mathbf{C}\{t\}$ as

$$\xi_q = t^{\overline{\beta_q}} + \sum_{j > \overline{\beta_q}} \gamma_{q,j} t^j .$$

Let us consider the one-parameter family of parametrizations

$$u_0 = t^m\,, \quad u_1 = t^{\overline{\beta_1}} + \sum_{j>\overline{\beta_1}} v^{j-\overline{\beta_1}} \gamma_{1,j} t^j\,, \quad \dots, \quad u_g = t^{\overline{\beta_g}} + \sum_{j>\overline{\beta_g}} v^{j-\overline{\beta_g}} \gamma_{g,j} t^j\,.$$

The reader can check that for $v \neq 0$, the curve thus described is isomorphic to our original curve C. (hint: make the change of parameter $t = vt'$ in the ξ_q and the change of coordinates $u_j = v^{\overline{\beta_j}} u'_j$, and remember the definition of the ξ_j). For $v = 0$, we have the parametric description of the monomial curve.

So we have, in fact, described a map $\mathbf{C} \times \mathbf{C} \to \mathbf{C}^{g+1} \times \mathbf{C}$ which induces the identity on the second factors (with coordinate v). The image of this map is a surface, which is the total space of a deformation of the monomial curve, all of its fibers except the one for $v = 0$ being isomorphic to our plane curve C. It follows that the monomial curve is a specialization, in this family, of our plane curve. In this specialization the multiplicity and the semigroup remain constant; in a rather precise sense it is an equisingular specialization, or one may say that *the plane curve is an equisingular deformation of the monomial curve with the same semigroup.*

The same phenomenon can be also observed in the language of equations rather than parametrizations. Let us consider a one parameter family of equations for curves in $\mathbf{C}^{g+1}$, of the form

$$\begin{aligned}
F_1 = \quad & u_1^{n_1} - u_0^{\ell_0^{(1)}} - vu_2 = 0\,, \\
F_2 = \quad & u_2^{n_2} - u_0^{\ell_0^{(2)}} u_1^{\ell_1^{(2)}} - vu_3 = 0\,, \\
\vdots \quad & \qquad \vdots \\
F_{g-1} = \quad & u_{g-1}^{n_{g-1}} - u_0^{\ell_0^{(g-1)}} u_1^{\ell_1^{(g-1)}} \dots u_{g-2}^{\ell_{g-2}^{(g-1)}} - vu_g = 0\,, \\
F_g = \quad & u_g^{n_g} - u_0^{\ell_0^{(g)}} u_1^{\ell_1^{(g)}} \dots u_{g-1}^{\ell_{g-1}^{(g)}} = 0\,.
\end{aligned}$$

For $v = 0$ we get the equations of the monomial curve, and for $v \neq 0$ we get a curve which has semigroup Γ; this is a general heuristic principle of equisingularity: we have added to each equation of the monomial curve, homogeneous of degree $n_i\overline{\beta_i}$, a perturbation of degree $\overline{\beta_{i+1}} > n_i\overline{\beta_i}$, and this should not change the equisingularity class (the perturbation is "small" compared to the equation).

Notice that for each fixed $v \neq 0$ the curve described by the above equations is a plane curve: for simplicity take $v = 1$; then use the first equation to compute $u_2 = u_1^{n_1} - u_0^{\ell_0^{(1)}}$, substitute this in the next equation, and use this to compute u_3 as a function of u_0, u_1, and so on. Finally, the last equation gives us the equation of a plane curve of the form

$$\left(\cdots\left(\left(u_1^{n_1} - u_0^{\ell_0^{(1)}}\right)^{n_2} - u_0^{\ell_0^{(2)}} u_1^{\ell_1^{(2)}}\right)^{n_3} - \cdots\right)^{n_g} - u_0^{\ell_0^{(g)}} u_1^{\ell_1^{(g)}} \left(u_1^{n_1} - u_0^{\ell_0^{(1)}}\right)^{\ell_2^{(g)}} \cdots = 0\,.$$

The first consequence (see [7]) is that we can produce explicitely the equation of a plane curve with given characteristic exponents: compute the semigroup and its generators, and then write the equation above.

A more important fact is that one can show (*loc. cit*) that any plane curve with a given semigroup appears up to isomorphism as a fiber in a deformation depending on a finite number of parameters: it is a deformation of the monomial curve obtained by adding to the j-th equation a polynomial in the u_i's of order $> n_j\overline{\beta_j}$, where u_{j+1} appears linearly if $j < g$, and these polynomials can in principle be explicitely computed.

In fact it is shown in [7] that we can in this manner produce equations for *all* branches having the same semigroup (or equisingularity type) up to an analytic isomorphism.

In view of the constancy of the jacobian Newton polygon for equisingular families of plane branches, it is plausible that the special family above represents all degenerations of plane branches to the associated monomial curve, as far as the variation of jacobian Newton polyhedra are concerned. We shall therefore make computations for this family.

2 The Discriminant

Set $v = (v_1, \dots, v_{g-1})$, and consider the map

$$\begin{array}{rcl} \phi : \mathbf{C}^{g+1} \times \mathbf{C}^{g-1} & \longrightarrow & \mathbf{C}^{g+1} \times \mathbf{C}^{g-1} \\ (u_0, \dots, u_g, v) & \longmapsto & (u_0, F_1, \dots, F_g, v) \end{array}$$

in the coordinates $(t_0, \dots, t_g, v)$ on the right-hand copy of $\mathbf{C}^{g+1} \times \mathbf{C}^{g-1}$.

Let us first verify that the morphism ϕ is flat. We shall see below that we have even better. Indeed, it is a map between two non singular spaces, whose fiber over 0 is a complete intersection. The flatness follows. Since the special fiber has an isolated singularity at the origin, the critical subspace $\mathcal{C}$ of ϕ is finite over its image in $\mathbf{C}^{g+1} \times \mathbf{C}^{g-1}$, at least locally, by the Weierstrass preparation theorem. This image (or at least its germ at 0) is then a complex analytic space which is by definition the discriminant of the map ϕ ([6], §1).

Let us now compute the discriminant of the mapping ϕ as the image of critical subspace using the Fitting ideal of the algebra of the critical subspace $\mathcal{C}$ as in [6].

The critical subspace is defined by the ideal generated by the coefficients of the differential form

$$dF_1 \wedge \cdots \wedge dF_g \wedge dt_0 \wedge dt_1 \cdots \wedge dt_g \wedge dv_1 \cdots \wedge dv_{g-1}.$$

Since $F_i = u_i^{n_i} - u_0^{\ell_0^{(i)}} \cdots u_{i-1}^{\ell_{i-1}^{(i)}} - v_i u_{i+1}$ for $j < g$ and $F_g = u_g^{n_g} - u_0^{\ell_0^{(g)}} \cdots u_{g-1}^{\ell_{g-1}^{(g)}}$, we see that a generator for the ideal of the critical subspace can be taken of the form

$$\mathcal{C} = \beta_0 u_1^{n_1-1} \cdots u_g^{n_g-1} - \sum_{\alpha} c_\alpha v_1^{\alpha_1} \cdots v_{g-1}^{\alpha_{g-1}} u_0^{m_0(\alpha)} \cdots u_g^{m_g(\alpha)}$$

where each α_i is 0 or 1 and $(m_1(\alpha), \dots, m_g(\alpha)) \neq (n_1 - 1, \dots, n_g - 1)$.

Lemma 2.1. *Giving to the variable u_i the weight $\overline{\beta_i}$ and to v_j the (negative) weight $\beta_j - \beta_{j+1}$, the polynomial $\mathcal{C}$ is homogeneous of degree $\sum_{i=1}^{g}(n_i - 1)\overline{\beta}_i$.*

Proof. The statement follows directly from the homogeneity of the polynomials F_i and the computation of $\mathcal{C}$ as a jacobian determinant. ☐

Let us denote by $I_{\mathcal{C}}$ the ideal of $\mathbf{C}[u_0, \dots, u_g, t_0, \dots, t_g, v_1, \dots, v_{g-1}]$ generated by $(u_0 - t_0, F_1 - t_1, \dots, F_g - t_g, \mathcal{C})$. The generators constitute a regular sequence since their initial forms involve different variables.

Let us consider the $\mathbf{C}[t_0, \dots, t_g, v_1, \dots, v_{g-1}]$-module

$$\mathcal{O}_{\mathcal{C}} = \mathbf{C}[u_0, \dots, u_g, t_0, \dots, t_g, v_1, \dots, v_{g-1}]/I_{\mathcal{C}}.$$

Lemma 2.2. *The* $\mathbf{C}[t_0, \ldots, t_g, v_1, \ldots, v_{g-1}]$*-module*

$$\mathbf{C}[u_0, \ldots, u_g, t_0, \ldots, t_g, v_1, \ldots, v_{g-1}]/(u_0 - t_0, F_1 - t_1, \ldots, F_g - t_g)$$

is free and generated by the β_0 *images of the monomials* $u_1^{i_1} \cdots u_g^{i_g}$ *with* $0 \leq i_k \leq n_k - 1$.

Proof. It follows directly for the form of the equations because each of the equations expresses the corresponding $u_i^{n_i}$ as a linear combination with coefficients in $\mathbf{C}[t_0, \ldots, t_g, v_1, \ldots, v_{g-1}]$ of our generating monomials. □

If we identify t_0 and u_0 and set

$$N = \mathbf{C}[t_0, \ldots, t_g, v_1, \ldots, v_{g-1}, u_1, \ldots, u_g]/(F_1 - t_1, \ldots, F_g - t_g),$$

the $\mathbf{C}[t_0, \ldots, t_g, v_1, \ldots, v_{g-1}]$-module $\mathcal{O}_{\mathcal{C}}$ is the cokernel of the map of multiplication by $\mathcal{C}$ in N. By [6] again we have:

Proposition 2.3. *1. The discriminant* $\mathrm{Disc}(\phi)$ *of the morphism* ϕ *is (up to multiplication by a nonzero constant) the determinant of the matrix* M *of the multiplication in the free* $\mathbf{C}[t_0, \ldots, t_g, v_1, \ldots, v_{g-1}]$*-module* N *by the equation* $\mathcal{C}$ *of the critical subspace.*

2. Giving to the variable t_j *the weight* $n_j\overline{\beta_j}$ *and to* v_k *the (negative) weight* $\beta_k - \beta_{k+1}$*, the polynomial* $\Delta = \mathrm{Disc}(\phi) \in \mathbf{C}[t_0, \ldots, t_g, v_1, \ldots, v_{g-1}]$ *is homogeneous of degree*

$$\deg\Delta = \overline{\beta_0}(\sum_{i=1}^{g}(n_i - 1)\overline{\beta_i}).$$

Proof. The first part of the assertion follows directly from §1 of [6]. For the second part, first note that if we give to u_i the weight $\overline{\beta}_i$, the free $\mathbf{C}[t_0, \ldots, t_g, v_1, \ldots, v_{g-1}]$-module N is graded when the variables are given the weights of the proposition since the equations F_i are homogeneous.

We now apply Lemma 1 of §1 of [6]; in view of Lemma 2.1, if we want the morphism of multiplication by $\mathcal{C}$ to be homogeneous of degree 0, setting $A = \mathbf{C}[t_0, \ldots, t_g, v_1, \ldots, v_{g-1}]$ and $d_{i_1,\ldots,i_g} = \sum_{k=1}^{g} i_k\overline{\beta}_k$, we may write the first copy of N as

$$N = \bigoplus_{i_1,\ldots,i_g} A[d_{i_1,\ldots,i_g}],$$

where $A[s]$ is the A-module A regraded (shifted) by giving 1 the degree s, and then we must write the second copy of N as

$$N = \bigoplus_{i_1,\ldots,i_g} A[d_{i_1,\ldots,i_g} - \sum_{k=1}^{g}(n_k - 1)\overline{\beta}_k]$$

where $0 \leq i_k \leq n_k - 1$. The result follows immediately from *loc.cit.* which states that the degree of the determinant is the sum of the differences of the shifts in the first and second copies over all values of $i_1, \ldots, i_g$. □

Remark 2.4. • In what follows, we shall constantly use the fact that the Fitting image definition of the discriminant commutes with base change and in particular with restriction over subspaces of the target space (see [6]).

• We denote by τ_i the exponent of t_i in a monomial, and by υ_j the exponent of v_j; then Proposition 2.3 means that all the monomials appearing in the equation of the discriminant satisfy (setting $n_0 = 1$):

$$\sum_{i=0}^{g} n_i \overline{\beta}_i \tau_i + \sum_{j=1}^{g-1} (\beta_j - \beta_{j+1}) \upsilon_j = \overline{\beta}_0 (\sum_{i=1}^{g} (n_i - 1) \overline{\beta}_i).$$

3 Curves with Two Characteristic Pairs

The purpose of this section is the computation of the Newton polyhedron in the coordinates (t_0, t_1, t_2) of the discriminant of the morphism ϕ in the case of two characteristic pairs, both for $v = 0$ and v nonzero.

If $g = 2$ the morphism ϕ is defined by the equations

$$u_0 - t_0 = 0\,, \quad u_1^{n_1} - u_0^{\ell_0^{(1)}} - v u_2 - t_1 = 0\,, \quad u_2^{n_2} - u_0^{\ell_0^{(2)}} u_1^{\ell_1^{(2)}} - t_2 = 0\,. \qquad (1)$$

Identifying u_0 with t_0, we have the equations

$$u_1^{n_1} - t_0^{\ell_0^{(1)}} - v u_2 - t_1 = 0\,, \quad u_2^{n_2} - t_0^{\ell_0^{(2)}} u_1^{\ell_1^{(2)}} - t_2 = 0\,,$$

and the equation of the critical subspace is

$$\mathcal{C} = \beta_0 u_1^{n_1 - 1} u_2^{n_2 - 1} - \ell_1^{(2)} v t_0^{\ell_0^{(2)}} u_1^{\ell_1^{(2)} - 1} = 0\,.$$

In view of Proposition 2.3, we have to compute the matrix of multiplication by $\mathcal{C}$ in the basis $e_{i,j} = u_1^i u_2^j$, $0 \leq i \leq n_1 - 1$, $0 \leq j \leq n_2 - 1$ for the $\mathbf{C}[t_0, t_1, t_2, v]$-module $N = \mathbf{C}[t_0, t_1, t_2, v, u_1, u_2]/(F_1 - t_1, F_2 - t_2)$.

Remark 3.1. If $\ell_1^{(2)} = 0$, which is the case for example if $\Gamma = \langle 6, 8, 27 \rangle$, the critical subspace is $\mathcal{C} = \beta_0 u_1^{n_1 - 1} u_2^{n_2 - 1}$, and the computation is simpler but has to be conducted a little differently, introducing $B = t_0^{\ell_0^{(2)}} + t_2$ and the conclusion is the same. *We will present the computations in the case where* $\ell_1^{(2)} \geq 1$.

In order to write down the matrix of a presentation of the $\mathbf{C}\{t_0, t_1, t_2, v\}$-module $\mathcal{O}_{\mathcal{C}}$, we have to compute modulo the ideal $(F_1 - t_1, F_2 - t_2)$ the effect of the multiplication by $\mathcal{C}$ on the generators $e_{i,j}$. The matrix can be presented by blocks, each block corresponding to a fixed value of j. Our matrix is constructed as an $n_2 \times n_2$ matrix of blocks of size n_1; when j is fixed and we fix also the block j' in which we look at the relations, the situation can be represented by an $n_1 \times n_1$-matrix $M_{j,j'}$ whose elements are indexed by (i, i').

For $j = 0$, if $i = 0$, the relation is the equation $\mathcal{C} = 0$, which we write as

$$e_{n_1-1,n_2-1} - (1-c)vT_0 e_{\ell_1^{(2)}-1,0} = 0,$$

where $c = 1 - \frac{\ell_1^{(2)}}{\beta_0}$ and we set for simplicity $T_0 = t_0^{\ell_0^{(2)}}$.

For $j = 0$ and $1 \leq i \leq n_1 - 1$, we have two cases: if $i < n_1 - \ell_1^{(2)} + 1$, we obtain the relation:

$$Ae_{i-1,n_2-1} + vt_2 e_{i-1,0} + cvT_0 e_{\ell_1^{(2)}+i-1,0} = 0,$$

where we set for simplicity $A = t_0^{\ell_0^{(1)}} + t_1$.

If $i \geq n_1 - \ell_1^{(2)} + 1$, we obtain the relation:

$$Ae_{i-1,n_2-1} + vt_2 e_{i-1,0} + cvT_0 A e_{\ell_1^{(2)}+i-1-n_1,0} + cv^2 T_0 e_{\ell_1^{(2)}+i-1-n_1,1} = 0.$$

This gives us our first line of blocks: For $j = 0$ the relations involve elements in the blocks $j' = 0$, $j' = 1$ and $j' = n_2 - 1$.

For $j' = 0$ the matrix is:

$$\mathbf{M_{0,0}} = \begin{pmatrix}
0 & 0 & \dots & -(1-c)vT_0 & 0 & 0 & \dots & 0 \\
vt_2 & 0 & \dots & & cvT_0 & 0 & \dots & 0 \\
0 & vt_2 & \dots & 0 & & cvT_0 & \dots & \\
\vdots & & & & \vdots & & \vdots & \\
& & & & & & & cvT_0 \\
cvAT_0 & 0 & \dots & vt_2 & \dots & & 0 & 0 \\
\vdots & & & & & & \vdots & \vdots \\
0 & cvAT_0 & 0 & \dots & \dots & vt_2 & 0 & 0 \\
0 & \dots & cvAT_0 & \dots & 0 & \dots & vt_2 & 0
\end{pmatrix}$$

where the nonzero elements are aligned on parallels to the second diagonal, the first nonzero element of the first line is in column $\ell_1^{(2)}$ and the last nonzero element of the first column is in line $n_1 - \ell_1^{(2)} + 2$.

For $j' = 1$ the matrix is:

$$\mathbf{M_{0,1}} = \begin{pmatrix} 0 & 0 & \dots & & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & & 0 & \dots & \\ \vdots & & & & \vdots & & \vdots & \\ & & & & & & & 0 \\ cv^2T_0 & 0 & 0 & \dots & & 0 & 0 & \\ \vdots & & & & & & \vdots & \vdots \\ 0 & cv^2T_0 & 0 & \dots & \dots & \dots & 0 & 0 \\ 0 & \dots & cv^2T_0 & \dots & 0 & \dots & 0 & 0 \end{pmatrix}$$

where the nonzero elements are aligned on parallels to the second diagonal, the first and only nonzero element of the last line is in column $\ell_1^{(2)} - 1$ and the last nonzero element of the first column is in line $n_1 - \ell_1^{(2)} + 2$.

For $j' = n_2 - 1$ the matrix is

$$\mathbf{M_{0,n_2-1}} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ A & 0 & \dots & & 0 & 0 & \dots & 0 \\ 0 & A & \dots & 0 & & 0 & \dots & \\ \vdots & & & & \vdots & & \vdots & \\ & & & & & & & 0 \\ 0 & 0 & \dots & A & \dots & & 0 & 0 \\ \vdots & & & & & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & A & 0 & 0 \\ 0 & \dots & 0 & \dots & 0 & \dots & A & 0 \end{pmatrix}.$$

For $1 \leq j < n_2 - 1$, the relations involve elements in the blocks $j' = j - 1$, $j' = j$ and $j' = j + 1$. They are as follows:

- For $i = 0$, we have

$$t_2 e_{n_1-1,j-1} + cvT_0 e_{\ell_1^{(2)}-1,j} + AT_0 e_{\ell_1^{(2)}-1,j-1} = 0.$$

- For $0 < i < n_1 - \ell_1^{(2)} + 1$, we have

$$At_2 e_{i-1,j-1} + vt_2 e_{i-1,j} + AT_0 e_{i+\ell_1^{(2)}-1,j-1} + cvT_0 e_{i+\ell_1^{(2)}-1,j} = 0.$$

- For $i \geq n_1 - \ell_1^{(2)} + 1$, we have:

$$\begin{aligned} &At_2 e_{i-1,j-1} + vt_2 e_{i-1,j} + A^2 T_0 e_{i+\ell_1^{(2)}-n_1-1,j-1} \\ &\qquad +(1+c)vAT_0 e_{i+\ell_1^{(2)}-n_1-1,j} + cv^2 T_0 e_{i+\ell_1^{(2)}-n_1-1,j+1} = 0. \end{aligned}$$

For $j' = j - 1$, the matrix is:

$$\mathbf{M_{j,j-1}} = \begin{pmatrix} 0 & 0 & \dots & AT_0 & 0 & \dots & 0 & t_2 \\ At_2 & 0 & \dots & & AT_0 & 0 & \dots & 0 \\ 0 & At_2 & \dots & 0 & & AT_0 & \dots & \\ \vdots & & & & \vdots & & \vdots & \\ & & & & & & & AT_0 \\ A^2T_0 & 0 & \dots & At_2 & \dots & & 0 & \\ \vdots & & & & & & & \\ 0 & & 0 & \dots & \dots & At_2 & 0 & 0 \\ 0 & \dots & A^2T_0 & \dots & 0 & & At_2 & 0 \end{pmatrix}$$

where the nonzero elements are aligned on parallels to the second diagonal, the first nonzero element of the first line is in column $\ell_1^{(2)}$ and the last nonzero element of the first column is in line $n_1 - \ell_1^{(2)} + 2$.
For $j' = j$, the matrix is:

$$\mathbf{M_{j,j}} = \begin{pmatrix} 0 & 0 & \dots & cvT_0 & 0 & 0 & \dots & 0 \\ vt_2 & 0 & \dots & & cvT_0 & 0 & \dots & 0 \\ 0 & vt_2 & \dots & 0 & & cvT_0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots & \vdots \\ & & & & & & & cvT_0 \\ (c{+}1)vAT_0 & 0 & \dots & vt_2 & \dots & & 0 & 0 \\ \vdots & & & & & & & \vdots \\ 0 & (c{+}1)vAT_0 & 0 & \dots & \dots & vt_2 & 0 & 0 \\ 0 & \dots & (c{+}1)vAT_0 & \dots & 0 & & vt_2 & 0 \end{pmatrix}$$

where the nonzero elements are aligned on parallels to the second diagonal, the first nonzero element of the first line is in column $\ell_1^{(2)}$ and the last nonzero element of the first column is in line $n_1 - \ell_1^{(2)} + 2$.
For $j' = j + 1$, the matrix is:

$$\mathbf{M_{j,j+1}} = \begin{pmatrix} 0 & 0 & \dots & & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & & 0 & \dots & \\ \vdots & & & & \vdots & & \vdots & \\ & & & & & & & 0 \\ cv^2T_0 & 0 & \dots & 0 & \dots & & 0 & 0 \\ \vdots & & & & & & & \\ 0 & cv^2T_0 & 0 & \dots & \dots & \dots & 0 & 0 \\ 0 & \dots & cv^2T_0 & \dots & 0 & & 0 & 0 \end{pmatrix}$$

where the nonzero elements are aligned on parallels to the second diagonal, the first and only nonzero element of the last line is in column $\ell_1^{(2)}-1$.

For $j=n_2-1$, the relations involve elements in the blocks $j'=n_2-1$, $j'=n_2-2$, $j'=0$ and $j'=1$. They are as follows:

- For $i=0$, we have

$$t_2e_{n_1-1,n_2-2}+cvT_0e_{\ell_1^{(2)}-1,n_2-1}+AT_0e_{\ell_1^{(2)}-1,n_2-2}=0.$$

- For $i>0$ and $i\leq n_1-\ell_1^{(2)}$, we have:

$$At_2e_{i-1,n_2-2}+AT_0e_{\ell_1^{(2)}+i-1,n_2-2}+vt_2e_{i-1,n_2-1}+cvT_0e_{\ell_1^{(2)}+i-1,n_2-1}=0.$$

- For $n_1-\ell_1^{(2)}+1\leq i\leq 2(n_1-\ell_1^{(2)})$, we have:

$$\begin{aligned}At_2e_{i-1,n_2-2}\,&+A^2T_0e_{\ell_1^{(2)}+i-n_1-1,n_2-2}+(1+c)AT_0ve_{\ell_1^{(2)}+i-n_1-1,n_2-1}\\&+vt_2e_{i-1,n_2-1}+cv^2t_2e_{\ell_1^{(2)}+i-n_1-1,0}+cv^2T_0e_{2\ell_1^{(2)}+i-n_1-1,0}=0.\end{aligned}$$

- For $i\geq 2(n_1-\ell_1^{(2)})+1$, we have:

$$\begin{aligned}At_2e_{i-1,n_2-2}\,&+A^2T_0e_{\ell_1^{(2)}+i-n_1-1,n_2-2}+(1+c)vAT_0e_{\ell_1^{(2)}+i-n_1-1,n_2-1}\\&+vt_2e_{i-1,n_2-1}+cv^2t_2e_{\ell_1^{(2)}+i-n_1-1,0}+cv^2AT_0e_{2\ell_1^{(2)}+i-2n_1-1,0}\\&+cv^3T_0e_{2\ell_1^{(2)}+i-2n_1-1,1}=0.\end{aligned}$$

For $j'=0$, the matrix is:

$$\mathbf{M_{n_2-1,0}}=\begin{pmatrix}0&0&\dots&0&0&0\\\vdots&\vdots&\dots&\vdots&&\dots\\&&&0&&\\cv^2t_2&0&\dots&&cv^2T_0&0\\\vdots&cv^2t_2&&&\vdots&\vdots\\0&\dots&&&&cv^2T_0\\cv^2AT_0&&&\dots&\dots&0\\\vdots&\vdots&\dots&\vdots&&\dots\\0&cv^2AT_0&\dots&cv^2t_2&&0\end{pmatrix}$$

where the nonzero elements are aligned on parallels to the second diagonal, the first line with nonzero entries is the line with number $n_1-\ell_1^{(2)}+2$, and its last nonzero element is in column $\ell_1^{(2)}$. The last nonzero element of the first column is in line $2(n_1-\ell_1^{(2)}+1)$.

For $j' = 1$, the matrix is:

$$\mathbf{M_{n_2-1,1}} = \begin{pmatrix}
0 & 0 & \dots & & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & & 0 & \dots & \\
\vdots & & & & \vdots & & \vdots & \\
 & & & & & & & 0 \\
cv^3T_0 & 0 & \dots & 0 & \dots & & 0 & 0 \\
\vdots & & & & & & & \\
0 & cv^3T_0 & 0 & \dots & \dots & \dots & 0 & 0 \\
0 & \dots & cv^3T_0 & \dots & 0 & & 0 & 0
\end{pmatrix}$$

where the first nonzero element of the first column is on line $2(n_1 - \ell_1^{(2)} + 1)$.
For $j' = n_2 - 2$, the matrix is:

$$\mathbf{M_{n_2-1,n_2-2}} = \begin{pmatrix}
0 & 0 & \dots & AT_0 & 0 & \dots & 0 & t_2 \\
At_2 & 0 & \dots & & AT_0 & 0 & \dots & 0 \\
0 & At_2 & \dots & 0 & & AT_0 & \dots & 0 \\
\vdots & & & & \vdots & & \vdots & \vdots \\
 & & & & & & & AT_0 \\
A^2T_0 & 0 & \dots & At_2 & \dots & & 0 & 0 \\
\vdots & & & & & & \vdots & \\
0 & A^2T_0 & 0 & \dots & \dots & At_2 & 0 & 0 \\
0 & \dots & A^2T_0 & \dots & 0 & 0 & At_2 & 0
\end{pmatrix}$$

where the first nonzero element of the first line is in column $\ell_1^{(2)}$ and the last nonzero element of the first column is in line $n_1 - \ell_1^{(2)} + 2$.
For $j' = n_2 - 1$, the matrix is:

$$\mathbf{M_{n_2-1,n_2-1}} = \begin{pmatrix}
0 & 0 & \dots & cvT_0 & & 0 & \dots & 0 \\
vt_2 & 0 & \dots & & cvT_0 & 0 & \dots & 0 \\
0 & vt_2 & \dots & 0 & & cvT_0 & \dots & 0 \\
\vdots & & & & \vdots & & \vdots & \\
 & & & & & & & cvT_0 \\
(1+c)vAT_0 & 0 & \dots & vt_2 & \dots & & 0 & \\
\vdots & \vdots & & & & & & \\
0 & (1+c)vAT_0 & 0 & \dots & \dots & vt_2 & 0 & 0 \\
0 & \dots & (1+c)vAT_0 & \dots & 0 & & vt_2 & 0
\end{pmatrix}$$

where the first nonzero element of the first line is in column $\ell_1^{(2)}$ and the last nonzero element of the first column is in line $n_1 - \ell_1^{(2)} + 2$.

Finally, the matrix M of the presentation of the $\mathbf{C}[t_0, t_1, t_2, v]$-module $\mathcal{O}_{\mathcal{C}}$ is described by the blocks $M_{j,j'}$:

$$\mathbf{M} = \begin{pmatrix} M_{0,0} & M_{0,1} & 0 & \dots & & 0 & 0 & M_{0,n_2-1} \\ \vdots & \vdots & \dots & \vdots & & \dots & & \vdots \\ \vdots & \vdots & \dots & \vdots & & \dots & & \vdots \\ 0 & & M_{j-1,j-2} & M_{j-1,j-1} & M_{j-1,j} & & & 0 \\ 0 & & & M_{j,j-1} & M_{j,j} & M_{j,j+1} & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & & \dots & & \vdots \\ \vdots & \vdots & \dots & \vdots & & \dots & & \vdots \\ M_{n_2-1,0} & M_{n_2-1,1} & 0 & \dots & 0 & & M_{n_2-1,n_2-2} & M_{n_2-1,n_2-1} \end{pmatrix}.$$

We are going to get information about the determinant of the matrix of relations between the generators $e_{i,j}$ using this decomposition into blocks.

Lemma 3.2. *For $v = 0$, the determinant of the matrix M is given by:*

$$\det M = A^{n_1-1} \left(\det M_{j,j-1}\right)^{n_2-1} .$$

Proof. For $v = 0$ the only nonzero blocks are M_{0,n_2-1} and the $M_{j,j-1}$ which are all equal. By expanding the determinant of M successively along the last n_1 columns, we find that it is equal to A^{n_1-1} times the determinant of the matrix $\overline{M}$ of size $\beta_0 - n_1$ obtained by deleting the first n_1 lines and the last n_1 columns of M. That matrix is subdivided into blocks $M_{j,j'}$, among which the only nonzero ones are the $n_2 - 1$ blocks $M_{j,j-1}$, which are all equal; therefore they commute and we can compute the determinant of $\overline{M}$ as the product of the determinants of the blocks (see [1], §9, Lemme 1). The result follows. $\square$

Lemma 3.3. *For $v = 0$ and $t_2 = 0$, the discriminant is equal to*

$$T_0^{n_1(n_2-1)} A^{n_2(n_1-1)+\ell_1^{(2)}(n_2-1)}.$$

As a consequence, the Newton polyhedron of the discriminant of ϕ contains as an edge the segment joining the two points

$$P_1 = (n_1(n_2-1)\ell_0^{(2)}, n_2(n_1-1) + \ell_1^{(2)}(n_2-1), 0)$$

and

$$P_2 = ((n_1-1)\overline{\beta}_1 + (n_2-1)\overline{\beta}_2, 0, 0).$$

Proof. The first part of the statement follows directly from Lemma 3.2. Since the Newton polyhedron of the discriminant is necessarily entirely on one side of any of the coordinate hyperplanes $\tau_i = 0$ or $v = 0$, its intersection with one of them is necessarily a face. If, upon intersecting with the other hyperplane, we find a segment, that segment is necessarily an edge. We apply this to $v = 0$ and $\tau_2 = 0$ and use the expressions of the $n_i\overline{\beta}_i$ in terms of $\overline{\beta}_k$ with $k < i$ to compute the coordinates of the points P_i. This proves the second part. $\square$

Lemma 3.4. *For $v = 0$ the Newton polyhedron of the discriminant of $\phi|_{v=0}$ lies entirely on one side of ("above") the hyperplane*

$$n_2\tau_0 + \overline{\beta}_1\tau_1 = (n_1 - 1)n_2\overline{\beta}_1 \,.$$

Proof. Using the result of Lemma 3.2 and the expression of $M_{j,j-1}$, we see that it suffices to show that the exponents (τ_0, τ_1) of the monomials appearing in the determinant of M and arising from the term $A^{n_1-1}t_2^{n_2}$ in the determinant of $M_{j,j-1}$ satisfy the inequality $n_2\tau_0+\overline{\beta}_1\tau_1 \geq (n_1-1)n_2\overline{\beta}_1$. Indeed all the other monomials contain higher exponents of t_0 or t_1. This amounts to studying the exponents of t_0 and t_1 appearing in the expansion of $A^{n_1-1}A^{(n_1-1)(n_2-1)} = A^{(n_1-1)n_2}$. But these are terms $t_0^{i\ell_0^{(1)}}t_1^j$ with $i + j = (n_1 - 1)n_2$. Substituting in the equation of our hyperplane and remembering that by definition $n_2\ell_0^{(1)} = \overline{\beta}_1$ gives the result. □

Lemma 3.5. *The Newton polyhedron of the discriminant of ϕ contains as an edge the segment joining the two points*

$$P_2 = ((n_1 - 1)\overline{\beta}_1 + (n_2 - 1)\overline{\beta}_2, 0, 0) \quad \textit{and} \quad P_3 = ((n_1 - 1)\overline{\beta}_1, 0, n_1(n_2 - 1)).$$

Proof. Since the Newton polyhedron of the discriminant is contained in the hyperplane of homogeneity, its intersection with the coordinate plane $\tau_1 = 0, v = 0$ is contained in a line. By convexity this line is a segment. By the same argument as above it is an edge of the Newton polyhedron of the discriminant. We are going to determine its extremities by seeking the points of maximum and minimum value of τ_0. We apply this to $v = 0$ and $\tau_1 = 0$, which means that we compute the expression of the discriminant for $v = 0$ and $t_1 = 0$ using lemma 3.2 and the expression of $M_{j,j-1}$ to seek the maximum value of the exponent of t_0, which is obtained by taking the product of the AT_0 and A^2T_0 in the expansion of the determinant of $M_{j,j-1}$, and its minimum value, obtained by taking the term $A^{n_1-1}t_2^{n_1}$ in that expansion. Finally we use the expressions of the $n_i\overline{\beta}_i$ in terms of $\overline{\beta}_k$ with $k < i$ to compute the coordinates of the points P_i. □

Lemma 3.6. *The Newton polyhedron of the discriminant of ϕ contains as an edge the segment $\overline{P_3P_4}$, where $P_4 = (0, n_2(n_1 - 1), n_1(n_2 - 1))$.*

Proof. Again, use Lemma 3.2 and observe that for $v = t_0 = 0$ the determinant of $M_{j,j-1}$ is equal to the monomial $t_1^{n_1-1}t_2^{n_1}$. Use the expressions of the $n_i\overline{\beta}_i$ in terms of $\overline{\beta}_k$ with $k < i$ to compute the coordinates of the point P_4. On the other hand, it follows from Lemma 3.2 that the Newton polyhedron for $v = 0$ is entirely on one side of the hyperplane $\tau_2 = n_1(n_2 - 1)$. It meets it in the two points P_3, P_4 which are in different coordinate planes, therefore along the edge $\overline{P_3P_4}$. □

Lemma 3.7. *The segment $\overline{P_1P_4}$ is an edge of the Newton polyhedron of ϕ and the plane containing it and parallel to the τ_1-axis supports a non-compact face of the Newton polyhedron.*

Proof. The equation of the hyperplane parallel to the t_1-axis and containing $\overline{P_1P_4}$ is:

$$\mathcal{H} : \tau_0 + \ell_0^{(2)}\tau_2 = n_1(n_2-1)\ell_0^{(2)}.$$

The expression given above for the matrix M shows that the products appearing in the expansion of its determinant are all up to a constant factor of the form $v^\delta A^\alpha T_0^\beta t_2^\gamma$ with $\beta+\gamma \geq n_1(n_2-1)$. The result follows because this implies by a direct computation that the Newton polyhedron of the discriminant of ϕ is entirely on one side of $\mathcal{H}$. □

Proposition 3.8. *The Newton polyhedron of the discriminant of the map ϕ restricted to $v=0$ has one compact face which is is the convex hull of the points $P_1, P_2.P_3, P_4$ and two non compact faces, the plane parallel to the τ_1 axis and containing the segment $\overline{P_1P_4}$ and the plane parallel to the τ_2-axis and containing the segment $\overline{P_3P_4}$.*

Proof. It follows from the previous lemmas since we know by Proposition 2.3 that the compact face of the Newton polyhedron for $v=0$ is contained in the plane $\overline{\beta}_0\tau_0 + n_1\overline{\beta}_1\tau_1 + n_2\overline{\beta}_2\tau_2 = \overline{\beta}_0((n_1-1)\overline{\beta}_1 + (n_2-1)\overline{\beta}_2)$. □

Lemma 3.9. *The convex hull of the points P_3, P_4, P_5, where $P_5 = (0,0,\overline{\beta}_0-1)$ is a face of the Newton polyhedron of ϕ.*

Proof. Taking $t_0 = t_1 = 0$, the determinant of the matrix M reduces to the monomial $v^{(n_1-1)n_2}t_2^{\overline{\beta}_0-1}$, which corresponds to P_5. To prove the lemma it suffices to show that the Newton polyhedron is entirely on one side of ("above") the hyperplane $\overline{\beta}_0\tau_0 + n_1\overline{\beta}_1\tau_1 + \overline{\beta}_0\overline{\beta}_1\tau_2 - \overline{\beta}_0(\overline{\beta}_0-1)\overline{\beta}_1 = 0$ determined by the points P_3, P_4, P_5.

Given a point P with coordinates $(\tau_0, \tau_1, \tau_2, \upsilon)$ satisfying the relation of homogeneity

$$\overline{\beta}_0\tau_0 + n_1\overline{\beta}_1\tau_1 + n_2\overline{\beta}_2\tau_2 + (\beta_1-\beta_2)\upsilon - \overline{\beta}_0((n_1-1)\overline{\beta}_1 + (n_2-1)\overline{\beta}_2) = 0$$

of Proposition 2.3, we must check that it gives a positive value to

$$H_2(\tau_0,\tau_1,\tau_2) = \overline{\beta}_0\tau_0 + n_1\overline{\beta}_1\tau_1 + \overline{\beta}_0\overline{\beta}_1\tau_2 - \overline{\beta}_0(\overline{\beta}_0-1)\overline{\beta}_1.$$

A short computation using the identities between the β_j and $\overline{\beta}_k$ after eliminating τ_0 and τ_1 by substracting the homogeneity relation from H_2 and rewriting (modulo that relation)

$$\begin{aligned} H_2(\tau_0,\tau_1,\tau_2) \;=\; & \overline{\beta}_0((n_1-1)\overline{\beta}_1 + (n_2-1)\overline{\beta}_2) - \overline{\beta}_0(\overline{\beta}_0-1)\overline{\beta}_1 \\ & + (\beta_2-\beta_1)\upsilon + (\overline{\beta}_0\overline{\beta}_1 - n_2\overline{\beta}_2)\tau_2\,, \end{aligned}$$

shows that for such a point we have the equality

$$H_2(\tau_0, \tau_1, \tau_2) = (\beta_2 - \beta_1)(\overline{\beta}_0(n_2 - 1) + \upsilon - n_2\tau_2),$$

so that we have the required inequality if and only if $n_2\tau_2 - \upsilon \leq \overline{\beta}_0(n_2 - 1)$.

But this is always true since the greatest possible exponent of t_2 in the determinant of M is $\overline{\beta}_0 - 1$, corresponding to the terms in the subdiagonal of M, and by looking again at the matrix one sees that $\tau_2 - \upsilon \leq n_2 - 1$ since there are only $n_2 - 1$ occurrences of t_2 without a factor υ. □

Proposition 3.10. *The Newton polyhedron of the discriminant of the map ϕ for a fixed $\upsilon \neq 0$ has two compact faces, which are respectively the convex hulls of $P_1, P_2.P_3, P_4$ and of P_3, P_4, P_5 and one non compact face, which is that of Lemma 3.7.*

Proof. The statement follows from the previous lemmas. □

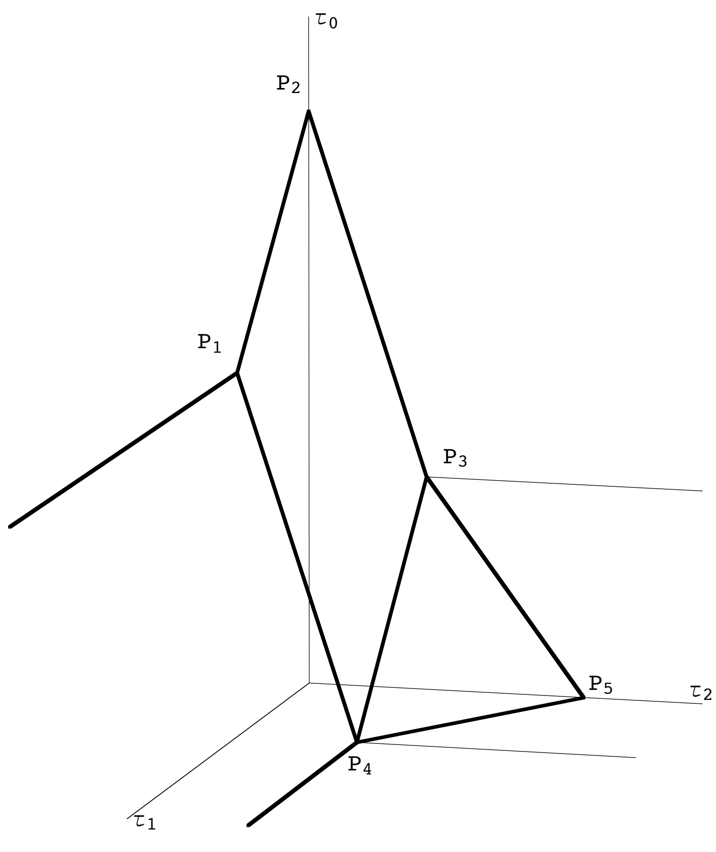

Figure 1: Newton polyhedron of the discriminant of ϕ.

Figure 1 gives an idea for the shape of the Newton polyhedron for $v \neq 0$. The non-compact face which appears for $v = 0$ is suggested in thinner lines.

The intersection with $\tau_1 = 0$ is the jacobian Newton polygon of the plane branch.

1. $P_1 = (n_1(n_2-1)\ell_0^{(2)},\ n_2(n_1-1) + \ell_1^{(2)}(n_2-1),\ 0)$,
2. $P_2 = ((n_1-1)\overline{\beta}_1 + (n_2-1)\overline{\beta}_2,\ 0,\ 0)$,
3. $P_3 = ((n_1-1)\overline{\beta}_1,\ 0,\ n_1(n_2-1))$,
4. $P_4 = (0,\ n_2(n_1-1),\ n_1(n_2-1))$,
5. $P_5 = (0,\ 0,\ \overline{\beta}_0 - 1)$.

4 A Question of Genericity

The linear form u_0 is *not* general with respect to the monomial curve defined by the vanishing of $f_1 = u_1^{n_1} - u_0^{\ell_0^{(1)}}$ and $f_2 = u_2^{n_2} - u_0^{\ell_0^{(2)}} u_1^{\ell_1^{(2)}}$; this is attested by the fact that the critical space of the map (u_0, f_1, f_2) is not reduced, contradicting a known result on polar varieties (see [8], Chap. IV). Therefore it could be that the Newton polyhedron that we obtain for $v = 0$ is not really the jacobian Newton polyhedron of the monomial curve. We are going to verify that in fact it is.

The method is to check that considering the critical subspace with respect to a general linear form $u_0 + \sigma u_1 + \tau u_2$ affects the matrix of our presentation only by adding terms whose effect on the determinant is to possibly add exponents which can be seen to be above the Newton polyhedron computed for u_0. Therefore those terms do not modify the Newton polyhedron.

A direct computation shows that modulo the equation $f_1 = 0$ we have

$$\begin{aligned} df_1 \wedge df_2 = {} & n_1 n_2 u_1^{n_1-1} u_2^{n_2-1} du_1 \wedge du_2 - \ell_0^{(1)} n_2 u_0^{\ell_0^{(1)}-1} u_2^{n_2-1} du_0 \wedge du_2 \\ & + (n_1 \ell_0^{(2)} + \ell_0^{(1)} \ell_1^{(2)}) u_0^{\ell_0^{(1)} + \ell_0^{(2)} - 1} u_1^{\ell_1^{(2)}-1} du_0 \wedge du_1 \,. \end{aligned}$$

In fact, the computation gives

$$\begin{aligned} df_1 \wedge df_2 = {} & n_1 n_2 u_1^{n_1-1} u_2^{n_2-1} du_1 \wedge du_2 - \ell_0^{(1)} n_2 u_0^{\ell_0^{(1)}-1} u_2^{n_2-1} du_0 \wedge du_2 \\ & + u_0^{\ell_0^{(2)}-1} u_1^{\ell_1^{(2)}-1} (n_1 \ell_0^{(2)} u_1^{n_1} + \ell_0^{(1)} \ell_1^{(2)} u_0^{\ell_0^{(1)}}) du_0 \wedge du_1, \end{aligned}$$

but modulo f_1, we can replace $u_1^{n_1}$ by $u_0^{\ell_0^{(1)}}$. From this it follows, using the definitions of the $\ell_k^{(j)}$, that

$$\begin{aligned} df_1 \wedge df_2 \wedge du_0 &= \overline{\beta}_0 u_1^{n_1-1} u_2^{n_2-1} du_0 \wedge du_1 \wedge du_2, \\ df_1 \wedge df_2 \wedge du_1 &= \overline{\beta}_1 u_0^{\ell_0^{(1)}-1} u_2^{n_2-1} du_0 \wedge du_1 \wedge du_2, \\ df_1 \wedge df_2 \wedge du_2 &= \overline{\beta}_2 u_0^{\ell_0^{(1)} + \ell_0^{(2)} - 1} u_1^{\ell_1^{(2)}-1} du_0 \wedge du_1 \wedge du_2, \end{aligned}$$

the last equation being read mod.f_1. The equation of our critical subspace with respect to a general linear form therefore now reads, modulo f_1,

$$\overline{\beta}_0 u_1^{n_1-1} u_2^{n_2-1} + \sigma\overline{\beta}_1 u_0^{\ell_0^{(1)}-1} u_2^{n_2-1} + \tau\overline{\beta}_2 u_0^{\ell_0^{(1)}+\ell_0^{(2)}-1} u_1^{\ell_1^{(2)}-1} = 0.$$

Since σ and τ are now assumed to be "general" constants, we may simplify this to

$$u_1^{n_1-1} u_2^{n_2-1} + \sigma u_0^{\ell_0^{(1)}-1} u_2^{n_2-1} + \tau u_0^{\ell_0^{(1)}+\ell_0^{(2)}-1} u_1^{\ell_1^{(2)}-1} = 0.$$

Since $u_0 = t_0$, this means that we have to study which effect adding the multiplication by $\sigma t_0^{\ell_0^{(1)}-1} u_2^{n_2-1} + \tau t_0^{\ell_0^{(1)}+\ell_0^{(2)}-1} u_1^{\ell_1^{(2)}-1}$ has on our matrix and its determinant for $v = 0$.

Using the same method as above, we see that the submatrices $M_{j,j'}$ which are affected are M_{0,n_2-1}, which becomes

$$\tilde{\mathbf{M}}_{\mathbf{0,n_2-1}} = \begin{pmatrix}
\sigma t_0^{\ell_0^{(1)}-1} & 0 & \dots & 0 & 0 & 0 & \dots & 1 \\
A & \sigma t_0^{\ell_0^{(1)}-1} & \dots & & 0 & 0 & \dots & 0 \\
0 & A & \sigma t_0^{\ell_0^{(1)}-1} & 0 & & 0 & \dots & \\
\vdots & & & & \vdots & & \vdots & \\
& & & & & & & 0 \\
0 & 0 & \dots & A & \dots & & 0 & 0 \\
\vdots & & & & & & \vdots & \vdots \\
0 & 0 & 0 & \dots & \dots & A & \sigma t_0^{\ell_0^{(1)}-1} & 0 \\
0 & \dots & 0 & \dots & 0 & \dots & A & \sigma t_0^{\ell_0^{(1)}-1}
\end{pmatrix},$$

and $M_{j,j}$, for $j \geq 0$, which becomes (remember that $v = 0$)

$$\tilde{\mathbf{M}}_{\mathbf{j,j}} = \tau \begin{pmatrix}
0 & 0 & \dots & t_0^{\ell_0^{(1)}+\ell_0^{(2)}-1} & 0 & \dots & 0 \\
0 & 0 & \dots & & t_0^{\ell_0^{(1)}+\ell_0^{(2)}-1} & \dots & 0 \\
\vdots & & & & \vdots & \vdots & \vdots \\
& & & & & & t_0^{\ell_0^{(1)}+\ell_0^{(2)}-1} \\
At_0^{\ell_0^{(1)}+\ell_0^{(2)}-1} & 0 & \dots & 0 & \dots & 0 & 0 \\
\vdots & & & & & & \vdots \\
0 & \dots & 0 & \dots & \dots & 0 & 0 \\
0 & \dots At_0^{\ell_0^{(1)}+\ell_0^{(2)}-1} & & \dots & 0 & 0 & 0
\end{pmatrix}.$$

(the nonzero elements are aligned on parallels to the second diagonal, the first nonzero element in the first line is in column $\ell_1^{(2)}$, and the first nonzero element in the first column is in line $n_1 - \ell_1^{(2)} + 2$), and, finally, the matrix $M_{j,j-1}$, which becomes (setting $U_0 = t_0^{\ell_0^{(1)}-1}$)

$$\tilde{\mathbf{M}}_{\mathbf{j,j-1}} = \begin{pmatrix} \sigma U_0 t_2 & 0 & \dots & AT_0 & \sigma T_0 U_0 & \dots & 0 & t_2 \\ At_2 & \sigma U_0 t_2 & \dots & & AT_0 & \sigma T_0 U_0 & \dots & 0 \\ 0 & At_2 & \dots & 0 & & AT_0 & \dots & \\ \vdots & & & & \vdots & & \vdots & \sigma T_0 U_0 \\ \sigma t_0^{\ell_0^{(1)}} T_0 U_0 & & & & & & & AT_0 \\ A^2 T_0 & \sigma t_0^{\ell_0^{(1)}} T_0 U_0 & \dots & At_2 & \dots & & 0 & \\ \vdots & & & & & & & \\ 0 & & 0 & \dots & \dots & At_2 & \sigma U_0 t_2 & 0 \\ 0 & \dots & A^2 T_0 & \sigma t_0^{\ell_0^{(1)}} T_0 U_0 & 0 & & At_2 & \sigma U_0 t_2 \end{pmatrix}.$$

The matrix $\tilde{M}(0)$ corresponding to a general linear form and $v = 0$ has the following structure: it is described by the blocks $\tilde{M}_{j,j'}$:

$$\tilde{\mathbf{M}}(\mathbf{0}) = \begin{pmatrix} \tilde{M}_{0,0} & 0 & 0 & \dots & & 0 & 0 & \tilde{M}_{0,n_2-1} \\ \tilde{M}_{1,0} & \tilde{M}_{1,1} & 0 & 0 & & \dots & & 0 \\ \vdots & \vdots & \dots & \vdots & & \dots & & \vdots \\ 0 & & \tilde{M}_{j-1,j-2} & \tilde{M}_{j-1,j-1} & 0 & & & 0 \\ 0 & & & \tilde{M}_{j,j-1} & \tilde{M}_{j,j} & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & & \dots & & \vdots \\ \vdots & \vdots & \dots & \vdots & & \dots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & & \tilde{M}_{n_2-1,n_2-2} & \tilde{M}_{n_2-1,n_2-1} \end{pmatrix}.$$

Let us set $E = \det \tilde{M}_{0,n_2-1}$ and, using the fact that the matrices $\tilde{M}_{j,j}$ and $\tilde{M}_{j,j-1}$ are in fact independant of j, write $D = \det \tilde{M}_{j,j}$ and $S = \det \tilde{M}_{j,j-1}$. We can now use the Laplace expansion (see [1, §8]) of the determinant with respect to the last n_1 columns, we obtain (neglecting signs) an expression $\pm\det\tilde{M} = E\det M_1 \pm D\det N_1$. Then we notice that we can again use the Laplace expansion with respect to the last n_1 lines or n_1 columns, and we obtain

$$\pm \det \tilde{M}(0) = ES^{n_2-1} \pm D^{n_2}.$$

The discriminant D is easy to compute and equal to

$$D = \tau^{n_1} A^{\ell_1^{(2)}-1} t_0^{n_1(\ell_0^{(1)}+\ell_0^{(2)}-1)}.$$

We can check that the exponent of t_0 appearing in D^{n_2} is larger than an exponent already appearing in the discriminant for $\sigma = \tau = 0$. Therefore, it does not affect the Newton polyhedron.

In the expression that we have seen above in Lemma 3.2 for $\ell = u_0$ and $v = 0$, the power of t_0 which appears is

$$(n_1 - 1)\ell_0^{(1)} + (n_2 - 1)\big(n_1(\ell_0^{(1)} + \ell_0^{(2)}) + (\ell_1^{(2)} - 1)\ell_0^{(1)}\big).$$

We only have to prove the inequality

$$n_1n_2(\ell_0^{(1)}+\ell_0^{(2)}-1)+n_2(\ell_1^{(2)}-1)\ell_0^{(1)} \\ \geq (n_1-1)\ell_0^{(1)}+(n_2-1)\big(n_1(\ell_0^{(1)}+\ell_0^{(2)})+(\ell_1^{(2)}-1)\ell_0^{(1)}\big).$$

After some rewriting, it comes down to

$$\ell_1^{(2)}\ell_0^{(1)}+n_1\ell_0^{(2)}-n_1n_2\geq 0.$$

But if we remember that we have the equality

$$\ell_0^{(2)}\overline{\beta}_0=n_2\overline{\beta}_2-\ell_1^{(2)}\overline{\beta}_1=(n_2-1)\overline{\beta}_2+\overline{\beta}_2-\ell_1^{(2)}\overline{\beta}_1$$

and the fact that $\ell_1^{(2)}<n_1$, we get $\ell_0^{(2)}\geq n_2$, and this suffices to prove our inequality.

Let us now deal with ES^{n_2-1}: The exponent of the diagonal term in E, equal to $\sigma^{n_1}t_0^{n_1(\ell_0^{(1)}-1)}$, is larger than the exponent of $t_0^{(n_1-1)\ell_0^{(1)}}$ which appears in A^{n_1-1}, because $\ell_0^{(1)}>n_1$. So we can forget about that diagonal term in E.

Next, let us consider S^{n_2-1}: Our polyhedron for $v=0$ is bounded by the three hyperplanes

1. $\overline{\beta}_0\tau_0+n_1\overline{\beta}_1\tau_1+n_2\overline{\beta}_2\tau_2=\overline{\beta}_0((n_1-1)\overline{\beta}_1+(n_2-1)\overline{\beta}_2)$
2. $n_2\tau_0+\overline{\beta}_1\tau_1=(n_1-1)n_2\overline{\beta}_1$
3. $\tau_0+\ell_0^{(2)}\tau_2=n_1(n_2-1)\ell_0^{(2)}$

Calling L_1,L_2,L_3 the linear forms appearing in the left-hand side of these three equations, for each L_i we seek successively in each column of the matrix $\tilde{M}_{j,j-1}$ the terms which give it the lowest value and which it is possible to choose in the expansion of the discriminant, and then check that such a choice gives rise in ES^{n_2-1} to exponents which are above the corresponding support hyperplane of our polyhedron.

For example, the linear form L_1 takes as minimum value in the first $\ell_1^{(2)}$ columns the value $\overline{\beta}_0(2\ell_0^{(1)}-1+\ell_0^{(2)})$ which corresponds to $\sigma t_0^{\ell_0^{(1)}}T_0U_0$, and on the last $n_1-\ell_1^{(2)}$ columns the minimal value $\overline{\beta}_0(\ell_0^{(1)}-1+\ell_0^{(2)})$ which corresponds to σT_0U_0. This gives us a term $t_0^{n_1(\ell_0^{(1)}-1+\ell_0^{(2)})+\ell_1^{(2)}\ell_0^{(1)}}$ in S and, therefore, exponents $\big((n_2-1)(n_1(\ell_0^{(1)}-1+\ell_0^{(2)})+\ell_1^{(2)}\ell_0^{(1)})+(n_1-1-i)\ell_0^{(1)},i,0\big)$ in the expansion of $S^{n_2-1}A^{n_1-1}$.

Since $n_1\overline{\beta}_1=\ell_0^{(1)}\overline{\beta}_0$, it suffices to check the inequality on L_1 for $i=0$. This means to verify the inequality

$$(n_2-1)\big((n_2-1)(n_1(\ell_0^{(1)}-1+\ell_0^{(2)})+\ell_1^{(2)}\ell_0^{(1)})\big)+(n_1-1)\ell_0^{(1)} \\ \geq (n_1-1)\overline{\beta}_1+(n_2-1)\overline{\beta}_2\,.$$

We can now use the equalities $\overline{\beta}_2 = n_1\ell_0^{(2)} + \ell_0^{(1)}\ell_1^{(2)}$ and $n_2\ell_0^{(1)} = \overline{\beta}_1$ which follow from the definitions to rearrange the terms on the left into

$$(n_2-1)\overline{\beta}_2 + (n_1-1)\overline{\beta}_1 + (n_2-1)(\ell_0^{(1)} - n_1 + 2)$$

and prove that the inequality follows from $\ell_0^{(1)} > n_1$.

If we now take L_2, the term giving the minimal value in each column of $\tilde{M}_{j,j-1}$ is $\sigma U_0 t_2$. This gives a term with $\tau_0 = n_1(n_2-1)(\ell_0^{(1)}-1)+(n_1-1)\ell_0^{(1)}$, which again gives the same value to L_2 as all the other terms coming from A^{n_1-1}. So we have to prove the inequality

$$n_1n_2(n_2-1)(\ell_0^{(1)}-1) + (n_1-1)n_2\ell_0^{(1)} \geq (n_1-1)n_2\overline{\beta}_1.$$

Again, using $n_2\ell_0^{(1)} = \overline{\beta}_1$, we can rearrange the left-hand side of this inequality into $(n_1-1)n_2\overline{\beta}_1 + (\ell_0^{(1)} - n_1)n_2(n_2-1)$, and the result then follows from $\ell_0^{(1)} > n_1$. The last case is left to the reader. From these computations one finally deduces that the Newton polyhedron with respect to the linear form u_0 is indeed the general one for $v=0$.

5 The Information is Constant

To conclude let us check that the Newton polyhedra for $v=0$ and for $v \neq 0$ both contain the same information, namely the semigroup of the plane branch, or equivalently its Puiseux characteristic, its equisingularity type, or its topological type.

First, it follows from the description of the polyhedra that they are both determined by the generators of the semigroup; the numbers $\overline{\beta}_i$, n_1, n_2 and $\ell_k^{(j)}$ are all determined by the semigroup. The Newton polyhedron for $v \neq 0$ contains as a plane section the jacobian Newton polyhedron of the plane branch which is known to determine the equisingularity type, so that its datum is equivalent to that of the equisingularity type, or the semigroup. It is also easy to check directly that its knowledge gives us the generators of the semigroup: the point P_5 gives us $\overline{\beta}_0 = n_1n_2$, so that from the homogeneity relation of Proposition 2.3 we know $n_1\overline{\beta}_1$ and $n_2\overline{\beta}_2$. But once we know $\overline{\beta}_0$ the coordinates of the point P_3 give us n_1 and n_2, and we are done.

It remains to verify that no information is lost when $v=0$. Let us collect the information that we have: First we have the homogeneity relation for $v=0$:

$$\overline{\beta}_0\tau_0 + n_1\overline{\beta}_1\tau_1 + n_2\overline{\beta}_2\tau_2 = \overline{\beta}_0((n_1-1)\overline{\beta}_1 + (n_2-1)\overline{\beta}_2).$$

It gives us the coefficients up to a multiplicative rational factor.

The point P_4 gives us $n_1 - n_2 = d$ by difference of its second and third coordinates. Substituting in the second coordinate we find that it is equal

to $n_2(n_2+d-1)$, so that we know the product and the difference of n_2 and n_2+d-1. Since d is known, we now know n_2, hence also n_1 and their product $\overline{\beta}_0$. From the homogeneity equation we can finally deduce $\overline{\beta}_1$ and $\overline{\beta}_2$.
So the information is indeed constant, with two different encodings.

Questions: It is to be hoped that for any number of characteristic pairs, the Newton polyhedron for $v \neq 0$ has exactly g compact faces, which intersect the plane $\tau_1 = \cdots = \tau_{g-1} = 0$ along the jacobian Newton polygon of the plane branch, and that the information contained in the Newton polyhedron for $v = 0$ is still equivalent to the knowledge of the semigroup of the branch.

More generally, one can hope that given a branch, plane or not, such that the monomial curve with the same semigroup is a complete intersection, the jacobian Newton polyhedron associated to the map defined by the equations of the branch and a general linear form encodes the semigroup.

Acknowledgements

This research was partially supported by the "Acción Integrada Hispano-Francesa" HF2003-0048 and the Spanish research projects MMCyT BFM 2001-2251 and MEC PN MTM2004-00958.

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Depth and Differential Forms

Helmut A. Hamm

Abstract

The purpose of this paper is to show that certain theorems about the de Rham cohomology of smooth complex projective varieties are still valid for the lower cohomology groups if we pass from a smooth variety to a local complete intersection, still using the ordinary de Rham complex. This concerns the isomorphism between singular and de Rham cohomology, Hodge decomposition, Lefschetz theorem on hyperplane section and Gysin sequence. Moreover, the Akizuki-Nakano theorem can be generalized to this case.

Introduction

Let X be a smooth complex algebraic variety. Then the singular cohomology of X can be expressed by holomorphic forms: $H^k(X;\mathbb{C}) \simeq \mathbb{H}^k(X, \Omega_X^\bullet)$, where $\Omega_X^\bullet$ denotes the complex of sheaves of holomorphic differential forms on X and $\mathbb{H}^k(X, \Omega_X^\bullet)$ the k-th hypercohomology group of this complex: de Rham cohomology. The key ingredient is the holomorphic Poincaré lemma.

If X is singular the Poincaré lemma fails, in general, so singular and de Rham cohomology are different. But for local complete intersections there is still a coincidence for small k.

This circumstance has already been observed long ago in singularity theory, when differential forms were used in order to study the topology of an isolated singularity: If we look locally at a hypersurface with isolated singularity the Picard-Lefschetz monodromy is a very useful invariant. Brieskorn [5] showed that the characteristic polynomial of this monodromy can be calculated analytically, using differential forms. Greuel [12] generalized this in his thesis to complete intersections. A technical difficulty for him was that he had to generalize the so-called de Rham lemma. He managed this problem

1991 *Mathematics Subject Classification.* 14F40, 14M10, 32S20, 32S35

Key words. Depth, de Rham cohomology, Akizuki-Nakano theorem, Lefschetz theorem, Gysin sequence, Hodge decomposition

using an extension theorem due to Scheja [20], the prerequisite is here some depth condition. In this context he realized that the Poincaré lemma extends in a weaker form to local complete intersections.

We will use the generalized de Rham lemma here as a starting point in order to compare geometric and analytic Lefschetz theorems; by this we mean theorems which compare the cohomology of a projective variety and certain hypersurfaces, in particular hyperplane sections, where cohomology is taken in the sense of singular resp. holomorphic de Rham cohomology. In fact, the Lefschetz theorem for holomorphic de Rham cohomology is closely related to a generalized Akizuki-Nakano theorem. Finally we will discuss to which extent there is a Gysin sequence for de Rham cohomology. We will include Hodge-theoretic aspects, using the du Bois complex [6].

The results underline the impact of the generalization obtained by Greuel in his thesis on the study of lower de Rham cohomology groups.

So, let us replace the assumption that X is smooth by the one that X is locally a complete intersection. When speaking of complex algebraic varieties we mean a a separated scheme of finite type over $Spec\,\mathbb{C}$. The strength of the results will depend on the dimension of the singular locus of X. In this paper we denote the latter by Σ_X and put $\dim\emptyset := 0$.

If one considers functions, i.e. differential forms of degree zero, one can work under a more general assumption, see [16]; this case is especially interesting in view of the Picard group.

In particular, we will prove (with $\Sigma := \Sigma_X$):

Theorem 0.1 (Akizuki-Nakano Theorem). *Let X be a complex projective variety which is locally a complete intersection of dimension n, $\mathcal{F}$ an ample line bundle on X. Then $H^q(X, \Omega_X^p \otimes \mathcal{F}^{-1}) = 0$ for $p + q < n - \dim\Sigma$.*

Theorem 0.2. *Let X be a complex projective variety which is locally a complete intersection of dimension n. Then*

$$H^k(X;\mathbb{C}) \simeq \mathbb{H}^k(X, \Omega_X^\bullet) \simeq \oplus_{p+q=k} H^q(X, \Omega_X^p)$$

for $k < n - \dim\,\Sigma$.

Note that we have the first isomorphism already by Greuel's thesis [12].

Theorem 0.3 (Lefschetz theorem for differential forms). *Let $X \subset \mathbb{P}_m$ be a complex projective variety which is locally a complete intersection of dimension n, H a hyperplane in $\mathbb{P}_m$ which is sufficiently general, $Y := X \cap H$, $\mathrm{codim}_X Y = 1$. Then:*

a) The mapping $\mathbb{H}^r(X, \Omega_X^\bullet) \to \mathbb{H}^r(Y, \Omega_Y^\bullet)$ is bijective for $r < n - \dim\Sigma_Y - 1$.

b) The mapping $H^q(X,\Omega^p_X) \to H^q(Y,\Omega^p_Y)$ is bijective for $p+q < n - \dim\Sigma_Y - 1$.

Theorem 0.4. *Under the hypothesis of Theorem 0.3, we have long exact Gysin sequences:*

$$\begin{aligned} \ldots \to\ & H^q(X,\Omega^p_X) \to H^q(X,\Omega^p_X(\log Y)) \to H^q(Y,\Omega^{p-1}_Y) \\ \to\ & H^{q+1}(X,\Omega^p_X) \to \ldots \end{aligned}$$

for $p \le n - \dim\Sigma$, and

$$\begin{aligned} \ldots \to\ & \mathbb{H}^k(X,\Omega^\bullet_X) \to \mathbb{H}^k(X,\Omega^\bullet_X(log\,Y)) \overset{Res}{\to} \mathbb{H}^{k-1}(Y,\Omega^\bullet_Y) \\ \to\ & \mathbb{H}^{k+1}(X,\Omega^\bullet_X) \to \ldots \to \mathbb{H}^{n-\dim\Sigma-2}(Y,\Omega^\bullet_Y)\,. \end{aligned}$$

1 A de Rham Lemma for Complete Intersections

This section presents known results in a form which is suitable for the following.

Let U be an open neighbourhood of 0 in $\mathbb{C}^m$, $f_1,\ldots,f_k : U \to \mathbb{C}$ holomorphic, $X := \{z \in U \mid f_1(z) = \ldots = f_k(z) = 0\}$, $\dim X = n := m-k$, so X is a complete intersection in U. Let Σ_X be the singular locus of X. Note:

$$\Omega^p_{X,0} = \Omega^p_{\mathbb{C}^m,0}/(f_1,\ldots,f_k,df_1,\ldots,df_k)$$

We want to show:

Theorem 1.1. *depth $\Omega^p_{X,0} \ge n-p$ for $p \le n - \dim\Sigma_X$.*

Proof. Put $\Omega^p := \Omega^p_{\mathbb{C}^m,0}$. Note that depth $\Omega^p = m$. We have:

$$0 \longrightarrow \Omega^p/(f_1,\ldots,f_{j-1}) \xrightarrow{f_j} \Omega^p/(f_1,\ldots,f_{j-1}) \longrightarrow \Omega^p/(f_1,\ldots,f_j) \longrightarrow 0$$

so inductively: depth $\Omega^p/(f_1,\ldots,f_j) \ge m-j$. In particular,

$$\operatorname{depth}\Omega^p/(f_1,\ldots,f_k) \ge n\,. \tag{1}$$

So, it is sufficient to prove part b) of the following Proposition 1.2 for $j=k$ (cf. [12, Lemma 1.6]). □

Proposition 1.2. *Let $1 \le j \le k$ and $0 \le p \le n - \dim\Sigma_X$.*

a) $df_j\wedge : \Omega^{p-1}/(f_1,\ldots,f_k,df_1,\ldots,df_j) \to \Omega^p/(f_1,\ldots,f_k,df_1,\ldots,df_{j-1})$ is injective, *b) depth* $\Omega^p/(f_1,\dots,f_k,df_1,\dots,df_j)\geq n-p$.

Proof. Induction on j, for fixed j on p. The case $j=0$ is covered by (1), so assume that $j>0$. Again, the case $p=0$ is clear, because

$$\operatorname{depth}\Omega^0/(\boldsymbol{f},df_1,\dots,df_j)=\operatorname{depth}\Omega^0/(\boldsymbol{f})\geq n$$

by (1) (here, $\boldsymbol{f}$ stands for $f_1,\dots,f_k$). Step from $p-1$ to p ($p>0$):

a) Let V be on open Stein neighbourhood of 0, then we have

$$\begin{array}{ccc} H^0(V,\Omega_V^{p-1}/(\boldsymbol{f},df_1,\dots,df_j)) & \xrightarrow{df_j\wedge} & H^0(V,\Omega_V^{p}/(\boldsymbol{f},df_1,\dots,df_{j-1})) \\ \downarrow & & \downarrow \\ H^0(V\setminus\Sigma,\Omega_V^{p-1}/(\boldsymbol{f},df_1,\dots,df_j)) & \xrightarrow{df_j\wedge} & H^0(V\setminus\Sigma,\Omega_V^{p}/(\boldsymbol{f},df_1,\dots,df_{j-1})) \end{array}$$

By the induction hypothesis, $\operatorname{depth}\Omega_V^{p-1}/(\boldsymbol{f},df_1,\dots,df_j)>n-p$, so the left vertical is injective by Scheja's extension theorem, see below. The injectivity of the lower horizontal is obvious. So the upper horizontal is injective, too.

b) We have an exact sequence

$$\begin{aligned} 0 \longrightarrow\ & df_j\wedge\Omega^{p-1}/(\boldsymbol{f},df_1,\dots,df_j)\longrightarrow\Omega^p/(\boldsymbol{f},df_1,\dots,df_{j-1}) \\ \longrightarrow\ & \Omega^p/(\boldsymbol{f},df_1,\dots,df_j)\longrightarrow 0 \end{aligned}$$

and $df_j\wedge\Omega^{p-1}/(\boldsymbol{f},df_1,\dots,df_j)\simeq\Omega^{p-1}/(\boldsymbol{f},df_1,\dots,df_j)$ by part a).

By the induction hypothesis, $\operatorname{depth}\Omega^{p-1}/(\boldsymbol{f},df_1,\dots,df_j)\geq n-p+1$, $\operatorname{depth}\Omega^p/(\boldsymbol{f},df_1,\dots,df_{j-1})\geq n-p$, which implies our assertion. □

Theorem 1.3. *(Scheja [20]) Let* $\mathcal{S}$ *be a coherent analytic sheaf on a complex space* X, Z *a closed analytic subset, then the mapping*

$$H^k(X,\mathcal{S})\longrightarrow H^k(X\setminus Z,\mathcal{S})$$

is bijective for $k<$ *depth* $\mathcal{S}-\dim Z-1$ *and injective for* $k=$ *depth* $\mathcal{S}-\dim Z-1$, *i.e.* $H_Z^k(X,\mathcal{S})=0$ *for* $k\leq$ *depth* $\mathcal{S}-\dim Z-1$.

Corollary 1.4. *If* X *is a complex space which is locally a complete intersection and* $\operatorname{codim}_X\Sigma_X\geq 1$ *the space* X *is reduced.*

Proof. Let $j:X\setminus\Sigma_X\to X$ be the inclusion. Then $\mathcal{O}_X\to j_*\mathcal{O}_{X\setminus\Sigma_X}$ is injective by Theorem 1.3. □

2 Application: Complete Intersections in $\mathbf{P}_m$

Let $f_1, \ldots, f_k \in \mathbb{C}[Z_0, \ldots, Z_m]$ be homogeneous of degree $d_1, \ldots, d_k$. Let X be the subvariety of $\mathbb{P}_m$ defined by $f_1 = \ldots = f_k = 0$, $\dim X = n := m - k$, so X is a complete intersection. Here, we put $\dim \emptyset := 0$.

Theorem 2.1. *a) Let $p \leq n - \dim \Sigma_X$. Then $H^q(\mathbb{P}_m, \Omega^p_{\mathbb{P}_m}) \to H^q(X, \Omega^p_X)$ is bijective for $q < n - p$ and injective for $q = n - p$.*

b) The mapping $\mathbb{H}^r(\mathbb{P}_m, \Omega^\bullet_{\mathbb{P}_m}) \to \mathbb{H}^r(X, \Omega^\bullet_X)$ is bijective for $r < n - \dim \Sigma_X$ and injective for $r = n - \dim \Sigma_X$.

Proof. Put $\Omega^p := \Omega^p_{\mathbb{P}_m}$.

a) Suppose $s < 0$. It is well-known that $H^q(\mathbb{P}_m, \Omega^p(s)) = 0$ for $q < m$. Let us look at the exact sequence

$$\begin{aligned} 0 \quad &\longrightarrow \quad \Omega^p(s - d_j)/(f_1, \ldots, f_{j-1}) \xrightarrow{f_j} \Omega^p(s)/(f_1, \ldots, f_{j-1}) \\ &\longrightarrow \quad \Omega^p(s)/(f_1, \ldots, f_j) \longrightarrow 0\,. \end{aligned}$$

Inductively, we obtain $H^q(\mathbb{P}_m, \Omega^p(s)/(f_1, \ldots, f_j)) = 0$ for each $q < m - j$. Thus, $H^q(X, \Omega^p(s)/(f_1, \ldots, f_k)) = 0$ for $q < n$.

Let $p \leq n - \dim \Sigma_X$. By Proposition 1.2, we have an exact sequence

$$\begin{aligned} 0 \quad &\to \quad \Omega^{p-1}(s - d_j)/(\boldsymbol{f}, df_1, \ldots, df_j) \xrightarrow{df_j \wedge} \Omega^p(s)/(\boldsymbol{f}, df_1, \ldots, df_{j-1}) \\ &\to \quad \Omega^p(s)/(\boldsymbol{f}, df_1, \ldots, df_j) \to 0 \end{aligned}$$

(again, $\boldsymbol{f}$ stands for $f_1, \ldots, f_k$).

Inductively, we get $H^q(X, \Omega^p(s)/(\boldsymbol{f}, df_1, \ldots, df_j)) = 0$ for $q < n - p$. Now we obtain for $s = 0$, using the above exact sequences,

$$H^q(\mathbb{P}_m, \Omega^p/(f_1, \ldots, f_{j-1})) \longrightarrow H^q(\mathbb{P}_m, \Omega^p/(f_1, \ldots, f_j))$$

is bijective for $q < m - j$ and injective for $q = m - j$,

$$H^q(X, \Omega^p/(\boldsymbol{f}, df_1, \ldots, df_{j-1})) \longrightarrow H^q(X, \Omega^p/(\boldsymbol{f}, df_1, \ldots, df_j))$$

is bijective for $q < n - p$ and injective for $q = n - p$.

b) Let $\mathcal{K}^p := \ker\big(\Omega^p \to \Omega^p/(\boldsymbol{f}, df_1, \ldots, df_k)\big)$. By a), $H^q(\mathbb{P}_m, \mathcal{K}^p)$ vanishes for $q \leq n - p$, $p \leq n - \dim \Sigma_X$, in particular for $p + q \leq n - \dim \Sigma_X$. Therefore, $\mathbb{H}^r(\mathbb{P}_m, \mathcal{K}^\bullet) = 0$ for $r \leq n - \dim \Sigma_X$, which implies our assertion. $\square$

3 Relation to Residues

Now we turn to a projective variety X which is *locally* a complete intersection of dimension n. Let Y be an effective Cartier divisor on X. Note that $\Sigma_X \cap Y \subset \Sigma_Y$. We want to look at logarithmic differential forms on X with respect to Y and their residues along Y:

We have an invertible sheaf $\mathcal{O}_X(Y)$ which is associated with Y. Note that $\mathcal{O}_X(kY) \simeq \mathcal{O}_X(Y)^k$ (k-th tensor power). Let $i : Y \hookrightarrow X$ and $j : X \setminus Y \hookrightarrow X$ be the inclusions. The sheaf $\mathcal{O}_X(kY)$ can be regarded as a subsheaf of $j_*\mathcal{O}_{X\setminus Y}$, and there is an induced mapping $\Omega^p_X \otimes \mathcal{O}_X(kY) \to j_*\Omega^p_{X\setminus Y}$. Let $\Omega^p_X(kY)$ be the image of this mapping.

Lemma 3.1. *Let $p < n - \dim \Sigma_X \cap Y$.*

a) The canonical mapping $\Omega^p_X \longrightarrow j_*\Omega^p_{X\setminus Y}$ is injective.

b) The canonical mapping $\Omega^p_X \otimes \mathcal{O}_X(kY) \longrightarrow \Omega^p_X(kY)$ is bijective.

Proof. a) Let U be an open Stein neighbourhood of x. Then we have

$$\begin{array}{ccc} H^0(U,\Omega^p_X) & \longrightarrow & H^0(U\setminus Y,\Omega^p_X) \\ \downarrow & \nearrow & \\ H^0(U\setminus \Sigma_X\cap Y,\Omega^p_X) & & \end{array}$$

For $p < n - \dim \Sigma_X \cap Y$, the left vertical is injective, because of Theorem 1.1 and 1.3, as well as the diagonal arrow, so the upper horizontal, too.

b) The surjectivity is clear. If U is an open Stein neighbourhood of x, we have:

$$\begin{array}{ccc} H^0(U,\Omega^p_X\otimes\mathcal{O}(kY))) & \longrightarrow & H^0(U,\Omega^p_X(kY)) \\ \downarrow & & \downarrow \\ H^0(U\setminus\Sigma_X\cap Y,\Omega^p_X\otimes\mathcal{O}(kY))) & \longrightarrow & H^0(U\setminus\Sigma_X\cap Y,\Omega^p_X(kY)) \end{array}$$

For $p < n - \dim \Sigma_X \cap Y$, the left vertical is injective, because of Theorem 1.1 and 1.3. The lower horizontal is injective, so the upper one, too. $\square$

So, we can describe $\Omega^p_X(kY)$ as the sheaf of p-forms which are holomorphic on $X \setminus Y$ and have at most a pole of order k along Y if $k > 0$ resp. which are holomorphic on X and vanish along Y of order at least $-k$ if $k < 0$.

Now we can define homomorphisms $\lambda_Y : \Omega^{p-1}_Y \to \Omega^p_X(Y)|_Y$ as follows, where $|_Y$ denotes the analytical restriction to Y:

The sheaf $\mathcal{O}_X(-Y)$ is locally free; let U be a Zariski open subset of X such that $\mathcal{O}_X(-Y)|_U \subset \mathcal{O}_U$ is generated by a function f. Then the homomorphism $\wedge\frac{df}{f} : \Omega^{p-1}_Y|_{Y\cap U} \longrightarrow \Omega^p_X(Y)|_{Y\cap U}$ is independent of the choice of f. Therefore, we get the desired homomorphism λ_Y.

By the de Rham lemma (Proposition 1.2 a), we get:

Lemma 3.2. *For $p < n - \dim \Sigma_Y$, $\lambda_Y : \Omega_Y^{p-1} \longrightarrow \Omega_X^p(Y)|_Y$ is injective.*

On the other hand, we may look at the sheaf $\Omega_X^p(\log Y)$ of logarithmic p-forms on X along Y. We have a mapping $\phi : \Omega_X^p(\log Y) \to i_*(\Omega_X^p(Y)|_Y)$, the image coincides with the image of λ_Y, so we can define $\mathrm{Res} : \Omega_X^p(\log Y) \longrightarrow i_*\Omega_Y^{p-1}$ by $\phi(\omega) = \lambda_Y(\mathrm{Res}\,\omega)$ (residue homomorphism) as soon as $p < n - \dim \Sigma_Y$.

Lemma 3.3. *For $p < n - \dim \Sigma_Y$, the following sequences are exact:*

a) $0 \longrightarrow \Omega_Y^{p-1} \otimes \mathcal{O}_X(-Y)|_Y \xrightarrow{\lambda_Y} \Omega_X^p|_Y \longrightarrow \Omega_Y^p \longrightarrow 0,$

b) $0 \longrightarrow \Omega_X^p \longrightarrow \Omega_X^p(\log Y) \xrightarrow{Res} i_*\Omega_Y^{p-1} \longrightarrow 0.$

Note that indeed the first mapping in b) is injective, as can be seen from the commutative diagram

$$\begin{array}{ccc} H^0(U, \Omega_X^p) & \xrightarrow{\hspace{3em}} & H^0(U, \Omega_X^p(\log Y)) \\ \downarrow & & \downarrow \\ H^0(U \setminus \Sigma \cap Y, \Omega_X^p) & \longrightarrow & H^0(U \setminus \Sigma \cap Y, \Omega_X^p(\log Y)) \end{array}$$

where U is a Stein neighbourhood of some point in X. See the proof of Lemma 3.1.

4 A Generalized Akizuki-Nakano Theorem

We want to study the cohomology of differential forms on a projective variety which is only locally a complete intersection. Here a theorem of Lefschetz type is useful which will be proved later on. A prerequisite is a generalization of the theorem of Akizuki-Nakano which will be proved in this section.

Let X be a complex projective variety and $\Sigma := \Sigma_X$ its critical locus.

Recall that the classical Akizuki-Nakano (or Kodaira-Nakano) theorem says the following ([1], [11, p. 155]): *Let X be a smooth complex projective variety of pure dimension n, $\mathcal{F}$ an ample line bundle on X, then $H^q(X, \Omega_X^p \otimes \mathcal{F}^{-1}) = 0$ for $p + q < n$.*

First, we can prove the following generalization, using results which are already known from the literature: Let X be a complex projective variety which is locally a complete intersection of dimension n, $\mathcal{F}$ an ample line bundle on X. Then $H^q(X, \Omega_X^p \otimes \mathcal{F}^{-1}) = 0$ for $p + q < n - \dim \Sigma - 1$.

This can be proved in the following way: First, there is an Akizuki-Nakano theorem for non-complete algebraic manifolds which implies that

$$H^q(X \setminus \Sigma, \Omega_X^p \otimes \mathcal{F}^{-1}) = 0 \quad \text{for} \quad p + q < n - \dim \Sigma - 1$$

see [18, Theorem I] or [9, Cor. 6.15]. By Scheja's theorem (Theorem 1.3) and Theorem 1.1, we know that

$$H^q_\Sigma(X, \Omega^p_X \otimes \mathcal{F}^{-1}) = 0 \quad \text{for} \quad p + q \leq n - \dim \Sigma - 1\,.$$

This implies our assertion.

But, we want to show that we can admit $p+q < n - \dim \Sigma$. First, we want to prove that the classical Akizuki-Nakano theorem still holds if X is a local complete intersection with isolated singularities, at least if $\mathcal{F}$ admits a non-trivial section which has a smooth divisor. Here we will make use of the following topological Lefschetz theorem:

Theorem 4.1. *(see [14, Th. 1.1.1]) Suppose that X is a projective variety of pure dimension n with isolated singularities, Y a smooth hypersurface in X such that $X \setminus Y$ is affine. Then the mapping $H^r(X \setminus \Sigma; \mathbb{C}) \to H^r(Y; \mathbb{C})$ is bijective for $r < n-1$ and injective for $r = n-1$.*

Theorem 4.2. *Let X be a complex projective variety which is locally a complete intersection of dimension n, $\mathcal{F}$ an ample line bundle on X which admits a section τ which corresponds to a smooth divisor Y. Assume that X has only isolated singularities. Then $H^q(X, \Omega^p_X \otimes \mathcal{F}^{-1}) = 0$ for $p+q < n$.*

Proof. Note that the multiplication by τ yields an isomorphism $\mathcal{O}_X(Y) \simeq \mathcal{F}$. So we assume $\mathcal{F} = \mathcal{O}_X(Y)$. By Lemma 3.1, $\Omega^p_X \otimes \mathcal{F}^{-1} \simeq \Omega^p_X(-Y)$.

Let $\pi : \tilde{X} \longrightarrow X$ a resolution of singularities such that $\tilde{\Sigma} := \pi^{-1}(\Sigma)$ is a divisor with normal crossings. Look at the mappings

$$H^q(X, \Omega^p_X) \longrightarrow H^q(\tilde{X}, \Omega^p_{\tilde{X}}(\log \tilde{\Sigma})) \longrightarrow H^q(U, \Omega^p_X) \longrightarrow H^q(Y, \Omega^p_Y)\,.$$

Here, U denotes a closed neighbourhood of Y in X whose complement is Stein, and $\Omega^\bullet_{\tilde{X}}(\log \tilde{\Sigma})$ is the logarithmic de Rham complex, see [7].

We may assume that $p < n$. By Theorem 1.1, $\operatorname{depth} \Omega^p_X \geq n - p$, which implies that $H^q_c(X \setminus U, \Omega^p_X) = 0$ for $q < n - p$, see [4, I Theorem 3.6]. So, $H^q(X, \Omega^p_X) \to H^q(U, \Omega^p_X)$ is bijective for $q < n-p-1$ and injective for $q = n-p-1$.

Now, $H^r(\tilde{X} \setminus \tilde{\Sigma}; \mathbb{C}) \simeq H^r(X \setminus \Sigma; \mathbb{C})$, and $H^r(X \setminus \Sigma; \mathbb{C}) \to H^r(Y; \mathbb{C})$ is bijective for $r < n-1$ and injective for $r = n-1$, by Theorem 4.1, since $X \setminus Y$ is affine; see [17, Prop. II 2.1]. The same holds if we pass to Gr^p_F, where F denotes the canonical Hodge filtration. Now,

$$\operatorname{Gr}^p_F H^{p+q}(\tilde{X} \setminus \tilde{\Sigma}; \mathbb{C}) \simeq H^q(\tilde{X}, \Omega^p_{\tilde{X}}(\log \tilde{\Sigma}))\,,$$

and $\operatorname{Gr}^p_F H^{p+q}(Y; \mathbb{C}) \simeq H^q(Y, \Omega^p_Y)$. Thus, the mapping

$$H^q(\tilde{X}, \Omega^p_{\tilde{X}}(\log \tilde{\Sigma})) \longrightarrow H^q(Y, \Omega^p_Y)$$

is bijective for $q < n-p-1$ and injective for $q = n-p-1$. It follows that $H^q(X,\Omega_X^p) \to H^q(Y,\Omega_Y^p)$ is bijective for $q < n-p-1$ and injective for $q = n-p-1$.

Now, we consider the mappings

$$H^q(X,\Omega_X^p) \longrightarrow H^q(Y,\Omega_X^p|_Y) \longrightarrow H^q(Y,\Omega_Y^p)\,.$$

Look at the exact sequence (see Lemma 3.3 a)

$$0 \longrightarrow \Omega_Y^{p-1} \otimes \mathcal{O}_X(-Y)|_Y \xrightarrow{\lambda_Y} \Omega_X^p|_Y \longrightarrow \Omega_Y^p \longrightarrow 0\,.$$

Now Y is smooth of dimension $n-1$, and $\mathcal{F}|_Y$ is ample, too. By the classical Akizuki-Nakano theorem, we have $H^q(Y,\Omega_Y^{p-1} \otimes \mathcal{O}_X(-Y)|_Y) = 0$ for $q < (n-1)-(p-1) = n-p$, because $\mathcal{O}_X(-Y) \simeq \mathcal{F}^{-1}$ and $\mathcal{F}^{-1}|_Y$ is ample. So $H^q(X,\Omega_X^p|_Y) \to H^q(Y,\Omega_Y^p)$ is bijective, $q < n-p-1$, and injective, $q = n-p-1$. Altogether, we get that $H^q(X,\Omega_X^p) \to H^q(Y,\Omega_X^p|_Y)$ is bijective for $q < n-p-1$ and injective for $q = n-p-1$, i.e. $H^q(X,\Omega_X^p(-Y)) = 0$ for $q < n-p$, because of the exact sequence

$$0 \longrightarrow \Omega_X^p(-Y) \longrightarrow \Omega_X^p \longrightarrow \Omega_X^p|_Y \longrightarrow 0\,.$$

□

Next, we want to treat the case where X may have non-isolated singularities:

Proof of Theorem 0.1. Induction on $s := \dim\Sigma$. We may assume $s < n$. Let us treat the induction step, i.e. prove the theorem for $\dim\Sigma = s$ $(s \geq 0)$ under the hypothesis that it is true if $\dim\Sigma < s$:

a) First, let us assume that $\mathcal{F}$ admits a section which corresponds to a divisor Y which is smooth outside $Y \cap \Sigma$, so $\mathcal{F} \simeq \mathcal{O}_X(Y)$, and such that $\dim Y \cap \Sigma = s-1$ if $s > 0$, resp. $Y \cap \Sigma = \emptyset$ if $s = 0$. The case $s = 0$ being handled in Theorem 4.2, we assume $s > 0$.

We have $\operatorname{depth}\Omega_X^p \geq n-p$ for $p \leq n-s$. Hence, $H^q(X,\Omega_X^p \otimes \mathcal{F}^{-k}) = 0$ for $k \gg 0$, $q < n-p$, $p \leq n-s$, so for $q \leq n-p-s$. See [10, Exp. XII Cor. 1.4] or [4, IV Cor. 3.3].

We want to show by induction on $-k$ that $H^q(X,\Omega_X^p \otimes \mathcal{F}^{-k})$ vanishes for $k \geq 1$. So, let $k \geq 1$ be given. Note that $\mathcal{F}^{-k} \simeq \mathcal{O}_X(-kY) \supset \mathcal{O}_X((-k-1)Y)$. Moreover, $\Omega_X^p \otimes \mathcal{O}_X(-kY) \simeq \Omega_X^p(-kY)$ by Lemma 3.1. Thus, we get an exact sequence

$$0 \longrightarrow \Omega_X^p \otimes \mathcal{O}((-k-1)Y) \longrightarrow \Omega_X^p \otimes \mathcal{O}(-kY) \longrightarrow \Omega_X^p \otimes \mathcal{O}(-kY)|_Y \longrightarrow 0\,,$$

which corresponds to the exact sequence

$$0 \longrightarrow \Omega_X^p((-k-1)Y) \longrightarrow \Omega_X^p(-kY) \longrightarrow \Omega_X^p(-kY)|_Y \longrightarrow 0\,.$$

It remains to show $H^q(Y, \Omega_X^p \otimes \mathcal{F}^{-k}|_Y) = 0$ for $k > 0$, $p + q < n - s - 1$.

But by Lemma 3.3 a), we have an exact sequence

$$0 \longrightarrow \Omega_Y^{p-1} \otimes \mathcal{O}_X(-Y)|_Y \longrightarrow \Omega_X^p|_Y \longrightarrow \Omega_Y^p \longrightarrow 0\,.$$

Since, by the induction hypothesis,

$$H^q(Y, \Omega_Y^p \otimes (\mathcal{F}|_Y)^{-k}) = H^q(Y, \Omega_Y^{p-1} \otimes \mathcal{O}_X(-Y)|_Y \otimes (\mathcal{F}|_Y)^{-k}) = 0\,,$$

$H^q(Y, \Omega_X^p \otimes \mathcal{F}^{-k}|_Y)$ vanishes, provided that $p + q < n - s$.

b) Now, assume only that $\mathcal{F}$ is ample. Then there is an $l > 0$ such that $\mathcal{F}^l$ is very ample. We use essentially the same trick as in [19]: We may assume that X is embedded in some $\mathbb{P}_m$ and that $\mathcal{F}^l = \mathcal{O}_{\mathbb{P}_m}(1)|_X$. Take a general linear form on $\mathbb{P}_m$. In this way, we find a section σ in $\mathcal{F}^l$ such that the zero set of σ is transverse to the regular part of the zero section of $\mathcal{F}^l$ and $\dim \Sigma_1 = s - 1$ if $s > 0$, resp. $\Sigma_1 = \emptyset$ if $s = 0$, where $\Sigma_1 := \Sigma \cap \{\sigma = 0\}$. Look at the cyclic covering $f : X' \to X$ of X with degree l branched along the zero locus of σ. More precisely, $X' := \{v \in \mathcal{F} \mid v^l = \sigma(\pi(v))\}$. Note that X' is smooth outside $\Sigma' = f^{-1}(\Sigma)$.

Now $f^*(\mathcal{F})$ admits a non-trivial section τ with $\tau^l = f^*(\sigma)$, and X' is locally a complete intersection; the divisor of τ is smooth outside Σ', see [19]. Furthermore, $f^*\mathcal{F}$ is ample. So, by the special case which is already proved, $H^q(X', \Omega_{X'}^p \otimes f^*(\mathcal{F})^{-1}) = 0$ for $q < n - p - s$. Let

$$\Sigma_1' := \Sigma' \cap \{\sigma = 0\}\,,$$

then $\dim \Sigma_1' = s - 1$ if $s > 0$, resp. $\Sigma_1' = \emptyset$ if $s = 0$. For $p \leq n - s$, we have $\operatorname{depth} \Omega_{X'}^p \geq n - p$, so $H^q(X' \setminus \Sigma_1', \Omega_{X'}^p \otimes f^*(\mathcal{F})^{-1}) = 0$ for $q < n - p - s$, by Theorem 1.3 (the case $s = 0$ is obvious). By the adjunction formula,

$$\begin{aligned} H^q(X' \setminus \Sigma_1', \Omega_{X'}^p \otimes f^*(\mathcal{F})^{-1}) &= H^q(X \setminus \Sigma_1, f_*(\Omega_{X'}^p \otimes f^*(\mathcal{F})^{-1})) \\ &= H^q(X \setminus \Sigma_1, f_*(\Omega_{X'}^p) \otimes \mathcal{F}^{-1})\,. \end{aligned}$$

Now, the Galois group of the covering $f : X' \to X$ acts on $f_*\Omega_{X'}^p$. Outside Σ_1, Ω_X^p coincides with the invariant part of $f_*\Omega_{X'}^p$. So we get

$$0 = H^q(X \setminus \Sigma_1, \Omega_X^p \otimes \mathcal{F}^{-1}) = H^q(X, \Omega_X^p \otimes \mathcal{F}^{-1})\,,$$

using Scheja's Theorem 1.3. □

5 A Preliminary Lefschetz Theorem for Differential Forms

Let X be a complex projective variety which is locally a complete intersection of dimension n and Σ_X its singular locus. Let Y be an effective Cartier divisor in X such that $\mathcal{O}_X(Y)$ is ample.

Lemma 5.1. *a) The mapping* $H^q(X,\Omega_X^p) \to H^q(Y,\Omega_X^p|_Y)$ *is bijective for* $q < n-p-\dim\Sigma_X - 1$ *and injective for* $q = n-p-\dim\Sigma_X - 1$.

b) The mapping $H^q(Y,\Omega_X^p|_Y) \to H^q(Y,\Omega_Y^p)$ *is bijective for* $q < n-p-\dim\Sigma_Y - 1$ *and injective for* $q = n-p-\dim\Sigma_Y - 1$.

Proof. a) We may assume $p < n - \dim\Sigma_X$. We have an exact sequence:

$$0 \longrightarrow \Omega_X^p(-Y) \longrightarrow \Omega_X^p \longrightarrow \Omega_X^p|_Y \longrightarrow 0\,.$$

By Lemma 3.1, $\Omega_X^p \otimes \mathcal{O}_X(-Y) \simeq \Omega_X^p(-Y)$. Thus, by the generalized Akizuki-Nakano Theorem 0.1, $H^q(X,\Omega_X^p(-Y)) = 0$ for $q < n-p-\dim\Sigma_X$. So, we get the desired assertion.

b) We may assume that $p < n - \dim\Sigma_Y$. By Lemma 3.3 a), we have an exact sequence

$$0 \longrightarrow \Omega_Y^{p-1} \otimes \mathcal{O}_X(-Y)|_Y \longrightarrow \Omega_X^p|_Y \longrightarrow \Omega_Y^p \longrightarrow 0\,.$$

By the generalized Akizuki-Nakano Theorem 0.1, $H^q(Y,\Omega_Y^{p-1} \otimes \mathcal{O}_X(-Y)|_Y)$ vanishes for $q < n-p-\dim\Sigma_Y$, which proves the assertion. □

Remark 5.2. Using the same method of proof we can show that

$$H^q(Y,\Omega_X^p|_Y \otimes \mathcal{F}^{-r}) = 0\,, \quad r > 0\,,$$

provided that $q < n-p-\dim\Sigma_X - 1$, if $\mathcal{F}$ is ample.

Corollary 5.3. *Let* $\Sigma := \Sigma_X \cup \Sigma_Y$.

a) The mapping $H^q(X,\Omega_X^p) \to H^q(Y,\Omega_Y^p)$ *is bijective for* $q < n-p-\dim\Sigma - 1$ *and injective for* $q = n-p-\dim\Sigma - 1$.

b) The mapping $\mathbb{H}^r(X,\Omega_X^\bullet) \to \mathbb{H}^r(Y,\Omega_Y^\bullet)$ *is bijective for* $r < n-\dim\Sigma-1$, *and injective for* $r = n-\dim\Sigma - 1$.

Proof. a) follows from Lemma 5.1. For b), put $\mathcal{K}^p := \ker(\Omega_X^p \to i_*\Omega_Y^p)$, where $i : Y \hookrightarrow X$ denotes the inclusion. Then, a) implies that $H^q(X,\mathcal{K}^p) = 0$ for $p+q < n-\dim\Sigma$. So, $\mathbb{H}^r(X,\mathcal{K}^\bullet) = 0$, $r < n-\dim\Sigma$, which proves the assertion. □

For a modified version, see Theorem 7.1.

Remark 5.4. (Theorem 2.1 revisited) As in Section 2, let X be the subvariety of $\mathbb{P}_m$ defined by $f_1 = \ldots = f_k = 0$, where f_j is a homogeneous polynomial of degree d_j. Let $\dim X = n := m-k$, let Σ be the singular locus of X, and let $(c_1,\ldots,c_k)$ be a non-zero regular value of $(f_1,\ldots,f_k) : \mathbb{C}^{m+1} \to \mathbb{C}^k$. Put

$$F_j(z_0,\ldots,z_{m+1}) := f_j(z_0,\ldots,z_m) - c_j z_{m+1}^{d_j}\,, \quad j = 1,\ldots,k\,,$$

let Z_j be the subvariety of $\mathbb{P}_{m+1}$ defined by $F_1 = \ldots = F_j = 0$, and let Σ_j be the singular locus of Z_j, $j = 1, \ldots, k$. Then $\Sigma_k \simeq \Sigma$ by the choice of $(c_1, \ldots, c_k)$, so $\dim \Sigma_j \leq \dim \Sigma + k - j$. By Corollary 5.3 a), we know that the arrows

$$H^q(\mathbb{P}_{m+1}, \Omega^p_{\mathbb{P}_{m+1}}) \to H^q(Z_1, \Omega^p_{Z_1}) \to \ldots \to H^q(Z_k, \Omega^p_{Z_k}) \to H^q(X, \Omega^p_X)$$

are bijective for $q < n - p - \dim \Sigma$ and injective for $q = n - p - \dim \Sigma$.

Since $H^q(\mathbb{P}_{m+1}, \Omega^p_{\mathbb{P}_{m+1}}) \simeq H^q(\mathbb{P}_m, \Omega^p_{\mathbb{P}_m})$ for $q \leq m$ we obtain that the mapping $H^q(\mathbb{P}_m, \Omega^p_{\mathbb{P}_m}) \to H^q(X, \Omega^p_X)$ is bijective for $q < n - p - \dim \Sigma$ and injective for $q = n - p - \dim \Sigma$. So we get a weakened version of Theorem 2.1 a). Similarly, we obtain Theorem 2.1 b).

We can deduce Theorem 0.1 from Corollary 5.3. In fact, practically we did so at the end of the proof of Theorem 4.2 for $\dim \Sigma = 0$.

Induction on $s = \dim \Sigma$. We can assume that $\mathcal{F}$ admits a non-trivial section τ which corresponds to a divisor Y which is smooth outside the singular locus of X and which satisfies $\dim \Sigma_Y = s - 1$ if $s > 0$, resp. $\Sigma_Y = \emptyset$ if $s = 0$. We may assume $p < n - s$. By Lemma 3.3 a) we have an exact sequence

$$0 \longrightarrow \Omega^{p-1}_Y \otimes \mathcal{O}_X(-Y)|_Y \longrightarrow \Omega^p_X|_Y \longrightarrow \Omega^p_Y \longrightarrow 0\,.$$

By induction hypothesis, $H^q(Y, \Omega^{p-1}_Y \otimes \mathcal{O}_X(-Y)|_Y) = 0$, $q < n-p-s$, so the mapping $H^q(Y, \Omega^p_X|_Y) \longrightarrow H^q(Y, \Omega^p_Y)$ is bijective for $q < n - p - s - 1$ and injective for $q = n - p - s - 1$.

By Corollary 5.3 a), we have that the composition of the mappings

$$H^q(X, \Omega^p_X) \longrightarrow H^q(Y, \Omega^p_X|_Y) \longrightarrow H^q(Y, \Omega^p_Y)$$

is bijective for $q < n - p - s - 1$ and injective for $q = n - p - s - 1$. Altogether, we get that the mapping $H^q(X, \Omega^p_X) \to H^q(X, \Omega^p_X|_Y)$ is bijective for $q < n - p - s - 1$ and injective for $q = n - p - s - 1$. Starting from the exact sequence

$$0 \longrightarrow \Omega^p_X(-Y) \longrightarrow \Omega^p_X \longrightarrow \Omega^p_X|_Y \longrightarrow 0\,,$$

we see that $H^q(X, \Omega^p_X(-Y)) = 0$, $q < n - p - s$, which implies our assertion.

6 De Rham Cohomology and Singular Cohomology

If X is a complex manifold the holomorphic Poincaré lemma implies that the de Rham cohomology and the singular cohomology coincide:

$$H^r(X; \mathbb{C}) \simeq \mathbb{H}^r(X, \Omega^\bullet_X)\,.$$

Note that $H^r(X;\mathbb{C}) \simeq H^r(X,\mathbb{C}_X)$, where $\mathbb{C}_X$ is the constant sheaf of complex numbers on X.

We will show that a weaker comparison theorem holds for local complete intersections. It yields a bridge between the Lefschetz theorem for de Rham cohomology and a corresponding one for singular cohomology and allows to understand the Lefschetz theorem for de Rham cohomology from a geometric point of view.

We will first make use of a local topological Lefschetz theorem:

Theorem 6.1. *Let X be the representative of a germ of a complete intersection of dimension n at 0 in $\mathbb{C}^m$, Z a closed analytic subset of X. Replace X by its intersection by a small open ball around 0. Then $\tilde{H}^r(X\setminus Z;\mathbb{C}) = 0$ for $r < n - \dim Z - 1$.*

Proof. See [15, Theorem 1.4]. □

Theorem 6.2. *Let X be a complex space which is locally a complete intersection of dimension n, Σ the singular locus. Then the mapping $H^k(X,\mathbb{C}_X) \longrightarrow \mathbb{H}^k(X,\Omega_X^\bullet)$ is bijective for $k < n - \dim\Sigma$.*

Note that the mapping $H^k(X,\mathbb{C}_X) \to \mathbb{H}^k(X,\Omega_X^\bullet)$ is always injective by the theorem of Bloom-Herrera [2].

Proof. By Scheja's Theorem 1.3 and Theorem 1.1, we have: $H^q_\Sigma(X,\Omega^p_X) = 0$ for $q < n - p - \dim\Sigma$, so $\mathbb{H}^k_\Sigma(X,\Omega^\bullet_X) = 0$ for $k < n - \dim\Sigma$. This means that the mapping $\mathbb{H}^k(X,\Omega^\bullet_X) \to \mathbb{H}^k(X\setminus\Sigma,\Omega^\bullet_X)$ is bijective for $k < n - \dim\Sigma - 1$ and injective for $k = n - \dim\Sigma - 1$.

Now, $H^k(X\setminus\Sigma,\mathbb{C}_X) \simeq \mathbb{H}^k(X\setminus\Sigma,\Omega^\bullet_X)$ by the holomorphic Poincaré lemma.

On the other hand, if U is a suitable neighbourhood of $x \in X$, the maping $H^k(U,\mathbb{C}_X) \to H^k(U\setminus\Sigma,\mathbb{C}_X)$ is bijective for $k < n - \dim\Sigma - 1$ and injective for $k = n - \dim\Sigma - 1$, by Theorem 6.1.

So, $\mathcal{H}^k_\Sigma(\mathbb{C}_X) = 0$, $k < n - \dim\Sigma$, which implies that $H^k_\Sigma(X,\mathbb{C}_X) = 0$, $k < n - \dim\Sigma$. This means that $H^k(X,\mathbb{C}_X) \to H^k(X\setminus\Sigma,\mathbb{C}_X)$ is bijective for $k < n - \dim\Sigma - 1$ and injective for $k = n - \dim\Sigma - 1$.

Finally, we look at the commutative diagram

$$\begin{array}{ccc} H^k(X,\mathbb{C}_X) & \longrightarrow & H^k(X\setminus\Sigma,\mathbb{C}_X) \\ \downarrow & & \downarrow \\ \mathbb{H}^k(X,\Omega^\bullet_X) & \longrightarrow & \mathbb{H}^k(X\setminus\Sigma,\Omega^\bullet_X)\,. \end{array}$$

For $k < n - \dim\Sigma - 1$, the horizontals and the right vertical are bijective, so the left vertical, too. But this is not sufficient for our purpose. Thus, we argue in a different way, using the theorem of Bloom-Herrera instead of Theorem 6.1, as in the proof of [12, Satz 4.4]:

Suppose that $k < n - \dim \Sigma$. Then the lower horizontal is injective and the right vertical is bijective. Furthermore, by [2], the left vertical is injective, and there is a natural splitting $r : \mathbb{H}^k(X, \Omega_X^\bullet) \to H^k(X, \mathbb{C}_X)$ of the mapping $i : H^k(X, \mathbb{C}_X) \to \mathbb{H}^k(X, \Omega_X^\bullet)$, i.e. $r \circ i = \mathrm{id}$. Let $c \in \mathbb{H}^k(X, \Omega_X^\bullet)$. Then $c - i(r(c))$ is mapped onto $0 \in \mathbb{H}^k(X \setminus \Sigma, \Omega_X^\bullet)$ since the right vertical is bijective. By the injectivity of the lower horizontal we get that $c - i(r(c)) = 0$, so the left vertical is surjective, hence bijective. Altogether this implies our theorem. □

Corollary 6.3. *(cf. [12, Satz 4.4]) Let X be as in Theorem 6.2. Then the complex*

$$0 \longrightarrow \mathbb{C}_X \longrightarrow \Omega_X^0 \longrightarrow \ldots \longrightarrow \Omega_X^{n-\dim \Sigma}$$

is exact.

Proof. Let $\mathcal{F}^\bullet$ be the complex $0 \to \mathbb{C}_X \to \Omega_X^0 \to \Omega_X^1 \to \ldots$ so that $\mathcal{F}^0 = \mathbb{C}_X$. Let G be the following decreasing filtration of this complex: $G_k\mathcal{F}^\bullet = \mathcal{F}^\bullet$ for $k < 0$, $G_0\mathcal{F}^p := 0$ for $p \leq 0$, $G_0\mathcal{F}^p := \Omega_X^{p-1}$ for $p > 0$, and $G_k\mathcal{F}^\bullet := 0$ for $k > 0$. Then the long exact cohomology sequence for

$$0 \longrightarrow G_0/G_1 \longrightarrow G_{-1}/G_1 \longrightarrow G_{-1}/G_0 \longrightarrow 0$$

can be identified with

$$\longrightarrow \mathcal{H}^q(\mathcal{F}^\bullet) \longrightarrow \mathcal{H}^q(\mathcal{C}^\bullet) \longrightarrow \mathcal{H}^q(\Omega_X^\bullet) \longrightarrow \mathcal{H}^{q+1}(\mathcal{F}^\bullet) \longrightarrow,$$

where $\mathcal{C}^\bullet$ is the complex with $\mathcal{C}^0 = \mathbb{C}_X$ and $\mathcal{C}^p = 0$ for $p \neq 0$. Let U be an open contractible Stein neighbourhood of $x \in X$. Now apply Theorem 6.2 and the following remark to U instead of X. □

Remark 6.4. a) Corollary 6.3 implies Theorem 6.1 in the case $Z = \Sigma$, see [12, Bemerkung after Satz 4.4].

b) From Corollary 6.3, we may deduce Theorem 6.2 again: It is sufficient to show that $\mathbb{H}^k(X, \mathcal{F}^\bullet) = 0$, $k \leq n - \dim \Sigma$, which follows from $\mathcal{H}^k(\mathcal{F}^\bullet) = 0$, $k \leq n - \dim \Sigma$.

In the proof of the following theorem we will make use of the global Lefschetz theorem for local complete intersections:

Theorem 6.5. *(see [13] or [15, Theorem 3.4.1]) Let X be a projective variety which is locally a complete intersection of dimension n, Y a hypersurface in X, $X \setminus Y$ Stein. Then the mapping $H^r(X;\mathbb{C}) \longrightarrow H^r(Y;\mathbb{C})$ is bijective for $r < n-1$ and injective for $r = n-1$.*

Remember that $H^r(X;\mathbb{C})$ carries a canonical mixed Hodge structure, see [8]. Let F be the corresponding Hodge filtration.

Theorem 6.6. *Let X be a complex projective variety which is locally a complete intersection of dimension n, Σ the singular locus. Then we have:*

a) For $k < n - \dim\Sigma$, the canonical mixed Hodge structure on $H^k(X;\mathbb{C})$ is pure of weight k,

b) $H^q(X,\Omega^p_X) \simeq \mathrm{Gr}^p_F H^{p+q}(X;\mathbb{C})$ for $p+q < n - \dim\Sigma$.

Proof. Induction on $n = \dim X$. Let X be contained in $\mathbb{P}_m$. Induction step (from $n-1$ to n): If X is smooth the statement is well-known. Otherwise, take a general hyperplane L in $\mathbb{P}_m$. Put $Y := X \cap L$. By the Lefschetz theorem (Theorem 6.5), $H^k(X;\mathbb{C}) \longrightarrow H^k(Y;\mathbb{C})$ is injective for $k < n$ and bijective for $k < n-1$. Let $k < n - \dim\Sigma$. The mixed Hodge structure on $H^k(Y;\mathbb{C})$ is pure of weight k: this is obvious for $\dim\Sigma = 0$ because Y is then smooth, for $\dim\Sigma > 0$ it is true because of the induction hypothesis. So, the mixed Hodge strucure on $H^k(X;\mathbb{C})$ is pure of weight k, too. This implies a).

Now, let $(\overline{\Omega}^\bullet_X, F)$ be the filtered de Rham complex in the sense of du Bois [6]. It is uniquely defined in the derived category of filtered complexes $D_{diff}(X)$. Note that there is a morphism $(\Omega^\bullet_X,\sigma) \to (\overline{\Omega}^\bullet_X, F)$ which is a quasi-isomorphism if X is smooth, see [6, 4.1]; here σ denotes the stupid filtration (filtration bête). Furthermore, the associated spectral sequence with E_1 terms $E_1^{pq}(X,\overline{\Omega}^\bullet_X) = \mathbb{H}^q(X, \mathrm{Gr}^p_F \overline{\Omega}^\bullet_X[p])$ degenerates at E_1, and we have $E_\infty^{pq}(X,\overline{\Omega}^\bullet_X) = \mathrm{Gr}^p_F H^{p+q}(X;\mathbb{C})$ where the latter is taken in the sense of Deligne's theory [8], see [6, Th. 4.5].

We have also a spectral sequence with $E_1^{pq}(X,\Omega^\bullet_X) := H^q(X,\Omega^p_X)$. So, we get a commutative diagram of E_1 terms:

$$\begin{array}{ccc} H^q(X,\Omega^p_X) & \longrightarrow & H^q(Y,\Omega^p_Y) \\ \downarrow & & \downarrow \\ \mathbb{H}^q(X,\mathrm{Gr}^p_F\,\overline{\Omega}^\bullet_X[p]) & \longrightarrow & \mathbb{H}^q(Y,\mathrm{Gr}^p_F\,\overline{\Omega}^\bullet_Y[p]) \end{array}$$

Assume $p+q < n - \dim\Sigma$. By induction hypothesis the right vertical arrow is an isomorphism, and by Corollary 5.3 a) the upper horizontal is injective. So the left vertical is injective, too.

Note that we always have $\dim E_\infty^{pq} \le \dim E_1^{pq}$. For $k < n - \dim\Sigma$, we have $H^k(X;\mathbb{C}) \simeq \mathbb{H}^k(X,\Omega^\bullet_X)$, so

$$\begin{aligned} \dim H^k(X;\mathbb{C}) &= \sum_{p+q=k} \dim E_\infty^{pq}(X,\Omega^\bullet_X) \le \sum_{p+q=k} \dim H^q(X,\Omega^p_X) \\ &\le \sum_{p+q=k} \dim \mathbb{H}^q(X,\mathrm{Gr}^p_F\,\overline{\Omega}^\bullet_X[p]) = \dim H^k(X;\mathbb{C}), \end{aligned}$$

which implies our assertion. $\square$

Note that Theorems 6.2 and 6.6 imply Theorem 0.2.

Remark 6.7. Under the hypothesis of Theorem 6.6, we get for $k = p + q < n - \dim \Sigma - 1$: $H^k(X;\mathbb{C}) \simeq H^k(X \setminus \Sigma;\mathbb{C})$ and $H^q(X, \Omega^p_X) \simeq H^q(X \setminus \Sigma, \Omega^p_X)$, see the proof of Theorem 6.2. Consequently, we have $E_1^{pq} = E_\infty^{pq}$ for the Hodge spectral sequence on $X \setminus \Sigma$, which fits to [3, Theorem 6.1], see also [9, Cor. 6.14] . In fact, we have $\mathrm{Gr}^p_F\, H^k(X \setminus \Sigma;\mathbb{C}) \simeq H^q(X \setminus \Sigma, \Omega^p_X)$ for the canonical mixed Hodge structure, this follows from [9], loc.cit.

Furthermore, the mixed Hodge structure on $H^k(X\setminus\Sigma;\mathbb{C})$ is pure of weight k, by Theorem 6.6. This can also be seen as follows: $\mathrm{Gr}^W_q\, H^k(X \setminus \Sigma;\mathbb{C}) = 0$ for $q < k$ because $X \setminus \Sigma$ is smooth, as well as for $q > k$ because $H^k(X;\mathbb{C}) \simeq H^k(X \setminus \Sigma;\mathbb{C})$ and X is compact.

7 A Lefschetz Theorem for Differential Forms on Local Complete Intersections

Now let us discuss Corollary 5.3 again. Let X be a complex projective variety which is locally a complete intersection of dimension n, Y an effective Cartier divisor in X such that $X \setminus Y$ is Stein. We get the following generalization of Corollary 5.3:

Theorem 7.1. *a) The mapping $\mathbb{H}^r(X, \Omega^\bullet_X) \to \mathbb{H}^r(Y, \Omega^\bullet_Y)$ is bijective for $r < n - \dim \Sigma_Y - 1$ and injective for $r \leq n - \dim \Sigma_X - 1$.*

b) The mapping $H^q(X, \Omega^p_X) \to H^q(Y, \Omega^p_Y)$ is bijective for $p + q < n - \dim \Sigma_Y - 1$, and injective for $p + q \leq n - \dim \Sigma_X - 1$.

Proof. a) Let us look at the following commutative diagram:

$$\begin{array}{ccc} H^k(X, \mathbb{C}_X) & \longrightarrow & H^k(Y, \mathbb{C}_Y) \\ \downarrow & & \downarrow \\ \mathbb{H}^k(X, \Omega^\bullet_X) & \longrightarrow & \mathbb{H}^k(Y, \Omega^\bullet_Y)\,. \end{array}$$

The upper horizontal is bijective for $k < n - 1$ and injective for $k = n - 1$, by Lefschetz (Theorem 6.5). By Theorem 6.2, the left vertical is bijective for $k < n - \dim \Sigma_X$, the right one is bijective for $k < n - \dim \Sigma_Y - 1$; note that the right one is always injective by the theorem of Bloom and Herrera. This implies our assertion. Note that $\dim \Sigma_X \leq \dim \Sigma_Y + 1$.

b) We use the same diagram as above. Note that the assumption implies that the vertical arrows are bijective for $k = p + q < n - \dim \Sigma_Y - 1$. Now, apply Theorem 6.6. In this way, we get the first statement. As for injectivity, note that if $H^q(X, \Omega^p_X) \to H^q(Y, \Omega^p_Y)$ fails to be injective for some (p, q) with $p + q \leq n - \dim \Sigma_X - 1$ as above, the rank of $\mathbb{H}^k(X, \Omega^\bullet_X) \to \mathbb{H}^k(Y, \Omega^\bullet_Y)$ would be less than $\dim \mathbb{H}^k(X, \Omega^\bullet_X)$ for $p + q = k$, which contradicts a). □

In particular, we obtain Theorem 0.3. Note that, in fact, the hypotheses of Theorem 0.3 imply that Y is an effective Cartier divisor on X.

Remark 7.2. From Theorem 7.1 we may deduce Corollary 5.3. As indicated in Section 5, we can therefore deduce the theorem of Akizuki-Nakano (Theorem 0.1). Unfortunately, we have already used the results of Section 4 and Section 5 in the proof of Theorem 6.6, the reader is invited to look for a different argument.

We can now derive Theorem 2.1 b) from Theorem 7.1 in a simpler way than at the end of Section 5, using the spaces $X \cap \{f_1 = \ldots = f_j = 0\}$.

8 Logarithmic de Rham Cohomology and the Gysin Sequence

Let X be a complex projective variety which is locally a complete intersection of dimension n, Σ the singular locus of X, Y an effective Cartier divisor on X such that the singular locus of Y is $Y \cap \Sigma$, $\dim Y \cap \Sigma = \dim \Sigma - 1$.

In Section 3, we have introduced mappings $\mathrm{Res} : \Omega^p_X(\log Y) \to i_*\Omega^{p-1}_Y$ for $p \leq n - \dim \Sigma$, where $i : Y \hookrightarrow X$ is the inclusion. If X, Y are smooth we get a corresponding homomorphism $\mathrm{Res} : \Omega^\bullet_X(\log Y) \to i_*\Omega^{\bullet-1}_Y$, which induces mappings

$$\mathrm{Res} : \mathbb{H}^k(X, \Omega^\bullet_X(\log Y)) \longrightarrow \mathbb{H}^{k-1}(Y, \Omega^\bullet_Y)\,.$$

In general, we define the latter differently and only for $k < n - \dim \Sigma$: in such a way that the diagram

$$\begin{array}{ccc} \mathbb{H}^k(X, \Omega^\bullet_X(\log Y)) & \longrightarrow & \mathbb{H}^{k-1}(Y, \Omega^\bullet_Y) \\ \downarrow & & \downarrow \\ \mathbb{H}^k(X \setminus \Sigma, \Omega^\bullet_X(\log Y)) & \to & \mathbb{H}^{k-1}(Y \setminus \Sigma, \Omega^\bullet_Y) \end{array}$$

is commutative. Note that the right vertical is bijective, see the proof of Theorem 6.2.

Theorem 8.1. *We have long exact Gysin sequences:*

a) $$\cdots \longrightarrow H^q(X, \Omega^p_X) \longrightarrow H^q(X, \Omega^p_X(\log Y)) \xrightarrow{Res} H^q(Y, \Omega^{p-1}_Y) \longrightarrow H^{q+1}(X, \Omega^p_X) \longrightarrow \ldots \quad \textit{for} \ \ p \leq n - \dim \Sigma\,,$$

b) $$\cdots \longrightarrow \mathbb{H}^k(X, \Omega^\bullet_X) \longrightarrow \mathbb{H}^k(X, \Omega^\bullet_X(\log Y)) \xrightarrow{\mathrm{Res}} \mathbb{H}^{k-1}(Y, \Omega^\bullet_Y) \longrightarrow \mathbb{H}^{k+1}(X, \Omega^\bullet_X) \longrightarrow \ldots \longrightarrow \mathbb{H}^{n - \dim \Sigma - 2}(Y, \Omega^\bullet_Y)\,.$$

Proof. a) follows from Lemma 3.3 b). For b), put $s := n - \dim \Sigma + 1$, and let $F^\bullet$ be the stupid filtration. By Lemma 3.3 b), we have an exact sequence

$$0 \to \Omega_X^\bullet/F^s \to \Omega_X^\bullet(\log Y)/F^s \xrightarrow{\mathrm{Res}} i_*\Omega_Y^{\bullet-1}/F^s \to 0\,,$$

where $i : Y \hookrightarrow X$ is the inclusion. Now, we look at the corresponding long exact hypercohomology sequence. Note that $\mathbb{H}^k(X, F^s\Omega_X^\bullet) = 0$, $k \le n - \dim \Sigma$, so $\mathbb{H}^k(X, \Omega_X^\bullet) \simeq \mathbb{H}^k(X, \Omega_X^\bullet/F^s)$ for $k < n - \dim \Sigma$. Similarly, we can argue for the other terms. □

Of course, Theorem 8.1 implies Theorem 0.4.

Theorem 8.2. *There is a long exact Gysin sequence*

$$\begin{aligned} \ldots \;\longrightarrow\;& H^k(X;\mathbb{C}) \longrightarrow H^k(X \setminus Y;\mathbb{C}) \longrightarrow H^{k-1}(Y;\mathbb{C}) \\ \longrightarrow\;& H^{k+1}(X;\mathbb{C}) \longrightarrow \ldots \longrightarrow H^{n-\dim\Sigma-2}(Y;\mathbb{C})\,. \end{aligned}$$

Proof. We start from the long exact cohomology sequence for the pair $(X \setminus Y \cap \Sigma, X \setminus Y)$. By excision, we have

$$\begin{aligned} H^{k+1}(X \setminus Y \cap \Sigma, X \setminus Y;\mathbb{C}) &\simeq H^{k+1}(X \setminus \Sigma, X \setminus Y \cup \Sigma;\mathbb{C})\,, \\ H^{k+1}(X \setminus \Sigma, X \setminus Y \cup \Sigma;\mathbb{C}) &\simeq H^{k-1}(Y \setminus \Sigma;\mathbb{C}) \simeq H^{k-1}(Y;\mathbb{C})\,, \end{aligned}$$

and $H^k(X;\mathbb{C}) \simeq H^k(X \setminus Y \cap \Sigma;\mathbb{C})$ for $k < n - \dim \Sigma$, by Theorem 6.1. See the proof of Theorem 6.2. □

Remember that $H^k(X \setminus Y;\mathbb{C}) \simeq \mathbb{H}^k(X, \Omega_X^\bullet(\log Y))$ if X is smooth. In our case, we can derive:

Theorem 8.3. $H^k(X \setminus Y;\mathbb{C}) \simeq \mathbb{H}^k(X \setminus Y, \Omega_{X\setminus Y}^\bullet) \simeq \mathbb{H}^k(X, \Omega_X^\bullet(\log Y))$ *for* $k < n - \dim \Sigma$.

Proof. Let $k < n - \dim \Sigma$, $s := n - \dim \Sigma + 1$. By Theorem 6.2, we have

$$H^k(X \setminus Y;\mathbb{C}) \simeq \mathbb{H}^k(X \setminus Y, \Omega_{X\setminus Y}^\bullet) \simeq \mathbb{H}^k(X \setminus Y \cap \Sigma, j_*\Omega_{X\setminus Y}^\bullet)\,,$$

where $j : X \setminus Y \hookrightarrow X \setminus Y \cap \Sigma$ is the inclusion. Note that j is affine.

As in the proof of Theorem 8.1, we have

$$\mathbb{H}^k(X \setminus Y \cap \Sigma, j_*\Omega_{X\setminus Y}^\bullet) \simeq \mathbb{H}^k(X \setminus Y \cap \Sigma, j_*\Omega_{X\setminus Y}^\bullet/F^s)$$

Now we want to show that

$$\mathbb{H}^k(X \setminus Y \cap \Sigma, j_*\Omega_{X\setminus Y}^\bullet/F^s) \simeq \mathbb{H}^k(X \setminus Y \cap \Sigma, \Omega_{X\setminus Y\cap\Sigma}^\bullet(\log Y \setminus \Sigma)/F^s)\,.$$

We have a commutative diagram with exact columns:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Omega^\bullet_{X\setminus Y\cap\Sigma}/F^s & \longrightarrow & \Omega^\bullet_{X\setminus Y\cap\Sigma}/F^s \\
\downarrow & & \downarrow \\
\Omega^\bullet_{X\setminus Y\cap\Sigma}(\log Y\setminus\Sigma)/F^s & \longrightarrow & j_*\Omega^\bullet_{X\setminus Y}/F^s \\
\downarrow & & \downarrow \\
(\Omega^\bullet_{X\setminus Y\cap\Sigma}(\log Y\setminus\Sigma)/\Omega^\bullet_{X\setminus Y\cap\Sigma})/F^s & \longrightarrow & (j_*\Omega^\bullet_{X\setminus Y}/\Omega^\bullet_{X\setminus Y\cap\Sigma})/F^s \\
\downarrow & & \downarrow \\
0 & & 0\,.
\end{array}$$

It is sufficient to show that $\mathbb{H}^k(X\setminus Y\cap\Sigma, (\Omega^\bullet_{X\setminus Y\cap\Sigma}(\log Y\setminus\Sigma)/\Omega^\bullet_{X\setminus Y\cap\Sigma})/F^s) \simeq \mathbb{H}^k(X\setminus Y\cap\Sigma, (j_*\Omega^\bullet_{X\setminus Y}/\Omega^\bullet_{X\setminus Y\cap\Sigma})/F^s)$, or that

$$\begin{aligned}
&\mathbb{H}^k(X\setminus Y\cap\Sigma, \Omega^\bullet_{X\setminus Y\cap\Sigma}(\log Y\setminus\Sigma)/\Omega^\bullet_{X\setminus Y\cap\Sigma}) \\
&\qquad\simeq\ \mathbb{H}^k(X\setminus Y\cap\Sigma, j_*\Omega^\bullet_{X\setminus Y}/\Omega^\bullet_{X\setminus Y\cap\Sigma})\,.
\end{aligned}$$

But the sheaves in question are concentrated on $Y'=Y\setminus\Sigma$, so we have a commutative diagram

$$\begin{array}{ccc}
\mathbb{H}^k(X\setminus Y\cap\Sigma, \Omega^\bullet_{X\setminus Y\cap\Sigma}(\log Y')/\Omega^\bullet_{X\setminus Y\cap\Sigma}) & \rightarrow & \mathbb{H}^k(X\setminus Y\cap\Sigma, j_*\Omega^\bullet_{X\setminus Y}/\Omega^\bullet_{X\setminus Y\cap\Sigma}) \\
\downarrow & & \downarrow \\
\mathbb{H}^k(X\setminus\Sigma, \Omega^\bullet_{X\setminus\Sigma}(\log Y')/\Omega^\bullet_{X\setminus\Sigma}) & \longrightarrow & \mathbb{H}^k(X\setminus\Sigma, j'_*\Omega^\bullet_{X\setminus Y\cup\Sigma}/\Omega^\bullet_{X\setminus\Sigma})
\end{array}$$

with bijective vertical arrows, where $j': X\setminus Y\cup\Sigma\hookrightarrow X\setminus\Sigma$ is the inclusion. Hence, we must show that the lower horizontal is bijective, i.e. that

$$\mathbb{H}^k(X\setminus\Sigma, \Omega^\bullet_{X\setminus\Sigma}(\log Y\setminus\Sigma)/\Omega^\bullet_{X\setminus\Sigma}) \simeq \mathbb{H}^k(X\setminus\Sigma, j'_*\Omega^\bullet_{X\setminus Y\cup\Sigma}/\Omega^\bullet_{X\setminus\Sigma})\,.$$

But $X\setminus\Sigma$ is smooth. Look at the long exact hypercohomology sequences attached to the exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^\bullet_{X\setminus\Sigma} & \longrightarrow & \Omega^\bullet_{X\setminus\Sigma}(\log Y\setminus\Sigma) & \longrightarrow & \Omega^\bullet_{X\setminus\Sigma}(\log Y\setminus\Sigma)/\Omega^\bullet_{X\setminus\Sigma} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^\bullet_{X\setminus\Sigma} & \longrightarrow & j'_*\Omega^\bullet_{X\setminus Y\cup\Sigma} & \longrightarrow & j'_*\Omega^\bullet_{X\setminus Y\cup\Sigma}/\Omega^\bullet_{X\setminus\Sigma} & \longrightarrow & 0
\end{array}$$

Note that the first two verticals yield isomorphisms for the hypercohomology groups, so the right one, too.

Next we want to show that

$$\mathbb{H}^k(X\setminus Y\cap\Sigma, \Omega^\bullet_{X\setminus Y\cap\Sigma}(\log Y\setminus\Sigma)/F^s) \simeq \mathbb{H}^k(X, \Omega^\bullet_X(\log Y)/F^s)\,.$$

But this is clear since depth $\Omega^p_X(\log Y) \geq n-p$ for $p \leq n - \dim\Sigma$: note that depth $\Omega^p_X \geq n-p$, depth $\Omega^{p-1}_Y \geq n-p$, so depth $\Omega^p_X(\log Y) \geq n-p$ by Lemma 3.3 b).

Finally, it is clear that $\mathbb{H}^k(X, \Omega^\bullet_X(\log Y)/F^s) \simeq \mathbb{H}^k(X, \Omega^\bullet_X(\log Y))$. Altogether, we conclude

$$\begin{aligned} H^k(X \setminus Y; \mathbb{C}) &\simeq \mathbb{H}^k(X \setminus Y \cap \Sigma, j_*\Omega^\bullet_{X\setminus Y}) \simeq \mathbb{H}^k(X \setminus Y \cap \Sigma, j_*\Omega^\bullet_{X\setminus Y}/F^s) \\ &\simeq \mathbb{H}^k(X \setminus Y \cap \Sigma, \Omega^\bullet_{X\setminus Y \cap \Sigma}(\log Y \setminus \Sigma)/F^s) \\ &\simeq \mathbb{H}^k(X, \Omega^\bullet_X(\log Y)/F^s) \simeq \mathbb{H}^k(X, \Omega^\bullet_X(\log Y))\,. \qquad \square \end{aligned}$$

Theorem 8.3 explains the relation between Theorem 8.1 and 8.2. Furthermore, we can sharpen Theorem 8.2 and, therefore, 8.1 b) under some transversality condition:

Theorem 8.4. *Assume that Y is a tranversal section of X in the stratified sense. Then there are long exact Gysin sequences:*

$$\ldots \to H^k(X;\mathbb{C}) \longrightarrow H^k(X\setminus Y;\mathbb{C}) \longrightarrow H^{k-1}(Y;\mathbb{C}) \longrightarrow H^{k+1}(X;\mathbb{C}) \longrightarrow \ldots$$

and

$$\begin{aligned} \ldots \quad &\longrightarrow \quad \mathbb{H}^k(X,\Omega^\bullet_X) \longrightarrow \mathbb{H}^k(X, \Omega^\bullet_X(\log Y)) \xrightarrow{\text{Res}} \mathbb{H}^{k-1}(Y, \Omega^\bullet_Y) \\ &\longrightarrow \quad \mathbb{H}^{k+1}(X,\Omega^\bullet_X) \longrightarrow \ldots \longrightarrow \mathbb{H}^{n-\dim\Sigma}(X, \Omega^\bullet_X(\log Y))\,. \end{aligned}$$

Proof. The first exact sequence follows from the exact cohomology sequence of the pair $(X, X\setminus Y)$ since $H^{k+1}(X, X\setminus Y;\mathbb{C}) \simeq H^{k-1}(Y;\mathbb{C})$.

To get the second sequence, put $K := n - \dim\Sigma$. For $k \leq n - \dim\Sigma$, we have the following commutative diagram whose upper and lower row are exact:

$$\begin{array}{ccccc} \cdots \longrightarrow H^{k-2}(Y;\mathbb{C}) & \longrightarrow & H^k(X;\mathbb{C}) & \longrightarrow & H^k(X\setminus Y;\mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow \\ \cdots \longrightarrow \mathbb{H}^{k-2}(Y,\Omega^\bullet_Y) & & \mathbb{H}^k(X,\Omega^\bullet_X) & \longrightarrow & \mathbb{H}^k(X,\Omega^\bullet_X(\log Y) \\ \downarrow & & \downarrow & & \downarrow \\ \cdots \to \mathbb{H}^{k-2}(Y,\Omega^\bullet_Y/F^K) & \to & \mathbb{H}^k(X,\Omega^\bullet_X/F^{K+1}) & \to & \mathbb{H}^k(X,\Omega^\bullet_X(\log Y)/F^{K+1}). \end{array}$$

The left verticals are bijective, the others injective except maybe the vertical at the top on the right. A diagram chase shows that the middle row can be extended to an exact sequence. $\square$

Acknowledgements

The author is grateful to the Deutsche Forschungsgemeinschaft for financial support.

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The Geometry of the Versal Deformation

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Abstract

In this paper, we give a survey on known results about the versal deformation of an isolated complex hypersurface singularity. We recall the interest in the geometry of these deformations and we give some open problems in relation with the local topology of isolated hypersurface singularities.

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be the germ at the point 0 of a complex analytic function, also denoted by f and defined on an open neighbourhood U of 0 in $\mathbb{C}^{n+1}$. Assume that f has an isolated critical point at 0. Weierstrass Preparation Theorem tells us that this hypothesis is equivalent to the finiteness of the complex dimension of the vector space

$$\mathbb{C}\{z_0, \ldots, z_n\}/(\partial f/\partial z_0, \ldots, \partial f/\partial z_n)$$

quotient of the complex algebra of convergent power series at 0 by the ideal generated by the partial derivatives of f. We shall denote by μ this complex dimension which is called the Milnor number of f at 0. Since f vanishes at 0, this is also equivalent to the finiteness of the complex dimension of

$$\mathbb{C}\{z_0, \ldots, z_n\}/(f, \partial f/\partial z_0, \ldots, \partial f/\partial z_n).$$

We shall denote by τ this complex dimension and call it the Tjurina number of f at 0.

In [19], G. Tjurina has introduced the following unfolding of f:

$$\Phi\colon (\mathbb{C}^{n+1} \times \mathbb{C}^{\tau-1}, 0) \to (\mathbb{C} \times \mathbb{C}^{\tau-1}, 0)$$

1991 *Mathematics Subject Classification.* 14B05, 14B07, 14F35, 32B30, 32G11, 32S30, 32S40, 32S55

Key words. Hypersurface singularity, monodromy, discriminant, Milnor number, fundamental group

defined by

$$\Phi(z_0,\dots,z_n,\lambda_1,\dots,\lambda_{\tau-1}) = (F(z_0,\dots,z_n,\lambda_1,\dots,\lambda_{\tau-1}),\lambda_1,\dots,\lambda_{\tau-1})\,,$$

with

$$F(z_0,\dots,z_n,\lambda_1,\dots,\lambda_{\tau-1}) = f(z_0,\dots,z_n) + \lambda_1 s_1 + \ldots + \lambda_{\tau-1}s_{\tau-1}$$

where the images of $1, s_1, \dots, s_{\tau-1}$ in

$$\mathbb{C}\{z_0,\dots,z_n\}/(f,\partial f/\partial z_0,\dots,\partial f/\partial z_n)$$

give a complex basis of this finite dimensional vector space. G. Tjurina proved that such an unfolding is a mini-versal deformation of the hypersurface singularity $(f^{-1}(0),0)$ with base space of smallest dimension, i.e. if $\varphi : (X,0) \to (S,0)$ is a deformation of $(f^{-1}(0),0)$, there is a complex analytic map $\sigma\colon (S,0) \to (\mathbb{C}\times\mathbb{C}^{\tau-1},0)$ such that φ is the pull-back by σ of Φ and the derivative of σ at 0 is uniquely defined.

A deformation of the hypersurface singularity $(f^{-1}(0),0)$ is called a versal deformation of $(f^{-1}(0),0)$, when any deformation of $(f^{-1}(0),0)$ is obtained by pull-back from the given versal deformation. A versal deformation of f is a mini-versal deformation of f if its base space has the dimension τ. Any versal deformation of a hypersurface singularity $\Psi : \mathbb{C}^N \to \mathbb{C}^P$, is isomorphic to a mini-versal deformation $\Phi\times\mathrm{Id} : \mathbb{C}^{n+\tau}\times\mathbb{C}^{N-n-\tau} \to \mathbb{C}^{\tau}\times\mathbb{C}^{N-n-\tau}$, extended by a trivial deformation.

In particular the function f itself is a deformation of the hypersurface $(f^{-1}(0),0)$ and is obviously obtained as a pull-back from Φ.

The singularity $(f^{-1}(0),0)$ might have several mini-versal deformations, but there are all isomorphic.

In this survey we shall show how much one can recover of the singularity of $(f^{-1}(0),0)$ from the geometry of a mini-versal deformation.

1 On the Local Topology of Isolated Hypersurface Singularities

In his book [12], J. Milnor has given most of the basic features which describe the local topology of an isolated hypersurface singularities. As in the introduction we consider a germ f at 0 of a complex analytic function with an isolated singularity at 0. Whenever there will be no confusion, we shall still denote by f a representative of this germ in a small neighbourhood U of 0. We denote by X the hypersurface $f^{-1}(0)$.

1.1 Local Conic Structure

Let $B_\varepsilon(0)$ the real $(2n+2)$-ball of $\mathbb{C}^{n+1}$ centered at 0 with radius ε. Let $S_\varepsilon(0)$ be the real $(2n+1)$-sphere boundary of $B_\varepsilon(0)$. J. Milnor showed that, there is some $\varepsilon_0 > 0$, such that for any ε, $\varepsilon_0 > \varepsilon > 0$, the sphere $S_\varepsilon(0)$ is transverse to X in $\mathbb{C}^{n+1}$. As a consequence, for $\varepsilon_0 > \varepsilon > 0$, the manifolds $X \cap S_\varepsilon(0)$ are diffeomorphic to each other and one obtains (see [12, Theorem 2.10]):

Theorem 1.1. *There is $\varepsilon_1 > 0$ such that, for any ε, $\varepsilon_1 > \varepsilon > 0$, the pair $(B_\varepsilon(0), B_\varepsilon(0) \cap X)$ is homeomorphic to the real cone from 0 on the pair $(S_\varepsilon(0), S_\varepsilon(0) \cap X)$.*

Therefore, the local topology of an isolated hypersurface singularity is given by the embedding of the local link $S_\varepsilon(0) \cap X$ in $S_\varepsilon(0)$, when ε is small enough. The following result shows that the complement of $S_\varepsilon(0) \cap X$ in $S_\varepsilon(0)$ fibers over the circle $\mathbb{S}^1$.

1.2 The Local Fibration

One of the most important properties of the local link $S_\varepsilon(0) \cap X \subset S_\varepsilon(0)$ is given in the (see [12, Theorem 4.8]):

Theorem 1.2. *There is $\varepsilon_2 > 0$, such that, for any ε, $\varepsilon_2 > \varepsilon > 0$, the quotient $f/|f|$ induces a locally trivial smooth fibration ψ_ε of $S_\varepsilon(0) \setminus S_\varepsilon(0) \cap X$ over $\mathbb{S}^1$. These fibrations are fiber isomorphic to each other.*

In fact, this fibration can be seen in another way:

Theorem 1.3. *There is $\varepsilon_3 > 0$ such that, for any ε, $\varepsilon_3 > \varepsilon > 0$, there exists $\eta_\varepsilon > 0$ such that, for any η, $\eta_\varepsilon > \eta > 0$, the map $\psi_{\varepsilon,\eta}$ induced by f from the intersection $B_\varepsilon(0)^\circ \cap f^{-1}(\partial D_\eta(0))$ of the open ball $B_\varepsilon(0)^\circ$ with the inverse image by f of the circle $\partial D_\eta(0)$, boundary of the disc $D_\eta(0)$, into $\partial D_\eta(0)$ is a locally trivial smooth fibration. For these choices of ε and η, the fibrations $\psi_{\varepsilon,\eta}$ are fiber isomorphic to each other and there are isomorphic to ψ_ε.*

This local fibration is also called the Milnor fibration of f at 0. The fibers of this local fibration being diffeomorphic to each other, a fiber of a Milnor fibration is called a Milnor fiber of f at 0.

1.3 The Local Monodromy

The fibrations ψ_ε (resp. $\psi_{\varepsilon,\eta}$) are given by a diffeomorphism h of a Milnor fiber $\mathbb{F}$, say $\mathbb{F} = \psi_\varepsilon^{-1}(1)$ (resp. $\mathbb{F} = \psi_{\varepsilon,\eta}^{-1}(\eta)$) onto itself called a geometric monodromy of the fibration. Of course, this diffeomorphism is not unique, but the isotopy class of this diffeomorphism is well-defined.

The geometric monodromy defines an automorphism on the homology (or on the cohomology) of the fiber called the monodromy of f at 0. In fact, J. Milnor made the following observation: since f has an isolated singularity, the homology groups of the fiber $\mathbb{F}$ vanish in dimensions $\neq 0, n$, namely we have (see [12, Theorem 7.2]):

Theorem 1.4. *Suppose that the germ f of complex analytic functions has an isolated singularity at 0, a Milnor fiber of f at 0 has a trivial k-homology for $k \neq 0, n$. When $n \neq 0$, the 0-th homology group is a cyclic free abelian group and the n-th homology group is a free abelian group of rank μ, with*

$$\mu = \dim_{\mathbb{C}} \mathbb{C}\{z_0, \ldots, z_n\}/(\partial f/\partial z_0, \ldots, \partial f/\partial z_n).$$

In the case $n = 0$, the reduced 0-th homology group is free abelian of rank $\mu = m - 1$, where m is the multiplicty of f at 0.

The number μ is called the Milnor number, or the Milnor multiplicity, of f at 0.

1.4 Exotic Spheres

From the work of M. Kervaire and J. Milnor one can decide when a local link $S_\varepsilon(0) \cap X \subset S_\varepsilon(0)$ gives a manifold $S_\varepsilon(0) \cap X$ which is homeomophic to a $(2n-1)$-dimensional sphere, but not diffeomorphic to the standard sphere, i.e. is an exotic sphere. Namely we have the following result (see [12, Theorem 8.5 and Remark 8.7]):

Theorem 1.5. *If $n \neq 2$ the manifold $S_\varepsilon(0) \cap X$ is homeomorphic to a sphere $\mathbb{S}^{2n-1}$ if and only if the value at 1 of the characteristic polynomial of the local monodromy of f at 0 is ± 1. In which case the differentiable structure of $S_\varepsilon(0) \cap X$ is given by the signature of the intersection pairing on the middle homology of a Milnor fiber if n is even or by the Kervaire invariant if n is odd.*

Therefore, it appears that the computation of the monodromy and the intersection pairing on the middle homology of a Milnor fiber are important to characterize the differentiable structure of the local link.

In the case of the quasi-homogeneous polynomials of the form $\sum_{i=0}^{n} z_i^{a_i}$, where the a_i's are integers ≥ 2, these computations were explicitly made by F. Pham in [15]. Using Pham's results E. Brieskorn (see [2]) could show, for instance, that the local link at 0 of the singularity defined by

$$x^2 + y^2 + z^2 + t^3 + u^{6k-1} = 0,$$

for $k = 1, \ldots, 28$, gives the 28 topological spheres of dimension 7 which bound a parallelizable manifold.

Below, we shall explain an idea of F. Pham (see [16]) to compute the intersection pairing on the middle homology of a Milnor fiber using the geometry of the versal deformation of the germ f.

2 An Example

Let us consider the simple case of the singularity defined by $f = z_0^{\mu+1} + \sum_1^n z_i^2$. In this case $\mu = \tau$ and we can choose $F = z_0^{\mu+1} + \sum_1^n z_i^2 + \lambda_{\mu-1} z_0^{\mu-1} + \ldots + \lambda_1 z_0$. The critical locus C of the deformation Φ of f is defined by

$$\partial F/\partial z_0 = \ldots = \partial F/\partial z_n = 0.$$

Therefore, C is non-singular. The discriminant D_μ of Φ is the image $\Phi(C)$ of C by Φ and is defined by the discriminant polynomial Δ_μ of the polynomial in z_0 given by $z_0^{\mu+1} + \lambda_{\mu-1} z_0^{\mu-1} + \ldots + \lambda_1 z_0 - \lambda_0$, where λ_0 is the coordinate of $\mathbb{C} \times \{0\}$ in $\mathbb{C} \times \mathbb{C}^{\mu-1}$.

A representative of the germ Φ defines a complex analytic map from an open neighbourhood $\mathcal{U}$ of 0 in $\mathbb{C}^{n+1} \times \mathbb{C}^{\mu-1}$ onto an open neighbourhood $\mathcal{V}$ of 0 in $\mathbb{C} \times \mathbb{C}^{\mu-1}$ which has no critical points over the complement of D_μ in $\mathcal{V}$. Since the fiber of Φ over 0 has an isolated singular point at 0, it is transverse to small spheres $S_\varepsilon(0)$ $(1 \gg \varepsilon > 0)$ of $\mathbb{C}^{n+1} \times \mathbb{C}^{\mu-1}$. For such a given ε, there is $\eta_\varepsilon > 0$, such that, for any η, $\eta_\varepsilon > \eta > 0$, the fibers of Φ over points $m \in \mathcal{V}$, such that $\|0m\|$, distance from 0 to m, is $< \eta$, are transverse to $S_\varepsilon(0)$. Using Ehresmann Lemma, it is easy to show that Φ induces a locally trivial fibration of $B_\varepsilon(0) \cap \Phi^{-1}(B_\eta(0) \setminus D_\mu)$ over $B_\eta(0) \setminus D_\mu$. Fibers of this fibration are Milnor fibers of f at 0. In particular the n-th homology groups of these fibers defined a local system over $B_\eta(0) \setminus D_\mu$. This local system corresponds to a representation of the fundamental group $\pi_1(B_\eta(0) \setminus D_\mu, m_0)$ of $B_\eta(0) \setminus D_\mu$ at a point $m_0 \in B_\eta(0) \setminus D_\mu$ in the n-th homology group $H_n(\phi^{-1}(m_0), \mathbb{Z})$:

$$\rho \colon \pi_1(B_\eta(0) \setminus D_\mu, m_0) \to Aut(H_n(\phi^{-1}(m_0), \mathbb{Z})).$$

For $\gamma \in \pi_1(B_\eta(0) \setminus D_\mu, m_0)$ the automorphism $\rho(\gamma)$ is called the monodromy of $H_n(\phi^{-1}(m_0), \mathbb{Z})$ along γ.

In [1], E. Artin showed that the fundamental group of the complement of D_μ in $\mathbb{C} \times \mathbb{C}^{\mu-1}$ is isomorphic to the group $B_{\mu+1}$ of braids with $\mu + 1$ strings. Since D_μ is defined by a quasi-homogeneous equation Δ_μ, the local fundamental group of the complement of D_μ in $\mathbb{C} \times \mathbb{C}^{\mu-1}$ at 0 is isomorphic to the fundamental group of the complement of D_μ in $\mathbb{C} \times \mathbb{C}^{\mu-1}$. Therefore, by choosing adequately the open neighbourhood $\mathcal{V}$ in $\mathbb{C} \times \mathbb{C}^{\mu-1}$, the fundamental group of the complement of D_μ in $\mathcal{V}$ is isomorphic to the local fundamental group at 0 of the complement of D_μ in $\mathbb{C} \times \mathbb{C}^{\mu-1}$, so it is also isomorphic to the group $B_{\mu+1}$ of braids with $\mu + 1$ strings. In fact, using the conic structure theorem proved in [5] for non-isolated singularities, one can prove that open balls centered at 0 with small radiuses are a system of good neighbourhoods in the sense of Prill (see [17]), so for η small enough, an open ball $B_\eta(0)$ of $\mathbb{C} \times \mathbb{C}^{\mu-1}$ centered at 0 with radius η is an adequate neighbourhood of 0 in $\mathbb{C} \times \mathbb{C}^{\mu-1}$ in which the complement of D_μ has a fundamental group isomorphic to the braid group $B_{\mu+1}$.

Let us choose a point $m_0 \in B_\eta(0) \setminus D_\mu$. Let L be a complex line through m_0 which intersects D_μ transversally in $B_\eta(0)$ at μ points. The fundamental group $\pi_1(L \setminus D_\mu, m_0)$ of $L \setminus D_\mu$ at m_0 is a free group generated by μ free generators. One can choose a distinguished basis e_i, $1 \leq i \leq \mu$, of $H_n(\phi^{-1}(m_0), \mathbb{Z})$ associated to generators γ_i, $1 \leq i \leq \mu$, of $\pi_1(B_\eta(0) \setminus D_\mu, m_0)$ images of free generators of the free group $\pi_1(L \setminus D_\mu, m_0)$ such that, for $1 \leq i < j \leq \mu$, $\gamma_i\gamma_j = \gamma_j\gamma_i$, if $|i-j| > 1$ and $\gamma_i\gamma_{i+1}\gamma_i = \gamma_{i+1}\gamma_i\gamma_{i+1}$, such that for any $e \in H_n(\phi^{-1}(m_0), \mathbb{Z})$, we have

$$\rho(\gamma_i)(e) = e + (-1)^{\frac{(n+1)(n+2)}{2}}(e, e_i)e_i,$$

where $(.\,,.)$ is the intersection pairing on $H_n(\phi^{-1}(m_0), \mathbb{Z})$.

The relations between the generators γ_i, $1 \leq i \leq \mu$, of the fundamental group gives relations between the monodromies $\rho(\gamma_i)$. We find, for $|i-j| > 1$,

$$(e_i, e_j) = 0$$

and, for $1 \leq i \leq \mu - 1$:

$$(e_i, e_{i+1}) = \pm 1$$

and, in fact $+1$ for a good choice of the orientation of the vanishing cycles e_i, $1 \leq i \leq \mu$.

In this example, an explicit presentation of the fundamental group of the complement of the discriminant D_μ has given a computation of the intersection form of the Milnor fiber.

Our aim is to have a similar result for the germ of a general holomorphic function with an isolated singularity.

3 Geometry of a Versal Deformation

As above, we assume that $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is the germ of a complex analytic function with an isolated critical point at 0. It is clear from the definition of the mini-versal deformation of $(f^{-1}(0), 0)$ that its isomorphism class only depends on the local ring $\mathcal{O}_{f^{-1}(0),0}$ of the hypersurface f^{-1} at the point 0. In fact, two complex hypersurfaces isolated singularities are isomorphic if and only if their mini-versal deformations are isomorphic. Therefore, we should recover analytic invariants of the hypersurface singularity from its mini-versal deformation. For instance:

Lemma 3.1. *The multiplicity of the discriminant of the versal deformation of $(f^{-1}(0), 0)$ equals the Milnor multiplicity of f at 0.*

Proof. Suppose that the multiplicity of f at 0 is ≥ 3. Let ε small enough, such that f has no critical point in the punctured ball $B_\varepsilon(0) \setminus \{0\}$. For a general $(a_0, \ldots, a_n) \in \mathbb{C}^{n+1}$ small enough, J. Milnor observed that the function $F_{\mathbf{a}} := f + \sum_0^n a_i z_i$ has only ordinary quadratic critical points in $B_\varepsilon(0) \setminus \{0\}$ with distinct values by $F_{\mathbf{a}}$. One may consider the function $F_{\mathbf{a}}$ as the restriction of the mini-versal deformation of $(f^{-1}(0), 0)$ over a line $\lambda - a_0 = \ldots = \lambda - a_n = 0$. The critical points of $F_{\mathbf{a}}$ being ordinary quadratic with distinct values, it means that this line is transverse to the discriminant D of Φ. The number of intersection points is the multiplicity of D at 0 and J. Milnor indicated that it is also the Milnor multiplicity of f at 0 (see [12], [11] or [14]).

When f has multiplicity 2, we can find local coordinates of $\mathbb{C}^{n+1}$ such that

$$f(z_0, \ldots, z_n) = \sum_0^{k-1} z_i^2 + g(z_k, \ldots, z_n),$$

where g has multiplicity ≥ 3. In this case, we consider the function $f + \sum_k^n a_i z_i$, which has μ distinct ordinary critical point, whenever $(a_k, \ldots, a_n)$ is general and small enough in $\mathbb{C}^{n-k+1}$.

Since a versal deformation is obtained from a mini-versal deformation by extending it by a trivial deformation the preceding result extends trivially to versal deformations. □

There is another way to find Milnor number. First, observe:

Lemma 3.2. *Let Φ a versal deformation of f. The critical locus $C(\Phi)$ of Φ is non-singular.*

Proof. Let us assume that the multiplicity of f at 0 is ≥ 3, i.e. $f \in \mathfrak{M}^3$, where $\mathfrak{M}$ is the maximal ideal of the local ring $\mathcal{O}_{\mathbb{C}^{n+1},0}$ of germs of holomorphic functions of $\mathbb{C}^{n+1}$ at the origin 0. In this case we can choose a base of

$$\mathbb{C}\{z_0, \ldots, z_n\}/(f, \partial f/\partial z_0, \ldots, \partial f/\partial z_n)$$

such that the representatives of this base are $1, z_0, \ldots z_n, s_{n+2}, \ldots, s_\tau$. Therefore, the germ of map $\Phi_0 \colon (\mathbb{C}^{n+1}, 0) \times (\mathbb{C}^{\tau-1}, 0) \to (\mathbb{C}, 0) \times (\mathbb{C}^{\tau-1}, 0)$ defined by

$$\Phi_0(z_0, \ldots, z_n, \lambda_1, \ldots, \lambda_{\tau-1}) = \big(F(z_0, \ldots, z_n, \lambda_1, \ldots, \lambda_{\tau-1}), \lambda_1, \ldots, \lambda_{\tau-1}\big),$$

with

$$\begin{aligned} &F(z_0, \ldots, z_n, \lambda_1, \ldots, \lambda_{\tau-1}) = \\ &\qquad f(z_0, \ldots, z_n) + \lambda_1 x_0 + \ldots \lambda_{n+1} x_n + \lambda_{n+2} s_{n+2} + \ldots + \lambda_{\tau-1} s_{\tau-1} \end{aligned}$$

is a mini-versal deformation. It is enough to prove that the critical locus of Φ_0 is non-singular to prove it for any mini-versal deformations of f, since that any mini-versal deformation of f is isomorphic to one of them, say Φ_0.

One can easily find that the critical locus $C(\Phi_0)$ of Φ_0 is given by

$$\partial f/\partial x_0 + \lambda_1 + \sum_{n+2}^{\tau-1} \lambda_i \partial s_i/\partial x_0 = \ldots = \partial f/\partial x_n + \lambda_{n+1} + \sum_{n+2}^{\tau-1} \lambda_i \partial s_i/\partial x_n = 0.$$

Since these equations are obviously of multiplicity 1 with transverse initial terms, the implicit function theorem immediately gives that $C(\Phi_0)$ is non-singular. Since any other mini-versal deformation Φ of f is isomorphic to Φ_0, the critical locus $C(\Phi)$ of Φ is also non-singular.

When f has multiplicity 2, as noticed above, we can find local coordinates of $\mathbb{C}^{n+1}$ such that

$$f(z_0, \ldots, z_n) = \sum_0^{k-1} z_i^2 + g(z_k, \ldots, z_n),$$

where g has multiplicity ≥ 3. Now, we observe that a mini-versal deformation of f is obtained from a mini-versal deformation of g. Considering the equations, it is easy to deduce the non-singularity of the critical space of a mini-versal deformation of f from the fact that the critical space of a mini-versal deformation of g, which has multiplicity ≥ 3, is non-singular according to the preceding proof.

As for the preceding result, since a versal deformation is obtained from a mini-versal deformation by extending it by a trivial deformation the regularity of the critical locus extends trivially to versal deformations. □

Now we have:

Proposition 3.3. *The map φ from $(C(\Phi), 0)$ onto $(\mathbb{C}^{\tau-1}, 0)$ induced by the composition of the mini-versal deformation Φ of f and the projection of $(\mathbb{C}, 0) \times (\mathbb{C}^{\tau-1}, 0)$ onto $(\mathbb{C}^{\tau-1}, 0)$ is a ramified covering of degree μ, the Milnor number of f at* 0.

Proof. First, notice that φ is a finite morphism. By the geometric version of Weierstrass preparation theorem (see [9]), it is enough to prove that φ is quasi-finite, i.e. the point 0 is isolated in the fiber of φ over 0. This is an immediate consequence of the fact that the origin 0 is an isolated critical point of f. Since $C(\phi)$ is not singular, the morphism φ is also flat. Any holomorphic flat finite morphism $\psi : (X, 0) \to (Y, 0)$ from a non-singular germ $(X, 0)$ into a non-singular germ $(Y, 0)$ is a ramified covering of degree equal to the complex dimension of the complex vector space $\mathcal{O}_{X,0}/\psi^* \mathfrak{M}_{y,0}$, where $\psi^* \colon \mathcal{O}_{Y,0} \to \mathcal{O}_{X,0}$

is the homomorphism induced by ψ and $\mathfrak{M}_{Y,0}$ is the maximal ideal of $\mathcal{O}_{Y,0}$. It remains to prove that the degree of the map φ is μ.

As noticed before the degree of φ equals $\dim_{\mathbb{C}} \mathcal{O}_{C(\Phi),0}/\varphi^*\mathfrak{M}_{\mathbb{C}^{\tau-1},0}$. One can verify that the local ring $\mathcal{O}_{C(\Phi),0}/\varphi^*\mathfrak{M}_{\mathbb{C}^{\tau-1},0}$ is isomorphic to

$$\mathcal{O}_{\mathbb{C}^{n+1},0}/(\partial f/\partial x_0, \ldots, \partial f/\partial x_n)\,,$$

therefore the degree of φ is μ. $\square$

There are also other interesting features of the geometry of the versal deformation. For instance, B. Teissier [18] proved that

Theorem 3.4. *Let $(\Delta(\Phi), 0)$ be the discriminant of Φ, image by Φ of the critical space $(C(\Phi), 0)$ of a versal deformation of f. The morphism Φ induces a map $n : (C(\Phi), 0) \to (\Delta(\Phi), 0)$ which is the normalization of $(\Delta(\Phi), 0)$ and also its Nash modification.*

In fact, one of the most interesting result that we shall need is the following:

Proposition 3.5. *Let Φ be a versal deformation of f. There is a non-empty Zariski open subset Ω_1 of the Grassmannian manifold of complex planes through 0 in $\mathbb{C}^\tau$ such that, for any $P \in \Omega_1$, there is a representative $\Delta(\Phi)$ of the discriminant $(\Delta(\Phi), 0)$ closed in an open neighbourhood U of 0 in $\mathbb{C}^\tau$, and an open neighbourhood V of 0 in $\mathbb{C}^\tau$ such for all $v \in V$ except in a closed analytic subset of V, the translate $P + v = P_v$ intersects $\Delta(\Phi)$ into a curve having only cusps and nodes as singularities.*

A theorem from [8] gives a way to compute the local fundamental group of the complement of an analytic germ of hypersurface $(H, 0)$ in $(\mathbb{C}^N, 0)$ at the point 0. First let H be a representative of the germ $(H, 0)$ in an open neighbourhood U of 0 in $\mathbb{C}^N$. Then, notice that, for $\varepsilon > 0$ small enough, the fundamental group of the complement of H in $B_\varepsilon(0)^\circ$, the open ball of $\mathbb{C}^N$ centered at 0 with radius ε, is isomorphic to the local fundamental group of the complement of an analytic germ of hypersurface $(H, 0)$ in $(\mathbb{C}^N, 0)$ at the point 0.

In [8], we have the following result:

Theorem 3.6. *Let H be a closed complex hypersurface in an open neighbourhood U of $0 \in H$ in $\mathbb{C}^N$. There is an open dense Zariski open set Ω_2 in the Grassmann space of 2-planes in $\mathbb{C}^N$, such that, for any $P \in \Omega_2$, there is an open neighbourhood V of 0 in $\mathbb{C}^N$ such for all $v \in V$ except in a closed analytic subset of V, the translate $P + v = P_v$ intersects H in such a way that, for any $\varepsilon > 0$ small enough, the fundamental group of $B_\varepsilon(0)^\circ \cap P_v \setminus H \cap P_v$ is isomorphic to the fundamental group of $B_\varepsilon(0)^\circ \setminus H$. Moreover, the fundamental group of $B_\varepsilon(0)^\circ \cap P \setminus H \cap P$ surjects on the fundamental group of $B_\varepsilon(0)^\circ \setminus H$.*

For simplicity, we shall denote by $P_v = P + v$ a general plane of $\mathbb{C}^\tau$ passing near 0, with $P \in \Omega_1 \cap \Omega_2$, $v \in V$, and such that, for $\varepsilon > 0$ small enough, the fundamental group of $B_\varepsilon(0)^\circ \cap P_v \setminus \Delta(\Phi) \cap P_v$ is isomorphic to the local fundamental group of the complement of $(\Delta(\Phi), 0)$ in $(\mathbb{C}^N, 0)$ and the intersection curve $B_\varepsilon(0)^\circ \cap P_v \cap \Delta(\Phi)$ has only cusps and nodes as singularities. And, we assume that the number of these cusps and nodes is the minimum for $P \in \Omega_1 \cap \Omega_2$. The plane P through 0 will also be denoted by P_0. For a general plane P_v of $\mathbb{C}^\tau$ passing near 0, the plane P is called a general plane of $\mathbb{C}^\tau$.

Since the fundamental group of the local complement of a plane curve in the plane at a cusp singularity is the group of braids with two strings, and at a nodal point is an abelian group of two generators, the preceding proposition and theorem shows that, conjugates of some generators of the local fundamental group of the complement of $(\Delta(\Phi), 0)$ in $\mathbb{C}^\tau$ at 0 either commute or satisfy a 2-braid relation.

Unfortunately, this is not enough to obtain a workable presentation of the local fundamental group of the complement of $(\Delta(\Phi), 0)$ in $(\mathbb{C}^\tau, 0)$, for instance, in view of performing an calculation of the intersection matrix as in the example presented above (from F. Pham).

Furthermore, we need also to know the number of cusps and nodes in a general plane section of $\Delta(\Phi)$ near 0.

4 Geometry of the Discriminant

In order to calculate the fundamental group of the local complement of $(\Delta(\Phi), 0)$ in $(\mathbb{C}^\tau, 0)$, we saw that we need to understand the geometry of a general plane section of $\Delta(\Phi)$ and the geometry of a section of $\Delta(\Phi)$ by a general plane section passing near 0.

The first information on a general plane section passing near 0 is the number of cusps and the number of nodes in the curve intersection.

In [10], we give a formula to get the number of cusps in that general section:

Proposition 4.1. *Let $\Delta(\Phi)$ be a small representative of the discriminant $(\Delta(\Phi), 0)$ of a versal deformation $\Phi : \mathbb{C}^{n+\tau} \to \mathbb{C}^\tau$ of f. Let P_v be a general plane of $\mathbb{C}^\tau$ passing near 0. The number of cusps of $\Delta(\Phi) \cap P_v$ equals $\mu + \mu_1 - 1$, where μ is the Milnor number of f at 0 and μ_1 is the Milnor number of the complete intersection curve $\Phi^{-1}(P_0) \cap C(\Phi)$ at 0.*

In the preceding proposition we have used the notion of Milnor number of a complete intersection X with an isolated singularity at a point 0. Let us define this notion.

Let $f_1 = \ldots = f_k = 0$ be equations of the complete intersection X in an open neighbourhood U of 0 in $\mathbb{C}^{n+k}$. So X is the fiber over 0 of $\varphi : \mathbb{C}^{n+k} \to \mathbb{C}^k$, where $\varphi = (f_1, \ldots, f_k)$. Let $B_\varepsilon(0)^\circ$ be an open ball centered with radius $1 \gg \varepsilon > 0$, such that $B_\varepsilon(0)^\circ \subset U$ and $B_\varepsilon(0)^\circ \cap X \setminus \{0\}$ has no singular point. Then, for $(t_1, \ldots, t_k)$ small enough and outside the discriminant of φ, the space $\varphi^{-1}(t_1, \ldots, t_k) = \mathbb{F}_{\mathbf{t}}$ is non-singular and has only homology in dimension 0 or n. The rank of the n-th reduced homology group of $\mathbb{F}_{\mathbf{t}}$ is an integer called the Milnor number of X at 0 (see [7]).

In the case $(X, 0)$ is a curve, there is another interpretation of the Milnor number due to R.O. Buchweitz and G.-M. Greuel (see [4]).

Let $\mathcal{O}_{X,0}$ be the local ring of the curve $(X, 0)$. Let $\overline{\mathcal{O}}_{X,0}$ be the integral closure of $\mathcal{O}_{X,0}$ in its total ring of fraction. The quotient $\overline{\mathcal{O}}_{X,0}/\mathcal{O}_{X,0}$ is a complex vector space of finite complex dimension $\delta(X, 0)$. Let $r(X, 0)$ be the number of branches of $(X, 0)$, i.e. the number of analytically irreducible components of $(X, 0)$. The Milnor number of X at 0 is

$$\mu(X, 0) = 2\delta(X, 0) - r(X, 0) + 1.$$

In [12], J. Milnor showed that this two definitions coincide when X is a plane curve. Another result gives the number of nodes knowing the number of cusps:

Proposition 4.2. *Let $\Delta(\Phi)$ a small representative of the discriminant $(\Delta(\Phi), 0)$ of a versal deformation $\Phi : \mathbb{C}^{n+\tau} \to \mathbb{C}^\tau$ of f. Let P_v be a general plane of $\mathbb{C}^\tau$ passing near 0. Let κ be the number of cusps of $\Delta(\phi) \cap P_v$. The number of nodes of $\Delta(\phi) \cap P_v$ equals*

$$\frac{\mu + \mu(\Delta(\phi) \cap P_v, 0) - 1 - 3\kappa}{2}$$

where μ is the Milnor number of f at 0 and $\mu(\Delta(\phi) \cap P_v, 0)$ is the Milnor number of $(\Delta(\phi) \cap P_v, 0)$.

Proof. It is enough to prove this proposition when Φ is a mini-versal deformation. Then, this result is consequence of the result of Teissier quoted above: since the normalisation of $\Delta(\Phi)$ coincides with the Nash modification, the limits of tangent spaces of the discriminant $\Delta(\Phi)$ at 0 is reduced to the hyperplane $\{0\} \times \mathbb{C}^{\tau-1}$. Therefore, the tangent cone of the curve $(\Delta(\phi) \cap P, 0)$ is $L = P \cap \{0\} \times \mathbb{C}^{\tau-1}$. The projection p of $\mathbb{C}^\tau$ onto $\{0\} \times \mathbb{C}^{\tau-1}$ induces a map p of $(\Delta(\phi) \cap P, 0)$ onto $(L, 0)$ which is transverse.

The discriminant of p consists only of the point 0, with multiplicity $\mu + \mu(\Delta(\phi) \cap P_v, 0) - 1$.

The sum of the discriminant numbers of the projection p_v induced by p of $\Delta(\phi) \cap P_v$ onto the line $L_v = P_v \cap \{0\} \times \mathbb{C}^{\tau-1}$ is also equal to $\mu + \mu(\Delta(\phi) \cap P_v, 0) - 1$, since discriminants are invariant by base change. The projection p_v

being transverse at any point of $\Delta(\phi) \cap P_v$, only singular points of $\Delta(\phi) \cap P_v$ contribute to the discriminant of p_v. Since we have ν nodes et κ cusps, the discriminant of p_v is $3\kappa + 2\nu$, because the discriminant of nodes is 2 and the discriminant of cusps is 3, so $\mu + \mu(\Delta(\phi) \cap P_v, 0) - 1 = 3\kappa + 2\nu$, which proves our proposition. $\square$

Unfortunately, so far, these are the only features on the geometry of general plane sections of the discriminant which are known.

In the case of section by 3-linear spaces near 0, in [13], some similar formulae have been obtained on the number of "deeper" singularities.

5 Open Problems

5.1 Fundamental Groups

The first unsolved problem is the computation of the fundamental group of the complement of the discriminant of a versal deformation.

At least do this computation for versal deformations of a particular class of holomorphic functions with isolated singularity. In [3], Brieskorn solved the case of simple singularities. One may consider the class of non-degenerate hypersurfaces.

One can consider the local fundamental group of the complement of the discriminant in a general plane section. This might be easier than the preceding problem, but as we saw above, the local fundamental group of the complement of the discriminant is a quotient of this latter one.

In the case of the function $z_0^{\mu+1} + \ldots + \sum_1^n z_i^2$ the general plane section of the discriminant of a versal deformation is a curve whose equisingularity type is the one of $\xi^\mu + \eta^{\mu+1}$. Since the fundamental group of the local complement of this curve is generated by two generators, we obtain that the braid group B_μ is generated by two generators.

We are lead to the following problem: determine the equisingularity class of a general plane section of the discriminant of a versal deformation.

5.2 Monodromy

The preceding problem about the fundamental group is aimed to compute the representation of this fundamental group in the n-th homology of a general fiber of a versal deformation of f. It remains to calculate this representation.

Is it possible to get the monodromy of the Milnor fibration of f from this representation?

5.3 Intersection Form

In the above example from F. Pham, the initial idea was to have an explicit way to have the intersection form on the Milnor fiber of f at 0. Is there a way to obtain the intersection form of the Milnor fiber from a presentation of the fundamental group of the complement of the discriminant of a versal deformation?

5.4 Topology of the Discriminant

A general problem is to understand the topology of the discriminant of a versal deformation. By Milnor theory, the topology of a hypersurface is essentially given by its Milnor fibration. Another natural problem is to determine the Milnor fiber of the discriminant. The geometrical monodromy of this Milnor fiber is related to the computation of the local fundamental group of the complement of the discriminant.

5.5 Complete Intersections

In the case of complete intersections. There is also a versal deformation for complete intersection with isolated singularity as it was shown by G. Tjurina ([19]).

The preceding observations can be repeated for the case of the discriminant of versal deformation of a complete intersection with isolated singularity. The main difference is that in this case the limit of tangent spaces at 0 of the discriminant is not unique. In [6], G.-M. Greuel and the author proposed a study of the number of cusps and nodes of a general plane section near 0 of the discriminant. No other studies have been made in this case of complete intersections.

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21 Years of SINGULAR Experiments in Mathematics

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Dedicated to Gert-Martin Greuel on the occasion of his 60th birthday

Abstract

This article gives some overview on SINGULAR, a computer algebra system for polynomial computations with special emphasis on the needs of commutative algebra, algebraic geometry and singularity theory, which has been developed under the guidance of G.-M. Greuel, G. Pfister and the second author [31]. We draw the bow from SINGULAR's early years to its latest features. Moreover, we present some explicit calculations, focusing on applications in singularity theory.

Introduction

By the development of effective computer algebra algorithms and of powerful computers, algebraic geometry and singularity theory (like many other disciplines of pure mathematics) have become accessible to experiments. Computer algebra may help

- to discover unexpected mathematical evidence, leading to new conjectures or theorems, later proven by traditional means,
- to construct interesting objects and determine their structure (in particular, to find counter-examples to conjectures),
- to verify negative results such as the non-existence of certain objects with prescribed invariants,
- to verify theorems whose proof is reduced to straightforward but tedious calculations,

1991 *Mathematics Subject Classification.* 13 , 14Q , 14B05, 32S10, 14E15
Key words. Computational algebraic geometr , singularit theor , computer algebra s s tem, mathematical e periments

- to solve enumerative problems, and
- to create data bases.

In fact, in the last decades, there is a growing number of research articles in algebraic geometry and singularity theory originating from explicit computations (such as [1] and [46] in this volume).

What abilities of a computer algebra system are needed to become a valuable tool for algebraic geometry and, in particular, for singularity theory? First of all, the system needs an efficient representation of polynomials with exact coefficients. Then, it should provide a fast implementation of Buchberger's Gröbner basis algorithm and its variants for standard basis computations in local rings (Mora's tangent cone algorithm, etc.). This makes available the basic ideal operations such as computing intersections and quotients of ideals and elimination ideals. Combining Gröbner (standard) bases with combinatoric evaluation leads to tools for computing (local) dimensions and multiplicities of ideals (in particular, Milnor and Tjurina numbers and intersection multiplicities), and for Hilbert functions of graded ideals.

This functionality should be extended to (graded) submodules of free modules; the system should provide commands for computing syzygies and (graded) free resolutions, and for computing cokernels of module homomorphisms (the basic constructions of homological algebra).

Based on multivariate factorization, the system should provide us with tools for computing the radical and the primary decomposition of an ideal. There should be some tools for visualization (or at least for communicating with visualization programs).

Moreover, for particular applications such as computing sheaf cohomology and direct image sheaves, or computing with D-modules, the system should be able to treat some important non-commutative structures, too.

It would be desirable to have user-written extensions (libraries) for specialized computational tasks and for automating repeated experiments.

All this is provided by the computer algebra system SINGULAR, which has been developed over the past two decades under the guidance of G.-M. Greuel, G. Pfister and the second author.

In this article, we review some history of SINGULAR, starting with the original motivation for setting up such a system, and drawing the bow to SINGULAR's latest version 3.0 (with its new features, including absolute primary decomposition, newly implemented algorithms for Gröbner basis computations, and, in particular, the ability to compute in non-commutative GR-algebras).

We explain some mathematical background for computing in local rings, focusing on the relation between computations with power series and with rational functions. By presenting explicit computations, we introduce some of the most important SINGULAR commands and libraries for applications in singularity theory. These examples include the computation of basic invariants such as the Milnor and Tjurina number or the spectrum of a hypersurface singularity as well as studying families of singularities, and the resolution of singularities.

1 The Early Years

In this section, we give some historical overview on SINGULAR's development.

1.1 First Steps

The birth of SINGULAR can be dated back to about 1982, when G.-M. Greuel and G. Pfister tried to generalize K. Saito's theorem saying that, *for a germ* $(X,0)$ *of an isolated complex hypersurface singularity, the following are equivalent:*

(a) $(X,0)$ *is quasi-homogeneous (that is, has a good* $\mathbb{C}^*$*-action).*

(b) $\mu(X,0) = \tau(X,0)$.

(c) *The Poincaré complex of* $(X,0)$ *is exact.*

Trying to extend this theorem to complete intersection curve singularities, they only succeeded in proving the equivalence of (a) and (b). They expected that (b) and (c) are, indeed, not equivalent for general complete intersection curve singularities. As the exactness of the Poincaré complex could be expressed as an equality of dimensions of $\mathcal{O}_{X,0}$-modules (see Example 2.4), there was hope that a counter-example could be found by experimenting with a computer (some trials showed that the needed computations were too hard for doing them manually). In those days, however, there was no computer algebra system available which could compute Milnor numbers, Tjurina numbers of $\mathcal{O}_{X,0}$-ideals and the dimensions of $\mathcal{O}_{X,0}$-modules (for non-trivial examples). Indeed, such a system requires the implementation of Mora's tangent cone algorithm for computing standard bases.

The first version of a standard basis algorithm (called `BuchMora`) was implemented in `BASIC` on a ZX-Spectrum by K.P. Neuendorf (born Schemmel) and G. Pfister in 1983. This implementation allowed them to compute first examples and to get some idea about the mathematics behind the examples. But, it was not yet sufficient for finding a counter-example.

The real development started in 1984 with an implementation of Mora's tangent cone algorithm in `Modula-2` on an Atari computer at the Humboldt-University in (East-)Berlin (by G. Pfister and a group of students including the second author). After a while, a list of counter-examples was produced (see [43]). At that time, the system could only compute with coefficients in a small prime field $\mathbb{F}_p$. However, the experiments showed which examples are candidates for being a counter-example and how the computations in characteristic 0 should look like. The proof was then given manually.

1.2 Learning to Speak: Language and Scripts

In the early 1990s the system's "home-town" moved to Kaiserslautern and the system was baptised Singular. A general standard basis algorithm was implemented in `C`, and Singular was ported to `Unix`, `MS-DOS`, and `MacOS`. A programming language and an interpreter were added, and the possibility to write scripts (later libraries) to obtain additional functionality by combining the building blocks provided by the system. The first draft of the language and the corresponding grammar were produced by W. Neumann, who helped converting the existing sources to `C/C++`, too. Today, you may still find some remainings of the eighties in the sources. The first publication on Singular [24] carries the names of Singular's development team in this period.

The driving force behind the development of Singular at that time was the hope to disprove *Zariski's multiplicity conjecture*: In his retiring address to the AMS [50], Zariski asked whether two complex hypersurface singularities $\{f = 0\}$ and $\{g = 0\}$ with the same embedded topological type have the same multiplicity. The statement has been known to hold for curves and for semi-quasihomogeneous singularities. Potential counter-examples have Milnor number > 1000. To make such examples accessible to practical computations, much more sophisticated strategies in the implementation of Buchberger's (resp. Mora's) algorithm than the ones used that far were needed.

After trying many examples without finding a counter-example, enough experience was gained to prove a partial positive result (see [25, Chapter 5]).

1.3 Kindergarten and Elementary School

Continuous extensions (such as univariate and multivariate polynomial factorization[1], gcd computations[1], links) and refinements led in 1997 to the release of Singular version 1.0 and in 1998 to the release of version 1.2 (with much faster implementations of standard and Gröbner bases algorithms, including a Hilbert driven version; libraries for primary decomposition, integral closure of rings, etc.).

[1] ro i e b the `Singular-Factory` librar , which was programme b R Stobbe, J Schmi t an essollen, an which is also use b `Macaulay2`

1.4 The Teenage Years

O. Bachmann contributed with a lot of computer science know-how to the project — he programmed a customized memory manager, better data representations for monomials and a lot of optimizations (minor changes with large effects). Jointly with the second author, major parts of the Singular kernel were rewritten. As a result, for most computations, Version 2.0 (released in 2001) was much faster than the previous versions of SINGULAR. This was also caused by new and better implemented algorithms (e.g., for computing resolutions and determinants, see [5, 6]). We also learned many tricks from others: geobuckets [49] were originally designed to speed up `Macaulay`, bit vector support for monomials (for speeding up divisibility tests [9]) appeared first in a paper of the `CoCoA` group.

Besides these internal changes, Singular 2.0 offered many new features and functionalities (which were partly already incorporated in the 1.3 series). For instance, a native `Windows` distribution with the usual `Windows` installer, an `Emacs` user interface, a new help system (based mainly on `HTML`), etc. Many new people started using SINGULAR with version 2.0. Besides applications in algebraic geometry and singularity theory, there were also surprising applications: see [7] for an application in group theory. Other applications came from microelectronics and from economics. For these tasks a symbolic-numerical solver was programmed into SINGULAR.

At the ISSAC 2004 conference in Santander, SINGULAR was awarded the Jenks Prize for Excellence in Software Engineering.

Version 3.0 was developed in parallel to the continuated work on the 2.0.x series. Some of its new features appeared in (inofficial) versions 2.1.x. The official release of SINGULAR 3.0 was at the MEGA conference in May 2005.

1.5 Friends

Some of the building blocks of SINGULAR have been borrowed from elsewhere:

- the Gnu Multiple Precision Library (`GMP`, `http://www.swox.com/gmp/`) provides a fast arithmetic for long integers and long floating point numbers.
- the `NTL` library of Victor Shoup (`http://shoup.net/ntl/`) is often used for operations on univariate polynomials, such as factorization and gcd computation.
- `surf` (see `http://surf.sourceforge.net/`) is used for visualizing real algebraic varieties (curves in $\mathbb{R}^2$, surfaces in $\mathbb{R}^3$ and hyperplane sections on such surfaces). It provides the drawing abilities for SINGULAR.
- some others are used by libraries (for example, `resgraph.lib` uses tools from the `graphiz` packages, see `http://www.graphviz.org/`)

2 Applications in Singularity Theory

Usually, singularity theory deals with (germs of) complex spaces rather than with (germs of) algebraic varieties. That is, we have to compute with (formal or convergent) power series rather than with polynomials or rational functions. And, we have to compute with complex coefficients rather than with coefficients in some computational field.

In a computer algebra system like SINGULAR, however, we can basically compute only in polynomial rings $K[\boldsymbol{x}] = K[x_1, \dots, x_n]$ or in specific[2] localizations thereof, where K is some computational field (such as $\mathbb{Q}$ and $\mathbb{F}_p$ or a finite or transcendental extension thereof). Nevertheless, for many questions, computations in such rings are sufficient for deducing properties of complex space germs. Some theoretical background is given in Section 2.1. In Section 2.2, we list the most important SINGULAR libraries for applications in singularity theory. Some of these are used in Section 2.3 where we turn to explicit examples.

2.1 Computing in Local Rings

The implementation of localizations of polynomial rings is based on fixing specific orderings on the monomials in $K[\boldsymbol{x}]$.

We call any semigroup ordering on the set of monomials $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\boldsymbol{\alpha} \in \mathbb{N}^n$, a *monomial ordering*. It is called a *global ordering* if $1 < x_i$ for each i, and a *local ordering* if $1 > x_i$ for each i. Otherwise, it is called a *mixed ordering*. Given a monomial order $>$ on $K[\boldsymbol{x}]$, the set $S_>$ of all polynomials $u \in K[\boldsymbol{x}]$ whose largest (or *leading*) *term* $\mathrm{L}_>(u)$ is a non-zero constant is multiplicatively closed. We define $K[\boldsymbol{x}]_>$ to be the localization of $K[\boldsymbol{x}]$ at $S_>$.

If $I \subset K[\boldsymbol{x}]_>$ is an ideal, a finite subset $\mathcal{G} = \{f_1, \dots, f_r\} \subset I$ is called a *standard basis* iff the leading terms $\mathrm{L}_>(f_i)$, $i = 1, \dots, r$, generate the *leading ideal* $\mathrm{L}_>(I) := \langle \mathrm{L}_>(f) \mid f \in I \setminus \{0\} \rangle \subset K[\boldsymbol{x}]$. Then $\mathcal{G}$ is also a generating set for I as an ideal of $K[\boldsymbol{x}]_>$. It can be computed by a variant of Buchberger's algorithm which is due to Mora (for local orderings) respectively Greuel and Pfister, and independently Gräbe (for arbitrary orderings), see [30, Section 1.7]. The corresponding SINGULAR command is `std`.

From a standard basis, we may read all information encoded in the leading ideal $\mathrm{L}_>(I)$. For instance, we may read the dimension (resp. vector space dimension) of the quotient $K[\boldsymbol{x}]_>/I$, since

$$\dim K[\boldsymbol{x}]_>/I = \dim K[\boldsymbol{x}]/L_>(I)\,, \quad \bigl(\dim_K K[\boldsymbol{x}]_>/I = \dim_K K[\boldsymbol{x}]/L_>(I)\bigr)\,.$$

The concept of standard bases extends to submodules M of free $K[\boldsymbol{x}]_>$-modules, providing us with an algorithm for computing syzygies (kernels of

[2] See 30, E ample 1 5 3 for e amples

module homomorphisms). The corresponding SINGULAR command is `syz`. Successively computing syzygies of syzygies (and minimizing the result by Gaussian elimination), we can compute a (minimal) free resolution of M (`mres`). Alternatively, make use of Schreyer's algorithm (`sres`), see [30, Section 2.5].

If $K \subset L$ is a field extension, and if I is an ideal of $L[\boldsymbol{x}]_>$ generated by polynomials $f_1, \ldots, f_r$ with coefficients in K, then the standard basis algorithm applied to $f_1, \ldots, f_r$ yields standard basis elements for I which are also defined over K. In particular, this allows us to derive information on the (local) dimension and multiplicity of complex varieties by computations over $\mathbb{Q}$ (respectively over some number field K). Similarly for information on free resolutions of finitely generated $K[\boldsymbol{x}]_>$-modules such as the Betti numbers.

Note, however, that a prime ideal of $K[\boldsymbol{x}]_>$ need not generate a prime ideal of $L[\boldsymbol{x}]_>$. From a computational point of view, this is reflected by the fact that for computing a primary decomposition, algorithms for polynomial factorization are needed in addition to standard basis techniques. In contrast to Buchberger's algorithm, the algorithms for polynomial factorization and their results are highly sensitive to the coefficient field (see Section 3.3).

In SINGULAR, it is possible to compute with ideals of $K[\boldsymbol{x}]_>$ and with finitely generated $K[\boldsymbol{x}]_>$-modules (given either by a presentation matrix or as a submodule of a free $K[\boldsymbol{x}]_>$-module with a fixed basis). In fact, computations over $K[\boldsymbol{x}]_>$ are mimicked by considering the elements of $S_>$ as units in standard basis computations which entirely take place in $K[\boldsymbol{x}]$. To emphasize this point, we say that $K[\boldsymbol{x}]_>$ is the ring *implemented by the monomial ordering* $>$. We illustrate the behavior of standard basis computations by a simple example:

Example 2.1. We consider the ideal generated by $x^2 + x = x(x+1)$ in two different rings. The ring `R` implements $\mathbb{Q}[x]$, while `S` implements $\mathbb{Q}[x]_{\langle x \rangle}$:

```
> ring R = 0, x, dp;    // global ordering: x>1
> ideal I = x2+x;
> std(I);
_[1]=x2+x
> ring S = 0, x, ds;    // local ordering: 1>x
> ideal I = x2+x;
> std(I);
_[1]=x
```

The first entry `0` in the defining list for the rings refers to the characteristic of the coefficient field, the second entry to the variables, and the third entry to the chosen monomial ordering. For `R`, we have chosen the degree reverse lexicographic ordering $>_{\mathtt{dp}}$, which is a global monomial ordering, defined by

$$\boldsymbol{x}^{\boldsymbol{\alpha}} >_{\mathtt{dp}} \boldsymbol{x}^{\boldsymbol{\beta}} :\Longleftrightarrow \deg \boldsymbol{x}^{\boldsymbol{\alpha}} > \deg \boldsymbol{x}^{\boldsymbol{\beta}} \text{ or } \quad (\deg \boldsymbol{x}^{\boldsymbol{\alpha}} = \deg \boldsymbol{x}^{\boldsymbol{\beta}} \text{ and the last non-zero entry of } \boldsymbol{\alpha} - \boldsymbol{\beta} \text{ is negative}).$$

For `S`, we have chosen the *negative degree reverse lexicographic order* $>_{\mathtt{ds}}$, which is a local monomial ordering, defined by

$$\boldsymbol{x}^{\boldsymbol{\alpha}} >_{\mathtt{ds}} \boldsymbol{x}^{\boldsymbol{\beta}} :\Longleftrightarrow \deg \boldsymbol{x}^{\boldsymbol{\alpha}} < \deg \boldsymbol{x}^{\boldsymbol{\beta}} \text{ or } \quad (\deg \boldsymbol{x}^{\boldsymbol{\alpha}} = \deg \boldsymbol{x}^{\boldsymbol{\beta}} \text{ and the last non-zero entry of } \boldsymbol{\alpha} - \boldsymbol{\beta} \text{ is negative}).$$

Together with the above observation on the role of the coefficient field, the following proposition allows us to deduce (numerical) information on complex space germs from computations over $\mathbb{Q}[\boldsymbol{x}]_{\langle \boldsymbol{x} \rangle}$. For a proof, see [30, Cor. 7.4.6].

Proposition 2.2. *Let $I \subset K[\boldsymbol{x}]_{\langle \boldsymbol{x} \rangle}$ be an ideal.*

(1) The map $K[\boldsymbol{x}]_{\langle \boldsymbol{x} \rangle}/I \to K[[\boldsymbol{x}]]/IK[[\boldsymbol{x}]]$ induced by the natural inclusion $K[\boldsymbol{x}]_{\langle \boldsymbol{x} \rangle} \subset K[[\boldsymbol{x}]]$ is a faithfully flat injection. In particular, a sequence $0 \to M' \to M \to M'' \to 0$ of $K[\boldsymbol{x}]_{\langle \boldsymbol{x} \rangle}/I$-modules is exact iff the induced sequence of $K[[\boldsymbol{x}]]/IK[[\boldsymbol{x}]]$-modules is exact.

(2) If $K[\boldsymbol{x}]_{\langle \boldsymbol{x} \rangle}/I$ is a finite dimensional K-vector space, then the inclusion $K[\boldsymbol{x}]_{\langle \boldsymbol{x} \rangle}/I \subset K[[\boldsymbol{x}]]/IK[[\boldsymbol{x}]]$ is an isomorphism of local K-algebras. In particular, both vector spaces have the same dimension and a common basis represented by monomials.

If $K = \mathbb{R}$ or $\mathbb{C}$, the analogous statements hold for $K\{\boldsymbol{x}\}$ in place of $K[[\boldsymbol{x}]]$.

Remark 2.3. Of course, for some questions we cannot get around computing with power series. A typical example is the factorization problem in $K[[x_1, \ldots, x_n]]$. We do not know any system which could effectively solve this problem for $n \geq 3$ variables. [3]

For more details and for a careful introduction into Singular and its programming language, we refer to [30, 12].

2.2 Libraries for Singularity Theory

The Singular package comes with many user-written libraries which are valuable for applications in singularity theory. When turning to explicit computations in the next section, we will apply several commands which are provided by such libraries. Here, we just give an overview on the functionality:

[3]For 2, the `hnexpansion` comman from `hnoether.lib` computes implicitl an irre ucible factori ation in $\overline{\mathbb{Q}}$ $\boldsymbol{x}$, $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$

Library	**Purpose**
`sing.lib`	Computing invariants of singularities
`classify.lib`	Arnold's classification of singularities
`mondromy.lib`	Compute the monodromy of an isolated hypersurface singularity
`gmssing.lib`	Invariants related to the Gauß-Manin system of an isolated hypersurface singularity
`gmspoly.lib`	Invariants related to the Gauß-Manin system of a cohomologically tame polynomial
`primdec.lib`	Algorithms for computing (absolute) primary decomposition and radical of ideals
`mprimdec.lib`	Primary decomposition of modules
`normal.lib`	Normalization of affine rings, geometric genus of projective curves
`resolve.lib`	Resolution of singularities
`reszeta.lib`	Applications of resolution of singularities (intersection form, Denef-Loeser zeta function)
`resgraph.lib`	Display tree of charts of the resolution
`deform.lib`	Computing miniversal deformations
`hnoether.lib`	Hamburger-Noether (Puiseux[4]) expansion of reduced plane curve singularities
`equising.lib`	Equisingularity ideal and equisingular strata of (families of) plane curve singularities
`spcurve.lib`	Deformations and invariants of Cohen-Macaulay codimension 2 singularities
`qhmoduli.lib`	Moduli spaces of semi-quasihomogeneous isolated hypersurface singularities
`finvar.lib`	Compute invariant rings of finite groups
`brnoeth.lib`	Brill-Noether algorithm for solving the Riemann-Roch problem (for plane curves), Weierstraß semigroup, and applications to AG-codes

2.3 Examples

The first example makes us return to SINGULAR's origin. It is one example showing that the exactness of the Poincaré complex

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_{C,0} \xrightarrow{d} \Omega^1_{C,0} \xrightarrow{d} \Omega^2_{C,0} \xrightarrow{d} \Omega^3_{C,0} \longrightarrow 0$$

of a complete intersection curve singularity $(C, 0)$ does not imply that the curve singularity is quasihomogeneous.

[4] he concept of amburger Noether e pansions replaces the concept of uiseu e pan sions for coefficient el s of positi e characteristic n characteristic 0, a uiseu e pansion can be e uce from a amburger Noether e pansion b a coor inate change of t pe $t \mapsto tu^{1/m}$, $u \in K$ t a unit

Example 2.4. Let $f = xy+z^4$ and $g = xz+y^5+yz^2$, and let $(C,0) \subset (\mathbb{C}^3,0)$ be defined by $f = g = 0$. We use SINGULAR to compute the Tjurina number

$$\tau(C,0) = \dim_{\mathbb{C}} T^1_{(C,0)} = \dim_{\mathbb{C}} \mathbb{C}\{x,y,z\}/\langle f,g,M_1,M_2,M_3\rangle\,,$$

where M_1, M_2, M_3 are the 2–minors of the Jacobian matrix of f and g:

```
> ring R = 0, (x,y,z), ds;    // compute in Q[x,y,z]_<x,y,z>
> poly f, g = xy+z4, xz+y5+yz2;
> ideal I = f, g;
> matrix J = jacob(I);        // Jacobian matrix
> ideal Tjur = I, minor(J,2);
> vdim(std(Tjur));            // compute K-dimension of R/Tjur
12
```

Alternatively, use the built-in command `tjurina` from `sing.lib`.

```
> LIB "sing.lib";
> tjurina(I);
12
```

Comparing it with the Milnor number ([26, Korollar 5.5], see also [41])

$$\begin{aligned}\mu(C,0) &= \dim_{\mathbb{C}} \Omega^1_{C,0}/d\mathcal{O}_{C,0} \\ &= \dim_{\mathbb{C}} \mathbb{C}\{x,y,z\}/\langle f,M_1,M_2,M_3\rangle - \dim_{\mathbb{C}} \mathbb{C}\{x,y,z\}/\left\langle \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}, \tfrac{\partial f}{\partial z}\right\rangle,\end{aligned}$$

we see that $(C,0)$ is not quasihomogeneous[5]:

```
> milnor(I);                  // from sing.lib
13
```

However, the Poincaré complex is exact. Indeed, as shown in [29], it is sufficient to check that $\mu(C,0) = \dim_{\mathbb{C}} \Omega^2_{C,0} - \dim_{\mathbb{C}} \Omega^3_{C,0}$. Note that $\dim_{\mathbb{C}} \Omega^3_{C,0} = 1$. Moreover, $\Omega^2_{C,0} = \Omega^2_{\mathbb{C}^3,0}\big/\big(\langle f,g\rangle \Omega^2_{\mathbb{C}^3,0} + df \wedge \Omega^1_{\mathbb{C}^3,0} + dg \wedge \Omega^1_{\mathbb{C}^3,0}\big)$ is isomorphic to $\mathcal{O}^3_{C,0}/M$, where $M \subset \mathcal{O}^3_{C,0}$ is generated by the six vectors

$$\big(\tfrac{\partial f}{\partial y}, \tfrac{\partial f}{\partial z}, 0\big),\ \big(\tfrac{\partial f}{\partial x}, 0, -\tfrac{\partial f}{\partial z}\big),\ \big(0, \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}\big),\ \big(\tfrac{\partial g}{\partial y}, \tfrac{\partial g}{\partial z}, 0\big),\ \big(\tfrac{\partial g}{\partial x}, 0, -\tfrac{\partial g}{\partial z}\big),\ \big(0, \tfrac{\partial g}{\partial x}, \tfrac{\partial g}{\partial y}\big):$$

```
> qring Q = std(I);       // quotient ring Q=R/I
> poly f = imap(R,f);     // map f from R to Q
> poly g = imap(R,g);
> module M = [diff(f,y),diff(f,z),0], [diff(f,x),0,-diff(f,z)],
.            [0,diff(f,x),diff(f,y)], [diff(g,y),diff(g,z),0],
.            [diff(g,x),0,-diff(g,z)],[0,diff(g,x),diff(g,y)];
> vdim(std(M));
14
```

[5] Generali ing Saito's result for isolate h persurface singularities, it is shown in 29 that $C,0$ is uasihomogeneous i $\mu\ C,0$ $\tau\ C,0$

From the output, we read that $\dim_{\mathbb{C}} \Omega^2_{C,0} = 14 = \mu(C,0) + \dim_{\mathbb{C}} \Omega^3_{C,0}$. □

The next example deals with determining the number of singularities of a given projective hypersurface:

Example 2.5. We use SINGULAR to check that the septic surface constructed by O. Labs and D. van Straten in [37] (see also the preamble of this proceedings volume) has, indeed, 99 nodes:

```
> ring R = (0,a), (x,y,w,z), dp; //the surface is defined over a
> minpoly = 7*a^3+7*a+1;          //primitive field extension of Q
> number a(1) = -12/7*a^2 - 384/49*a - 8/7;
> number a(2) = -32/7*a^2 + 24/49*a - 4;
> number a(3) = -4*a^2 + 24/49*a - 4;
> number a(4) = -8/7*a^2 + 8/49*a - 8/7;
> number a(5) = 49*a^2 - 7*a + 50;
> poly f = ((z+w)*(x2+y2)+a(1)*z3+a(2)*z2w+a(3)*zw2+a(4)*w3)^2*
.           (z+a(5)*w) - (x7-21x5y2+35x3y4-7xy6+7z*(x2+y2)^3
.                        -56z3*(x2+y2)^2+112z5*(x2+y2)-64z7);
> ideal I = jacob(f);             //f homogeneous => f in jacob(f)
> option(redSB);
> ideal J=std(I);
> degree(J);
// dimension (proj.)  = 0
// degree (proj.)   = 99
```

From the output, we read that, counted with multiplicity, the surface has 99 singularities. To get some feeling for the size of J, we make SINGULAR print the number of generators and the number of digits and letters used when displaying J:

```
> size(J);
73
> size(string(J));
208101
```

To convince ourselves that all singularities are simple nodes, we compute the non-nodal locus of the septic in $\mathbb{P}^3(\mathbb{C})$:

```
> matrix Hessian = jacob(jacob(f));      //Hessian matrix
> ideal NonNodal = minor(Hessian,2), J;
> NonNodal = std(NonNodal);
> degree(NonNodal);
// dimension (affine) = 0
// degree (affine)  = 810
```

From the output, we read that the non-nodal locus of the affice cone over C consists only of the origin. Thus, the non-nodal locus in $\mathbb{P}^3$ is empty. Note that these computations take only a few seconds. On the other hand, trying to compute the singular locus in the affine charts takes much longer (see [37] for such computations[6]). □

Our third example deals with families of singularities. We use SINGULAR to stratify the base space of the miniversal deformation of a simple hypersurface singularity with respect to the Tjurina number. Such a stratification may be used, for instance, for explicitly determining the adjacencies (possible deformations) of the given singularity.

The stratification problem can also be treated for Cohen-Macaulay codimension 2 singularities using SINGULAR. In this case, however, a modification of the standard basis algorithm is required. See [18] for details and, in particular, for the obtained adjacency diagram of simple isolated complete intersection space curve singularities (SINGULAR experiments helped to complete the previously known results by Giusti and Goryunov).

Example 2.6. For simplicity reasons, we consider the case of an A_k-singularity (we choose $k = 3$). To compute the miniversal deformation, we could use `versal` from `deform.lib`. However, since we know the result, we prefer to proceed as follows:

```
> ring R = 0, (x,y), ds;
> int k = 3;
> poly f = x^(k+1)+y^2;
> ideal Tjur = f, jacob(f);   //Tjurina ideal
> def kb = kbase(std(Tjur));  //vector space basis of R/Tjur
> ring Ra = 0, (x,y,a(1..k)), (ds(2),dp);
> def kb = imap(R,kb);
> poly F = imap(R,f);
> for (int i=1; i<=k; i++) { F = F+a(i)*kb[i]; }
> F;                          //the miniversal family
a(3)+x*a(2)+x^2*a(1)+y^2+x^4
```

Next, we compute a presentation matrix for the relative T^1 of the miniversal family as $\mathbb{Q}[a_1, \ldots, a_k]$-module (making use of [38, Proposition 4.5]):

```
> ring R1 = 0, (x,y,a(1..k)), (dp(2),dp); //need special ordering
> poly F = imap(Ra,F);
```

[6] f we ust want to chec whether the result is plausible, we coul compute o er a sufficientl large nite el , sa K $\mathbb{F}_{34511}$, where the pol nomial $7a^3 + 7a + 1$ has a root here, 17 Usuall , the result obtaine o er such a el coinci es with the result obtaine in characteristic 0, an the computations are much faster

```
> ideal jF = diff(F,x), diff(F,y);
> option(redSB);              //compute reduced Groebner bases
> jF = std(jF);
> matrix PresT1[k][k];
> for (i=1; i<=k; i++) {
.   PresT1[i,1..k] = transpose(coef(reduce(x^(i-1)*F,jF),xy))[2];
. }
```

Finally, we use the procedure `flatteningStrat` from `homolog.lib` to compute the strata of constant Tjurina number:

```
> LIB "homolog.lib";
> flatteningStrat(PresT1);
[1]:
   _[1]=4*a(1)^3*a(2)^2-16*a(1)^4*a(3)+27*a(2)^4
        -144*a(1)*a(2)^2*a(3)+128*a(1)^2*a(3)^2-256*a(3)^3
[2]:
   _[1]=9*a(2)^3-32*a(1)*a(2)*a(3)
   _[2]=3*a(1)*a(2)^2-8*a(1)^2*a(3)+32*a(3)^2
   _[3]=a(1)^2*a(2)+12*a(2)*a(3)
   _[4]=2*a(1)^3+9*a(2)^2-8*a(1)*a(3)
[3]:
   _[1]=a(3)
   _[2]=a(2)
   _[3]=a(1)
```

As we might have expected, we get that the Tjurina number is 1 along a hypersurface (having a swallowtail singularity), there is a 1-dimensional stratum in the base of the miniversal family over which each fibre has Tjurina number 2 (the singular locus of the swallowtail) and Tjurina number 3 holds precisely over the origin. If we consider the fibre over the point $(1, 0, \frac{1}{4})$, resp. over $(-\frac{3}{2}, 1, -\frac{3}{16})$, which both belong to the 1-dimensional stratum with Tjurina number 2, we see that the A_3-singularity is splitted up into two nodes, resp. deformed to an A_2-singularity.

```
> map phi = R1,x,y,1,0,1/4;
> primdecGTZ(slocus(phi(F))); //primary decomp. of singular locus
[1]:
   [1]:
      _[1]=2*x^2+1
      _[2]=y
   [2]:
      _[1]=2*x^2+1
      _[2]=y
> map psi = R1,x,y,-3/2,1,-3/16;
```

```
> primdecGTZ(slocus(psi(F)));
[1]:
   [1]:
      _[1]=4*x^2-4*x+1
      _[2]=y
   [2]:
      _[1]=2*x-1
      _[2]=y
```
□

The final example deals with resolution of singularities and the computation of spectral numbers. Here, we make use of the libraries `resolve.lib`, `reszeta.lib` and `gmssing.lib` which belong to the new features of SINGULAR 3.0 (see [20], [21], respectively [44], [45], for more on the algorithms implemented in these libraries):

Example 2.7. We compute the resolution of an isolated surface singularity (one of the examples in [20]):

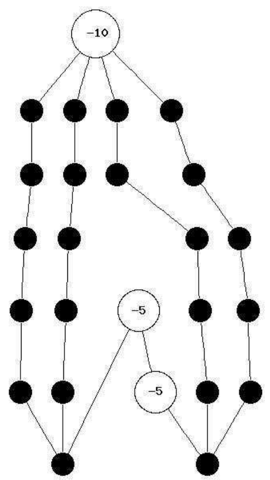
```
> LIB "resolve.lib";
> LIB "reszeta.lib";
> LIB "resgraph.lib";
> ring R = 0, (x,y,z), dp;
> poly f = x2y2z2+x7+y8+z8;
> list L = resolve(f);       //compute resolution of singularity
> size(L[1]); size(L[2]);
44
90
```

The list `L` consists of two list of rings. The first list of rings collects all information on the resolution in the 44 final charts (where the singularity is resolved). The second list collects all information on intermediate results of the resolution process. To get an overview on the resolution process, we use the `Restree` command (from `resgraph.lib`). At this writing, this command calls the external programs `dot` and `xv` (not included in the SINGULAR package) to produce a picture visualizing the tree of all the 90 charts considered in the resolution process. Before applying `Restree`, we have to use `collectDiv` (from `reszeta.lib`) to identify the exceptional divisors in different charts:

```
> list identED = collectDiv(L);
> ResTree(L,identED[1]);
```

We do not print the returned tree of charts here as it would be unreadable due to the limited amount of space available. Instead, we compute the intersection form and the genera of the exceptional divisors. Instead of printing the intersection matrix, we use `InterDiv` from `resgraph.lib` (which calls `dot` and `xv`) to get a graphic visualization:

```
> list iD = intersectionDiv(L);
> iD[2];            // genera of exceptional divisors
0,0,3,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0
> InterDiv(iD[1]);
```



In the diagram, each filled circle corresponds to an exceptional divisor with self intersection -2. Otherwise, the self intersection is displayed. The circles are joined by a line iff the corresponding exceptional divisors meet.

Next, we compute the Denef-Loeser zeta function:

```
> zetaDL(L,1);
[1]:
   (1120s3+1413s2+618s+98)/(784s4+1960s3+1764s2+686s+98)
```

To compute the spectral numbers (and their multiplicities) of the isolated surface singularity defined by f, we use `spectrum` from `gmssing.lib`:

```
> LIB "gmssing.lib";
> ring Rloc = 0, (x,y,z), ds;
> poly f = imap(R,f);
> list Sp = spectrum(f);
> size(Sp[1]); //number of pairwise different spectral numbers
70
> def SN = Sp[1]; SN[1..11];   //the first 11 spectral numbers
-1/2 -3/8 -5/14 -1/4 -13/56 -3/14 -1/8 -3/28 -5/56 -1/14 0
> def SM = Sp[2]; SM[1..11];   //and their multiplicities
1 2 1 3 2 1 4 2 2 1 6
```

As the numbers in `Sp[1]` are ordered by size, we see that there are precisely 10 pairwise different negative spectral numbers. The negative spectral numbers could also have been computed by applying the command `spectralNeg` (from `reszeta.lib`) to `L` in the ring `R`. □

For more examples of SINGULAR applications, see, for instance, [30], [12], [27], [28], [13], [42] or SINGULAR's home `http://www.singular.uni-kl.de`.

3 The Latest Features: SINGULAR 3.0

In the remaining part of this article, we shortly discuss some of the most important new features of SINGULAR 3.0 : non-commutative computations over GR-algebras, additional implementations of Gröbner basis algorithms, absolute primary decomposition, and the concept of dynamic modules (and name spaces).

3.1 Non-Commutative GR-Algebras

The SINGULAR kernel component PLURAL[7] allows us to compute Gröbner bases and syzygies over a large class of non-commutative algebras to which we refer as *GR-algebras* (here, GR stands for Gröbner-ready). GR-algebras are obtained from the free non-commutative algebra on $x_1, \ldots, x_n$ by imposing specific relations. We write $K\langle \boldsymbol{x}\rangle = K\langle x_1, \ldots, x_n\rangle$ for this free algebra. That is, $K\langle \boldsymbol{x}\rangle$ is the non-commutative graded K-algebra with K-vector space basis

$$\mathscr{B} = \left\{ x_{i_1} x_{i_2} \cdots x_{i_\nu} \,\middle|\, \nu \in \mathbb{N},\, 1 \leq i_\ell \leq n \text{ for all } \ell \right\},$$

where multiplication and grading are defined in the obvious way. We refer to the elements of $\mathscr{B}$ as *words*. Moreover, we set

$$\mathscr{M} := \left\{ \boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \boldsymbol{\alpha} \in \mathbb{N}^n \right\} \subset \mathscr{B}.$$

Since $\mathscr{M}$ can be identified with the set of monomials in $K[\boldsymbol{x}]$, each monomial order $>$ on $K[\boldsymbol{x}]$ induces a total order on $\mathscr{M}$ which we again denote by $>$. It, thus, makes sense to speak of the *leading term* $\mathrm{L}(h) = \mathrm{L}_>(h)$ of a K-linear combination h of words in $\mathscr{M}$.

Definition 3.1. A *G-algebra* R is the quotient of $K\langle \boldsymbol{x}\rangle$ by a two-sided ideal J_0 generated by elements of type

$$x_j x_i - c_{ij} x_i x_j - h_{ij}\,, \quad 1 \leq i < j \leq n\,, \tag{1}$$

where the c_{ij} are non-zero scalars in K, and where the h_{ij} are K-linear combinations of words in $\mathscr{M}$. Further, we require that

(G1) $c_{ik}c_{jk}h_{ij}x_k - x_k h_{ij} + c_{jk}x_j h_{ik} - c_{ij}h_{ik}x_j + h_{jk}x_i - c_{ij}c_{ik}x_i h_{jk} = 0$

for all $1 \leq i < j < k \leq n$, and that

[7] he name `PLURAL` results from the ob ious wor pla

(G2) there is a global monomial order $>$ on $K[\boldsymbol{x}]$ such that $x_i x_j > \mathrm{L}(h_{ij})$ for all i, j.

Each global monomial order on $K[\boldsymbol{x}]$ satisfying (G2) is called an *admissible monomial order* for R. We refer to the elements of $\mathscr{M}$ as *monomials* in R.

Note that the "rewriting relations" (1) together with (G2) imply that each element of R can be represented by a K-linear combination of monomials. More precisely, successively rewriting a word $w = x_{i_1} x_{i_2} \cdots x_{i_\nu}$ according to the relations (1), we get $w \equiv c\boldsymbol{x}^{\boldsymbol{\alpha}} + h \bmod J_0$, where $c \in K \setminus \{0\}$, $\boldsymbol{x}^{\boldsymbol{\alpha}}$ is the monomial obtained by rearranging the letters of w, and where h is a K-linear combination of monomials such that $\mathrm{L}(h) < \boldsymbol{x}^{\boldsymbol{\alpha}}$. The condition (G1) guarantees that this representation is uniquely determined. Indeed, for $i < j < k$, it guarantees that the result obtained by rewriting $x_k x_j x_i$ in terms of monomials does not depend on whether we first apply the rewriting relation for the pair (j, k) or for the pair (i, j). As a consequence, we get that $\mathscr{M}$ is a K-vector space basis for R (see [40]).

The latter observations allow us to extend the theory of Gröbner bases for ideals and modules over polynomial rings to a theory of left (right) Gröbner bases for left (right) ideals and modules over G-algebras. Also, division with remainder (normal forms) and Buchberger's algorithm can be extended to G-algebras. And, we may compute two-sided (that is, left and right) Gröbner bases for two-sided ideals, which allows us to implement quotients of G-algebras by two-sided ideals:

Definition 3.2. A *GR-algebra* A is the quotient $A = R/J$ of a G-algebra R by a two-sided ideal $J \subset R$. □

Examples of G-algebras include quasi-commutative polynomial rings (for example, the quantum plane with $yx = q \cdot xy$), universal enveloping algebras of finite dimensional Lie algebras [4, 39], positive (negative) parts of quantized enveloping algebras [36], some iterated Ore extensions, some non-standard quantum deformations [32, 33], Weyl algebras and quantizations of Weyl algebras, Witten's deformation of $U(\mathfrak{sl}_2)$, Smith algebras, conformal $\mathfrak{sl}_2$–algebras [8], some diffusion algebras [34] and many other.

Among the GR-algebras, you find exterior algebras, Clifford algebras, finite dimensional associative algebras given by structure constants [14] and many more.

Remark 3.3. At this writing, the implemented algorithms for non-commutative computations over GR-algebras allow us to compute:

- left normal forms and left Gröbner bases for left ideals/modules given by a finite set of generators (`reduce`, `std`);
- left syzygies and free resolutions of left ideals/modules (`syz`, `mres`);

- right normal forms, Gröbner bases, syzygies and free resolutions (by implementing the opposite algebra via `opp`, `opposite`);
- (left) Gröbner bases for two-sided ideals (modules) given by a finite set of generators (`twostd`);
- preimages of ideals under ring maps (`preimage`);
- intersection and quotients of ideals and modules (`intersect`, `modulo`);
- central elements and centralizers of elements (`center.lib`);
- central character decomposition of a module (`ncdecomp.lib`);
- Gelfand-Kirillov dimension (`gkdim.lib`).

Based on Singular's functionality for non-commutative computations, the library `sheafcoh.lib` provides commands for computing the cohomology of coherent sheaves via free resolutions over the exterior algebra[8]. Implementations of algorithms for computing direct image sheaves will be available soon. Further, a library for computations with D-modules is projected.

3.2 Gröbner Basis Computations

Gröbner Bases via slimgb. The study of J.-C. Faugère's F4 algorithm [17] for computing Gröbner bases of homogeneous ideals led to the development and implementation in Singular of a new variant of Buchberger's algorithm which is accessible via the `slimgb` command.

The F4 algorithm relies on a matrix representation for the ideal under consideration and a "structured" Gauß algorithm which tries to preserve the sparseness of the matrix in the process. Similarly[9], `slimgb` is based on an algorithm which is specifically designed to reduce the "weighted length" of the intermediate polynomials produced on its way (taking into account the size of the coefficients and the number of terms).

In implementing the algorithm, much experience and many tricks from the previous implementations of Gröbner basis algorithms in Singular have been used: for instance, the polynomial representation [6], geobuckets [49], and bit vector support for monomials (speeding up divisibility tests [9]).

It is still an experimental feature — we have to learn more about improving the applied strategies. However, timings on several benchmark examples

At this writing, custom built, fast implementations speciali ing on particular G an GR algebras such as the We l algebra or the e terior algebra are still missing in Singular As a result, the Singular implementation of the algorithm of Eisenbu , Flø sta , an Schre er 15 cannot et compete with its `Macaulay2` implementation

o ha e a fast implementation of the F4 algorithm itself woul re uire an implementation of its fast matri representation an , thus, a rewriting of most of the pol nomial arithmetic in Singular's ernel are very promising. At this writing, we can already say that typically the use of `slimgb` instead of `std` is advisable for Gröbner basis computations over transcendental extensions of prime fields.

It is planned to integrate the newly implemented algorithm into the Gröbner basis engine of SINGULAR's kernel, and to make use of variants of it for elimination and syzygy computations, too.

Gröbner Walk Algorithms. It is well-known that the performance of Buchberger's algorithm is sensitive to the choice of monomial order. A Gröbner basis computation with respect to a less favorable order such as the lexicographic order may easily run out of time or memory even in cases where a Gröbner basis computation with respect to a more efficient order such as $>_{\mathtt{dp}}$ is very well feasible. Gröbner walk (or, more generally, Gröbner basis conversion) algorithms take their cue from this observation.

The basic idea of a Gröbner walk conversion algorithm is to approach the target Gröbner basis in several steps, "walking" along a path through the Gröbner fan. In each step, a Gröbner basis computation with respect to an "intermediate" monomial order is performed. There are several strategies for choosing the path through the Gröbner fan, leading to different variants of the algorithm. See [11], [3], [2], and [48] for details.

At this writing, the SINGULAR implementation of the Gröbner walk algorithms is still affected in its efficiency by the internal limitations on weight vectors (while walking, each intermediate monomial order is defined as an order with an extra weight vector). Nevertheless, the commands provided by the library `grwalk.lib`, and the kernel command `frwalk` often yield a result in cases where a direct Gröbner basis computation fails.

Janet Bases. Let $\mathscr{M}$ be the set of all monomials in $K[\boldsymbol{x}] = K[x_1, \ldots, x_n]$, K any field, and let $\mathscr{F} \subset \mathscr{M}$ be a finite subset. Janet [35] introduced an (involutive) division by elements of $\mathscr{F}$: A monomial $\boldsymbol{x}^{\boldsymbol{\beta}} \in \mathscr{F}$ is called a *Janet divisor* of a term $c\boldsymbol{x}^{\boldsymbol{\alpha}}$ (w.r.t. $\mathscr{F}$) if $\boldsymbol{x}^{\boldsymbol{\beta}}$ divides $c\boldsymbol{x}^{\boldsymbol{\alpha}}$ and if the quotient $\boldsymbol{x}^{\boldsymbol{\alpha}}/\boldsymbol{x}^{\boldsymbol{\beta}}$ is a product of multiplicative variables with respect to $\boldsymbol{x}^{\boldsymbol{\beta}}$ in $\mathscr{F}$. Here, a variable x_i is called a *multiplicative variable with respect to* $\boldsymbol{x}^{\boldsymbol{\beta}}$ *in* $\mathscr{F}$ if for each $\boldsymbol{x}^{\boldsymbol{\gamma}} \in \mathscr{F}$ we have $\gamma_1 = \beta_1, \ \ldots\ , \gamma_{i-1} = \beta_{i-1}$ and $\gamma_i \leq \beta_i$. We write $x_i \in \mathrm{Mult}(\boldsymbol{x}^{\boldsymbol{\beta}}, \mathscr{F})$. Otherwise, x_i is called *non-multiplicative* (with respect to $\boldsymbol{x}^{\boldsymbol{\beta}}$ in $\mathscr{F}$).

Fixing a global monomial ordering $>$ on $K[\boldsymbol{x}]$, this concept of divisibility leads to a notion of division with remainder and of a Janet normal form: Let $\mathcal{J} = \{f_1, \ldots, f_r\} \subset K[\boldsymbol{x}]$ be a finite set, and let $f \in K[\boldsymbol{x}]$. Successively (Janet) dividing f by elements of $\mathcal{J}$, we get a remainder h such that no term of h has a Janet divisor in the set $\mathrm{LM}_>(\mathscr{F}) = \{\mathrm{LM}_>(f) \mid f \in \mathscr{F}\} \subset \mathscr{M}$.[10]

[1] ere, L $_>$ f enotes the *leading monomial* of f, that is, L$_>$ f cL $_>$ f for some scalar $c \in K$

We refer to h as a *Janet normal form* for f with respect to $\mathcal{J}$ and we write $h = \mathrm{NF}_J(f \mid \mathcal{J})$.

Definition 3.4. A finite set $\mathcal{J} \subset K[\boldsymbol{x}]$ is called a *Janet basis* (for the ideal $I \subset K[\boldsymbol{x}]$ generated by $\mathcal{J}$) if for each $f \in \mathcal{J}$ we have

(a) $\mathrm{NF}_J(f \cdot x_i, \mathcal{J}) = 0$ for each $x_i \notin \mathrm{Mult}\big(\mathrm{LM}_>(f), \mathrm{LM}_>(\mathcal{J})\big)$, and
(b) no term of f has a Janet divisor in the set $\mathrm{LM}_>(\mathcal{J}) \setminus \{\mathrm{LM}_>(f)\}$.

A Janet basis $\mathcal{J}$ is called *minimal* if each other Janet basis $\mathcal{J}'$ for I satisfies $\mathrm{LM}_>(\mathcal{J}) \subset \mathrm{LM}_>(\mathcal{J}')$.

Note that each Janet basis for I is also a Gröbner basis for I (with respect to $>$). But, not each Gröbner basis is a Janet basis. Also, a minimal Janet basis is usually far from being a minimal (reduced) Gröbner basis. However, a minimal (reduced) Gröbner basis can easily be computed from a Janet basis by removing the irrelevant generators.

Singular provides an implementation of the algorithm by V.P. Gerdt, Y.A. Blinkov and D.A. Yanovich [22, 23] for computing a minimal Janet basis. It is accessible by the `janet` command:

```
> ring R = 0, (x,y), dp;
> ideal J = x2y,x5;
> janet(J);   //compute minimal Janet basis
Length of Janet basis: 4
_[1]=x2y
_[2]=x3y
_[3]=x4y
_[4]=x5
> janet(J,1); //compute minimal Groebner basis via a Janet basis
Length of Janet basis: 4
_[1]=x2y
_[2]=x5
```

The computation of a Janet basis allows no choices in the selection of polynomials when reducing newly generated elements. It may avoid the intermediate expression swell which frequently appears when computing a Gröbner basis by Buchberger's algorithm. However, one can hardly predict when computing a minimal Gröbner basis via a Janet basis is superior to computing it by a variant of Buchberger's algorithm.

3.3 Absolute Primary Decomposition

The new Singular library `absfact.lib` by G. Lecerf provides the command `absFactorize` for absolutely factorizing a multivariate polynomial $f \in \mathbb{Q}[\boldsymbol{x}]$, that is, for computing its factorization over the algebraic closure $\overline{\mathbb{Q}}$. The underlying algorithm first factorizes f over $\mathbb{Q}$, say $f = f_1 \cdot \dots \cdot f_s$ (applying the `factorize` command). Then, it uses Trager's [47] idea to compute an absolutely irreducible factor of f_i by factorizing over some finite extension field L of $\mathbb{Q}$ which is chosen such that $\{f = 0\}$ has a smooth point with coordinates in L. Finally, a minimal extension field is determined making use of the Rothstein-Trager partial fraction decomposition algorithm (see [10] for more on this).

Relying on `absFactorize`, the library `primdec.lib` provides the new command `absPrimdecGTZ` for computing an absolute primary decomposition of an ideal $I \subset \mathbb{Q}[\boldsymbol{x}]$.

Absolute primary decomposition is used, for instance, in the context of resolution of singularities when computing the intersection matrix for the exceptional divisors (see Example 2.7).

3.4 Dynamic Modules

Besides writing libraries, a user has now an alternative way of adding new functionality to SINGULAR: writing a *dynamic module*.[11] In contrast to a library, which collects procedures written in the SINGULAR programming language, a dynamic module consists of procedures written in `C/C++`. A procedure in a dynamic module is not parsed each time the corresponding command is executed, but just once at compile time. This provides a great improvement in speed if the procedure contains loops with a large number of commands to be executed.

There are several reasons for using a dynamic module. For instance, one should use a dynamic module if one wants to

- access internal functions and data structures of SINGULAR's kernel;
- speed up a small, time critical routine;
- have access to external `C/C++` libraries or to other programs;
- separate large but rarely used program parts from SINGULAR's kernel (to keep the kernel of moderate size).
- replace some routines of SINGULAR's kernel by other routines (e.g., for testing different implementations, debugging, etc.).

An important example for the use of dynamic modules (by SINGULAR's programmers) are the basic polynomial operations. For the 15 most important routines, there exist 1651 implementations which are collected in four

[11] he implementation of namic mo ules is base on so calle " namic libraries" an the abilities of the namic lin er of the s stem

dynamic modules (`p_Procs_FieldGeneral.so`, `p_Procs_FieldIndep.so`, `p_Procs_FieldQ.so` and `p_Procs_FieldZp.so`). When defining a basering, Singular decides which of the modules has to be linked to the kernel.

Of course, writing a dynamic module is not as simple as writing a Singular library. Besides the knowledge of programming in `C/C++`, it requires also some knowledge about the way the Singular kernel works. We refer to [19] for information on how to write a dynamic module.

The user defined dynamic modules appear like a library. They can be loaded into a Singular session by applying the `load` command. Name conflicts are avoided by encapsulating all identifiers of a module into a separate *name space*.

Example 3.5. An example of a user written dynamic module is `kstd.so`, which provides the command `Kstd` for computing a partial standard basis (that is, in the process of computing a standard basis for a module all those vectors whose leading term involves `gen(i)`, $i \geq k$, for some given bound k, are ignored when forming the critical pairs: see [18] for details). It is based on a routine of the Singular kernel (used to compute syzygies, see [25]) which is not available via the interpreter.

Below, we print the important part of the module declaration file for `kstd.so`. It defines a procedure `kstd` which takes a `module h1` and an `int k` and, after checking the type of the arguments and performing some technical operations, it calls the standard basis engine via `kStd` with the appropriate arguments and returns the result to the interpreter (see [19] for details):

```
%{
#include "ideals.h"
#include "ring.h"
#include "kstd1.h"
#include "prCopy.h"
%}
package="kstd";
%procedures
module kstd ( module h1, int k)
{
  %declaration;
  ideal s_h1; ideal s_h3; int j;
  ring orig_ring; ring syz_ring;
  %typecheck;
  ....
  s_h3 = kStd(s_h1,NULL,testHomog,&w,NULL,k);
  ....
  %return = (void *)s_h3;
}
```

```
%C
```

The following SINGULAR session shows `kstd` at work:

```
> load("kstd.so");
> ring R = 0, x, dp;
> module M = [x,0], [x+1,x2+x], [0,x2];
> print(M);
x,x+1, 0,
0,x2+x,x2
> print(Kstd::kstd(M,1));      //partial standard basis
1,0, x,
x,x2,0
> print(std(M));               //(minimal) standard basis
x,1,
0,x
```

□

Acknowledgements

Development of the system SINGULAR was supported by the DFG and the Stiftung Rheinland-Pfalz für Innovation, which we kindly acknowledge.

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The Patchworking Construction in Tropical Enumerative Geometry

Eugenii Shustin

Dedicated to Gert-Martin Greuel on the occasion of his 60th birthday

Abstract

We prove two new patchworking theorems which describe deformations of algebraic curves inscribed into a family of algebraic surfaces such that the central surface is reducible and a general surface is irreducible. These theorems justify all known applications of tropical geometry to the enumeration of real and complex nodal curves on toric surfaces. They may serve for similar applications to the enumeration of curves with more complicated singularities. In addition, using the patchworking techniques, we classify certain planar deformations of non-planar curve singularities which appear as an element of the tropical approach to the enumeration of curves on surfaces.

1 Introduction

The rapid development of tropical algebraic geometry over the last years has led to interesting applications of singular algebraic curves in enumerative geometry, proposed by Kontsevich (see [6]). The first result in this direction has been obtained by Mikhalkin [7, 8], who counted curves with a given number of nodes on toric surfaces via lattice paths in convex lattice polygons. It has further been applied to the enumeration of real rational curves on Del Pezzo surfaces [3, 4, 16] and of complex rational curves in higher-dimensional toric varieties [9]. Patchworking naturally appears as a part of the tropical approach to enumerative geometry [8, 15, 16]. In our treatment of the patchworking, we follow the version of [15, 16]. The main goal of the paper is to present a new patchworking theorem, which not only covers the needs of [15, 16], where one

1991 *Mathematics Subject Classification.* 14H15, 14H20
Key words. singular algebraic curve, equisingular family, toric variety

counted nodal and cuspidal curves, but can be applied to the enumeration problem for curves with more complicated singularities.

In 1979–80, O. Viro [19, 20, 21, 22] invented a patchworking construction for real non-singular projective algebraic hypersurfaces. We would like to mention that almost all known topological types of real non-singular algebraic curves are realized in this way. In general, the initial data of the construction consist of

- $Y \to (\mathbb{C}, 0)$, a one-parametric flat family of algebraic varieties of dimension $n \ge 2$, with irreducible fibres Y_t, $t \ne 0$, and reduced reducible central fibre Y_0;
- a line bundle $\mathcal{L}$ on Y;
- a hypersurface $X_0 \subset Y_0$, being the zero locus of a section ξ_0 of $\mathcal{L}_0 = \mathcal{L}\big|_{Y_0}$.

The construction extends the section ξ_0 to a section ξ of $\mathcal{L}$, and thus, the hypersurface X_0 to a flat family of hypersurfaces $X_t \subset Y_t$, $t \in (\mathbb{C}, 0)$, such that the hypersurfaces X_t, $t \ne 0$, inherit some properties of X_0.

In the early 1990's the author suggested to use the patchworking construction for tracing properties of objects, defined by polynomials, other than in the original Viro method. For example, for tracing prescribed singularities of algebraic hypersurfaces [10, 13, 14], critical points of polynomials [12, 13], singular points and limit cycles of planar polynomial vector fields [5], resultants of bivariate polynomials [11].

In the present paper, we restrict ourselves to the case $n = 2$. We consider families of surfaces and curves in these surfaces, and we trace the property to possess a certain collection of singularities. As a result, we prove a patchworking theorem for curves on toric surfaces (Theorem 2.4, Section 2.3) which is sufficient for applications to a tropical calculation of Gromov-Witten and Welschinger invariants on toric surfaces as in [15, 16, 17]. Comparing with [18], we notice that the results of [18] are insufficient to treat the needs of tropical enumerative geometry, but they cover a wider class of situations. Furthermore, we generalize our result to the case of curves on arbitrary surfaces (Theorem 3.3, Section 3.3).

We notice that in [10, 13, 14] we always supposed that the components of the hypersurface X_0 are reduced and meet the singular locus of the variety Y_0 transversally. However, the degenerations which appear in the tropical enumeration of curves (see [15, 16]) can be more complicated. That is, X_0 may be non-reduced, and its components may be not transversal to the singular locus of Y_0. The novelty of the patchworking theorems presented in this paper is that, in the case $n = 2$, we allow curves X_0 with such properties.

Another result of the paper concerns local deformations of non-planar curve singularities, which appear in tropical limits of curves on surfaces. More precisely, in the above notation, we assume that $n = 2$, two components Y_0', Y_0'' of the surface Y_0 intersect along a line L, a point $p \in L$ is non-singular for L and an isolated singular point for a curve $X_0 \subset Y_0$. We ask about possible deformations of the germ (X_0, p) inscribed into the family Y_t, $t \in (\mathbb{C}, 0)$. Using Theorem 3.3, we determine how many nodes may appear in such deformations. This question naturally arises in the tropical count of nodal and cuspidal curves [8, 15], and potentially in a tropical enumeration of curves with more complicated singularities.

Throughout the paper, we work over the complex field $\mathbb{C}$. However, if the initial data of the patchworking construction are defined over the reals, then the resulting family is real as well. In particular, the embedded topology of the real point set of the constructed curves can be obtained precisely as in Viro's "gluing" procedure [19, 20, 21, 22] (cf. also [13, 14]). But we do not focus on this.

2 Patchworking of Curves on Toric Surfaces

2.1 Initial Data for Patchworking

Let Δ be a convex non-degenerate lattice polygon, lying in the nonnegative quadrant $\mathbb{R}^2_+ \subset \mathbb{R}^2$. By $\mathrm{Tor}(\Delta)$ we denote an associated toric variety. The sides of Δ determine toric divisors in $\mathrm{Tor}(\Delta)$, whose union we denote by $\mathrm{Tor}(\partial\Delta)$. By $\mathcal{L}(\Delta)$ denote the tautological line bundle on $\mathrm{Tor}(\Delta)$.

Let $\Delta = \Delta_1 \cup \ldots \cup \Delta_N$ be a subdivision of Δ into non-degenerate convex lattice polygons, which are linearity domains of a convex piece-wise linear function on Δ.

Let $a_{ij} \in \mathbb{C}$, $(i,j) \in \Delta \cap \mathbb{Z}^2$, be such that $a_{ij} \neq 0$ as far as (i,j) is a vertex of any polygon $\Delta_1, \ldots, \Delta_N$. The equations

$$f_k(x,y) := \sum_{(i,j)\in\quad k} a_{ij} x^i y^j = 0 \tag{1}$$

define curves $C_k \in |\mathcal{L}(\Delta_k)|$ of the toric surfaces $\mathrm{Tor}(\Delta_k)$, $k = 1, \ldots, N$.

We impose the following restrictions to the curves $C_1, \ldots, C_N$:

(C1) if C is a multiple component of C_k, $1 \le k \le N$, then C is defined by a binomial equation (in particular, $C \simeq \mathbb{P}^1$ and $C^2 = 0$), furthermore, C crosses any other component of C_k transversally and these intersections lie in the big torus $(\mathbb{C}^*)^2 = \mathrm{Tor}(\Delta_k) \backslash \mathrm{Tor}(\partial\Delta_k)$;

(C2) if σ is a side of Δ_k, and a point $p \in \mathrm{Tor}(\sigma)$ belongs to a non-multiple component of C_k, then in suitable local coordinates u, v in a neighborhood of p in $\mathrm{Tor}(\Delta_k)$, the line $\mathrm{Tor}(\sigma)$ is represented by $v = 0$, and the curve C_k is represented by

$$\varphi_{k,p}(u,v) := \sum_{i\,m(k,p)+jm \ge m\,m(k,p)} \alpha_{ij} u^i v^j = 0 \ , \qquad (2)$$

where $m(k,p), m$ are positive integers (m being the intersection number $(\mathrm{Tor}(\sigma) \cdot C_k)_p$), and the quasihomogeneous polynomial

$$\varphi^0_{k,p}(u,v) = \sum_{i\,m(k,p)+jm = m\,m(k,p)} \alpha_{ij} u^i v^j$$

is non-degenerate, i.e., has no critical points outside the origin.

To introduce additional data, consider all the pairs (k,p), where $1 \le k \le N$, $p \in \mathrm{Tor}(\sigma)$, $\sigma = \Delta_k \cap \Delta_l$ a common edge. We then construct a graph $\mathcal{G}$, whose vertices are the above pairs, and two pairs (k,p) and (l,q) are joined by an arc if (i) $k = l$ and p, q belong to the same multiple component of C_k, or (ii) $p = q \in \sigma$, $\sigma = \Delta_k \cap \Delta_l$. Denote by Θ the set of the connected components of $\mathcal{G}$. These components all are segments. We denote by Θ_s, $s = 0, 1, 2$, the set of the elements $\theta \in \Theta$ such that precisely s endpoints (k,p) of the graph θ correspond to isolated singular (or non-singular) points $p \in C_k$ such that $(C_k \cdot \mathrm{Tor}(\sigma))_p \ge 2$, $p \in \mathrm{Tor}(\sigma)$, $\sigma \subset \partial\Delta_k$. Further on we use the notation $(k,p) \in \theta \in \Theta_1 \cup \Theta_2$ to designate that (k,p) is an endpoint of the graph θ and p is an isolated singular point of C_k.

If $\theta \in \Theta_1$, $(k,p) \in \theta$, p is an isolated singular point of C_k, locally given by (2), then by a *deformation pattern* associated with θ we call any curve $C_\theta \subset \mathrm{Tor}(\Delta_\theta)$, where Δ_θ is the triangle with the vertices $(0,0)$, $(m,0)$, and $(0, m(k,p))$, defined by a polynomial $f_\theta(u,v)$, whose truncation to the edge $[(m,0),(0,m(k,p))]$ (i.e., the sum of monomials of the given polynomial, corresponding to the integral points in the edge) coincides with $\varphi^0_{k,p}(u,v)$, and the coefficient of u^{m-1} vanishes.

If $\theta \in \Theta_2$, $(k,p), (l,q) \in \theta$, p is an isolated singular point of C_k, locally given by (2), q is an isolated singular point of C_k, locally given by

$$\varphi_{l,q}(u,v) := \sum_{i\,m(l,q)+jm \ge m\,m(l,q)} b_{ij} u^i v^j = 0 \ ,$$

then by a *deformation pattern* associated with θ we call any curve $C_\theta \subset \mathrm{Tor}(\Delta_\theta)$, where Δ_θ is the triangle with the vertices $(0, -m(l,q))$, $(m,0)$, and $(0, m(k,p))$, defined by a polynomial f_θ, whose truncation to the edge $[(m,0),(0,m(k,p))]$ coincides with $\varphi^0_{k,p}(u,v)$, truncation to the edge $[(m,0),(0,-m(l,q))]$ coincides (up to a constant factor) with $\varphi^0_{l,q}(u,v^{-1})$, and the coefficient of u^{m-1} vanishes.

Notice only that the vanishing of the coefficient of u^{m-1} in f_θ is not a restriction, since can be achieved by a suitable shift $u \mapsto u + a$.

2.2 Transversality

Transversality of equisingular strata provides sufficient conditions for the patchworking (cf. [13, 14]).

Let $\mathcal{S}$ be a topological or (contact) analytic equivalence of isolated planar curve singular points. We intend to define the $\mathcal{S}$-transversality for triples $(\Delta_k, \Delta_k^-, C_k)$, $1 \le k \le N$, where Δ_k^- is a connected (or empty) union of some edges of Δ_k, and for deformation patterns.

Pick a triple $(\Delta_k, \Delta_k^-, C_k)$, $1 \le k \le N$.

Denote by $\mathrm{Sing}^{\mathrm{iso}}(C_k)$ the set of isolated singular points of C_k. If $p \in \mathrm{Sing}^{\mathrm{iso}}(C_k) \cap (\mathbb{C}^*)^2$, denote by $M^{\mathcal{S}}(C_k, p)$ the germ at C_k of the $\mathcal{S}$-equisingular stratum of (C_k, p) in $|\mathcal{L}(\Delta_k)|$. The (projective) Zariski tangent space to $M^{\mathcal{S}}(C_k, p)$ at C_k is formed by the curves $\{g = 0\} \in |\mathcal{L}(\Delta_k)|$, with $g \in I^{\mathcal{S}}(C_k, p) \subset \mathcal{O}_{\mathrm{Tor}(\ _k),p}$, where $I^{\mathcal{S}}(C_k, p)$ is the equisingular ideal or the Tjurina ideal (see [1, 2, 23]), according to whether $\mathcal{S}$ is the topological or analytic equivalence.

Let $p \in C_k \cap \mathrm{Tor}(\sigma)$ be an isolated singular or nonsingular point of C_k, where σ is an edge of Δ_k. Let u, v be local coordinates in a neighborhood of p in $\mathrm{Tor}(\Delta_k)$ as introduced in condition (C2), Section 2.1. The ideals

$$I_0^{\mathrm{sqh}}(C_k, p) = \left\{ g \in \mathcal{O}_{\mathrm{Tor}(\ _k),p} \;\middle|\; g = \sum_{i\,m(k,p) + jm \ge m\,m(k,p)} \beta_{ij} u^i v^j \right\},$$

$$I^{\mathrm{sqh}}(C_k, p) = I_0^{\mathrm{sqh}}(C_k, p) + \left\langle \frac{\partial \varphi_{k,p}}{\partial u} \right\rangle$$

naturally define the linear subsystems $M_0^{\mathrm{sqh}}(C_k, p)$, $M^{\mathrm{sqh}}(C_k, p)$ in $|\mathcal{L}(\Delta_k)|$, respectively.

Let C_k^{red} be the reduction of C_k, and let $p \in \mathrm{Sing}^{\mathrm{iso}}(C_k^{\mathrm{red}}) \backslash \mathrm{Sing}^{\mathrm{iso}}(C_k)$. Then p is an intersection point of two distinct components $\{g' = 0\}, \{g'' = 0\}$ of C_k having multiplicities m', m'', respectively, with $m' + m'' > 2$. Denote by $M^{eg}(C_k, z)$ the closure of the germ at C_k of the family of curves $C \in |\mathcal{L}(\Delta_k)|$, having $m'm''$ nodes in a neighborhood of p. This germ is smooth, provided that its (projective) Zariski tangent space at C_k has codimension $m'm''$ (cf. [15, Section 5.2]). More precisely

Lemma 2.1. *(i) The (projective) Zariski tangent space to $M^{eg}(C_k, p)$ at C_k is formed by the curves $\{g = 0\}$, $g \in \Lambda(\Delta_k)$, with $g \in I^{eg}(C_k, p) := \langle (g')^{m'}, (g'')^{m''} \rangle \subset \mathcal{O}_{\mathrm{Tor}(\ _k),p}$.*

(ii) Let Δ_k be a parallelogram with a pair of non-parallel edges σ_1, σ_2, the curve C_k given by $\{f_k = 0\}$, where f_k is a product of a monomial and binomials. Then the germ $M^{eg}(C_k) = \bigcap_z M^{eg}(C_k, z)$, where z runs over all intersection points of distinct components of C_k, is smooth of codimension

Area(Δ_k) *in* $\Lambda(\Delta_k)$*, and intersects transversally with the space of curves, defined by polynomials* f *with Newton polygon* Δ_k *such that* $f^{\sigma_i} = f_k^{\sigma_i}$, $i = 1, 2$.

Proof. (i) In a neighborhood of z, the curves $C \in M^{eg}(C_k, p)$ are unions of $m' + m''$ discs (counting multiplicities), and are represented by equations $((g')^{m'} + g_1')((g'')^{m''} + g_1'') = 0$ with sufficiently small g_1', g_1''; thus, the claim follows.

(ii) Observe that the number of the intersection points of $\{f_k^{\sigma_1} = 0\}$ and $\{f_k^{\sigma_2} = 0\}$ in $(\mathbb{C}^*)^2$ is equal to the area of the parallelogram Δ_k, divided by the lattice lengths of σ_1 and σ_2. Then we derive the required statement, when showing that f_k is the only polynomial with Newton polygon Δ_k, the fixed truncations on σ_1, σ_2, and belonging to the ideal $\langle f_k^{\sigma_1},\ f_k^{\sigma_2}\rangle_w \subset \mathcal{O}_{\mathbb{C}^2,w}$, for any point $w \in \{f_k^{\sigma_1} = 0\} \cap \{f_k^{\sigma_2} = 0\} \cap (\mathbb{C}^*)^2$, where $f_k^{\sigma_1}$ and $f_k^{\sigma_2}$ are the truncations to the edges σ_1, σ_2, respectively. The latter claim immediately follows from Bézout's theorem. $\square$

Definition 2.2. In the above notation, let Δ_k^+ be the union of the edges σ of Δ_k such that $\sigma \not\subset \Delta_k^-$. The triad $(\Delta_k, \Delta_k^-, C_k)$ is called $\mathcal{S}$-transversal, if all the germs

$$\begin{cases} M^{\mathcal{S}}(C_k, p), & p \in \mathrm{Sing}^{\mathrm{iso}}(C_k) \cap (\mathbb{C}^*)^2, \\ M^{eg}(C_k, p), & p \in \mathrm{Sing}^{\mathrm{iso}}(C_k^{\mathrm{red}}) \backslash \mathrm{Sing}^{\mathrm{iso}}(C_k), \\ M_0^{\mathrm{sqh}}(C_k, p), & p \in C_k \cap \mathrm{Tor}(\Delta_k^-) \text{ is not a non-isolated singular point}, \\ M^{\mathrm{sqh}}(C_k, p), & p \in C_k \cap \mathrm{Tor}(\Delta_k^+) \text{ is not a non-isolated singular point} \end{cases}$$

are smooth of expected dimension and intersect transversally in $|\mathcal{L}(\Delta_k)|$.

For $k = 1, \dots, N$, introduce the zero-dimensional scheme $Z_k \subset \mathrm{Tor}(\Delta_k)$, defined at the points $p \in C_k$ mentioned in Definition 2.2 by the ideals $I^{\mathcal{S}}(C_k, p)$, $I^{eg}(C_k, p)$, $I_0^{sqh}(C_k, p)$, $I^{sqh}(C_k, p)$, respectively.

Definition 2.3. Given $\theta \in \Theta_1 \cup \Theta_2$, $i = 1, 2$, a deformation pattern C_θ, corresponding to θ, is called $\mathcal{S}$-transversal, if the triad $(\Delta_\theta, \Delta_\theta^-, C_\theta)$ is $\mathcal{S}$-transversal, where Δ_θ^- is the union of those edges of Δ_θ which are neither vertical nor horizontal.

For various cohomological or numerical criteria of $\mathcal{S}$-transversality we refer to [15, Section 5.2].

2.3 Formulation of the Patchworking Theorem

The convex piece-wise linear functions on Δ, whose linearity domains are exactly $\Delta_1, \dots, \Delta_N$, form a cone, and we choose any generic such function, defined over $\mathbb{Q}$, then multiply it by a suitable integer in order to get a function

$\nu : \Delta \to \mathbb{R}$, which is integral-valued at integral points. We, moreover, assume that in all transformations and extensions of ν in the proof of Theorem 2.4, it remains integral-valued at integral points. Notice that then the faces of $\mathrm{Graph}(\nu)$ are $\mathbb{Z}$-congruent to $\Delta_1, \dots, \Delta_N$, respectively. The overgraph

$$\widetilde{\Delta} = \left\{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \;\middle|\; (\alpha, \beta) \in \Delta,\ \gamma \geq \nu(\alpha, \beta)\right\}$$

defines a toric three-fold Y with a natural projection $\mathrm{Tor}(\widetilde{\Delta}) \to \mathbb{C}$, whose fibres Y_t, $t \neq 0$, are isomorphic to $\mathrm{Tor}(\Delta)$, and the central fibre Y_0 splits into the union of $\mathrm{Tor}(\Delta_k)$, $k = 1, \dots, N$. The curves C_k, $k = 1, \dots, N$, from the initial data can be lifted to these components, forming a divisor $C^{(0)} \subset Y_0$, which we are going to extend up to a family $C^{(t)} \subset Y_t$, $t \in (\mathbb{C}, 0)$.

Denote by G the adjacency graph of the polygons $\Delta_1, \dots, \Delta_N$, and by $\mathcal{G}$ the set of orientations of G, which have no oriented cycles and induce a (natural) linear order on any element $\theta \in \Theta$. For $\Gamma \in \mathcal{G}$, denote by $\Delta_k^-(\Gamma)$ the union of those edges of Δ_k which correspond to arcs of G, which are Γ-oriented inside Δ_k. We assume that $\Delta_k^-(\Gamma)$ is connected for any $k = 1, \dots, N$. Denote by $\mathrm{Arc}(\Gamma)$ the set of ordered pairs (k, l), where Δ_k, Δ_l have a common edge, and the corresponding arc of Γ is emanating from Δ_k to Δ_l.

Theorem 2.4. *Under the assumptions of sections 2.1, suppose that all the given deformation patterns are $\mathcal{S}$-transversal, and there is $\Gamma \in \mathcal{G}$ such that every triad $(\Delta_k, \Delta_k^-(\Gamma), C_k)$ is $\mathcal{S}$-transversal, $k = 1, \dots, N$. Then there exists a flat family of curves $C^{(t)}$, $t \in (\mathbb{C}, 0)$, including the curve $C^{(0)} \subset Y_0$ and such that the curves $C^{(t)} \in |\mathcal{L}(\Delta)|$, $t \neq 0$, satisfy the following condition: there is an $\mathcal{S}$-equivalent 1-to-1 correspondence between* $\mathrm{Sing}(C^{(t)})$ *and the disjoint union of*

- *the sets* $\mathrm{Sing}^{\mathrm{iso}}(C_k) \cap (\mathbb{C}^*)^2$*,* $k = 1, \dots, N$*,*
- *the sets* $\mathrm{Sing}(C_\theta)$*,* $\theta \in \Theta_1 \cup \Theta_2$*,*
- *the set of* $\sum_{k=1}^N \sum_p \dim \mathcal{O}_{\mathbb{C}^2,p}/I^{eg}(C_k, p)$ *nodes, where p runs over* $\mathrm{Sing}(C_k^{\mathrm{red}}) \backslash \mathrm{Sing}^{\mathrm{iso}}(C_k)$*,* $k = 1, \dots, N$*.*

2.4 Proof of the Patchworking Theorem

Before presenting the argument in all detail, we would like to comment on the main ideas behind. Our treatment of the singularities of $C^{(0)}$, which lie in the big tori of the components of $Y_0 = \bigcup_i \mathrm{Tor}(\Delta_i)$, is the same as appears in [13, 14], where we show that the conditions for the $\mathcal{S}$-equisingular deformation are governed by independent sets of parameters, due to the transversality conditions imposed. This treatment is set forth in the following Steps 1-5. Our way to deform the singularities of $C^{(0)}$ lying on the intersection lines of the components of Y_0 and non-isolated singularities (Steps 6 and 7 below) geometrically can be viewed as a refinement of the patchworking data: we additionally blow up[1] Y at singular points of $C^{(0)}$ on $\mathrm{Sing}(Y_0)$ or along multiple components of $C^{(0)}$, which, in fact, brings the data to the situation with only non=isolated singularities and transverse intersection of $C^{(0)}$ with $\mathrm{Sing}(Y_0)$ as considered in [13, 14]. Geometry of such blow up is reflected in Figure 1 illustrating the formal consideration. In this connection we only point out that the statement of Theorem 2.4 does not reduce directly to the patchworking theorems in [13, 14] by performing blow up as indicated. The transversality conditions, required after blow up are more restrictive than in Theorem 2.4. For example, all the ideals $I^{\mathrm{sqh}}(C_k, p)$, participating in the transversality conditions, should be replaced by $I^{\mathrm{sqh}}_0(C_k, p)$, and the latter restriction does not apply to the problems, appearing in the tropical enumerative geometry [15, 16].

Step 1. Fix $k = 1, \ldots, N$. Denote by $\mathcal{P}(\Delta_k)$ the linear space of polynomials spanned by the monomials $x^i y^j$, $(i, j) \in \Delta_k \cap \mathbb{Z}^2$. We shall split this linear space into subspaces and choose specific bases in them. Put

$$\mathcal{P}(\Delta_k^-) = \mathrm{Span}\{x^i y^j,\ (i,j) \in \Delta_k^-\},\quad \overline{\mathcal{P}}(\Delta_k^-) = \mathrm{Span}\{x^i y^j,\ (i,j) \in \Delta_k \backslash \Delta_k^-\}\ .$$

The $\mathcal{S}$-transversality of the triad $(\Delta_k, \Delta_k^-, C_k)$ yields by Definition 2.2 the surjectivity of the map $\mathrm{pr}_k : \mathcal{P}(\Delta_k) \to H^0(Z'_k, \mathcal{O}_{Z'_k})$, where Z'_k is the part of the zero-dimensional scheme Z_k, supported at points on $\mathrm{Tor}(\partial\Delta_k)$, where Z_k is introduced after Definition 2.2. Since $f_k \in \mathrm{Ker}(\mathrm{pr}_k)$ and Δ_k^- is connected,

$$\mathrm{Ker}(\mathrm{pr}_k) = \mathrm{Span}\{f_k\} \oplus (\mathrm{Ker}(\mathrm{pr}_k) \cap \overline{\mathcal{P}}(\Delta_k^-))\ .$$

The variety germs $M^{\mathcal{S}}(C_k, p)$ and $M^{eg}(C_k, p)$, listed in Definition 2.2 and related to singular points p of C_k in $(\mathbb{C}^*)^2$, naturally lift to germs at f_k (f_k being defined in (1)) of varieties in $\mathcal{P}(\Delta_k)$. The latter germs we extend up to the germs $\widetilde{M}^{\mathcal{S}}(C_k, p)$ and $\widetilde{M}^{eg}(C_k, p)$ at f_k of the corresponding equisingular varieties in the space $\mathcal{P}(\Delta)$. Due to the $\mathcal{S}$-transversality condition, the intersection M_k of all the germs $\widetilde{M}^{\mathcal{S}}(C_k, p)$, $\widetilde{M}^{eg}(C_k, p)$, corresponding to the singular points of C_k in $(\mathbb{C}^*)^2$, is smooth of expected codimension $n_k = \sum_p \dim \mathcal{O}_{\mathbb{C}^2,z}/I^{\mathcal{S}}(C_k, p) + \sum_p \dim \mathcal{O}_{\mathbb{C}^2,p}/I^{eg}(C_k, p)$ in $\mathcal{P}(\Delta)$. Furthermore, M_k intersects transversally with $\mathrm{Ker}(\mathrm{pr}_k) \cap \overline{\mathcal{P}}(\Delta_k^-)$ in $\mathcal{P}(\Delta)$. That is, in a neighborhood of f_k, this germ is given by a system of analytic equations

$${}^{(k)}_{1}(F) = \ldots = {}^{(k)}_{n_k}(F) = 0\ , \tag{3}$$

[1] We use weighted blow ups (cf. [18])

where F stands for a variable polynomial in $\mathcal{P}(\Delta)$, and there is a set B_k of n_k linearly independent elements of $\mathrm{Ker}(\pi_k) \cap \overline{\mathcal{P}}(\Delta_k^-)$ such that

$$\det\left(\frac{\partial\{\ {}_i^{(k)}(F),\ i=1,\ldots,n_k\}}{\partial B_k}\right)\Bigg|_{F=F_k} \neq 0\ . \tag{4}$$

Moreover, the elements of B_k can be chosen as the disjoint union of sets $B_{k,p}$, $p \in \mathrm{Sing}(C_k^{\mathrm{red}})$ such that $B_{k,p}$ projects to a basis of the complex vector space $\mathcal{O}_{\mathbb{C}^2,p}/I^{\mathcal{S}}(C_k,p)$ or $\mathcal{O}_{\mathbb{C}^2,p}/I^{eg}(C_k,p)$, according as $p \in \mathrm{Sing}^{\mathrm{iso}}(C_k) \cap (\mathbb{C}^*)^2$ or $p \in \mathrm{Sing}(C_k^{\mathrm{red}}) \backslash \mathrm{Sing}^{\mathrm{iso}}(C_k)$, and, in addition $B_{k,p}$ projects to zero in any other such space corresponding to any point $p' \neq p \in \mathrm{Sing}(C_k^{\mathrm{red}}) \cap (\mathbb{C}^*)^2$.

Step 2. Under the hypotheses of Step 1, we choose a basis of $H^0(Z_k', \mathcal{O}_{Z_k'})$ reflecting its splitting

$$\bigoplus_{p \in C_k \cap \mathrm{Tor}(\ {}_k^-) \backslash C_k^{nr}} \mathcal{O}_{\mathrm{Tor}(\ {}_k),p}/I_0^{\mathrm{sqh}}(C_k,p) \oplus \bigoplus_{\substack{p \in C_k \cap \mathrm{Tor}(\ {}_k^+) \backslash C_k^{nr} \\ (C_k\ \mathrm{Tor}(\ {}_k^+))_p \geq 2}} \mathcal{O}_{\mathrm{Tor}(\ {}_k),p}/I^{\mathrm{sqh}}(C_k,p)\ , \tag{5}$$

where C_k^{nr} stands for the union of the multiple components of C_k.

For a point p, occurring in the above splitting, we have local coordinates u, v, introduced in condition (C2), Section 2.1, in which $I_0^{\mathrm{sqh}}(C_k,p)$ is generated by monomials lying on or above the segment $[(0,m(k,p)),(m,0)]$. Then we can choose the monomial basis

$$u^iv^j, \quad i \cdot m(k,p) + jm < m \cdot m(k,p)\ , \tag{6}$$

for $\mathcal{O}_{\mathrm{Tor}(\ {}_k),p}/I_0^{\mathrm{sqh}}(C_k,p)$, and the monomial basis

$$u^iv^j, \quad i \cdot m(k,p) + jm < m \cdot m(k,p), \quad (i,j) \neq (m-1,0)\ , \tag{7}$$

for $\mathcal{O}_{\mathrm{Tor}(\ {}_k),p}/I^{\mathrm{sqh}}(C_k,p)$.

Take a point $p \in C_k \cap \mathrm{Tor}(\Delta_k^+) \backslash C_k^{nr}$ such that $(C_k \cdot \mathrm{Tor}(\Delta_k^+))_z \geq 2$. In the local coordinates u, v, the ideal $I^{\mathrm{sqh}}(C_k,p)$ is generated by monomials lying on or above segment $[(0,m(k,p)),(m,0)]$ and by $\partial\varphi_k/\partial u$. We lift the monomial basis (7) to polynomials $\pi_{ij}^{(k,p)} \in \mathcal{P}(\Delta_k)$, which can be chosen obeying the following restrictions:

- $\pi_{ij}^{(k,p)}$ vanishes in all summands of (5) corresponding to points different from p,
- $\pi_{ij}^{(k,p)}$ belongs to $\mathcal{P}(\Delta_k \backslash \Delta_k^-)$,
- $\pi_{ij}^{(k,p)}$ with $j > 0$ does not contain the monomials u^av^b, $(a,b) \in \sigma$, where $p \in C_k \cap \mathrm{Tor}(\sigma)$.

Similarly, if $p \in C_k \cap \mathrm{Tor}(\Delta_k^-)$ such that $(C_k \cdot \mathrm{Tor}(\Delta_k^-))_p \ge 2$, then, in the local coordinates u, v, the ideal $I_0^{\mathrm{sqh}}(C_k, p)$ is generated by monomials lying on or above segment $[(0, m(k,p)), (m, 0)]$, and we lift the monomial basis (6) to polynomials $\pi_{ij}^{(k,p)} \in \mathcal{P}(\Delta_k)$, which can be chosen obeying the following restrictions:

- $\pi_{ij}^{(k,p)}$ vanishes in all summands of (5) corresponding to points $\ne p$,
- $\pi_{ij}^{(k,p)}$, $j > 0$, belongs to $\mathcal{P}(\Delta_k \backslash \Delta_k^-)$.

Step 3. Let $\theta \in \Theta$, $C_\theta \subset \mathrm{Tor}(\Delta_\theta)$ the given deformation pattern, defined by a polynomial f_θ with Newton polygon Δ_θ. Assume that $\mathcal{P}(\Delta_\theta)$ is embedded into some finite-dimensional linear space $\mathcal{V}$ of polynomials. The $\mathcal{S}$-transversality of the deformation pattern C_θ means that the germ at f_θ of the $\mathcal{S}$-equisingular stratum in $\mathcal{V}$, corresponding to $\mathrm{Sing}(C_\theta)$, is smooth of expected dimension (which we denote by n_θ), and is the intersection of smooth analytic transverse hypersurfaces $\ {}_1^\theta(F) = 0, \ \dots\ , \ \ {}_{n_\theta}^\theta(F) = 0$, $F \in \mathcal{V}$, and furthermore, there is the set B_θ of n_θ coefficients of monomials $(i,j) \in \Delta_\theta \backslash \Delta_\theta^-$, such that

$$\det \left(\frac{\partial \{\ \ {}_i^\theta(F),\ i = 1, \dots, n_\theta\}}{\partial \{B_\theta\}} \right) \Bigg|_{F = f_\theta} \ne 0\ .$$

Step 4. We intend to write a formula for the desired family of polynomials f with unknown coefficients, which then will be found as a solution to certain system of equations.

For $k = 1, \dots, N$, the restriction $\nu\big|_{\ \ k}$ is a linear function $\lambda_k(i,j) = \alpha_k i + \beta_k j + \gamma_k$. Introduce a $\mathbb{C}$-linear map

$$T_{k,t} : \mathbb{C}[x, y, x^{-1}, y^{-1}] \to \mathbb{C}[x, y, x^{-1}, y^{-1}], \quad T_{k,t}(x^i y^j) = x^i y^j t^{\lambda_k(i,j)},\ t \ne 0\ .$$

Put

$$\begin{aligned} f(x,y) = & \sum_{(i,j) \in\ } a_{ij} x^i y^j t^{\nu(i,j)} + \sum_{k=1}^N \sum_{h \in B_k} c_h T_{k,t}(h(x,y)) \\ & + \sum_{\substack{(k,p) \in \theta \in \Theta \\ p \in \mathrm{Tor}(\sigma),\ \sigma \subset\ {}_k^+}} \ \sum_{\substack{i\,m(k,p) + jm < m\,m(k,p) \\ (i,j) \ne (m-1, 0)}} t c_{ij}^{(k,p)} T_{k,t}(\pi_{ij}^{(k,p)}(x,y)) \\ & + \sum_{\substack{(k,p) \in \theta \in \Theta \\ p \in \mathrm{Tor}(\sigma),\ \sigma \subset\ {}_k^-}} \ \sum_{\substack{i\,m(k,p) + jm < m\,m(k,p) \\ j > 0}} t c_{ij}^{(k,p)} T_{k,t}(\pi_{ij}^{(k,p)}(x,y))\ , \end{aligned} \tag{8}$$

where all the coefficients $c_h = c_h(t)$, $c_{ij}^{(k,p)} = c_{ij}^{(k,p)}(t)$ are analytic functions in a neighborhood of zero.

Step 5. Pick $k=1,\dots,N$, and consider the polynomial

$$\hat{f}_k(x,y):=T_{k,t}^{-1}(f(x,y))=\sum_{(i,j)\in\quad {}_k}a_{ij}x^iy^j+\sum_{h\in B_k}c_hh(x,y)$$

$$+\sum_{(i,j)\in\quad\setminus\quad {}_k}a_{ij}x^iy^jt^{\nu(i,j)-\lambda_k(i,j)}+\sum_{l\neq k}\sum_{h\in B_l}c_hT_{k,t}^{-1}T_{l,t}(h(x,y))$$

$$+\sum_{\substack{(r,p)\in\theta\in\Theta\\ p\in\mathrm{Tor}(\sigma),\ \sigma\subset\quad {}^{+}_{r}}}\ \sum_{\substack{i\,m(r,p)+jm<m\,m(r,p)\\ (i,j)\neq(m-1,0)}}tc_{ij}^{(r,p)}T_{k,t}^{-1}T_{r,t}(\pi_{ij}^{(r,p)}(x,y))$$

$$+\sum_{\substack{(r,p)\in\theta\in\Theta\\ p\in\mathrm{Tor}(\sigma),\ \sigma\subset\quad {}^{-}_{r}}}\ \sum_{\substack{i\,m(r,p)+jm<m\,m(r,p)\\ j>0}}tc_{ij}^{(r,p)}T_{k,t}^{-1}T_{r,t}(\pi_{ij}^{(r,p)}(x,y))\ . \tag{9}$$

This is a deformation of $f_k(x,y)$ in $\mathcal{P}(\Delta)$.

Our first requirement about $\hat{f}_k(x,y)$ is that $\hat{f}_k\in M_k$, where $M_k\subset\mathcal{P}(\Delta)$ is the variety germ introduced in Step 1. Consequently, by (4), this can be expressed in the form

$$c_h=L_h^k\left(\{c_{h'}\ :\ h'\in B_r,\ (r,k)\in\mathrm{Arc}(\Gamma)\}\right)+O(t),\quad h\in B_k\ , \tag{10}$$

where L_h^k are linear functions with constant coefficients in their variables $c_{h'}$.

Step 6. Now, we intend to show how appear singularities of a deformation pattern in the constructed family of curves $\{f=0\}$. We consider the case $\theta\in\Theta_2$ and assume that the graph θ has more than one arc. The other cases can be treated similarly, and even in a simpler manner.

Let $(k,p),(l,q)\in\theta$, $p\in\mathrm{Tor}(\sigma)$, $\sigma\subset\partial\Delta_k$, $q\in\widetilde{\sigma}$, $\widetilde{\sigma}\subset\partial\Delta_l$, $\sigma\neq\widetilde{\sigma}$, and denote $m:=(C_k\cdot\mathrm{Tor}(\sigma))_p\geq 2$. By assumption and the definition of θ, the polygons Δ_k, Δ_l are joined by a well ordered sequence of polygons such that any two neighboring polygons in the sequence have a common edge, and all these common edges, among them σ and $\widetilde{\sigma}$, are parallel.

Let $\Delta_1,\dots,\Delta_s$, $s\geq 1$, be the sequence of polygons joining Δ_k and Δ_l (see Figure 1 (a)). Geometrically, this means that the points p and q are joined in $\bigcup_{i=1}^N\mathrm{Tor}(\Delta_i)$ by a sequence of non-singular rational components $C_1'\subset C_1$, $\dots$, $C_s'\subset C_s$, such that each component C_i' appears in C_i with multiplicity m. Without loss of generality assume that ν is constant on $\sigma,\widetilde{\sigma}$ and on all parallel to them edges of $\Delta_1,\dots,\Delta_s$. Perform the following coordinate change $(x,y)\mapsto(x'',y'')$:

- Let M_σ be an affine automorphism of $\mathbb{Z}^2$ which makes σ horizontal (see Figure 1 (b)). This corresponds to a monomial coordinate change $x=(x')^a(y')^b$, $y=(x')^c(y')^d$ in $f(x,y)$. The truncation of the new polynomial $f'(x',y')$ on the edge σ is a polynomial in x', multiplied

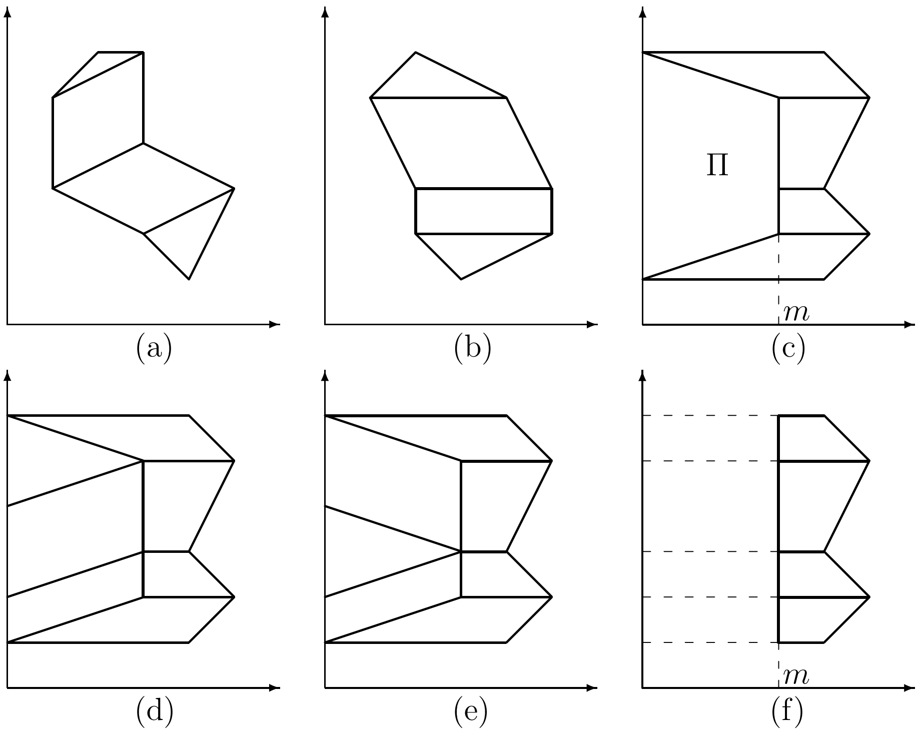

Figure 1: Deformation of multiple components and singular points lying on $\mathrm{Sing}(Y_0)$.

by a monomial in x', y', and with coefficient, analytically depending on t. Reducing the above monomial in x', y' and a common power of t in the coefficients, we put $t = 0$ in that truncation and obtain a complex polynomial $P_0(x')$, which has a root $\xi \neq 0$ of multiplicity m, corresponding to the point p;

- Assuming, without loss of generality, that ν is zero along σ, we perform the shift $x' = x'' + \xi$, $y' = y''$, and put $f''(x'', y'') = f'(x', y')$.

The Newton polygons $\Delta''_k, \Delta''_l, \Delta''_1, \dots, \Delta''_s$ of the respective polynomials

$$\begin{aligned} f''_k(x'', y'') &= \sum_{(i,j)\in \ ''_k} a''_{ij}(x'')^i(y'')^j := f_k(x, y)\,, \\ f''_l(x'', y'') &= \sum_{(i,j)\in \ ''_l} a''_{ij}(x'')^i(y'')^j := f_l(x, y)\,, \\ f''(x'', y'') &= \sum_{(i,j)\in \ ''_u} a''_{ij}(x'')^i(y'')^j := f\ \ (x, y)\,, \quad u = 1, \dots, s, \end{aligned}$$

will be located as shown in Figure 1 (c), surrounding the trapeze Π with vertices $(0, r_s - m(l, q))$, $(0, r_0 + m(k, p))$, (m, r_0), (m, r_s). Denote the ordinates of the horizontal edges of $\Delta''_1, \dots, \Delta''_s$ by $r_0 > r_1 > \dots > r_s$ (see Figure 1 (f)). The function ν induces, via the above coordinate change, linear affine functions $\lambda''_k, \lambda''_l, \dots, \lambda''_1, \lambda''_s$ on the polygons $\Delta''_k, \Delta''_l, \Delta''_1, \dots, \Delta''_s$, respectively, which together form a convex function ν'', non-linear on the union of any two of the polygons. Due to generality of ν, there is a unique extension of ν'' on Π as a convex piece-wise linear function with linearity domains inside Π being parallelograms and a translate of the triangle Δ_θ (see, for example, Figure 1 (d,e)).

The linear map Ψ, which takes any polynomial $g(x,y)$ to $g''(x'',y'')$ along the above coordinate change, and then projects $g''(x'',y'')$ to the space $\mathcal{P}(\Pi_0)$, where Π_0 is obtained from the trapeze Π by removing the non-vertical edges and the right vertical edge, induces the isomorphisms

$$\begin{aligned}\mathrm{Span}&\left(\left\{\pi^{(k,p)}_{ij} \;\middle|\; \begin{array}{l} i\cdot m(k,p)+jm < m\cdot m(k,p), \\ (i,j)\ne(m-1,0)\end{array}\right\} \cup \left\{\frac{\partial f''_k}{\partial x''}\right\}\right) \\ &\simeq\ \mathrm{Span}\left\{(x'')^\alpha (y'')^\beta \;\middle|\; \beta \ge r_0,\ m(k,p)\alpha + m(\beta - r_0) < m\cdot m(k,p)\right\},\end{aligned}$$

$$\begin{aligned}\mathrm{Span}&\left\{\pi^{(l,q)}_{ij} \;\middle|\; j>0,\ i\cdot m(l,q)+jm < m\cdot m(l,q)\right\} \\ &\simeq\ \mathrm{Span}\left\{(x'')^\alpha (y'')^\beta \;\middle|\; \beta < r_s,\ m(l,q)\alpha + m(r_s-\beta) < m\cdot m(l,q)\right\},\end{aligned}$$

and, for each $u = 1, \dots, s$,

$$\mathrm{Span}\left(\bigcup_{z\in C'_u} B_{\ ,z}\right) \simeq \mathrm{Span}\left\{(x'')^\alpha (y'')^\beta \;\middle|\; 0\le\alpha<m,\ r_{\ } \le \beta < r_{\ -1}\right\}.$$

The latter isomorphism statement comes from the fact that the monomials $(x'')^\alpha (y'')^\beta$, $0 \le \alpha < m, r_{\ } \le \beta < r_{\ -1}$, project to a basis of the complex vector space $\bigoplus_{z\in C'_u} \mathcal{O}_{\mathbb{C}^2,z}/I^{eg}(C_{\ }, z)$, whereas $\bigcup_{z\in C'_u} B_{\ ,z}$ projects to a basis of the aforementioned space $\bigoplus_{z\in C'_u} \mathcal{O}_{\mathbb{C}^2,z}/I^{eg}(C_{\ }, z)$ by construction (see Step 1).

Put $\widetilde{\nu}(i,j) = \max\{\lambda''_k(i,j), \lambda''_1(i,j), \dots, \lambda''_s(i,j), \lambda''_l(i,j)\}$. Any coefficient $A_{ij}(t)$, $(i,j)\in\Pi_0$, of $f''(x'',y'')$ starts with an exponent of t, greater than $\widetilde{\nu}(i,j)$. Furthermore, we have

$$\Psi(f''(x'',y'')) = t\left(\sum_{(i,j)\in\Pi_0} (L_{1,ij} + O(t))t^{\widetilde{\nu}(i,j)} + L_2\cdot\frac{\partial f''}{\partial x''}\right),$$

where $L_{1,ij}$ is a linear function with constant coefficients depending on the parameters

$$\left\{\begin{array}{ll} c^{k,p}_{\alpha\beta}, & \text{where } m(k,p)\alpha + m\beta < m\cdot m(k,p),\ (\alpha,\beta)\ne(m-1,0)\ , \\ c^{l,q}_{\alpha\beta}, & \text{where } m(l,q)\alpha + m\beta < m\cdot m(l,q)\ , \\ c_h, & \text{where } h \in \bigcup_{z\in C'_1} B_{1,z} \cup \ldots \cup \bigcup_{z\in C'_s} B_{s,z}\ , \end{array}\right. \qquad (11)$$

and

$$\left\{ c_h, \quad \text{where } h \in \bigcup_{(\ ,k)\in\mathrm{Arc}(\Gamma)} B \ \cup \bigcup_{(\ ,l)\in\mathrm{Arc}(\Gamma)} B \ \cup \bigcup_{\alpha=1}^{s} \bigcup_{(\ ,\alpha)\in\mathrm{Arc}(\Gamma)} B \right\}, \tag{12}$$

and L_2 is a linear function with constant coefficients depending on the parameters

$$\left\{ \begin{array}{l} \{c_h,\ h \in B_k\}, \quad \{c_h,\ h \in \bigcup_{(s,k)\in\mathrm{Arc}(\Gamma)} B_s\}, \\ \{c_{\alpha\beta}^{(k,p)}),\ m(k,p)\alpha + m\beta < m \cdot m(k,p),\ (\alpha,\beta) \neq (m-1,0)\}, \\ \{c_{\alpha\beta}^{(k,z)},\ z \in C_k \cap \mathrm{Tor}(\partial\Delta_k),\ z \neq p\}, \\ \{c_{\alpha\beta}^{(\ ,z)},\ z \in \bigcup_{(\ ,k)\in\mathrm{Arc}(\Gamma)} (C\ \ \cap \mathrm{Tor}(\partial\Delta\ \ ))\} \end{array} \right. . \tag{13}$$

Using the isomorphism induced by Ψ, we conclude that there exist

$$\left\{ \begin{array}{ll} c_{\alpha\beta}^{(k,p)}, & m(k,p)\alpha + m\beta < m \cdot m(k,p),\ (\alpha,\beta) \neq (m-1,0)\ , \\ c_h, & h \in \bigcup_{z\in C'_1} B_{1,z} \cup \ldots \cup \bigcup_{z\in C'_s} B_{s,z}\ , \\ c_{\alpha\beta}^{(l,q)}, & j > 0,\ m(l,q)\alpha + m\beta < m \cdot m(l,q)\ , \end{array} \right. \tag{14}$$

and an analytic function $\tau(t)$, vanishing at zero, such that

$$\Psi(f''(x'' + t\tau(t), y'') = \sum_{(i,j)\in\Pi_0} d_{ij}^{\theta}(t) t^{\nu''(i,j)} (x'')^i (y'')^j\ , \tag{15}$$

where d_{ij}^{θ} are analytic in a neighborhood of zero. Formally, (15) reduces to a system of equations for the variables (14)

$$\left\{ \begin{array}{l} c_{ij}^{(k,p)} = L_{ij}^{(k,p)} + O(t)\ , \\ \qquad i \cdot m(k,p) + jm < m \cdot m(k,p),\ (i,j) \neq (m-1,0)\ , \\ c_{ij}^{(l,q)} = L_{ij}^{(l,q)} + O(t), \quad j > 0,\ i \cdot m(l,q) + jm < m \cdot m(l,q)\ , \end{array} \right. \tag{16}$$

$$c_h = L_h + O(t), \quad h \in \bigcup_{z\in C'_1} B_{1,z} \cup \ldots \cup \bigcup_{z\in C'_s} B_{s,z}\ , \tag{17}$$

where $L_{ij}^{(k,p)}, L_h, L_{ij}^{(l,q)}$ are linear function with constant coefficients, whose variables are d_{ij}^{θ}, $(i,j) \in$
Pi_0, and additionally

- for $L_{ij}^{(k,z)}$,

$$\left\{ \begin{array}{ll} c_h, & \text{where} \quad h \in \bigcup_{(s,k)\in\mathrm{Arc}(\Gamma)} B_s\ , \\ c_{\alpha\beta}^{(k,z)}, & \text{where} \quad z \in \mathrm{Tor}(\Delta_k^-),\ (C_k \cdot \mathrm{Tor}(\Delta_k^-))_z \geq 2\ , \end{array} \right. \tag{18}$$

- for L_h, $h \in B\ $, $1 \le u \le s$,

$$c_{\alpha\beta}, \quad \text{where} \quad (\alpha, \beta) \in \Delta^- \ , \tag{19}$$

and

$$\begin{cases} c_h, & \text{where} \quad h \in B_k \ , \\ c_h, & \text{where} \quad h \in \bigcup_{(v,k)\in\mathrm{Arc}(\Gamma)} B_v \ , \\ c^{(k,p)}_{\alpha\beta}, & \text{where} \quad m(k,p)\alpha + m\beta < m \cdot m(k,p) \ , \\ & \qquad\qquad (\alpha,\beta) \ne (m-1, 0) \ , \\ c^{(k,z)}_{\alpha\beta}, & \text{where} \quad z \in C_k \cap \mathrm{Tor}(\partial\Delta_k), \ z \ne p \ , \\ c^{(s,z)}_{\alpha\beta}, & \text{where} \quad z \in \bigcup_{(v,k)\in\mathrm{Arc}(\Gamma)} (C_v \cap \mathrm{Tor}(\partial\Delta_v)) \end{cases} \tag{20}$$

- for $L^{(l,q)}_{ij}$,

$$\begin{cases} b_h(t), & \text{where} \quad h \in \bigcup_{(v,l)\in\mathrm{Arc}(\Gamma)} B_v \ , \\ c^{(v,z)}_{\alpha\beta}, & \text{where} \quad (v,l) \in \mathrm{Arc}(\Gamma), \ z \in C_l \cap \mathrm{Tor}(\Delta^-_l) \ , \end{cases} \tag{21}$$

  and (20).

Our demands on $d^\theta_{ij}(t)$, $(i,j) \in \Pi_0$ are as follows. Without further confusion we identify Δ_θ with the triangle in the subdivision of Π. Let $(m, r\ )$, $(0, r\ + m(k,p))$, $(0, r\ - m(l,q))$ be its vertices. We write $d^\theta_{ij}(t) = d^\theta_{ij}(0) + e^\theta_{ij}(t)$, where $e^\theta_{ij}(0) = 0$, and suppose that

- for any parallelogram P in the subdivision of Π, when equating $\nu''\big|_P = 0$ (by subtracting a suitable linear function) and letting $t = 0$ in $f''(x'', y'')$, and using (15), we obtain the polynomial

$$f_P(x'', y'') := \sum_{(i,j)\in P\cap\Pi_0} d^\theta_{ij}(0)(x'')^i(y'')^j + \sum_{(i,j)\in P\setminus\Pi_0} a''_{ij}(x'')^i(y'')^j \ , \tag{22}$$

  which must be a product of a monomial and binomials;

- when similarly equating $\nu''\big|_{\ \theta} = 0$ and letting $t = 0$ in $f''(x'', y'')$, and using (15), we obtain the polynomial

$$\sum_{(i,j)\in\ \ \theta\cap\Pi_0} d^\theta_{ij}(0)(x'')^i(y'')^j + a''_{mr_u}(x'')^m(y'')^{r_u} \ ,$$

  which must be proportional to f_θ.

All this, clearly, determines $d^\theta_{ij}(0)$, $(i,j) \in \Pi_0$, uniquely.

Next we impose conditions on $e^\theta_{ij}(t)$, $(i,j)\in\Pi_0$. Namely, pick $v=1,\dots,s$ and consider the parallelogram P_v from the subdivision of Π, whose vertices are

$$(m,r_{v-1}),\ (m,r_v),\ (0,r_{v-1}+m(k,p)),\ (0,r_v+m(k,p))$$

if $v\le u$ (i.e., the parallelogram lies above the triangle Δ_θ, Figure 1 (e)), or

$$(m,r_{v-1}),\ (m,r_v),\ (0,r_{v-1}-m(l,q)),\ (0,r_v-m(l,q))$$

if $v>u$ (i.e., the parallelogram lies below the triangle Δ_θ, Figure 1 (e)). Let us again equate $\nu''\big|_{P_v}=0$ in $f''(x'',y'')$. Then we obtain

$$f''(x'',y'')=\sum_{(i,j)\in P_v\cap\Pi_0} d^\theta_{ij}(t)(x'')^i(y'')^j+\sum_{(i,j)\in P_v\backslash\Pi_0} a''_{ij}(x'')^i(y'')^j+O(t)\ . \quad (23)$$

For $t=0$, it specializes to the polynomial f_{P_v} defined as in (22), which is a product of a monomial and binomials, and our demand is that the polynomial (23) belongs to the variety $M^{eg}(f_{P_v})$ as $t\ne0$, the latter variety being defined in Lemma 2.1(ii). Furthermore, Lemma 2.1(ii) yields that this requirement can be expressed by a system of equations

$$e^\theta_{ij}=L^\theta_{ij}+O(t),\quad (i,j)\in P'_v\ , \qquad (24)$$

where P'_v is obtained from P_v by removing its upper and right edges, if $v\le u$, or removing the lower and right edges, if $v>u$ (see Figure 1 (e)), and L^θ_{ij} are linear function with constant coefficients in the variables $e^\theta_{\alpha\beta}:=d^\theta_{\alpha\beta}-d^\theta_{\alpha\beta}(0)$ as (α,β) ranges over P'_v, and the variables c_h, $h\in\bigcup_{(\alpha,v)\in\mathrm{Arc}(\Gamma)}B_\alpha$.

At last, we equate $\nu''\big|_{\ \ \theta}=0$ in $f''(x'',y'')$. Then

$$f''(x''+t\tau(t),y'')=\sum_{(i,j)\in\ \ \theta} d^\theta_{ij}(t)(x'')^i(y'')^j+a''_{mr_u}(x'')^m(y'')^{r_u}+O(t)$$

represents a one-parameter deformation of the pattern $(y'')^{r_u}f_\theta(x'',y'')$, which we want to be $\mathcal{S}$-equisingular with respect to the singularities of C_θ in $\mathbb{C}^2$. As pointed out in Step 4, this can be expressed by a system of equations

$$e^\theta_{ij}=L^\theta_{ij}+O(t),\quad (i,j)\in B_\theta+r\ \ , \qquad (25)$$

where L^θ_{ij} are linear functions with constant coefficients in the variables $e^\theta_{\alpha\beta}$, $(\alpha,\beta)\in\Pi_0\backslash(B_\theta+r\ \ )$, and c_h, $h\in\bigcup_{(\alpha,\ \ )\in\mathrm{Arc}(\Gamma)}B_\alpha$.

The variables in the systems (24), $v=1,\dots,s$, and (25) are naturally ordered so that, for $t=0$, each variable depends linearly only on the preceding variables; hence, by the implicit function theorem, this bunch of equations can be resolved with respect to e^θ_{ij}, $(i,i)\in\Pi_0$. We then plug the solution obtained into system (16), (17), noticing that, in this substitution, the variables $c^{(v,z)}_{\alpha\beta}$, mentioned in (21), enter the terms $O(t)$ for all v, z, α, β.

Step 7. We should like to comment on the deformation of multiple components corresponding to an element $\theta \in \Theta_0$. In this case, we have a sequence of polygons $\Delta_1, \ldots, \Delta_s$ such that the common edges $\sigma_i = \Delta_i \cap \Delta_{i+1}$, $i = 1, \ldots, s-1$, are parallel to each other as well as to an edge $\sigma_0 \subset \Delta_1 \cap \partial\Delta$, and to an edge $\sigma_s \subset \Delta_s \cap \partial\Delta$. The given curves $C_i \in |\mathcal{L}(\Delta_i)|$, $i = 1, \ldots, s$, have multiple components $C'_i \subset C_i$ of the same multiplicity $m \geq 2$ and such that $C'_i \cap C'_{i+1} \cap \mathrm{Tor}(\sigma_i) \neq \emptyset$, $i = 1, \ldots, s-1$. In this case, we take care on deformation of the only intersection points of C'_i with other components of C_i in the torus $(\mathbb{C}^*)^2 \subset \mathrm{Tor}(\Delta_i)$. The condition that these deformations are in the strata $M^{eg}(C_i, z)$, $z \in C'_i \cap \mathrm{Sing}(C^{\mathrm{red}}_i)$, is included in equation (5) introduced in Step 1. It remains to explain that the non-isolated singular points indeed turn into collections of nodes. Applying the coordinate change $(x, y) \mapsto (x'', y'')$ as defined in Step 6, we obtain that the polygons $\Delta_1, \ldots, \Delta_s$ turn into polygons $\Delta''_1, \ldots, \Delta''_s$ with the left vertical sides on the same line (shown by solid lines in Figure 1 (f)). Then, similarly to the reasoning of Step 6, we extend the convex function ν'' by appending rectangle linearity domains as depicted by dashes in Figure 1 (f), introduce the variables $d_{ij}(t)$ and, finally, impose the requirement that the truncations of the polynomial $f''(x'', y'')\big|_{t=0}$ (normalized by the vanishing of ν'' on the respective rectangle) on any horizontal edge of any rectangle is square-free. Actually, this freedom in the choice of $d_{ij}(t)$ comes from the fact that we can freely move the component C'_i of C_i, $1 \leq i \leq s$, separating an m-multiple component into m distinct components.

Step 8. Before we join all the equation obtained in the preceding steps into one system, we should like to notice that some equations may be dependent, and hence must be removed from the system, since we finally intent to apply the implicit function theorem. Namely, the system of equations (16), (17) obtained in Step 6 is, in fact, included in the system (16), (10). Indeed, in our setting, (10) takes the form

$$c_h = L^\alpha_h \left(\left\{ c_{h'} \,\middle|\, h' \in B_v,\ (v, \alpha) \in \mathrm{Arc}(\Gamma) \right\} \right) + O(t), \quad h \in B_\beta,\ \beta = 1, \ldots, s\ . \tag{26}$$

By the implicit function theorem, we can resolve the system (16), (17) with respect to the variables in the left-hand side, then we substitute the expressions for $c^{k,p}_{ij}$, $c^{l,q}_{ij}$ into (17). The right-hand sides of the resulting system (17) depend on the same bunch of variables as in (26), and, by our construction, the system (26) implies the property that the distinct multiple components of any of the curves $C_1, \ldots, C_s$ do not glue up with each other and with any other component in a neighborhood of $\bigcup_{\gamma=1}^s (\mathrm{Sing}(C^{\mathrm{red}}_\gamma) \backslash \mathrm{Sing}^{\mathrm{iso}}(C_\gamma)$ along the deformation defined by $f(x, y)$. In turn, the system (17) simply expresses the latter property for some of the multiple components of $C_1, \dots, C_s$. Hence, the claim follows, and we get rid of all equations (17), including instead the equations (26) in the final system.

Step 9. We have expressed all the conditions imposed to the required polynomial $f(x, y)$ as systems of equations:

- (10) for all $k = 1, \dots, N$,
- (16) for all elements $\theta \in \Theta_1 \cup \Theta_2$.

The orientation Γ induces an ordering of the variables in the above united system so that, for $t = 0$, each variable is expressed only via strongly preceding variables, and hence the system can be resolved by the implicit function theorem.

Geometrically, the imposed conditions mean that, for each point $z \in \mathrm{Sing}^{\mathrm{iso}}(C_k) \cap (\mathbb{C}^*)^2$, $k = 1, \dots, N$, and for each $z \in \mathrm{Sing}(C_\theta)$, $\theta \in \Theta_1 \cup \Theta_2$, the polynomial $f(x, y)$ induces an $\mathcal{S}$-equisingular one-parametric deformation of the germ at z. Furthermore, each point $p \in \mathrm{Sing}(C_k^{\mathrm{red}}) \backslash \mathrm{Sing}^{\mathrm{iso}}(C_k)$, $1 \le k \le N$, bears $\dim \mathcal{O}_{\mathbb{C}^2,p}/I^{eg}(C_k, p)$ nodes, because the curves $C^{(t)} \subset \mathrm{Tor}(\Delta)$ have no multiple components (the curve $C^{(t)}$ crosses $\mathrm{Tor}(\partial\Delta)$ with multiplicity 1 at each point by the assumptions of Section 2.1).

At last, notice that $C^{(t)}$ has no non-isolated singularities,and, moreover, no singularities other than listed in the theorem. Indeed, the non-isolated singularities disappear due to the separation of multiple components as explained in Steps 6 and 7: this reflects the fact that the truncations of $f''(x'', y'')\big|_{t=0}$ (normalized by equating $\nu'' = 0$ on the respective parallelograms or rectangles) on the non-vertical edges of the parallelograms or rectangles are square-free polynomials by construction. The other singular points of $C^{(0)}$ on $\mathrm{Sing}(Y_0)$ like, for example, a point $z \in \mathrm{Tor}(\sigma) \cap C_k \cap C_l$, $\sigma = \Delta_k \cap \Delta_l$, with $(C_k \cdot \mathrm{Tor}(\sigma))_z = 1$, bear no singular points of $C^{(t)}$, $t \ne 0$, in view of [15, Lemma 3.2]. $\square$

2.5 Family of Deformations

The deformation $C^{(t)}$, $t \in (\mathbb{C}, 0)$, obtained in Theorem 2.4, is not unique. In fact, the polynomial $f(x, y)$ describing the deformation depends analytically not only on t, but also on other parameters, whose number is equal to the codimension of the germ at $C^{(t)}$ of the $\mathcal{S}$-equisingular stratum in $|\mathcal{L}(\Delta)|$ as $t \ne 0$. It is not difficult to extract these parameters from the proof of Theorem 2.4. Some particular subsets of such parameters are quite easy to describe:

For example, take any set B of the vertices of the polygons $\Delta_1, \dots, \Delta_N$ such that, for each $k = 1, \dots, N$, either $|B \cap \Delta_k| \le 3$, or $B \cap \Delta_k \subset \Delta_k^-(\Gamma)$.

Then the theorem provides a family of curves $C^{(t)}$, which can be defined by a polynomial

$$f(x,y) = \sum_{(i,j)\in} (a_{ij} + c_{ij}) x^i y^j t^{\nu(i,j)} \ , \tag{27}$$

where $c_{ij} = c_{ij}(t)$ are analytic functions vanishing at zero, $(i,j) \in \Delta$, and

$$c_{ij}(t) = \ {}^{B}_{ij}\left(\left\{c_{kl}(t) \,\middle|\, (k,l) \in B\right\}, t\right), \quad (i,j) \in \Delta \cap \mathbb{Z}^2 \backslash B \ , \tag{28}$$

with certain analytic functions ${}^{B}_{ij}$, $(i,j) \in \Delta \cap \mathbb{Z}^2 \backslash B$.

To obtain the required family, it is sufficient to show that the polynomials $h \in \bigcup_{k=1}^N B_k$ and all the polynomials $\pi_{ij}^{(k,z)}$, introduced in Step 2 of the proof of Theorem 2.4, can be chosen so that they do not contain monomials $x^\alpha y^\beta$, $(\alpha, \beta) \in B$. Indeed, if the latter holds, then the solution to the system considered in Step 9 of the proof of Theorem 2.4 depends on the coefficients c_ω, $\omega \in B$, as free parameters.

Pick $k = 1, \ldots, N$. If $B \cap \Delta_k \subset \Delta_k^-(\Gamma)$, then by construction, the polynomials $h \in B_k$ and the polynomials $\pi_{ij}^{(k,p)}$, $j > 0$, do not contain the monomials $x^\alpha y^\beta$, $(\alpha, \beta) \in B$. Assume that $|B \cap \Delta_k| \le 3$. Then, by applying a transformation of type $\mathbb{C}[x,y] \ni g(x,y) \mapsto ag(bx, cy)$, $a, b, c \in \mathbb{C}^*$, we can freely vary the coefficients of $x^\alpha y^\beta$, $(\alpha, \beta) \in B \cap \Delta_k$, in any polynomial $g(x,y)$ with Newton polygon Δ_k. On the other hand, all the strata $M^{\mathcal{S}}(C_k, p)$, $M^{eg}(C_k, p)$, and $M^{\rm sqh}(C_k, p)$ which appear in Definition 2.2 are invariant with respect to the above transformations (close to the identity). Hence (cf. the proof of [15, Lemmas 3.4 and 5.5]) the polynomials $h \in B_k$ and all $\pi_{ij}^{(k,p)}$ can be chosen free of the monomials $x^\alpha y^\beta$, $(\alpha, \beta) \in B$.

3 Patchworking of Curves on Arbitrary Surfaces

In this section, we consider patchworking of reduced curves on arbitrary algebraic surfaces.

3.1 Initial Data for Patchworking

Let us be given

- a flat family of projective surfaces $Y \to (\mathbb{C}, 0)$, all surfaces being irreducible except for the reduced reducible central fiber $Y_0 = \cup_{i=1}^N Y_0^i$;
- a line bundle $\mathcal{L}$ on Y;

- a reduced curve $X_0 \subset Y_0$, belonging to the linear system $|\mathcal{L}_0|$ (where $\mathcal{L}_0 = \mathcal{L}\big|_{Y_0}$) which contains no component of the intersection lines $Y_0^i \cap Y_0^j$, $i \neq j$.

An additional assumption on the family $Y \to (\mathbb{C}, 0)$ is that the surfaces Y_t, $t \neq 0$, and Y_0^i, $i = 1, \ldots, N$, may have only isolated singularities, the intersections $Y_0^i \cap Y_0^j \cap Y_0^k$, $i < j < k$, are finite, and that, for any line $E_{ij} = (Y_0^i \cap Y_0^j)_{red}$, $i \neq j$, and any point $p \in E_{ij} \backslash \mathrm{Sing}(E_{ij})$, in a neighborhood of p the family $Y \to (\mathbb{C}, 0)$ is isomorphic to (a neighborhood of) the family

$$\mathrm{Spec}\ \mathbb{C}[x, y, z, t]/(xy - t^a) \to \mathrm{Spec}\ \mathbb{C}[t] \tag{29}$$

with some positive integer a and $Y_0^i = \{x = t = 0\}$, $Y_0^j = \{y = t = 0\}$.

Furthermore, we suppose that $X_0 \cap \mathrm{Sing}(\bigcup_{i,j} E_{ij}) = \emptyset$, and, for any point $p \in X_0 \cap E_{ij}$, in suitable local coordinates x, y in a neighborhood of p in Y_0^i, the line E_{ij} is represented by $y = 0$, and the curve X_0^i is represented by

$$f_p^i(x, y) := \sum_{km' + lm \geq mm'} \alpha_{kl} x^k y^l = 0 \ ,$$

where m', m are positive integers (m is the intersection number $(E_{ij} \cdot X_0^i)_p$), and the quasihomogeneous polynomial

$$\varphi_p^i(x, y) = \sum_{km' + lm \geq mm'} \alpha_{kl} x^k y^l$$

is non-degenerate, i.e., has no critical points outside the origin.

At last, for any point $p \in X_0 \cap E_{ij}$, we pick a deformation pattern, a curve C_p in the respective toric surface $\mathrm{Tor}(\Delta_p)$, where C_p and Δ_p are defined precisely as the respective objects C_θ and Δ_θ in Section 2.1.

Remark 3.1. In principle, one can assume that the curve X_0 is non-reduced, but the requirements to multiple components do not look natural as they appear in the toric case. Furthermore, we shall not need such a generalization in the next application.

3.2 Transversality

As in Section 2.2, we work with an equivalence $\mathcal{S}$ of isolated planar curve singular points and define the germs $M^{\mathcal{S}}(X_0^k, p)$ at X_0^k of the $\mathcal{S}$-equisingular strata in the linear system $|\mathcal{L}_0^k|$, where $\mathcal{L}_0^k = \mathcal{L}\big|_{Y_0^k}$, for the points $p \in \mathrm{Sing}(X_0^k) \setminus \mathrm{Sing}(Y_0)$. We also introduce the linear subsystems $M_0^{\mathrm{sqh}}(X_0^k, p)$ and $M^{\mathrm{sqh}}(X_0^k, p)$ in $|\mathcal{L}_0^k|$ for all the points $p \in X_0^k \cap \bigcup_{i \neq k} E_{ik}$ as was done in Section 2.2.

We say that $M_0^{\mathrm{sqh}}(X_0^k, p)$ (or $M^{\mathrm{sqh}}(X_0^k, p)$) is *of expected dimension*, if the codimension of $M_0^{\mathrm{sqh}}(X_0^k, p)$ (resp. of $M^{\mathrm{sqh}}(X_0^k, p)$) in $|\mathcal{L}_0^k|$ is equal to $\dim \mathbb{C}\{x, y\}/I_0^{\mathrm{sqh}}(X_0^k, p)$ (resp. to $\dim \mathbb{C}\{x, y\}/I_0^{\mathrm{sqh}}(X_0^k, p)$).

Definition 3.2. Let $(Y_0^k)^-$ be a connected (or empty) union of some lines E_{ik}, $i \ne k$. Denote by $(Y_0^k)^+$ the union of the remaining lines E_{ik}, $i \ne k$. The triad $(Y_0^k, (Y_0^k)^-, X_0^k)$ is called $\mathcal{S}$-transversal, if all the germs

$$\begin{cases} M^{\mathcal{S}}(X_0^k, p), & p \in \mathrm{Sing}(X_0^k) \setminus \mathrm{Sing}(Y_0)\ , \\ M_0^{\mathrm{sqh}}(X_0^k, p), & p \in X_0^k \cap (Y_0^k)^-\ , \\ M^{\mathrm{sqh}}(X_0^k, p), & p \in X_0^k \cap (Y_0^k)^+ \end{cases}$$

are smooth of expected dimension and intersect transversally in $|\mathcal{L}_0^k|$.

3.3 Patchworking Theorem

Denote by G the adjacency graph of the components $Y_0^1, \dots, Y_0^N$ of Y_0, and by $\mathcal{G}$ the set of orientations of G, which have no oriented cycles. For $\Gamma \in \mathcal{G}$, denote by $(Y_0^k)^-(\Gamma)$ the union of those lines E_{ik}, $i \ne k$, which correspond to arcs of G Γ-oriented inside Y_0^k. We assume that $(Y_0^k)^-(\Gamma)$ is connected for any $k = 1, \dots, N$.

Theorem 3.3. *In the above notations, and under the assumptions of Section 3.1, suppose that all the triples $\big(Y_0^k, (Y_0^k)^-(\Gamma), X_0^k\big)$, $k = 1, \dots, N$, are $\mathcal{S}$-transversal, and that all the deformations patterns C_p, $p \in X_0 \cap \mathrm{Sing}(Y_0)$ are $\mathcal{S}$-transversal in the sense of Definition 2.3. Then there exists a flat deformation X_t, $t \in (\mathbb{C}, 0)$, inscribed in the deformation Y_t, $t \in (\mathbb{C}, 0)$, such that there is an $\mathcal{S}$-equivalent 1-to-1 correspondence between $\mathrm{Sing}(C^{(t)})$ and the disjoint union of the sets $\mathrm{Sing}(X_0^k) \setminus \mathrm{Sing}(Y_0)$, $k = 1, \dots, N$, and the sets $\mathrm{Sing}(C_p)$, $p \in X_0 \cap \mathrm{Sing}(Y_0)$.*

Again, we point out that the assumptions of the above theorem are weaker than those in the patchworking theorems of [18]. The difference is that there we replace the ideals $I_0^{\mathrm{sqh}}(X_0^k, p)$, $p \in X_0^k \cap (Y_0^k)^+(\Gamma)$, by the bigger ideals $I^{\mathrm{sqh}}(X_0^k, p)$ for all $k = 1, \dots, N$.

Proof of Theorem 3.3. Our reasoning is a combination of the argument in the proof of Theorem 2.4 and that in the proof of [18, Theorems 2.8 and 3.1], so we shall omit details and only indicate some crucial points.

Namely, first note that the $\mathcal{S}$-transversality of a triple $(Y_0^k, (Y_0^k)^-(\Gamma), X_0^k)$ can be expressed as (see [18, formula (2)])

$$H^1\left(Y_0^k, \mathcal{L}_0^k \otimes \mathcal{J}_{Z_k} \otimes \mathcal{O}_{Y_0^k}\left(-\bigcup_{E_{ik} \subset (Y_0^k)^-(\Gamma)} E_{ik}\right)\right) = 0\ ,$$

where $\mathcal{J}_{Z_k}$ is the ideal sheaf of the zerodimensional scheme $Z_k \subset Y_0^k$, concentrated at $\mathrm{Sing}(X_0^k) \cup (X_0^k \cap \mathrm{Sing}(Y_0))$ and defined by the ideals

(i) $I^{\mathcal{S}}(X_0^k, p)$, $p \in \mathrm{Sing}(X_0^k) \backslash \mathrm{Sing}(Y_0)$,

(ii) $I_0^{\mathrm{sqh}}(X_0^k, p)$, $p \in X_0^k \cap (Y_0^k)^-(\Gamma)$,

(iii) $I^{\mathrm{sqh}}(X_0^k, p)$, $p \in X_0^k \cap (Y_0^k)^+(\Gamma)$.

Since Γ defines a partial order on the set of components $Y_0^1, \dots, Y_0^N$, using induction and standard cohomology arguments, we can show that, for the ideal sheaf $\mathcal{J}$ of a certain zerodimensional scheme in Y_0, we have $H^1(Y_0, \mathcal{L}_0 \otimes \mathcal{J}) = 0$, and hence $H^1(Y_0, \mathcal{L}_0) = 0$. This, in particular, implies that $\dim H^0(Y_t, \mathcal{L}_t) = \mathrm{const}$, $t \in (\mathbb{C}, 0)$, which means that all the sections of $\mathcal{L}_0$ are extendable up to sections of $\mathcal{L}$, and furthermore that

$$H^0(Y_0, \mathcal{L}_0) \simeq \bigoplus_{k=1}^{N} H^0 \left(Y_0^k, \mathcal{L}_0^k \otimes \mathcal{O}_{Y_0^k} \left(- \bigcup_{E_{ik} \subset (Y_0^k)^-(\Gamma)} E_{ik} \right) \right)$$

(cf. with the toric case treated in Section 2).

The rest of the proof goes along the lines of the proof of Theorem 2.4. For example, a local deformation of a singular point $p \in E_{ij} \cap X_0$ we represent by the following toric model. The germ (Y, p) we replace by a germ of the toric variety $(\mathrm{Tor}(\widetilde{\Delta}), z)$, where $\widetilde{\Delta}$ is the overgraph of a convex piece-wise linear function ν defined on some convex lattice polygon Δ with two linearity domains Δ', Δ'' and such that $|n_1 \times n_2| = a$, n_1, n_2 being the primitive integral gradients of ν, and a being the number from (29), and finally $z \in \mathrm{Tor}(\sigma)$, $\sigma = \Delta' \cap \Delta''$. The curve germs $(Y_0^i, p$, (Y_0^j, p) are replaced by suitable curve germs $(C', z) \subset \mathrm{Tor}(\Delta')$, $(C'', z) \subset \mathrm{Tor}(\Delta'')$. Then one follows the argument of Step 6 of the proof of Theorem 2.4, in which the trapeze Π coincides with the triangle Δ_p. □

4 Nodal Deformations of Non-Planar Isolated Curve Singularities

Consider a family of surfaces $\xi : Y \to (\mathbb{C}, 0)$ defined by (29). Let a curve $X_0 \subset Y_0$ have an isolated singularity at the origin. Denote by U a regular neighborhood of the germ $(Y_0, X_0, 0)$ in $\mathbb{C}^4$. We ask the question:

Given a flat deformation X_t, $t \in (\mathbb{C}, 0)$, of the germ $(X_0, 0)$ such that $X_t \subset Y_t = \xi^{-1}(t)$, $t \neq 0$. How many nodes may $X_t \cap U$ have?

In the case of a planar curve singularity, the sharp upper bound is given by the δ-invariant.

Denote by Y_0', Y_0'' the components of Y_0, by L their intersection line. Respectively put $X_0' = X_0 \cap Y_0'$, $X_0'' = X_0 \cap Y_0''$. Let $m = (X_0' \cdot L)_0 = (X_0'' \cdot L)_0$. By [15, Lemma 3.2], the number of nodes in question satisfies

$$\delta \le \delta(X_0', 0) + \delta(X_0'', 0) + m - \max\{r(X_0'), r(X_0'')\} \ , \tag{30}$$

where $\delta(*)$ stands for the δ-invariant, and $r(*)$ denotes the number of local branches at the origin. This bound, however is not sharp in general.

We give a sharp upper bound in the following statement. To formulate it, we introduce one more parameter. Let P_i, $i = 1, \ldots, r(X_0')$, be the set of local branches of X_0' at 0, and Q_i, $i = 1, \ldots, r(X_0'')$, the set of local branches of X_0'' at 0. Denote by n the maximal number of subsets in the disjoint splitting

$$\{1, \ldots, r(X_0')\} = \bigcup_{s=1}^{n} B_s', \quad \{1, \ldots, r(X_0'')\} = \bigcup_{s=1}^{n} B_s'' \ ,$$

such that

$$\sum_{i \in B_s'} (P_i \cdot L)_0 = \sum_{j \in B_s''} (Q_j \cdot L)_0, \quad s = 1, \ldots, n \ . \tag{31}$$

Theorem 4.1. *Under the above assumptions, for any flat family of curves X_t, $t \in (\mathbb{C}, 0)$, inscribed into the family Y_t, $t \in (\mathbb{C}, 0)$, the number of nodes of $X_t \cap U$ does not exceed*

$$\delta(X_0, 0) := \delta(X_0', 0) + \delta(X_0'', 0) + m - r(X_0') - r(X_0'') + n \ . \tag{32}$$

Furthermore, there exists a deformation $X_t \subset Y_t$, $t \in (\mathbb{C}, 0)$, such that the number of nodes of $X_t \cap U$ is equal to $\delta(X_0, 0)$.

Example 4.2. We illustrate the difference between (30) and (32) by the following simple example. Assume that X_0' has two non-singular local branches at 0, one transversal to L, and the other intersecting L with multiplicity 3; in turn, let X_0'' have a tacnode at 0 with tangent L, that is, have two non-singular local branches which both are quadratically tangent to L. The bound (30) reads as $\delta \le 1 + 2 + 4 - 2 = 5$, whereas the bound (32) says that $\delta \le 1 + 2 + 4 - 2 - 2 + 1 = 4$. Here, $n = 1$ since the only possible relation of type (31) involves all the local branches: $1 + 3 = 2 + 2$.

Proof. (1) We start with the upper bound.

The branches P_i, Q_j are topological discs, and the loops ∂P_i, ∂Q_j are linked (positively, when fixing the standard orientations) in Y with Y_0'' and Y_0', respectively (see the proof of [15, Lemma 3.2]). Since Y_t, $t \neq 0$ are disjoint to Y_0, in any deformation X_t, $t \in (\mathbb{C}, 0)$, the discs P_i glue up with the discs Q_j by some handles and vice versa.

Assume that, in a given deformation of X_0, the branches P_i, $i \in S' \subset \{1, \dots, r(X_0')\}$ glue up precisely with Q_j, $j \in S'' \subset \{1, \dots, r(X_0'')\}$ into a connected immersed surface Σ. This means, in particular, that the subgerm $\left(\bigcup_{i \in S'} P_i \cup \bigcup_{j \in S''} Q_j, 0\right)$ deforms in a flat family. Hence,

$$\sum_{i \in S'} (P_i \cdot L)_0 = \sum_{j \in S''} (Q_j \cdot L)_0, \quad s = 1, \dots, n$$

(see for example, part (3) of the proof of [9, Theorem 8.3]). Thus, the normalization $\check{\Sigma}$ satisfies $\chi(\check{\Sigma}) \le 2 - \#S' - \#S''$, which then yields

$$\chi(\check{X}_t) \le 2n - r(X_0') - r(X_0'') \ . \tag{33}$$

Modelling the deformation X_t by a patchworking of two curves of a large degree as was done in the proof of [15, Lemma 3.2], we easily obtain that the number of nodes of $X^t \cap U$ is equal to

$$\begin{aligned} \delta &= \delta(X_0', 0) + \delta(X_0'', 0) + m + \frac{\chi(\check{X}_t) - r(X_0') - r(X_0'')}{2} \\ &\overset{(33)}{\le} \delta(X_0', 0) + \delta(X_0'', 0) + m - r(X_0') - r(X_0'') + n \ . \end{aligned}$$

(2) For a given germ $(X_0, 0)$, we shall construct a deformation X_t such that

$$\chi(\check{X}_t) = 2 - r(X_0') - r(X_0'') \ . \tag{34}$$

This is sufficient for the sharpness of (32), since in general, we can split a given germ $(X_0, 0)$ into n subgerms following the splitting (31), and then deform each subgerm separately, keeping (34). This condition, in turn, is equivalent to the required sharpness as explained in the preceding part of the proof.

In the construction of a deformation satisfying (34), we use Theorem 3.3. Without loss of generality, assume that X_0', X_0'' are plane algebraic curves of a sufficiently large degree d having only one singular point, the surface Y_0 is split into two toric surfaces $\mathrm{Tor}(\Delta'), \mathrm{Tor}(\Delta'')$, where

$$\Delta' = \mathrm{conv}\{(0,0), (0,d), (d,0)\}, \quad \Delta'' = \mathrm{conv}\{(0,0), (0,-d), (d,0)\}$$

and $Y_0' \cap Y_0'' = \mathrm{Tor}(\sigma)$, $\sigma = \Delta' \cap \Delta''$ (see Figure 2), and that the family Y_t is modelled by the family $\mathrm{Tor}(\widetilde{\Delta}) \to \mathbb{C}$, where

$$\widetilde{\Delta} = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \ : \ \gamma \ge \nu(\alpha, \beta), \ (\alpha, \beta) \in \Delta := \Delta' \cup \Delta''\} \ ,$$

$$\nu : \Delta \to \mathbb{R}, \quad \nu\Big|_{\,'} = 0, \ \nu\Big|_{\,''}(\alpha, \beta) = -a\beta \ .$$

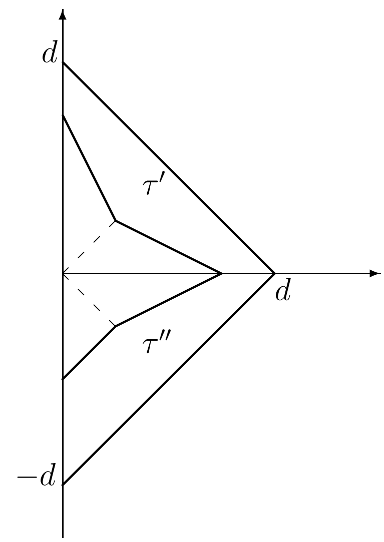

Figure 2: Deformations of non-planar curve singularities

Next we refine this family of surfaces and the curves X_0', X_0'' in order to satisfy the hypotheses of Theorem 3.3.

Consider the Newton diagrams of the singular points of X_0', X_0'' located at the origin, corresponding to the common vertex $(0,0)$ of Δ', Δ''. The curves X_0', X_0'' lift to the toric surfaces $\mathrm{Tor}(\tau'), \mathrm{Tor}(\tau'')$, where τ', τ'' are the parts of Δ', Δ'', respectively lying right to the Newton diagrams (see Figure 2). Now we restrict the function ν to $\tau' \cup \tau''$ and then extend it back to $\Delta' \cup \Delta'' \backslash (\tau' \cup \tau'')$ as a convex piece-wise linear function ν_1 so that the linearity domains of the extension will be the triangles obtained by cutting along the segments, joining the origin with the vertices of the Newton diagrams (shown by dashes in Figure 2). We then pass to the new surface family $\mathrm{Tor}(\widetilde{\Delta}_1) \to \mathbb{C}$, where $\widetilde{\Delta}_1$ is the overgraph of ν_1, and respectively we define a curve in the central fibre, taking X_0', X_0'', and the following curves $C_1, \ldots, C_s$ in the toric surfaces $\mathrm{Tor}(T_1), \ldots, \mathrm{Tor}(T_s)$, where $T_1, \ldots, T_s$ are all the triangles of the subdivision with vertex at the origin:

- the curve $C_i \in |\mathcal{L}(T_i)|$ is nodal, rational, and smooth along $\mathrm{Tor}(\partial T_i)$, $i = 1, \ldots, s$;
- if $\sigma_i = T_i \cap T_{i+1}$ is a common edge, then $C_i \cap \mathrm{Tor}(\sigma_i) = C_{i+1} \cap \mathrm{Tor}(C_{i+1})$ is one point, $1 \le i < s$;
- if $\sigma_i' = T_i \cap \tau'$ is a common edge, then $C_i \cap \mathrm{Tor}(\sigma_i') = X_0' \cap \mathrm{Tor}(\sigma_i')$ scheme-theoretically, that is, $(C_i \cdot \mathrm{Tor}(\sigma_i'))_p = (X_0' \cdot \mathrm{Tor}(\sigma_i'))_p$ for any common point $p \in \mathrm{Tor}(\sigma_i')$, and the same holds for the edges $\sigma_i'' = T_i \cap \tau''$.

The existence of such curves is an easy exercise (cf. [15, Lemma 3.5]).

Assume that the singularities of X_0 on $\mathrm{Tor}(\sigma)$, $\sigma = T_i \cap (\tau' \cup \tau'')$, $i = 1, \dots, s$, are semiquasihomogeneous in the sense of Section 3.1. Then we choose deformation patterns for each point $p \in X_0 \cap \mathrm{Sing}(Y_0)$ presented by a rational nodal curve, and orient the adjacency graph G from τ', τ'' to T_i's, and between T_i's, say, clockwise. The transversality of the triads $(\tau', (\tau')^-(\Gamma), X_0')$ and $(\tau'', (\tau'')^-(\Gamma), X_0'')$ follows from the free choice of d to be arbitrarily large, the transversality of nodal deformation patterns follows from [15, Lemma 5.5 (i)], since the nodes do not contribute to the left-hand sides of the corresponding inequalities, whereas the right-hand sides are always positive. Similarly all the triads $(T_i, T_i^-(\Gamma), C_i)$ are transversal by [15, Lemma 5.4 (ii)], where one simply has to verify that $T_i^- \neq \partial T_i$. Hence Theorems 2.4 and 3.3 apply, and we obtain a deformation $X_t \cap U$ glued out of the deformation patterns and the curves $C_1, \dots, C_s$. One can easily check that (34) holds.

If there are non-semiquasihomogeneous singularities of X_0 on $\mathrm{Tor}(\sigma)$, $\sigma = T_i \cap (\tau' \cup \tau'')$, $i = 1, \dots, s$, then we apply the procedure, described above (which, in fact, is a partial embedded resolution) to any such singular point. Since the Milnor numbers of the singularities of X_0 strictly decrease each time, after finitely many resolutions we come to the data satisfying the conditions of Theorem 3.3. Namely, we orient the adjacency graph G of the respective blow ups $(\mathrm{Tor}(\tau'))^*$, $(\mathrm{Tor}(\tau''))^*$ to the adjacent components of Y_0. We orient the components which appear at some stage clockwise like the triangles $T_1, \dots, T_s$ above. At last, we always orient G from the newly appeared components of Y_0 to the components, built on the preceding steps. Again, the transversality of the triads $\big((\mathrm{Tor}(\tau'))^*, ((\mathrm{Tor}(\tau'))^*)^-(\Gamma), (X_0')^*\big)$ and $\big((\mathrm{Tor}(\tau''))^*, ((\mathrm{Tor}(\tau''))^*)^-(\Gamma), (X_0'')^*\big)$ is provided by a sufficiently large d. The transversality of nodal deformation patterns and of the triads like $(T_i, T_i^-(\Gamma), C_i)$ has been explained above. Thus, Theorem 3.3 applies, and we obtain a nodal deformation X_t. We leave it to the reader to verify the equality (34). □

Acknowledgements

The research was supported by the grant no. 465 04 from the Israel Science Foundation and by the Hermann-Minkowski Minerva center for Geometry at the Tel Aviv University. The main part of this work was basically done during the author's stay at the Universität Kaiserslautern, supported by the Bessel research award from the Alexander von Humboldt Stiftung Foundation. The author is very grateful to Universität Kaiserslautern for the hospitality. I would like to thank G.-M. Greuel and I. Itenberg for useful discussions.

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Adjunction Conditions for One-Forms on Surfaces in Projective Three-Space

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Abstract

We study the relation between a certain graded part of the Jacobian ring of a projective hypersurface and a certain graded quotient for the Hodge filtration of its primitive cohomology, in the case that the hypersurface has at most isolated singularities. We distinguish a class of singularities for which this relation is best possible. The only interesting examples occur in the surface case.

Introduction

Let $X \subset P = \mathbb{P}^{n+1}(\mathbb{C})$ be a smooth hypersurface given by a homogeneous polynomial F of degree d. We let

$$\mathcal{S} = \mathbb{C}[X_0, \ldots, X_{n+1}]\,, \quad \mathcal{S}_k = \{G \in \mathcal{S} \mid G \text{ homogeneous of degree } k\}$$
$$\mathcal{J}(F) = \text{ ideal in } \mathcal{S} \text{ generated by } \partial_0 F, \ldots, \partial_{n+1} F\,, \quad \mathcal{R} = \mathcal{S}/\mathcal{J}(F)\,.$$

Then, by Griffiths [10], we have isomorphisms

$$\mathcal{R}_{d(p+1)-n-2} \xrightarrow{\cong} H_0^{n-p,p}(X)$$

where H_0 denotes primitive cohomology.

We intend to investigate the relation between $\mathcal{R}_{d(p+1)-n-2}$ and the cohomology of X in the case where X has isolated singularities. More precisely we investigate for which singularities there is a direct relation between $\mathrm{Gr}_F^{n-p} H^n(X, \mathbb{C})$ and $\mathcal{R}_{d(p+1)-n-2}$.

It appears that $\mathcal{R}_{d(p+1)-n-2}$ is closely related with the cohomology groups of the sheaf ω_X^{n-p} of residues of logarithmic $n-p+1$-forms on $\mathbb{P}^{n+1}$, whereas $\mathrm{Gr}_F^{n-p} H^n(X, \mathbb{C})$ is related to the $(n-p)$-th graded piece of the *filtered de*

1991 *Mathematics Subject Classification.* 14B05, 14C30, 32C35
Key words. Adjunction, differentials

Rham complex $\tilde{\Omega}^\bullet_X$ of X. So our question boils down to a comparison of ω_X^{n-p} (the sheaf of *Barlet forms*) and $\mathrm{Gr}_F^{n-p}\tilde{\Omega}^\bullet_X$.

The case $p = 0$ is classical. The n-th graded part of the filtered de Rham complex of X is the sheaf of meromorphic n-forms on X which lift holomorphically to any resolution. For each isolated singular point x of X one has an ideal $\mathcal{I} \subset \mathcal{O}_{X,x}$ with the property that the residue along X of a rational $n+1$-form

$$\omega = \frac{A\Omega}{F}$$

on $\mathbb{P}^{n+1}$ with a first order pole along X extends holomorphically to any resolution if and only if $A_x \in I$ for each x. These conditions are called the *adjunction conditions* and rational singularities are characterized by the fact that they do not impose adjunction conditions, i.e. $\mathcal{I} = \mathcal{O}_{X,x}$.

In this paper we study the case $p = 1$. It will become clear that the main case of interest is the surface case. We come to a satisfactory picture in the case that X is a surface with only a certain class of singularities, which includes the rational double points and cusps. We also deal with the case of surfaces in weighted projective spaces. We will give examples which show that the analogous class of singularities in dimensions different from two is probably empty.

A different approach to this problem occurs in [7].

1 Differentials on Spaces with Quotient Singularities

We recall some facts concerning sheaves of holomorphic differentials on singular spaces.

Recall that a *V-manifold* is a complex analytic space which is locally isomorphic to the quotient of a complex ball by a finite group of biholomorphic transformations. Local models for n-dimensional V-manifolds are of the form B^n/G where B^n is the n-dimensional open unit ball and G is a *small* finite subgroup of $\mathrm{U}(n,\mathbb{C})$ i.e. no element of G has 1 as an eigenvalue of multiplicity $n-1$.

For a V-manifold X, one defines sheaves $\tilde{\Omega}^p_X$, $p \geq 0$, on X as follows. Let Σ be the singular locus of X. Because a V-manifold is normal, it has codimension at least two in X. Let $j : X \setminus \Sigma \to X$ denote the inclusion map, and put

$$\tilde{\Omega}^p_X := j_*\Omega^p_{X\setminus\Sigma}. \tag{1}$$

Theorem 1.1. *1. If X is a V-manifold and $\pi : \tilde{X} \to X$ a resolution of singularities, then $\tilde{\Omega}^p_X \simeq \pi_*\Omega^p_{\tilde{X}}$.*

2. *If $X = Y/G$ with Y a complex manifold and G a finite group of biholomorphic transformations of Y, then $\tilde{\Omega}^p_X \simeq (\rho_*\Omega^p_Y)^G$.*

See [17, Sect. 1]. Moreover, we have

Theorem 1.2. *For any V-manifold X the complex $\tilde{\Omega}^\bullet_X$ is a resolution of the constant sheaf $\mathbb{C}_X$ and if moreover X is compact algebraic, the Hodge spectral sequence*

$$E_1^{pq} = H^q(X, \tilde{\Omega}^p_X) \Longrightarrow H^{p+q}(X, \mathbb{C}) \tag{2}$$

degenerates at E_1; this gives the Hodge filtration on the cohomology of X.

For spaces with arbitrary singularities the properties of these sheaves of differentials cannot all be preserved. Depending on the property one prefers there is a different generalisation of the above complex.

2 The Filtered de Rham Complex

First let us focus on the Hodge-theoretic property. A filtered complex $(\tilde{\Omega}^\bullet_X, F)$ which resolves the constant sheaf $\mathbb{C}_X$ and such that the Hodge spectral sequence (2) degenerates at E_1 has been constructed by Du Bois [6]. In the general case however the filtration F is no longer cutting off of the complex such that the graded complex $\tilde{\Omega}^p_X := \mathrm{Gr}^p_F \tilde{\Omega}^\bullet[-p]$ would be a single sheaf placed in degree p, but $\tilde{\Omega}^p_X$ is actually a complex with cohomology sheaves which may be non-zero on the whole range $0 \le j \le n-p$. This filtered complex $(\tilde{\Omega}^\bullet_X, F)$ is called the *filtered de Rham complex* of X.

Let X' denote the *weak normalization* of X. It is a complex variety over X which is homeomorphic to X, and sections of $\mathcal{O}_{X'}$ over an open set of X consist of those continuous functions whose restriction to the regular locus of X are holomorphic.

Suppose that X has isolated singularities only and that $\pi : Y \to X$ is a good resolution of singularities. This means that the inverse image $\pi^{-1}(\Sigma) =: E$, where Σ is the singular locus of X, is a divisor with normal crossings on Y. Then the cohomology sheaves of the filtered de Rham complex of X can be described as follows:

- $H^0(\tilde{\Omega}^0_X) = \mathcal{O}_{X'}$;
- $H^q(\tilde{\Omega}^p_X) = R^q\pi_*\Omega^p_Y(\log E)(-E)$ when $(p,q) \neq (0,0)$.

Here $\Omega^p_Y(\log E)(-E)$ is the kernel of the natural map $\Omega^p_Y \to \Omega^p_E/\text{torsion}$, or alternatively, the twist of the sheaf $\Omega^p_Y(\log E)$ of logarithmic p-forms with the ideal sheaf $\mathcal{O}_Y(-E)$ of E. For $q > 0$ the sheaf $R^q\pi_*\Omega^p_Y(\log E)(-E)$ has support on Σ and its stalk at a point $x \in \Sigma$ has finite length $b^{p,q}$. The numbers $b^{p,q}$ are the *Du Bois invariants* of the isolated singularity $X, x)$. See [18].

3 Barlet Differentials

We refer to [11] and [1] for details for this section. If X is a possibly singular hypersurface in a complex manifold P of dimension $n+1$, one obtains a natural notion of holomorphic q-forms by considering residues of meromorphic $(q+1)$-forms on P with logarithmic poles along X. These forms are the sections of the sheaf ω_X^q first considered by Barlet; for $q = n$ it coincides with Grothendieck dualizing sheaf. See [8, Sect. 2,3] for a discussion of the differences between $\tilde{\Omega}_X^n$ and ω_X^n.

Let P be compact and let $\mathcal{L}$ be a line bundle on P. We consider a hypersurface $X = V(F) \subset P$ with at most isolated singularities, given as the zero set of a global section F of $\mathcal{L}$. Let $\Omega^k(\ell X)$ denote the sheaf of germs of meromorphic k-forms on P with poles of order at most k along X. We have inclusions $\Omega^k(\ell X) \subset \Omega^k((\ell+1)X)$ and differentials $d : \Omega^k(\ell X) \to \Omega^{k+1}((\ell+1)X)$. We define

$$\Omega^k(\log X) = \ker\bigl(d : \Omega^k(X) \to \Omega^{k+1}(2X)/\Omega^{k+1}(X)\bigr) .$$

If X is smooth then the map $d : \Omega^n(X) \to \Omega^{n+1}(2X)/\Omega^{n+1}(X)$ is surjective; if X has isolated singularities, then the cokernel of this map is a skyscraper sheaf supported at the singular points of X. Its stalk at $x \in X$ is canonically isomorphic to $\Omega_X^{n+1} \otimes \mathcal{L}^2$ which in turn is non-canonically isomorphic to the quotient of $\mathcal{O}_{X,x}$ by the ideal generated by a local equation f of X and the partial derivatives of f. This stalk has finite length $\tau(X, x)$, the *Tjurina number* of (X, x). Then

$$\chi(\Omega_X^{n+1} \otimes \mathcal{L}^2) = \tau := \sum_{x \in X} \tau(X, x)$$

where χ stands for the Euler-Poincaré characteristic of sheaves.

We have the resolution

$$0 \to \Omega^n(\log X) \to \Omega^n(X) \xrightarrow{d} \Omega^{n+1}(2X)/\Omega^{n+1}(X) \to \Omega_X^{n+1} \otimes \mathcal{L}^2 \to 0$$

Consider the sheaf ω_X^{n-1} on X defined by $\omega_X^{n-1} := \Omega^n(\log X)/\Omega^n$, and define

$$c(\mathcal{L}) := \chi(\Omega^n \otimes \mathcal{L}) - \chi(\Omega^n) - \chi(\Omega^{n+1} \otimes \mathcal{L}^2) + \chi(\Omega^{n+1} \otimes \mathcal{L}).$$

Theorem 3.1. $\chi(\omega_X^{n-1}) = \tau + c(\mathcal{L})$.

This follows immediately from the exact sequences above.

4 Smoothing: the Specialization Sequence

We keep the notations of the previous section. Moreover we suppose that P is a Kähler manifold. Let G be a global section of $\mathcal{L}$ which does not vanish at the singularities of X. Then there exists $\epsilon > 0$ such that for $t \in \mathbb{C}$ with $0 < |t| < \epsilon$ the hypersurface X_t given by $F + tG = 0$ is smooth. We have the exact sequence

$$0 \to H^1(X, \tilde{\Omega}_X^{n-1}) \to \mathrm{Gr}_F^{n-1} H^n(\psi\mathbb{C}) \to \mathrm{Gr}_F^{n-1} H^n(\phi\mathbb{C}) \to H^2(X, \tilde{\Omega}_X^{n-1}) \to 0 \tag{3}$$

obtained by taking Gr_F^{n-1} from the specialisation sequence of this smoothing (cf. [17, Sect. 3]). Note that it implies the following analogue to Theorem 3.1:

Theorem 4.1.

$$\chi(\tilde{\Omega}_X^{n-1}) = s_{n-1} + c(\mathcal{L})$$

where the invariant $s_{n-1} = \sum_{x \in X} s_{n-1}(X, x)$ *is a sum of local contributions from each singularity:* $s_{n-1}(X, x) = \dim \mathrm{Gr}_F^{n-1} H^n(\phi\mathbb{C})_x$.

Corollary 4.2. *Suppose that* $X \subset P$ *is a hypersurface with isolated singularities such that* $\mathrm{Gr}_F^{n-1} \tilde{\Omega}_X^{\bullet}[n-1] \simeq \omega_X^{n-1}$. *Then* $s_{n-1} = \tau$.

In the next section we will prove the converse of this corollary also holds.

Singularity Spectrum

For an isolated hypersurface singularity $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ we define its *Milnor module*

$$\Omega_f := \Omega^{n+1} / df \wedge \Omega^n.$$

It carries a decreasing filtration $V^\bullet$ indexed by rational numbers a. The singularity spectrum is defined in terms of the V-filtration on Ω_f as follows: for $b \in \mathbb{Q}$ let $d(b) := \dim_{\mathbb{C}} \mathrm{Gr}_V^b \Omega_f$. We put

$$\mathrm{Sp}(f) := \sum_{b \in \mathbb{Q}} d(b)(b) \in \mathbb{Z}[\mathbb{Q}]$$

where the latter is the integral group ring of the additive group of the rational numbers. It is called the *singularity spectrum of* f.

The Hodge numbers s_k of the Milnor fibre of f are expressed in terms of the singularity spectrum of f by the formula

$$s_k = \sum_{n-k-1 < b \leq n-k} d(b).$$

See [16] for details.

5 Local Comparison

In this section we will derive a direct relation between the Barlet and Du Bois differentials for a complete variety X with isolated singularities.

Let $\pi : Y \to X$ be a good resolution: if Σ is the set of singular points of X, then $\pi^{-1}\Sigma = E$ is a divisor with normal crossings on Y with smooth irreducible components. In this case the graded quotients of the filtered de Rham complex of X are given by

$$\tilde{\Omega}^p_X = R\pi_*\Omega^p_Y(\log E)(-E)$$

for $p \geq 1$ and $\tilde{\Omega}^0_X$ is the single complex associated to the sequence $R\pi_*\mathcal{O}_Y \to R\pi_*\mathcal{O}_E \to \mathbb{C}_\Sigma$. By [1] we have $\omega^q_X \simeq \mathrm{Hom}(\Omega^{n-q}_X, \omega^n_X)$, and $\omega^n_X = \omega_X$ is the dualizing sheaf.

As we are interested in the case $p = n-1$ the first case to consider is where X is a curve and $n = 1$. Then $\tilde{\Omega}^0_X = \mathcal{O}_{X'}$ where X' is the *weak normalization* of X. Let us compare this with the sheaf $\omega^0_X = \mathrm{Hom}(\Omega^1_X, \omega_X)$.

This question was first considered in the plane curve case by Kyoji Saito [15]. He proved that in this case the sheaf $\Omega^1(\log X)$ is locally free.

Lemma 5.1. *Let X be a plane curve. Then ω^0_X is torsion free of rank one. There is a natural injection $\tilde{\mathcal{O}}_X \to \omega^0_X$, the quotient $\omega^0_X/\mathcal{O}_X$ is concentrated in the singular locus of X and its stalk at $x \in X$ has length $\tau(X, x)$. Suppose $X = V(f) \subset (\mathbb{C}^2, 0)$ is a reduced quasi-homogeneous plane curve singularity: $w_x x f_x + w_y y f_y = f$. Then the forms $\frac{df}{f}$ and $\frac{w_y y dx - w_x x dy}{f}$ form a local basis of $\Omega^1(\log X)$. If f is not quasi-homogeneous, then $\frac{df}{f} \in (x, y)\Omega^1(\log X)$.*

Proof. On the regular locus of X we have an isomorphism between ω^0_X and $\mathcal{O}_X$ by the residue map. Let us check that it extends to the desired injection. This, and the fact that ω^0_X is torsion free, can be checked locally near every singular point. So consider the case of a reduced plane curve singularity $(X, 0) \subset (\mathbb{C}^2, 0)$ given by a squarefree function germ $f \in \mathbb{C}\{x, y\}$. We have $\mathbb{C}\{x, y\}dx \oplus \mathbb{C}\{x, y\}dy \subset \Omega^1(\log X)_0 \subset \mathbb{C}\{x, y\}dx/f \oplus \mathbb{C}\{x, y\}dy/f$ so writing $\mathcal{O} = \mathcal{O}_{X,0}$ we have $\omega^0_{X,0} \subset \mathcal{O}dx/f \oplus \mathcal{O}dy/f$, so it is a subsheaf of a locally free sheaf. Hence ω^0_X is torsion free, and its rank is the same as the rank of its restriction to the regular locus, which equals one.

The element $a\, dx/f + b\, dy/f$ with $a, b \in \mathcal{O}$ belongs to $\omega^0_{X,0}$ iff $a\eta - b\xi = 0$ where ξ, η are the images of f_x, f_y in $\mathcal{O}$ respectively. This equation has the obvious solution $a = \xi, b = \eta$ which corresponds to the germ df/f and hence to $1 \in \mathcal{O}$. The injection $\mathcal{O} \hookrightarrow \omega^0_X$ is therefore given by $c \mapsto c\xi\, dx/f + c\eta\, dy/f$.

To compute the length of the stalk of $\omega^0_X/\mathcal{O}_X$ at a singular point of X we use a global argument, even if the question is local. To this end, suppose that $x \in X$ is the unique singular point of a plane projective curve of degree d. Then the length of $\omega^0_X/\mathcal{O}_X$ at x is the same as the Euler Poincaré characteristic $\chi(\omega^0_X/\mathcal{O}_X) = \chi(\omega^0_X) - \chi(\mathcal{O}_X)$. Recall that $\chi(\omega^0_X) = \tau + 1 - c_d$ whereas $\chi(\mathcal{O}_X) = 1 - c_d$. Hence $\chi(\omega^0_X/\mathcal{O}_X) = \tau$.

To show that ω^0_X contains $\tilde{\mathcal{O}}_X$ observe that for any $\omega \in \Omega^1_X$ and $a \in \tilde{\mathcal{O}}_X$, the product $a\omega$ lies in $\Omega_{\tilde{X}}$ so has no residues; hence it belongs to ω_X, i.e. $a \in \mathrm{Hom}(\Omega^1_X, \omega_X)$.

The remaining statements are left to the reader. See also [15, Proof of Theorem 2.11]. □

Example 5.2. Let X be a projective plane curve with only ordinary double points. Then $\omega^0_X \simeq \tilde{\mathcal{O}}_X$. In particular $H^0(X, \mathcal{O}_X) \to H^0(X, \omega^0_X)$ is an isomorphism if and only if X is irreducible.

Recall that X' denotes the weak normalization of the curve X.

Theorem 5.3. *Suppose that X is a plane curve such that $\mathcal{O}_{X'} = \omega^0_X$. Then X is smooth.*

Proof. If $\mathcal{O}_{X'} = \omega^0_X$ then $\mathcal{O}_{X'}/\mathcal{O}_X$ and $\omega^0_X/\mathcal{O}_X$ have stalks of the same lengths at all singular points. Hence for such a singularity one has the equality $\delta - r + 1 = \tau$. By [5, Lemma 6.1.2 and Cor. 6.1.4] $\tau \geq \delta + m - r$ where m is the multiplicity. Hence, $m = 1$ so X has no singular point.

Here is another argument, based on the spectrum. Let $Q^f = \Omega^2/df \wedge \Omega^1$ with its spectral V-filtration. We have $fQ^f \subset V^{>0}Q^f$ so

$$\tau = \dim Q^f/fQ^f \geq \dim Q^f/V^{>0} = \delta$$

with equality iff $r = 1$ and $fQ^f = V^{>1}$. So X is an irreducible plane curve singularity, and multiplication by f gives an isomorphism $Q^f/V^{>0} \to V^{>0}$. If $\alpha_1, \ldots, \alpha_\delta$ are the positive spectral numbers of f in increasing order, then the spectrum of f is $-\alpha_\delta, \ldots, -\alpha_1, \alpha_1, \ldots, \alpha_\delta$. We find that $\alpha_j + \alpha_{\delta-j+1} \geq 1$ for all j. This implies that the surface singularity with equation $f(x, y) + z^2 = 0$ has geometric genus $p_g \geq \mu/4$, but Némethi [14] has shown that for such a surface singularity $p_g \leq \mu/6$. This means that the spectral numbers have to lie closer to the middle than forced by the condition $fQ^f = V^{>0}Q^f$.

Another argument is based on Hertling's conjecture [12] on the variance of the spectrum, which has been proved by Brélivet in the curve case [3]. If $\alpha_j + \alpha_{\delta-j+1} \geq 1$, then $\alpha_j^2 + \alpha_{\delta-j+1}^2 \geq \frac{1}{2}$ so $\sum_{i=1}^{\mu} \alpha_j^2 \geq \mu/4$. On the other hand, by Hertling's conjecture

$$\frac{1}{\mu} \sum_{i=1}^{\mu} \alpha_j^2 \leq \frac{1}{12}(\alpha_\mu - \alpha_1) \leq \frac{1}{6}.$$

□

Next we turn to the study of ω_X^{n-1} in the case $n \geq 2$. Note that ω_X^{n-1} coincides with Ω_X^{n-1} on $X \setminus \Sigma$. Moreover it fits in the exact sequence

$$0 \to \omega_X^{n-1} \to \Omega^n(X) \otimes \mathcal{O}_X \to \Omega^{n+1}(2X) \otimes \mathcal{O}_X$$

hence $\omega_X^{n-1} \simeq j_*\Omega_X^{n-1}$ where $j : X \setminus \Sigma \hookrightarrow X$. We see that $\omega_X^{n-1} \simeq \tilde{\Omega}_X^{n-1}$ if X is a V-manifold. Let us look for the class of singularities which one may admit for this to be true. Consider a good resolution $\pi : (Y, E) \to (X, x)$ of an isolated n-dimensional singularity. Define

$$q'(X,x) = \ell \left(j_*\Omega_X^{n-1} / \pi_*\Omega_Y^{n-1}(\log E)(-E) \right)_x$$

and the Du Bois invariant (cf. [18])

$$b^{n-1,1}(X,x) = \ell \left(R^1\pi_*\Omega_Y^{n-1}(\log E)(-E) \right)_x .$$

Then clearly one has the following theorem:

Theorem 5.4. *Let X be an n-dimensional complex space with only isolated singularities, with $n \geq 2$. The following are equivalent:*

1. $\omega_X^{n-1} = \tilde{\Omega}_X^{n-1}$;
2. $q'(X,x) = b^{n-1,1}(X,x) = 0$ *for each singular point of X.*

Indeed, if $q' = 0$ then $\omega_X^{n-1} = j_*\Omega_X^{n-1} = \pi_*\Omega_Y^{n-1}(\log E)(-E)$, and if moreover $b^{n-1,1} = 0$ then $R^1\pi_*\Omega_Y^{n-1}(\log E)(-E) = 0$ so

$$\pi_*\Omega_Y^{n-1}(\log E)(-E) = R\pi_*\Omega_Y^{n-1}(\log E)(-E) .$$

Conversely, the equality $\omega_X^{n-1} = \tilde{\Omega}_X^{n-1}$ implies equality of their Euler characteristics, whose difference is equal to $q' + b^{n-1,1}$.

Corollary 5.5. *Suppose that (X,x) is an isolated hypersurface singularity. Then $q'(X,x) + b^{n-1,1}(X,x) = \tau(X,x) - s_{n-1}(X,x)$.*

Next we investigate which surface singularities have $q'(X,x) = b^{1,1}(X,x) = 0$. First recall the following result of Wahl [20, Corollary 2.9] :

Theorem 5.6. *For a two-dimensional smoothable normal Gorenstein singularity with Milnor fibre F write $\mu = \mu_0 + \mu_+ + \mu_-$ from diagonalizing the intersection pairing on $H_2(F, \mathbb{R})$. Then $\tau \geq \mu_0 + \mu_- = \mu - (2p_g - 2g - b)$.*

Here, p_g is the geometric genus, b is the first Betti number of the dual graph of a good resolution and g is the sum of the genera of the irreducible components of its exceptional divisor.

Theorem 5.7. *For a two-dimensional smoothable normal Gorenstein singularity the following are equivalent:*

1. $b^{1,1} = q' = 0$;
2. $\tau = \mu_0 + \mu_-$ *and* $g = 0$.

Proof. In the surface case we have $\mu = s_0 + s_1 + s_2$ where $s_2 = p_g$ and $s_0 = p_g - g - b$. If $b^{1,1} = q' = 0$ then $\tau = s_1 = \mu - (2p_g - g - b) = \mu_0 + \mu_- - g$ hence by Theorem 5.6 we have $g = 0$ and $\tau = \mu_0 + \mu_-$. The converse implication is similar. □

Corollary 5.8. *The following surface singularities satisfy* $q' = b^{1,1} = 0$*:*

1. *rational double points (ADE-singularities)*
2. *cusps (singularities of type* T_{pqr} *with* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$*);*
3. *generic* μ*-constant deformations of* $z^2 + x^{2a+1} + y^{2a+2}$ *(those which have minimal Tjurina number).*

Indeed, the rational double points have $\mu = \tau = \mu_-$, whereas the cusps have $\mu_+ = 1 = \mu - \tau$. Finally, the last category of examples was considered in [20, Example 4.6] and shown by Zariski to have $3a(a+1) = \mu - 2p_g$.

Example 5.9. According to a computation using Singular, the singularity $x^7 + x^4y^2 + x^2y^4 + y^7 + z^2$ has $\mu = 27$ and $\tau = 23$. Moreover, $p_g = 3$ and $b = 2$, $g = 0$, and $\mu - (2p_g - b) = 27 - 4 = \tau$ so $q' = b^{1,1} = 0$.

Example 5.10. The singularity $x^3 + y^{10} + z^{19}$ is considered in [13, Sect. 5]. A generic μ-constant deformation has $\tau = 246$ whereas $\mu = 324$ and $p_g = 39$, $g = b = 0$. So $\mu - \tau = 2p_g$ and $q' = b^{1,1} = 0$.

Example 5.11. It is not always so that for a generic μ-constant deformation of a quasi-homogeneous surface singularity one has $q' = b^{1,1} = 0$. The exceptional unimodal non-quasi-homogeneous singularities have $\tau = \mu - 1$, $p_g = 1$ and $g = b = 0$. So $\mu_0 = 0$, $\mu_- = \mu - 2$.

Take the singularity $x^5 + y^{11} + z^2$, also considered in [13, Sect. 5]. It has $\mu = 40$ and $\tau_{\min} = 34$ whereas $p_g = 4$, $g = 0$. That $\mu - \tau \leq 6$ can be seen from the spectral numbers. The submodule fQ^f of $Q^f = \Omega^3/df \wedge \Omega^2$ is cyclic with generator $[f\omega] \in V^{>\frac{87}{110}} = V^{\frac{89}{110}}$. But then

$$[xf\omega] \in V^{\frac{111}{110}}, \quad [yf\omega] \in V^{\frac{99}{110}} = V^{\frac{101}{110}}, \quad [y^2 f\omega] \in V^{\frac{111}{110}}.$$

So the spectral numbers of the filtration of fQ^f induced by V have the gaps $\frac{91}{111}$ and $\frac{93}{110}$.

Question. Is this kind of lower bound for τ on the μ-constant stratum provided by the spectral numbers sharp?

Remark 5.12. The only example of an isolated hypersurface singularity in dimension $n \geq 3$ I know that satisfies $q' = b^{n-1,1} = 0$ is the ordinary double point in dimension three. If Hertling's conjecture is valid, I can prove that no such singularities exist in dimension ≥ 9, and no rational singularity in dimension ≥ 6.

Consider the special case of "double suspension" singularities $g = f(x, y) + zw$ with f squarefree. These are rational, and they belong to our class iff $\tau_f = \delta_f$. By the inequality $\tau \geq \delta + m - r$ for curve singularities this implies that $r = m$ so f is a μ-constant deformation of a homogeneous singularity of degree m. This has finite order monodromy and highest spectral number $1 - \frac{2}{m}$ and this implies that $V^{-\frac{2}{m}} Q^f$ is in the kernel of multiplication by f. Hence $\tau_f \geq \dim V^{-\frac{2}{m}} Q^f = \delta_f + 2m - 5$ so $\tau_f - \delta_f \geq 1$ unless $m = 2$ and we have the ordinary double point!

6 Application to Projective Hypersurfaces

In this section we come back to the problem mentioned in the introduction: investigate the relation between the cohomology of a projective hypersurface with isolated singularities and certain graded parts of its Jacobian ring.

We consider a hypersurface $X = V(F) \subset \mathbb{P}^{n+1}$ of degree d with at most isolated singularities. Recall that $\Omega^k(\ell X)$ is the sheaf of germs of meromorphic k-forms on $\mathbb{P}^{n+1}$ with poles of order at most k along X. We have inclusions $\Omega^k(\ell X) \subset \Omega^k((\ell+1)X)$ and differentiation $d : \Omega^k(\ell X) \to \Omega^{k+1}((\ell+1)X)$. We defined

$$\Omega^k(\log X) = \ker(d : \Omega^k(X) \to \Omega^{k+1}(2X)/\Omega^{k+1}(X).$$

If X is smooth then the map $d : \Omega^n(X) \to \Omega^{n+1}(2X)/\Omega^{n+1}(X)$ is surjective; if X has isolated singularities, then the cokernel $\Omega_X^{n+1}(2X)$ of this map is a skyscraper sheaf concentrated at the singular points of X. Its stalk at $x \in X$ is isomorphic to the quotient of $\mathcal{O}_{X,x}$ by the ideal generated by a local equation f of X and the partial derivatives of f. This stalk has finite length $\tau(X, x)$, the *Tjurina number* of (X, x).

By Bott's vanishing theorem [2]

$$H^i(\Omega^n(X)) = H^i(\Omega^{n+1}(2X)/\Omega^{n+1}(X)) = 0 \text{ for } i > 0,$$

so we have a resolution

$$0 \to \Omega^n(\log X) \to \Omega^n(X) \xrightarrow{d} \Omega^{n+1}(2X)/\Omega^{n+1}(X) \to \Omega_X^{n+1}(2X) \to 0$$

of $\Omega^n(\log X)$ by sheaves which are acyclic, hence the cohomology groups of the complex of global sections of these sheaves

$$0 \to H^0(\Omega^n(X)) \to H^0(\Omega^{n+1}(2X)/\Omega^{n+1}(X)) \to H^0(\Omega^{n+1}_X(2X)) \to 0$$

are isomorphic to the cohomology groups of $\Omega^n(\log X)$. Explicitly we have the complex

$$0 \to \mathcal{S}_{d-n-2} \xrightarrow{E} \mathcal{S}^{\oplus n+2}_{d-n-1} \xrightarrow{h} \mathcal{S}_{2d-n-2}/F\mathcal{S}_{d-n-2} \to H^0(\Omega^{n+1}_X(2X)) \to 0$$

where $E(B) = (X_0B, \dots, X_{n+1}B)$ (corresponding to the Euler vector field) and

$$h(A_0, \dots, A_{n+1}) = \sum_{i=0}^{n+1} A_i \frac{\partial F}{\partial X_i} \bmod F$$

If X is smooth, we have the residue exact sequence

$$0 \to \Omega^n \to \Omega^n(\log X) \to \Omega^{n-1}_X \to 0$$

by which the cohomology groups of $\Omega^n(\log X)$ are identified with the primitive cohomology groups $H^{n-1,1}_{\text{prim}}(X)$.

Lemma 6.1. *If X is smooth, then*

$$\sum_{n\geq 0} \dim \mathcal{R}_n t^n = \left(\frac{t^{d-1}-1}{t-1}\right)^{n+2}.$$

The proof uses the fact that the partials of F form a regular sequence in $\mathcal{S}$, so we have the Koszul complex resolving $\mathcal{R}$.

Proposition 6.2. *Let $\mathcal{R} = \mathcal{S}/\mathcal{J}(F)$ as in the introduction. Then*

$$\dim \mathcal{R}_{2d-n-2} = c_d - \dim H^0(X, \Omega^n(\log X)).$$

Moreover, the map $H^i(\Omega^n(logX)) \to H^i(\omega_X^{n-1})$ is an isomorphism for $i \neq n-1$ and we have the exact sequence

$$0 \to H^{n-1}(\Omega^n(\log X)) \to H^{n-1}(\omega_X^{n-1}) \to H^n(\Omega^n) \to 0.$$

Proof. Note that

$$\mathcal{R}_{2d-n-2} = \operatorname{coker}(\mathcal{S}^{\oplus n+2}_{d-n-1} \xrightarrow{h} \mathcal{S}_{2d-n-2}/F\mathcal{S}_{d-n-2})$$

and that $\ker(h) = \operatorname{im}(E)$ in the smooth case. Hence

$$c_d = \dim \mathcal{S}_{2d-n-2} - (n+2)\dim \mathcal{S}_{d-n-1} = \binom{2d-1}{n+1} - (n+2)\binom{d}{n+1}$$

and

$$\ker(h)/\mathrm{im}(E) = H^0(X, \Omega^n(\log X))) = \ker\left(\mathcal{S}_{d-n-1}^{\oplus(n+2)} \to \mathcal{S}_{2d-n-2}\right).$$

To prove the remaining statements, note that $H^i(\Omega^n) = 0$ for all $i \neq n$. So we only have to show that $H^n(\Omega^n) \to H^n(\Omega^n(\log X))$ is the zero map. We do this by induction on n. The case $n = 1$ is obvious. Let $n \geq 2$. Consider a general hypersurface $L \subset \mathbb{P}^{n+1}$; we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^n & \longrightarrow & \Omega^n(\log L) & \longrightarrow & \Omega^{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^n(\log X) & \longrightarrow & \Omega^n(\log L + X) & \longrightarrow & \Omega^{n-1}(\log X \cap L) & \longrightarrow & 0\,,
\end{array}$$

which gives rise to the commutative diagram

$$\begin{array}{ccc}
H^{n-1}(\Omega^{n-1}) & \xrightarrow{a} & H^{n-1}(\Omega^{n-1}(\log X \cap L)) \\
{\scriptstyle b}\downarrow & & \downarrow \\
H^n(\Omega^n) & \xrightarrow{c} & H^n(\Omega^n(\log X))
\end{array}$$

By Lefschetz' theory, the Gysin map b is an isomorphism, and by induction hypothesis a is the zero map. Hence c is the zero map. $\square$

Corollary 6.3. *Suppose that $H^0(\Omega^n(\log X)) = 0$. Then $\mathcal{R}_{2d-n-2}$ has the expected dimension c_d and we have the exact sequence*

$$0 \to H^1(\Omega^n(\log X)) \to \mathcal{R}_{2d-n-2} \to H^0(X, \Omega_X^{n+1}(2X)) \to H^2(\Omega^n(\log X)) \to 0 \tag{4}$$

Moreover $H^i(\Omega^n(\log X)) = 0$ for all $i \geq 2$.

From now on we suppose that $\omega_X^{n-1} \simeq \tilde{\Omega}_X^{n-1}$, i.e. $q' = b^{n-1,1} = 0$ for all singular points of X. Moreover we will suppose that $n \geq 2$, as in the case $n = 1$ there are no singular points on X. This guarantees that we have isomorphisms

$$H^i(\Omega^n(\log X)) \simeq \mathrm{Gr}_F^{n-1} H^{n-1+i}(X)_{\mathrm{prim}}. \tag{5}$$

Lemma 6.4. $H^i(\mathbb{P}^{n+1}, \Omega^n(\log X)) = 0$ *for all* $i \neq 1, 2$

Proof. This follows from (5) and the fact that for a hypersurface X in $\mathbb{P}^{n+1}$ with isolated singularities one has $H^k(\mathbb{P}^{n+1}, \mathbb{Q}) \simeq H^k(X, \mathbb{Q})$ for all $k \neq n, n+1, 2n$. $\square$

Corollary 6.5. $\dim \mathcal{R}_{2d-n-2} = c_d$. *Moreover we have the exact sequence*

$$0 \to H^1(\Omega^n(\log X)) \to \mathcal{R}_{2d-n-2} \to H^0(X, \Omega_X^{n+1}(2X)) \to H^2(\Omega^n(\log X)) \to 0 \tag{6}$$

Let G be a homogeneous form of degree d which does not vanish at the singularities of X. Then there exists $\epsilon > 0$ such that for $t \in \mathbb{C}$ with $0 < |t| < \epsilon$ the hypersurface X_t given by $F + tG = 0$ is smooth. We let $\mathcal{R}^t$ denote its Jacobian ring. Note that $\lim_{t\to 0} \mathcal{J}(F+tG)_k$ makes sense in the Grassmannian of $\mathcal{S}_k$, and that it contains $\mathcal{J}(F)_k$. Hence, as $\mathcal{R}_k$ and $\mathcal{R}^t_k$ have equal dimension, they are equal. So we have the exact sequence

$$0 \to H^1(X, \tilde{\Omega}^{n-1}_X)_{\mathrm{prim}} \to \mathcal{R}_{2d-n-2} \to \mathrm{Gr}^{n-1}_F H^n(\phi C) \to H^2(X, \tilde{\Omega}^{n-1}_X)_{\mathrm{prim}} \to 0 \tag{7}$$

Under our hypotheses, the sequences (6) and (7) are identical!

Remark 6.6. Our reasoning also applies to hypersurfaces in weighted projective spaces, as long as they are transverse to the singular strata (so have isolated singularities only at regular points of the ambient space).

Acknowledgements

Our interest in this problem was raised by a question of Remke Kloosterman.

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Sextic Surfaces with 10 Triple Points

Jan Stevens

To Gert-Martin Greuel on his 60th birthday

Abstract

All families of sextic surfaces with the maximal number of isolated triple points are found.

Evaluation of the conditions imposed by ten triple points requires the solution of complicated systems of equations. Thanks to Gert-Martin's efforts the computer algebra system `Singular` [3] is around, making such computations possible.

Surfaces in $\mathbb{P}^3(\mathbb{C})$ with isolated ordinary triple points have been studied in [2]. The results are most complete for degree six. A sextic surface can have at most ten triple points, and such surfaces exist. For up to nine triple points [2] contains a complete classification. In this note I achieve the same for ten triple points.

The study of sextics with nine triple points is easier, because they do lie on a quadric Q. Given such a sextic with equation F the general element of the pencil $\alpha F + \beta Q^3$ is again a sextic with nine isolated triple points. It turns out that such a pencil also contains reducible surfaces, which are much easier to construct. The same argument shows that a sextic with ten triple points is a degeneration of one with nine (simply choose a quadric through nine of the ten points).

Therefore one can look for sextics with ten triple points in each of the five families given in [2]. In fact it suffices to consider only those two, which have a rather nice description. The one-parameter family of examples [2] was found in the first family by imposing extra symmetry. The surfaces in the other family have the simplest equations of all. Nevertheless I could not find a single solution, because I was looking at the wrong place: as explained below, I made an unwarranted general position assumption. Different families of sextics are connected by Cremona transformations. By transforming the

1991 *Mathematics Subject Classification.* 14J10, 14J26
Key words. sextic surface, triple point, Cremona transformation

known example I found the right assumptions. The equations for a tenth triple point in the family become very simple, as I had hoped all the time.

The equations for the tenth point come from the ten second partial derivatives of the defining function. The families with nine triple points depend on seven or eight moduli. The unknown position of the tenth singular point adds three more variables. One gets a very complicated system, which only can be attacked by using the special structure of the equations. The main problem is to cut away unwanted solutions corresponding to surfaces with nonisolated singularities. In homogeneous equations I get rid of solutions lying in a hypersurface $Q=0$ by adding the inhomogeneous equation $Q-1$, computing a Gröbner base and finally homogenising again with the function Q.

The main result is that there are four different families of sextics with ten triple points, each depending on three moduli. They are distinguished by the number of (-1)-conics, which ranges from two to five.

1 Nine Triple Points

The clue to the classification of sextics with many triple points is the study of exceptional curves of the first kind on the minimal resolution. Let X be a sextic with isolated triple points and $\widetilde{X}$ its minimal resolution. Whenever the canonical divisor $K_{\widetilde{X}}$ is effective, any exceptional curve of the first kind E is automatically a component, as $K_{\widetilde{X}}\cdot E=-1$. Therefore E comes from a rational curve on X which is contained in the base locus of the system of quadrics through the triple points. Assume that X has nine triple points $P_1,\dots,P_9$. Let Q be the unique (irreducible) canonical quadric surface and let $K=Q\cdot X$ be the adjoint curve. The resolution $\widetilde{X}$ has exactly three disjoint (-1)-curves C_1,C_2,C_3 of degrees c_1,c_2,c_3 which are components of K. There are two possibilities: either $C_1+C_2+C_3=K$ or not. In the first case $\widetilde{X}$ is a $K3$ surface blown up in three points. By [2], Prop. 4.10, there are up to permutation three choices for the degrees:

$$(c_1,c_2,c_3)\in\{(2,2,8),(2,4,6),(4,4,4)\}.$$

In the second case one ends up with an effective canonical divisor after blowing down C_1, C_2 and C_3. Now $\widetilde{X}$ is the blowup of a minimal properly elliptic surface in three points and by [2], Prop. 4.9, up to permutation

$$(c_1,c_2,c_3)\in\{(2,2,2),(2,2,4)\}.$$

In all cases the curves C_i of degree c_i can be constructed as complete intersection of Q and a surface of degree $c_i/2$. In particular, if $c_i=2$, five points

type	surface	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
$(4,4,4)$	K_1	0	2	1	1	1	1	1	1	1
	K_2	1	0	2	1	1	1	1	1	1
	K_3	2	1	0	1	1	1	1	1	1
$(2,4,6)$	L_1	1	0	0	0	0	1	1	1	1
	K_2	0	2	1	1	1	1	1	1	1
	C_3	2	1	2	2	2	1	1	1	1
$(2,2,8)$	L_1	1	0	0	0	0	1	1	1	1
	L_2	0	1	1	1	0	0	0	1	1
	Q_3	2	2	2	2	3	2	2	1	1

Table 1: Multiplicities at the singular points in the $K3$-case

lie on a conic in a plane. Such a conic will be called a (-1)-conic. I call the triple (c_1, c_2, c_3) the *type* of the surface.

For $(c_1, c_2, c_3) \in \{(2,2,8), (2,4,6), (4,4,4)\}$ there exists a seven parameter family of sextic surfaces with nine triple points ([2], Thm. 4.13). Moreover X occurs in a pencil of the form

$$\begin{aligned} \alpha\, K_1K_2K_3 + \beta\, Q^3 = 0\ , &\quad \text{if } (c_1,c_2,c_3) = (4,4,4)\ , \\ \alpha\, L_1K_2C_3 + \beta\, Q^3 = 0\ , &\quad \text{if } (c_1,c_2,c_3) = (2,4,6)\ , \\ \alpha\, L_1L_2Q_3 + \beta\, Q^3 = 0\ , &\quad \text{if } (c_1,c_2,c_3) = (2,2,8)\ . \end{aligned}$$

Here Q is the unique canonical surface, L_i stands for a linear form, K_i for a singular quadric, C_3 defines a four nodal cubic and Q_3 a quartic surface with a triple point and six double points. The multiplicities of the three surfaces in the nine singular points are displayed in Table 1. Note that I do not distinguish between a surface and the form defining it, which I also call its equation. Figure 1 shows a surface of type $(4,4,4)$. The picture was made with Stephan Endraß' program `surf` [1].

For $(c_1, c_2, c_3) \in \{(2,2,2), (2,2,4)\}$ there exists an eight parameter family of sextic surfaces with nine triple points ([2], Thm. 4.14). Moreover X occurs in a web of the form

$$\begin{aligned} \alpha\, L_1L_2L_3C + \beta\, L_1L_2L_3HQ + \gamma\, Q^3 = 0\ , &\quad \text{if } (c_1,c_2,c_3) = (2,2,2)\ , \\ \alpha\, L_1L_2K_3Q' + \ \beta\, L_1L_2K_3Q\ + \gamma\, Q^3 = 0\ , &\quad \text{if } (c_1,c_2,c_3) = (2,2,4)\ . \end{aligned}$$

Again L_i stands for a linear form. In the case $(2,2,2)$ the plane H passes through the three triple points not lying on the double lines of $L_1L_2L_3$. The reducible cubic HQ is an element of the pencil of cubics through all points with double points in P_7, P_8 and P_9, and C is another such cubic. In the case $(2,2,4)$ the surface K_3 is a quadric cone and Q' is a smooth quadric not passing through P_6. The multiplicities in the nine triple points of the surfaces giving (-1)-curves are displayed in Table 2. A surface of type $(2,2,2)$ is shown in Figure 2.

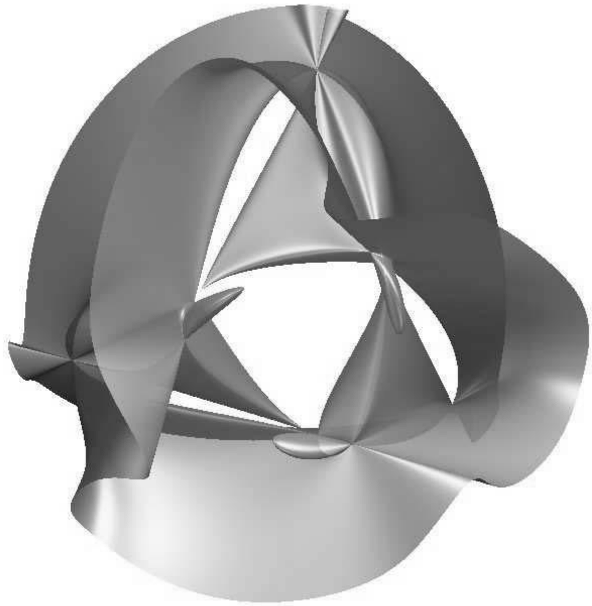

Figure 1: A surface of type $(4,4,4)$ with nine triple points

type	surface	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
$(2,2,2)$	L_1	0	0	1	1	1	1	1	0	0
	L_2	1	1	0	0	1	1	0	1	0
	L_3	1	1	1	1	0	0	0	0	1
$(2,2,4)$	L_1	0	0	1	1	1	1	1	0	0
	L_2	1	1	0	0	1	1	0	1	0
	K_3	1	1	1	1	0	1	1	1	2

Table 2: Multiplicities in the properly elliptic case

The three families of blown-up $K3$-surfaces are related via Cremona transformations. The ordinary plane Cremona transformation is the rational map defined by the linear system of conics through three points in general position. In suitable coordinates it can be given by the formula $(x:y:z) \mapsto (1/x:1/y:1/z)$. This formula generalises to higher dimensions. In particular, the space transformation, also known as *reciprocal transformation*,

$$(x:y:z:w) \mapsto \left(\frac{1}{x}:\frac{1}{y}:\frac{1}{z}:\frac{1}{w}\right)$$

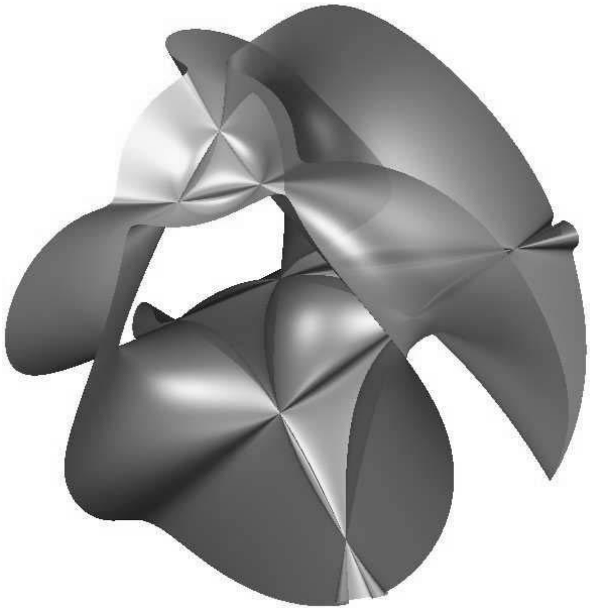

Figure 2: A surface of type $(2, 2, 2)$ with nine triple points

simultaneously blows up the vertices and blows down the faces of the coordinate tetrahedron. The vertices are called *fundamental points* of the reciprocal transformation. Let $X \subset \mathbb{P}^3$ by a surface of degree d not containing any of the coordinate planes. Let $m_1, \ldots, m_4$ be the multiplicities of X in the fundamental points. Then the image Y of X is a surface of degree $3d - m_1 - \cdots - m_4$. In many cases X will be singular in the fundamental points with singularities obtained from contracting the intersection curves of X with the coordinate planes.

Specifically, a reciprocal transformation with fundamental points P_1, P_2, P_4 and P_5 (see Table 1) will transform a surface of type $(4, 4, 4)$ into one of type $(2, 4, 6)$. To get from there to a surface of type $(2, 2, 8)$ one can apply a transformation with fundamental points P_2, P_5, P_6 and P_7. The two other families are also related via reciprocal transformations.

2 Families with Ten Triple Points

For a sextic surface X with ten isolated triple points $p_g(\widetilde{X}) = 0$ ([2], Cor. 4.6) so the ten points never lie on a quadric. Leaving out one point the remaining nine triple points determine a quadric Q. The general element of the pencil spanned by the sextic and Q^3 is a surface with nine isolated triple points and belongs therefore at least to one of the five families above.

Lemma 2.1. *A sextic with ten triple points belongs to the closure of the family of type* $(2,2,2)$ *or of the family of type* $(4,4,4)$.

Proof. No three triple points lie on a line ([2], Lemma 3.1). Two different (-1)-conics meet in two triple points ([2], Cor. 4.8). I study the planes containing (-1)-conics. If three planes have a line in common, there would be $2+3\cdot 3=11$ triple points; if four planes have a triple point in common, they contain $1+6+4=11$ points, again contradicting that the surface has ten triple points. The number of planes is at most six. If there are exactly six, then each triple point lies in three planes, and leaving out one of the points gives sextics with three planes, so of type $(2,2,2)$. If there are five planes, ten lines each contain two triple points, so five points lie in three planes and five only in two. Leaving out a point in only two planes gives a sextic of type $(2,2,2)$. If there are four planes and only one point lies in three of them, the fourth plane contains six points. So there are at least two points in three planes each. A plane containing them both has only four points on intersection lines so leaving out the fifth point in such a plane gives a sextic of type $(2,2,2)$. If there are three planes, they can contain at most nine points, so leaving out the point not on a plane keeps three planes. If there are only two planes one can leave out a point on the intersection line to get sextics without planes, so of type $(4,4,4)$. If there is only one plane leave out any point in that plane. □

2.1 Type $(2,2,2)$

I describe equations for the surfaces. After a change of coordinates one may assume that the three planes are the sides of the coordinate tetrahedron. The remaining coordinate transformations are given by diagonal matrices. On each axis in affine space lie two points and three additional ones lie on the triangle at infinity. I take them to be $P_7=(0:1:\lambda:0)$, $P_8=(\mu:0:1:0)$ and $P_9=(1:\nu:0:0)$. The equation has now the form

$$\alpha\, Q^3+\beta\, xyztQ+\gamma\, xyzK\ ,$$

with K is a four-nodal cubic passing through $(0:0:0:1)$. With notation slightly different from [2] I get

$$\begin{aligned} Q\ =\ & c_1c_2c_3t^2+t(b_1c_2c_3x+b_2c_1c_3y+b_3c_1c_2z)\\ &+c_2c_3x(\nu x-y-\mu\nu z)+c_1c_3y(\lambda y-z-\lambda\nu x)+c_1c_2z(\mu z-x-\lambda\mu y)\ ,\end{aligned}$$

$$\begin{aligned} K\ =\ & t^2(\lambda\nu c_1x+\lambda\mu c_2y+\mu\nu c_3z)\\ &+t\big(\lambda b_1x(\nu x-y-\mu\nu z)+\mu b_2y(\lambda y-z-\lambda\nu x)+\nu b_3z(\mu z-x-\lambda\mu y)\big)\\ &+(\nu x-y-\mu\nu z)(\lambda y-z-\lambda\nu x)(\mu z-x-\lambda\mu y)\ .\end{aligned}$$

I use the remaining freedom in coordinate transformations to place the putative tenth triple point in $(1:1:1:1)$. I compute in the affine chart $t=1$. The condition for a triple point is then that the function, its derivatives and the second order derivatives vanish at $(1,1,1)$. This gives ten equations which are linear in α, β and γ, so they may be eliminated: the maximal minors of the coefficient matrix have to vanish. One has

$$\frac{\partial Q^3}{\partial x} = 3Q^2 Q_x\ , \quad \frac{\partial^2 Q^3}{\partial x^2} = 3Q^2 Q_{xx} + 6QQ_x^2\ , \quad \frac{\partial^2 Q^3}{\partial x \partial y} = 3Q^2 Q_{xy} + 6QQ_xQ_y\ .$$

All these expressions are divisible by Q. Now I plug in $x=y=z=1$. From Q, I get

$$\begin{aligned} &Q(1,1,1) \\ &\quad = c_1c_2c_3 + c_2c_3(b_1+\nu-1-\mu\nu) + c_1c_3(b_2+\lambda-1-\lambda\nu) + c_1c_2(b_3+\mu-1-\lambda\mu)\ , \end{aligned}$$

an expression which I continue to denote by Q. One also gets expressions for all derivatives. Likewise

$$\begin{aligned} K \;=\; & \lambda\nu c_1 + \lambda\mu c_2 + \mu\nu c_3 \\ & + \lambda b_1(\nu-1-\mu\nu) + \mu b_2(\lambda-1-\lambda\nu) + \nu b_3(\mu-1-\lambda\mu) \\ & + (\nu-1-\mu\nu)(\lambda-1-\lambda\nu)(\mu-1-\lambda\mu)\ . \end{aligned}$$

Furthermore,

$$\begin{aligned} \left.\frac{\partial\, xyzK}{\partial x}\right|_{(1,1,1)} &= (yzK + xyzK_x)_{|(1,1,1)} = K + K_x\ , \\ \left.\frac{\partial^2\, xyzK}{\partial x^2}\right|_{(1,1,1)} &= (2yzK_x + xyzK_{xx})_{|(1,1,1)} = 2K_x + K_{xx}\ , \\ \left.\frac{\partial^2\, xyzK}{\partial x \partial y}\right|_{(1,1,1)} &= (zK + xzK_x + yzK_y + xyzK_{xy})_{|(1,1,1)} \\ &\qquad\qquad = K + K_x + K_y + K_{xy}\ . \end{aligned}$$

After dividing the first row by Q, which is allowed because the tenth triple point does not lie on the quadric Q, the matrix has the following form:

$$\begin{pmatrix} Q^2 & 3QQ_x & \dots & 3QQ_{xx}+6Q_x^2 & \dots & 3QQ_{xy}+6Q_xQ_y & \dots \\ Q & Q+Q_x & \dots & 2Q_x+Q_{xx} & \dots & Q+Q_x+Q_y+Q_{xy} & \dots \\ K & K+K_x & \dots & 2K_x+K_{xx} & \dots & K+K_x+K_y+K_{xy} & \dots \end{pmatrix}\ .$$

The vanishing of the maximal minors is the necessary condition for multiplicity 3 in the point $(1,1,1)$, but it is not sufficient for the existence of a surface with only isolated singularities. One has to cut away unwanted solutions, like

$Q = Q_x = Q_y = Q_z = 0$, which makes all minors vanish, but does not give isolated triple points. The minors are rather formidable expressions. I first simplify the matrix itself.

I start by subtracting $3Q$ times the second row from the first row to remove all second derivatives from the first row. After that I apply only column operations. Some experimentation with the matrix showed that it is possible to get two zeroes in one column. Observe that $Q_{xx}+2\nu Q_{xy}+\nu^2 Q_{yy} = 0$. Note that one can write $Q(x,y,z,t) = \frac{1}{2}Q_{xx}x^2+Q_{xy}xy+\cdots+\frac{1}{2}Q_{tt}t^2$, as the second derivatives are constants. The identity $Q_{xx}+2\nu Q_{xy}+\nu^2 Q_{yy} = 0$ now follows from the fact that the point $(1:\nu:0:0)$ lies on the quadric. The same point is a double point of the cubic K, so all first derivatives vanish, giving by the same argument that $(K_w)_{xx}+2\nu(K_w)_{xy}+\nu^2(K_w)_{yy} = 0$, where w is one of (x,y,z,t). Applying Euler's relation $3K = xK_x+yK_y+zK_z+tK_t$ in the point $(1:1:1:1)$ yields by adding that also $K_{xx}+2\nu K_{xy}+\nu^2 K_{yy} = 0$, where now K_{xx} again stands for a second derivative evaluated in $(1,1,1)$. Equivalent equations hold for the other second partials. Therefore one can get three columns with two zeroes by means of elementary column operations. To do this one needs to multiply one column, say the one containing containing Q_{xx}, with $1+\lambda\mu\nu$. The vanishing of this factor expresses that the three points P_7, P_8 and P_9 lie on a line, so multiplying may introduce new unwanted solutions, which have to be cut away later on in the computation. The result is

$$\begin{pmatrix} -2Q^2 & -Q^2 & \dots & Q^2-3QQ_x-3QQ_y+6Q_xQ_y & \dots & E_\nu & \dots \\ Q & Q_x & \dots & Q_{xy} & \dots & 0 & \dots \\ K & K_x & \dots & K_{xy} & \dots & 0 & \dots \end{pmatrix},$$

where E_ν is the first of three similar equations

$$\begin{aligned} E_\nu\colon\quad & (\nu^2+\nu+1)Q^2-3(\nu+1)Q(\nu Q_y+Q_x)+3(\nu Q_y+Q_x)^2\,, \\ E_\lambda\colon\quad & (\mu^2+\mu+1)Q^2-3(\mu+1)Q(\mu Q_x+Q_z)+3(\mu Q_x+Q_z)^2\,, \\ E_\mu\colon\quad & (\lambda^2+\lambda+1)Q^2-3(\lambda+1)Q(\lambda Q_z+Q_y)+3(\lambda Q_z+Q_y)^2\,. \end{aligned}$$

These equations have to hold, for if $E_\nu \neq 0$, then $\alpha = 0$ and the equation for the sextic is divisible by xyz. Considered as quadratic equation in Q and $\nu Q_y + Q_x$ the equation E_ν has discriminant $-3(\nu-1)^2$. The case $\nu = 1$ is excluded: if $\nu = 1$ then $0 = Q_x+Q_y-Q = c_1c_2(c_3+b_3+\mu)$, which means that the point $(0,0,1)$ is a triple point, which lies on the line through the tenth point $(1,1,1)$ and $P_9 = (1:1:0:0)$. Therefore no solution is defined over $\mathbb{R}$. One has to adjoin $\sqrt{-3}$ or what amounts to the same, the third roots of unity.

Factorising E_κ, $\kappa = \lambda,\mu,\nu$, gives linear equations, which express $Q_x + \nu Q_y$, $Q_y+\lambda Q_z$ and $Q_z+\mu Q_x$ as multiples of Q. To express Q_x, Q_y and Q_z themselves as multiples of Q one has to multiply with the determinant $1+\lambda\mu\nu$ of the system. After multiplying the fifth, sixth and seventh column of the matrix I use the first column to get zeroes on the first row in all other columns. This reduces the problem to the minors of a (2×6)-matrix.

The analysis up to this point is basically contained in [2]. To proceed further note that the three linear equations are in fact linear in $b_1c_2c_3$, $b_2c_1c_3$ and $b_3c_1c_2$. Therefore they can be used to eliminate the b_i. For the second row this is quite easy to do: by column operations remove the b_i from column 5, 6 and 7 and then take suitable linear combinations of columns 2, 3 and 4 with coefficients polynomials in (λ, μ, ν) such that the entries on the second row have the same coefficients at the $b_ic_ijc_k$ as the three equations. For the third row one has to first multiply with a quite complicated determinant, which leads to long expressions. At this stage the use of the computer becomes indispensable. The new second column turns out to be divisible by $\nu - 1$, and likewise the third by $\lambda - 1$, the fourth by $\mu - 1$. After division the entries $(2,2)$, $(2,3)$ and $(2,4)$ are equal, which means that one again gets columns with two zeroes, giving two equations. From the remaining (2×4)-matrix I take the 6 maximal minors. Now I have a system of 8 rather complicated equations in 6 variables. I still have to cut away unwanted solutions, those lying in $Q = 0$, $\lambda\mu\nu + 1 = 0$, $\lambda = 1$, $\mu = 1$, $\nu = 1$ and $c_i = 0$. This can be done in `Singular` as follows. First homogenise with an extra variable h. To cut away the solutions in $Q = 0$ adjoin the inhomogeneous equation $Q - 1$, where Q is made homogeneous with h, and compute a standard basis. Then homogenise again with Q. By doing the same for the other unwanted solutions one finally obtains equations of reasonably low degree. To do the calculation in reasonable time it is best to compute over a finite field $\mathbb{Z}/p\mathbb{Z}$ containing the third roots of unity. One can then try to lift the result to characteristic zero and check whether the guessed equations really solve the system.

Let ε be a primitive third root of unity. I first take the same root to solve the three equations E_κ:

$$\begin{aligned}
3(\nu Q_y + Q_x) - ((1-\varepsilon^2)\nu + (1-\varepsilon))Q &= 0\,,\\
3(\mu Q_x + Q_z) - ((1-\varepsilon^2)\lambda + (1-\varepsilon))Q &= 0\,,\\
3(\lambda Q_z + Q_y) - ((1-\varepsilon^2)\mu + (1-\varepsilon))Q &= 0\,.
\end{aligned}$$

By eliminating c_2 and c_3 I end up with one equation which is quadratic in c_1, so the solution space is three dimensional. The equations are rather involved, and I do not give them here.

A cyclic permutation of the variables (x, y, z) in the original configuration induces a cyclic permutation of each of the triples (b_1, b_2, b_3), (c_1, c_2, c_3) and (λ, μ, ν). A transposition of x and y has a more complicated effect on the coefficients. On the points P_7, P_8, P_9 it acts as $(0:1:\lambda:0) \mapsto (1:0:\lambda:0) = (1/\lambda:0:1:0)$, $(\mu:0:1:0) \mapsto (0:\mu:1:0) = (0:1:1/\mu:0)$ and $(1:\nu:0:0) \mapsto (\nu:1:0:0) = (1:1/\nu:0:0)$. The induced action on the coefficients is therefore

$((\lambda,\mu,\nu) \mapsto (1/\mu : 1/\lambda : 1/\nu)$. By also considering $P_1, \ldots, P_6$ one finds that $(c_1, c_2, c_3) \mapsto (c_2/\lambda\nu, c_1/\mu\nu, c_3/\lambda\mu)$ and $(b_1, b_2, b_3) \mapsto (b_2/\lambda\nu, b_1/\mu\nu, b_3/\lambda\mu)$. After clearing denominators in $Q(x,y,z)$ and $K(x,y,z)$. the equation $3(\nu Q_y + Q_x) - ((1-\varepsilon^2)\nu + (1-\varepsilon))Q = 0$ is transformed into $3(\nu Q_y + Q_x) - ((1-\varepsilon^2) + (1-\varepsilon)\nu)Q = 0$. By taking a particular normal form of the family I found two components, one with ε and one with ε^2, but the surfaces in those components are isomorphic. As the permutation of x and y is isotopic to the identity there is only one component (of dimension $3+15$) in the space of all sextics.

Now I take different roots of unity in the equations E_κ. By using permutations of (x,y,z) it suffices to consider:

$$\begin{aligned}
3(\nu Q_y + Q_x) - ((1-\varepsilon^2)\nu + (1-\varepsilon))Q &= 0\ , \\
3(\mu Q_x + Q_z) - ((1-\varepsilon^2)\lambda + (1-\varepsilon))Q &= 0\ , \\
3(\lambda Q_z + Q_y) - ((1-\varepsilon)\mu + (1-\varepsilon^2))Q &= 0\ .
\end{aligned}$$

I start the computation as described above. The two equations coming from the second row of the matrix factorise. Disregarding a factor $(\lambda\mu\nu+1)$ the equations are

$$\begin{gathered}
(\lambda\mu c_2 - c_3)\left(\nu c_1 c_2 c_3 - (\nu(\varepsilon^2+\lambda)c_1 c_3 - \nu c_2 c_3 - \varepsilon c_1 c_2)(\mu\nu - \nu + 1)\right)\ , \\
(\lambda\nu c_1 - c_2)\left(\mu c_1 c_2 c_3 - ((\varepsilon\nu+1)c_1 c_3 - \varepsilon^2\mu\nu c_2 c_3 - \mu c_1 c_2)(\lambda\mu - \mu + 1)\right)\ .
\end{gathered}$$

Applying a suitable transposition of the coordinates induces a transformation which sends the first equation to the second one with ε replaced by ε^2.

One finds one three dimensional solution by taking both long factors. One has $\mu c_2 = (\lambda\mu - \mu + 1)(\mu\nu - \nu + 1)$ and a quadratic equation in c_1, which I do not describe here.

Another three dimensional solution is found by taking the equations $\lambda\mu c_2 - c_3$ and $\mu c_1 c_2 c_3 - ((\varepsilon\nu+1)c_1 c_3 - \varepsilon^2\mu\nu c_2 c_3 - \mu c_1 c_2)(\lambda\mu - \mu + 1)$. The other possible choice gives a solution, isomorphic to the complex conjugate of this one. It might seem that one gets two different solutions, but as I shall show, the surfaces in question can also be written in a different way as a degeneration of a sextic of type $(2,2,2)$. Both solutions are slices of the same component in the space of all sextics. This time there is a linear equation for c_1:

$$c_1 + \varepsilon^2(\lambda\mu - \mu + 1)(\lambda\mu\nu + \varepsilon\mu\nu - \varepsilon\nu - \varepsilon^2) = 0\ .$$

One has already $c_3 = \lambda\mu c_2$ and one finds

$$(\lambda\mu\nu + \varepsilon\mu\nu + \varepsilon^2\nu - \varepsilon^2)c_3 + \varepsilon(\varepsilon\lambda\nu + \lambda - 1)c_1 = 0\ .$$

Finally, taking $c_3 = \lambda\mu c_2$ and $c_2 = \lambda\nu c_1$ gives a two dimensional solution consisting of two components, one of which lies inside the last component just found, and the other in the one obtained by interchanging the equations.

Proposition 2.2. *The family of sextic surfaces of type* $(2,2,2)$ *with nine triple points contains in its closure three different families of sextics with ten triple points, which contain three, four or five* (-1)*-conics.*

Proof. There are at most three different families. I distinguish between them with the number of (-1)-conics. A (-1)-conic not in a coordinate plane lies in a plane, whose intersection with one of the three coordinate planes can have at most two triple points. It has to contain at least two of the points P_1, $\dots$, P_6 on the coordinate axes, because P_7, P_8, P_9 and P_{10} are not coplanar. But if the plane contains a point on a coordinate axis, it contains only two other triple points on the coordinate planes through the point and therefore it contains the two points not in these planes. If there are three points of the points P_1, $\dots$, P_6 in the plane, it therefore contains again P_7, P_8, P_9 and P_{10}. Therefore there are only three possible planes which can contain a (-1)-conic, namely the planes through P_{10} and two of P_7, P_8 and P_9. The equation for the plane through P_7, P_8 and P_{10} is $\mu z - x - \lambda\mu y + \lambda\mu - \mu + 1$.

To determine the number of (-1)-conics in each family it suffices to do it for a specific example. One obtains three conditions by requiring that the points $(1:0:0:1)$, $(0:1:0:1)$ and $(0:0:1:1)$ are triple points. This gives the equations $c_1+b_1+\nu=0$, $c_2+b_2+\lambda=0$ and $c_3+b_3+\mu=0$. In the first family one finds $\lambda=\mu=\nu$, $c_1=c_2=c_3$, $\nu^4-3\varepsilon\nu^2+\varepsilon^2=(\nu^2+\varepsilon^2\nu-\varepsilon)(\nu^2-\varepsilon^2\nu-\varepsilon)$, $c_3^2+(1-\varepsilon^2)\nu^2-\varepsilon+1$. In the last family found above one gets $\lambda=\nu$, $\mu-2\varepsilon^2\nu+3$, $c_1=c_3$, $c_3+(\varepsilon^2-1)\nu+\varepsilon-1$, $\nu^2-\varepsilon\nu-\varepsilon^2$ and $c_2+(\varepsilon-\varepsilon^2)\nu$. For the third family this specialisation does not work, so a different one is needed. Checking in finite characteristic makes sure that there really exists a sextic with ten isolated triple points for these parameter values.

Now determine whether one of the three planes contains more than three triple points. The result is that the first family does not contain extra (-1)-conics. The second family contains one extra (-1)-conic, the plane through P_7, P_8 and P_{10}, which also contains a point on the x-axis and on the y-axis, with coordinates $(c_1:0:0:\nu)$ resp. $(0:c_2:0:l)$.

I specialise the third family by taking suitable values for λ, μ and ν. A good choice is $\lambda=\mu=-1$, $\nu=\varepsilon$. Compute the intersection points of the three planes with the coordinate axes and check whether they are triple points. The equations reduce to $3b_1+c_1+9\varepsilon$, $b_2+2\varepsilon-4$, $c_2-6\varepsilon+3$, $(\varepsilon+3)b_3+c_3-5\varepsilon-8$ and a quadratic equation for c_1, which does not factor in an easy way. For both values of c_1 the two points on the y-axis are given by $(y-3)(y+2\varepsilon-1)$, on the x-axis lies $(3,0,0)$ and on the z-axis $(0,0,\varepsilon+3)$. The result is that there are two extra (-1)-conics, the one through P_7, P_8 and P_{10} and the one through P_8, P_9 and P_{10}. □

Remark 2.3. The computation shows that there are no sextics with ten isolated triple points and six (-1)-conics. The arguments proving Lemma 2.1

do not exclude such a configuration. In fact one can take the three planes in the proof above and take as the points on the coordinate axes the intersection points with these planes. But a sextic with these isolated triple points occurs in a pencil, containing also the product of the six planes. The matrix above should then have rank one. The first 2×2 minor gives the equation $Q^2(Q - 2Q_x) = 0$, so together with the equations E_κ one finds $Q = 0$, contradicting the fact that the ten points do not lie on a quadric.

2.2 Type $(4, 4, 4)$

To complete the classification of sextics with ten triple points I look for a tenth triple point in the family of type $(4, 4, 4)$. Equations for the family are given in [2], which depend on seven parameters. It is convenient to work with more parameters, which then allows to take the tenth point in fixed position.

I take three quadratic cones K_i with vertices P_{i+1} at infinity such that K_i passes through P_{i-1} but not through P_i, where the indices are taken modulo 3. In general the quadrics intersect in eight distinct points. I require that two of them are the points $(0, 0, 0)$ and (u, v, w). The six remaining points will be the triple points of the sextic. I get

$$\begin{aligned} K_1 &= wx^2 + auz + bwx - (a + b + u)xz \,, \\ K_2 &= uy^2 + cvx + duy - (c + d + v)xy \,, \\ K_3 &= vz^2 + ewy + fvz - (e + f + w)yz \,. \end{aligned}$$

To compute Q, the quadric through $P_1, \ldots, P_9$, but not through $(0, 0, 0)$ and (u, v, w), note that the K_i lie in the ideal $(u - x, v - y, w - z)$. One can write

$$\begin{pmatrix} K_3 \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} 0 & (f + z)z & (e - z)y \\ (a - x)z & 0 & (b + x)x \\ (d + y)y & (c - y)x & 0 \end{pmatrix} \begin{pmatrix} u - x \\ v - y \\ w - z \end{pmatrix} \,.$$

Dividing the determinant of the matrix by xyz gives the inhomogeneous equation

$$Q = (a - x)(c - y)(e - z) + (b + x)(d + y)(f + z)$$

which is indeed the sought quadric. Note that my equations are homogeneous in the coefficients $a, \ldots, w$ and the affine coordinates x, y, z together.

The obvious thing to do now is to determine the conditions under which a surface $\lambda K_1 K_2 K_3 + \mu Q^3$ has a triple point in $(x, y, z) = (1, 1, 1)$. Despite great efforts I did not succeed in finding a single example. Finally I decided to compute the transformations which bring the known example from [2] (which is the same as the specific example in the first family above) into this family. The result was that the tenth point lies in the plane at infinity. In fact, a long, but doable computation with `Singular` shows that only solutions of the equations occur when $(1, 1, 1)$ lies on the quadric Q or one of the cones K_i.

Therefore, I search now under the

Assumption. The point $(1:1:1:0)$ is a triple point.

For the pencil $\alpha\, K_1K_2K_3 + \beta\, Q^3$ I compute all ten second partial derivatives and evaluate them in $(1:1:1:0)$. The resulting equations are linear in α and β, so I eliminate these variables and end up with a 2×10 matrix.

The vanishing of the minors of the matrix is again a necessary condition, for the existence of a sextic with ten triple points, but it is not sufficient for isolated triple points. Indeed, there are some easy to see 'false' solutions: if $K_1 = K_2 = K_3 = 0$, then the whole first row vanishes (I take at most second derivatives of the product $K_1K_2K_3$) and I get $\beta = 0$. Also, if $a + b = c + d = e + f = 0$, the second row vanishes. The ten points cannot lie on the quadric Q. I only want solutions with $Q \neq 0$, $K_1 \neq 0$, $K_2 \neq 0$ and $K_3 \neq 0$.

My equations are homogeneous in $a, \dots, z$. Moreover, the derivatives not involving t depend only on the sums $a + b$, $c + d$, $e + f$ and the u, v, w: note that $Q|_{t=0} = (a+b)yz+(c+d)xz+(e+f)xy$ and $K_1|_{t=0} = wx^2-(a+b+u)xz$. This means that I can start by analysing the six first columns. I cut away one after another the solutions lying in $Q = a + b + c + d + e + f = 0$, $K_1 = w - (a + b + u) = 0$, $K_2 = 0$ and $K_3 = 0$. To dispose of the solutions in a hyperplane $L = 0$ add the inhomogeneous equation $L = 1$ and compute a standard basis. Afterwards make the equations homogeneous again.

The computation with `Singular` gives twelve equations. They define two complex conjugate components. Eliminating a, c and e gives two equations

$$\begin{gathered}(b + d + f)^2 + (b + d + f)(u + v + w) + (u + v + w)^2\ ,\\ uv + uw + vw\ .\end{gathered}$$

Again one has to adjoin the third roots of unity. With ε a primitive third root of unity one finds two components, one of them given by

$$\begin{gathered}e + f + \varepsilon^2 v - \varepsilon w\\ c + d + \varepsilon^2 u - \varepsilon v\\ a + b + \varepsilon^2 w - \varepsilon u\\ b + d + f - \varepsilon(u + v + w)\\ uv + uw + vw\ .\end{gathered}$$

I give an explicit example: $v = w = 2$, $u = d = -1$, $b = 0$, so

$$\begin{aligned} K_1 &= 2x^2 - (\varepsilon+2)z + \varepsilon^2 xz\ , \\ K_2 &= -y^2 + 2\varepsilon x + y + \varepsilon^2 xy\ , \\ K_3 &= 2z^2 - 2\varepsilon^2 y + (6\varepsilon+2)z + 4\varepsilon^2 yz\ , \\ Q &= -(\varepsilon+2-x)(\varepsilon-y)(\varepsilon^2+z) + x(y-1)(z+3\varepsilon+1)\ . \end{aligned}$$

Then $27\,K_1K_2K_3 + 2\,Q^3$ has ten ordinary triple points. To find them it is convenient to compute in finite characteristic p. After some experimentation I found that for $p = 67$ with $\varepsilon = -30$ all points are defined over the base field.

Proposition 2.4. *The family of sextic surfaces of type* $(4,4,4)$ *contains in its closure one family of sextics with ten triple points, which each contain two* (-1)*-conics.*

Proof. One of the intersection points of the quadric cones K_i lies in the plane $t = 0$. To see this observe that $K_3|_{t=0} = z(vz + \varepsilon^2(v+w)y)$. By cyclic permutation one gets three lines $vz + \varepsilon^2(v+w)y$, $wx + \varepsilon^2(w+u)z$ and $uy + \varepsilon^2(u+v)z$. The condition that they pass though one point is

$$(u+v)(v+w)(w+u) + uvw = (u+v+w)(uv+uw+vw) = 0\ ,$$

which is satisfied on the component.

The intersection point is $(\varepsilon^2 uw : \varepsilon uv : vw : 0)$. Together with the tenth point $(1:1:1:0)$ it lies on the line $t = x + \varepsilon y + \varepsilon^2 z = 0$. One of the planes in the pencil of planes through this line contains three more triple points. It can be found by transforming the coordinates (x,y,z) into the eigenfunctions of cyclic permutation, making $x + \varepsilon y + \varepsilon^2 z$ into a coordinate and eliminating the others. The computation is best done in finite characteristic. Once the result is known one can find a derivation. One can observe the following factorisation modulo the ideal defining the component

$$uK_3 + \varepsilon^2 vK_1 + \varepsilon wK_2 \equiv (\varepsilon^2 vwx + wuy + \varepsilon uvz)(x + \varepsilon y + \varepsilon^2 z + e + \varepsilon d)\ .$$

In the affine chart $t \neq 0$ the six common points of the quadric cones lie therefore on two planes. The first factor contains the point (u,v,w), while the second factor is the sought plane of the pencil. Note also that $x + \varepsilon y + \varepsilon^2 z + e + \varepsilon d$ and $y + \varepsilon z + \varepsilon^2 x + a + \varepsilon f$ give the same plane.

Leaving out the point $P_9 = (\varepsilon^2 uw : \varepsilon uv : vw : 0)$ realises the surface in a different way as special element in a pencil of type $(4,4,4)$. A coordinate transformation brings it in standard form. To determine it requires the position of the three vertices, so I only computed in my specific example. I obtained values for the parameters $(a,\dots,f,u,v,w)$ and computed that they satisfy the equations for the complex conjugate component. This shows that there is only one family. $\square$

2.3 Cremona Transformations

To compute the effect of a Cremona transformation it is useful to know about other (-1)-curves on our surfaces. Each family lies also in the closure of other families of sextics with nine triple points. For explicit computations one needs to know the coordinates of the ten triple points. Therefore I use the specific examples in finite characteristic.

I start with the surface with two (-1)-conics. If I leave out P_1, then the surface has one (-1)-conic, so is of of type $(2,4,6)$ with the (-1)-conic the one determined above. The pencil has to contain the reducible surface $L_2K_1C_1$ with C_1 a cubic surface. In the example one finds an explicit equation for C_1. Leaving out P_7 or P_8 gives a surface with two (-1)-conics, which a priori can be of type $(2,2,8)$ or $(2,2,4)$. The explicit example shows that the first case occurs. Table 3 contains all the surfaces found in this way, with L_i planes, K_i quadric cones, C_i four-nodal cubics and Q_i quartics with one triple point and six nodes. Through each point pass 13 of the 16 surfaces and the reducible surface in the pencil obtained by leaving out this point is the union of the other three surfaces.

surface	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
L_1	1	1	1	0	0	0	0	0	1	1
L_2	0	0	0	1	1	1	0	0	1	1
K_1	0	2	1	1	1	1	1	1	1	0
K_2	1	0	2	1	1	1	1	1	1	0
K_3	2	1	0	1	1	1	1	1	1	0
K_4	1	1	1	0	2	1	1	1	0	1
K_5	1	1	1	1	0	2	1	1	0	1
K_6	1	1	1	2	1	0	1	1	0	1
C_1	0	1	2	1	1	1	2	2	1	2
C_2	2	0	1	1	1	1	2	2	1	2
C_3	1	2	0	1	1	1	2	2	1	2
C_4	1	1	1	0	1	2	2	2	2	1
C_5	1	1	1	2	0	1	2	2	2	1
C_6	1	1	1	1	2	0	2	2	2	1
Q_1	2	2	2	2	2	2	0	3	1	1
Q_2	2	2	2	2	2	2	3	0	1	1

Table 3: Multiplicities of the (-1)-curves in the case of two planes.

To get with a Cremona transformation again a surface with ten isolated triple points one has to take the four fundamental points such that no three lie in a plane. For the surfaces of type $(4,4,4)$ there are only a few possibilities, due to the symmetry in the configuration. One can compute the strict transform of each of the surfaces in Table 3 using the degree formula $3d - m_1 - \cdots - m_4$. The multiplicity of the transformed surface in one of the four image points is the degree of the exceptional curve, which is itself the image under a standard plane Cremona transformation of the intersection curve of the surface with the plane through the three opposite fundamental points: the new multiplicity m_1 is $2d - m_2 - m_3 - m_4$.

If one takes P_1, P_7, P_8 and P_9 as fundamental points the plane L_1 is transformed in a plane, as is the quadric K_3. One again gets a sextic with two (-1)-conics. The transform of each of the cubics C_4, C_5, C_6 is a quadric cone not passing through the new P_1 and simply through P_7, P_8 and P_9. So leaving out the new P_1 gives a surface of type $(4, 4, 4)$ again.

One gets three (-1)-conics by taking P_1, P_2, P_4 and P_7 as fundamental points. For four (-1)-conics one can take P_1, P_2, P_4 and P_5 as fundamental points.

surface	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
L_1	0	0	1	1	1	1	1	0	0	0
L_2	1	1	0	0	1	1	0	1	0	0
L_3	1	1	1	1	0	0	0	0	1	0
L_4	0	0	0	1	0	1	0	1	1	1
L_5	0	1	1	0	0	0	1	1	0	1
Q_1	2	2	1	1	2	2	2	0	2	3
Q_2	2	1	1	2	3	0	2	2	2	2
Q_3	2	1	2	0	2	2	2	1	3	2
Q_4	2	2	0	2	2	1	3	1	2	2
Q_5	3	0	2	1	2	1	2	2	2	2

Table 4: Multiplicities of the (-1)-curves in the case of five planes.

A surface with five (-1)-conics cannot be obtained directly with a reciprocal transformation. Instead I first study the configuration in more detail. Leaving out a point on two planes gives again surfaces of type $(2, 2, 2)$, whereas leaving out one of the five points on three planes leads to sextics of type $(2, 2, 8)$. There are five quartic surfaces Q_i with a triple point. Table 4 gives the multiplicities of the surfaces involved at the singular points.

A Cremona transformation with fundamental points P_1, P_5, P_7 and P_9 leads to the family with three (-1)-conics. This shows that all four families are related by Cremona transformations (obtained by composition of reciprocal transformations).

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Some Problems on Lagrangian Singularities

Duco van Straten

Abstract

Lagrangian varieties are objects of key interest in various areas of mathematics. The study of the basic singularities of lagrangian varieties was initiated by Arnol'd and Givental' in the 80's. We give an informal review some basic results, report on some newer developments and pose some open problems.

1 Definitions and Examples of Lagrangian Singularities

Consider a $2n$-dimensional complex manifold M with a symplectic form $\omega \in H^0(M, \Omega^2_M)$. This means that ω is *closed* and *non-degenerate*, thus providing an isomorphism $\Theta_M \xrightarrow{\approx} \Omega^1_M$. An n-dimensional analytic subspace $L \subset M$ is called *lagrangian subvariety* if ω vanishes on the regular locus $L \setminus \mathrm{Sing}(L)$. By a *lagrangian singularity* we will mean the germ $(L, 0) \subset (M, 0)$ of a lagrangian subvariety $L \subset M$ at some point $0 \in L$. There is a natural notion of isomorphy or equivalence of lagrangian singularities which is induced by symplectic mappings. A finite map $n : \Lambda \to M$ from an n-dimensional analytic space Λ is called a *lagrangian map* if $n^*(\omega)$ vanishes on $\Lambda \setminus \mathrm{Sing}(\Lambda)$. The image $n(\Lambda)$ of such a map is a lagrangian variety. Many examples of lagrangian singularities are constructed in this way.[1] By the theorem of Darboux, one can introduce local coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ on M such that

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

1991 *Mathematics Subject Classification.* 14B05, 14B12, 58F05, 32S40, 32S60

Key words. Lagrangian singularities

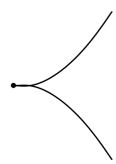
[1] At some places the names lagrangian singularity and lagrangian map are used to denote quite different notions.

So each lagrangian singularity can be considered as sitting in the standard symplectic space

$$(M,\omega) = (\mathbb{C}^{2n}, \sum_{i=1}^{n} dp_i \wedge dq_i).$$

For notational convenience we will follow in this paper the practice, common in singularity theory, of not always distinguishing between germs and representatives of them and write often L rather than $(L,0)$.

Example 1.1. Consider $M = \mathbb{C}^2$ with coordinates p, q and $\omega = dp \wedge dq$. Because the restriction of a two form to the smooth part of a curve is zero for dimensional reasons, any plane curve singularity C is automatically a lagrangian singularity. [2]

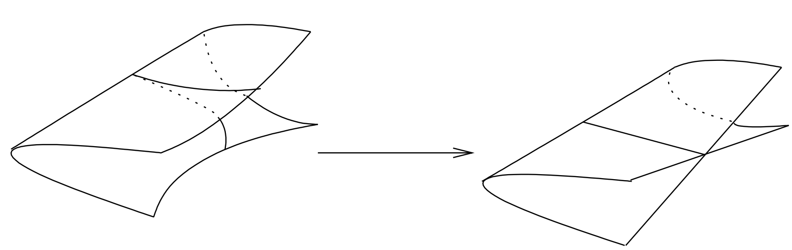


Nevertheless, lagrangian singularity theory is not the same as ordinary singularity theory, as the notion of isomorphy is not the usual one: in lagrangian singularity theory we only allow coordinate transformations that preserve the volume form ω. Let $\widetilde{C} \to C$ be the normalisation of the curve singularity C. The composition with the inclusion in the plane is a lagrangian map $\widetilde{C} \to \mathbb{C}^2$ with image C.

Example 1.2. Consider $M = \mathbb{C}^4 = \{(q_1, q_2, p_1, p_2)\}$ and let $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$. The map

$$n : \mathbb{C}^2 \longrightarrow \mathbb{C}^4; \quad (s,t) \mapsto (s, -st, \frac{1}{3}t^3, \frac{1}{2}t^2) = (q_1, q_2, p_1, p_2)$$

is lagrangian, as one computes that $n^*(\omega) = 0$. The image L is a lagrangian singularity, called the *open Whitney umbrella* [12], as it maps by forgetting the p_1-coordinate to the familiar Whitney umbrella in $\mathbb{C}^3 = \{(q_1, q_2, p_2)\}$, given by the equation $q_2^2 - 2p_2 q_1^2 = 0$.



[2]But notice that ω does not restrict to zero as a Kähler form on C.

L has an isolated singular point and the normalisation is $\mathbb{C}^2$. Hence L is not normal and not Cohen-Macaulay. The open umbrella is the simplest lagrangian corank one singularity. For any function $f(t)$ the map

$$(s,t) \mapsto (s, -st, \int tf'(t)dt, f(t))$$

is lagrangian.

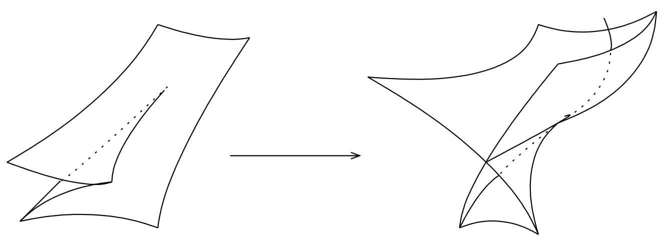
Example 1.3. Consider the set

$L := \{(a,b,c,e) \mid f := x^5 + ax^3 + bx^2 + cx + e \text{ has a root of multiplicity } \geq 3\}.$

Any $f \in L$ is of the form $(x-u)^3(x^2+3ux+v)$, hence L is the image of the mapping

$$n : \mathbb{C}^2 \longrightarrow \mathbb{C}^4; (u,v) \mapsto (v-6u^2, 8u^3-3uv, 3u^2v-3u^4, -u^3v) = (a,b,c,e).$$

One computes $n^*(\omega) = 0$, where $\omega = 3da \wedge de + dc \wedge db$, hence L is a lagrangian singularity and is called the *open swallowtail* [1]; taking the derivative $f \mapsto f'$ maps L to the ordinary swallowtail in (a,b,c)-space. As a polynomial of degree 5 can have only one root of multiplicity 3, the self-intersection of the ordinary swallowtail gets removed in L, whence the name.



Consider the parameter space $M = \{(a_1, a_2, \ldots, a_{2k})\}$ of odd degree polynomials $f = x^{2k+1} + a_1x^{2k-1} + \ldots + a_{2k}$ with root sum zero. The open swallowtail in M is the set

$$L := \{(a_1, a_2, \ldots, a_{2k}) \in M \mid f \text{ has a root of multiplicity } \geq k+1\}$$

It is lagrangian with respect to a naturally defined symplectic form on M, [11]. This variety turned up in the earliest researches of Hilbert on invariant theory, as L can be identified with the set of instable points for the action of SL_2 on binary forms. Givental' [13] showed that open swallowtails are Cohen-Macaulay.

Example 1.4. Consider $n : \mathbb{C}^2 \to \mathbb{C}^4$, $(u,v) \mapsto (u^3, u^2v, uv^2, v^3) = (x,y,z,t)$. As $d(u^3) \wedge d(v^3) = 3d(u^2v) \wedge d(uv^2)$, the mapping n is a lagrangian map for $\omega := dx \wedge dt - 3dy \wedge dz$. The image L is the quotient singularity $\mathbb{C}^2/(\mathbb{Z}/3)$, isomorphic to the cone of the rational normal curve of degree three. Hence L is a normal lagrangian surface singularity. I learned this example from P. Seidel.

Examples 1.5. Natural sources of examples of lagrangian singularities include *conormal varieties* and fibres of *completely integrable Hamiltonian systems*. Lagrangian varieties play an important role in the theory of *Frobenius manifolds* and more generally, F-manifolds, [13] and [18]. The general method for constructing lagrangian singularities is by the use of *generating functions*: consider a family $f : X \times B \to \mathbb{C}$ of functions of variables $x = (x_1, \dots, x_k)$, parametrised by variables $q = (q_1, q_2, \dots, q_n)$. The critical space $\mathcal{C} := \{(x, q) \mid \partial_x f(x, q) = 0\}$ is mapped via $(x, q) \mapsto (\partial_q f(x, q), q)$ to the cotangent space $T^*B = \{(p, q)\}$, which has a natural symplectic form ω. The image is a lagrangian subvariety $\mathcal{L} \subset T^*B$, which can also be seen as the image of the differential of a multi-valued function on B, whose graph in $B \times \mathbb{C}$ is the set of critical values $\{(q, f(x, q)) \mid (x, q) \in \mathcal{C}\}$, called the *front* of $\mathcal{L}$. But this is not the place to explain any of these things in more detail; the reder should consult [2].

2 Algebraic and Geometric Aspects

The symplectic structure of M gives rise to a *Poisson bracket* $\{-, -\} : \mathcal{O}_M \times \mathcal{O}_M \longrightarrow \mathcal{O}_M$ which is determined by the equation

$$df \wedge dg \wedge \omega^{n-1} = \{f, g\}\omega^n.$$

The main features of the Poisson bracket are

1. $\{-, -\}$ is skew-symmetric:

$$\{f, g\} = -\{g, f\}.$$

2. $\{f, -\}$ is a *derivation*:

$$\{f, g \cdot h\} = \{f, g\}h + \{f, h\}g.$$

The vector field $\{f, -\}$ is called the *Hamiltonian vector field* associated to f.

3. $\{-, -\}$ satisfies the *Jacobi-identity*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

The Poisson bracket gives rise to a bracket on the local ring $\mathcal{O} := \mathcal{O}_{(M,0)}$ which in local Darboux-coordinates is given as

$$\{f, g\} = \sum_{i=1}^{n} (\partial_{p_i} f \partial_{q_i} g - \partial_{q_i} f \partial_{p_i} g)$$

The vanishing ideal $\mathcal{I} \subset \mathcal{O}$ of a lagrangian singularity is characterised by the following properties:

1. $\mathcal{I}$ is an *involutive ideal*, that is, $\{\mathcal{I}, \mathcal{I}\} \subset \mathcal{I}$.
2. $\mathcal{I}$ is a radical ideal[3], $\sqrt{\mathcal{I}} = \mathcal{I}$.
3. $\dim(\mathcal{O}/\mathcal{I}) = \frac{1}{2} \dim \mathcal{O}$.

Example 2.1. (see [23], p.30) Consider the ideal generated by

$$f_1 = xz + \frac{3}{2}yt,\ f_2 = x^2 - \frac{9}{4}y^2z,\ f_3 = yz^2 + \frac{2}{3}xt,\ f_4 = z^3 - t^2,$$

which can be checked, for example using Singular [16], to define a reduced surface singularity $L \subset \mathbb{C}^4$. Let $\omega = dx \wedge dz + dy \wedge dt$. The Poisson brackets are

$$\{f_1, f_2\} = -2f_2, \quad \{f_1, f_3\} = \frac{1}{2}f_3, \quad \{f_1, f_4\} = 3f_4,$$
$$\{f_2, f_3\} = yf_1, \quad \{f_2, f_4\} = 6zf_1, \quad \{f_3, f_4\} = 0\,.$$

Hence, L is a lagrangian singularity, which in fact is just the open Whitney umbrella.

If $\mathcal{I}$ is an involutive ideal, then it follows from the derivative property that

$$\{\mathcal{I}^p, \mathcal{I}^q\} \subset \mathcal{I}^{p+q-1}.$$

From this it follows that the Poisson bracket descends to various brackets on quotients: most important for us are

$$\mathcal{I}/\mathcal{I}^2 \times \mathcal{O}/\mathcal{I} \longrightarrow \mathcal{O}/\mathcal{I}, \qquad \mathcal{I}/\mathcal{I}^2 \times \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{I}/\mathcal{I}^2.$$

If $f \in \mathcal{I}$ then $\{f, -\}$ is a derivation of $\mathcal{O}/\mathcal{I}$. If $f \in \mathcal{I}^2$, then this derivation is zero, so the first bracket is equivalent to the $\mathcal{O}$-linear map

$$\lambda : \mathcal{I}/\mathcal{I}^2 \longrightarrow Der(\mathcal{O}/\mathcal{I}, \mathcal{O}/\mathcal{I}), \quad f \longmapsto \{f, -\},$$

which I call the *fundamental map*. Geometrically, this expresses the fact that the Hamiltonian vector field is tangent to the variety defined by $\mathcal{I}$. On a smooth point of a lagrangian singularity the map λ is an isomorphism which identifies the conormal space to L with its tangent space.

A central problem in the theory of lagrangian singularities is certainly to determine which singularities do admit a lagrangian embedding in symplectic space. Obviously, the lagrangian condition puts very strong conditions.

[3] A *radical* involutive ideal $\mathcal{I}$ defines a coisotropic subvariety, but, as remarked by C. Hertling, *any* germ can appear as locus of an involutive ideal: for any ideal J the ideal $\mathcal{I} = J^2$ is involutive because by the derivation property

$$\{J^2, J^2\} \subset J\{J, J^2\} \subset J \cdot J\{J, J\} \subset J^2.$$

Definition 2.2. Let $L \subset M$ be lagrangian. We put

$$S_k(L) := \{q \in L \mid \operatorname{embdim}(L, q) = 2n - k\}$$

Here $\operatorname{embdim}(L, q)$ denotes the *embedding dimension* of L at q.

Proposition 2.3. *At a point $q \in S_k(L)$, $k > 0$, the germ (L, q) is of the form $(L', 0) \times (\mathbb{C}, 0)$ where $(L', 0)$ is lagrangian singularity.*

The reason is the following: a smooth hypersurface containing (L, q) gives a Hamiltonian flow that is non-vanishing at q, hence gives (L, q) a product structure. The space of integral curves is embedded as a lagrangian singularity in the Hamiltonian reduction of (M, q).

Let us mention some trivial consequences for lagrangian surface singularities in $M = \mathbb{C}^4$.

Corollary 2.4. *At a point of embedding dimension three, a lagrangian surface must have the form $(C, 0) \times (\mathbb{C}, 0)$, where $(C, 0)$ is a germ of a plane curve singularity. An isolated singular point must have embedding dimension four.*

The simplest normal surface singularities in $\mathbb{C}^4$ are the rational triple points, which were classified by M. Artin [3] and equations were given by G. Tjurina [27]. The cone over the rational normal curve of degree three is the simplest of these. The reducible lagrangian singularity given by the ideal

$$(q_1p_1, q_1q_2, p_2q_2) = (q_1, p_2) \cap (q_1, q_2) \cap (q_2, p_1)$$

consists of three planes and can be considered as the *limit* of the simplest series of triple points.

Problem 2.5. Do all rational triple points admit a realisation as a lagrangian singularity?

As the open Whitney umbrella shows, there exist interesting examples of non-Cohen-Macaulay isolated lagrangian singularities. Consider the mapping

$$n : \mathbb{C}^2 \longrightarrow \mathbb{C}^4, \quad (u, v) \longmapsto (u^2, v^2, uv^3, u^3v) = (x, y, z, t).$$

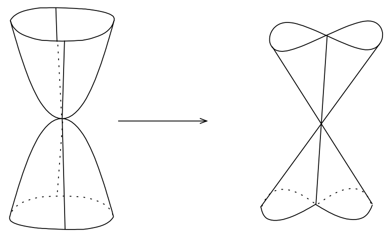
Its image $L = n(\mathbb{C}^2)$ is an isolated singularity, lagrangian with respect to the symplectic form $dx \wedge dz + dy \wedge dt$, with ideal generated by

$$f_1 = xz - yt,\ f_2 = x^3y - t^2,\ f_3 = x^2y^2 - zt,\ f_4xy^3 - z^2,$$

with Poisson brackets

$$\{f_1, f_2\} = -2f_2, \quad \{f_1, f_3\} = 0, \quad \{f_1, f_4\} = 2f_4,$$
$$\{f_2, f_3\} = -x^2f_1, \quad \{f_2, f_4\} = -6xyf_1, \quad \{f_3, f_4\} = -y^2f_1\,.$$

It maps to the surface $xy(x + y)^2 - w^2$ in (x, y, w)-space via $w = z + t$. I call L the *open double-cone*.



The normalisation $\widetilde{L}$ is isomorphic to an A_1-singularity and one has $\delta(L) := \dim_{\mathbb{C}}(\mathcal{O}\tilde{}/\mathcal{O}\ \ ) = 1$. (The function $uv = z/y$ is "missing" from $\mathcal{O}\ \ $.) We call L a $\delta = 1$-realisation of the A_1-singularity.

Problem 2.6. Do all rational double points X admit such a $\delta = 1$-realisation as lagrangian singularity in $\mathbb{C}^4$?

Complete intersection singularities are the simplest singularities from the view point of commutative algebra and those which are isolated (ICIS) are well studied in singularity theory [19], starting from Gert-Martin Greuel's classical paper [14].

Problem 2.7. (C. Hertling [18]) Does there exists a lagrangian isolated ICIS in $\mathbb{C}^4$? Note that this is the same as asking for an isolated Gorenstein lagrangian singularity.

To put this in perspective, we recall the following simple theorem, which links up this question with the Zariski-Lipman conjecture.

Theorem 2.8. *[25] Let $(L, 0)$ be a lagrangian complete intersection singularity, $\Sigma = \mathrm{Sing}(L)$ its singular locus. If $\mathrm{codim}(\Sigma) \geq 2$, then the module $\Theta\ \ $ of vector fields on $(L, 0)$ is* free.

Proof. For a general lagrangian singularity one has the following basic diagram:

$$\begin{array}{ccccccccccc} & & \mathcal{I}/\mathcal{I}^2 & \xrightarrow{d} & \Omega \otimes \mathcal{O} & \longrightarrow & \Omega^1 & \longrightarrow & 0 & & \\ & & \lambda \downarrow & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \Theta & \longrightarrow & \Theta \otimes \mathcal{O} & \longrightarrow & N & \longrightarrow & T^1 & \longrightarrow & 0 \end{array}$$

Here $\mathcal{I} \subset \mathcal{O} := \mathcal{O}_{(M,0)}$ is the ideal of $(L, 0)$ and $\mathcal{O}\ \ := \mathcal{O}/\mathcal{I}$. The top row is the defining exact sequence for the Kähler differentials on L, where we have put $\Omega := \Omega_{(M,0)}$. The bottom row is the dual of the top exact sequence, giving the exact sequence defining the space T^1. Here $\Theta := \Theta_{(M,0)} = \mathrm{Hom}_{\mathcal{O}}(\Omega, \mathcal{O})$. The vertical maps arise as follows. The symplectic form ω defines an isomorphism $\Omega \to \Theta$ which, tensored with $\mathcal{O}\ \ $, gives the middle vertical map. The left vertical map λ is the fundamental map discussed above. It associates to a function f vanishing on L its Hamiltonian vector field $\{f, -\}$ which is tangent to L. As a result there is an induced map

$$\rho : \Omega^1 \longrightarrow N \ .$$

Now the fact that L is *lagrangian* implies that λ and ρ are isomorphisms on a smooth point of L. For a reduced complete intersection the conormal module $\mathcal{I}/\mathcal{I}^2$ is free and the map d is injective. Hence the snake lemma gives an exact sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \xrightarrow{\lambda} \Theta \longrightarrow \mathrm{Tors}(\Omega^1) \longrightarrow 0.$$

But for a complete intersection with $\mathrm{codim}(\Sigma) \geq 2$ one has $\mathrm{Tors}(\Omega^1) = 0$,[14], [19]. Hence λ is an isomorphism and in particular, Θ is free. $\square$

The Zariski-Lipman conjecture claims that if the module of vector field is free, then the space is be smooth. This is known to be true if the $\mathrm{codim}(\Sigma) \geq 3$, [8] and in the quasi-homogeneous case, [21]. Another way of putting the result is as follows: if the Zariski-Lipman conjecture is true, then a non-smooth lagrangian complete intersection has to be singular in codimension one.

3 Deformations and Rigidity

There is an obvious notion of *deformation* of a lagrangian singularity L, which was worked out in [23]: a family over a base S consists of a flat map $\mathcal{L} \to S$, with an isomorphism of the fibre over $0 \in S$ with the given L, and a map $i : \mathcal{L} \to M_S = M \times S$ that is "fibre-wise lagrangian". In this way one obtains a corresponding deformation functor denoted by DefLag , which satisfies the Schlessinger conditions on **Art** if the tangent space DefLag $(\mathbb{C}[\epsilon])$ is finite dimensional.

There is a nice complex associated with an involutive ideal that was considered in [23], [24], [9], [10]. It is the standard complex of the Lie-algebroid $\mathcal{I}/\mathcal{I}^2$. It has terms $C^p := \mathrm{Hom}(\wedge^p(\mathcal{I}/\mathcal{I}^2), \mathcal{O})$ and the differential

$$\delta : C^p \longrightarrow C^{p+1}$$

is defined by

$$(\delta(\phi))(f_1, f_2, \ldots, f_{p+1}) = \sum_{1 \leq s \leq p+1} (-1)^s \{f_s, \phi(f_1, \ldots, \hat{f}_s, \ldots f_{p+1})\}$$
$$+ \sum_{1 \leq s < t \leq p+1} (-1)^{s+t-1} \phi(\{f_s, f_t\}, f_1, \ldots, \hat{f}_s, \ldots, \hat{f}_t, \ldots, f_{p+1}).$$

There is a natural map from the Kähler-deRham complex to this complex

$$(\Omega^\bullet, d) \longrightarrow (C^\bullet, \delta)$$

which extends the map $\rho : \Omega^1 \to N \ = C^1$ that appeared above. It is an easy exercise to establish the following relation between the complex $(C^\bullet, \delta)$ and DefLag .

Proposition 3.1. *([24])* DefLag $(\mathbb{C}[\epsilon]) = H^1(C^\bullet)$.

Example 3.2. Consider the case of a plane curve singularity $L = C \subset \mathbb{C}^2$, defined by an equation $f = 0$. In this case one has $N \ = \mathcal{O}$ and the complex is identified with the complex

$$\mathcal{O} \xrightarrow{\{f,-\}} \mathcal{O}$$

which in turn can be identified with the complex

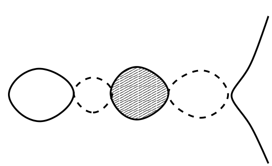
$$\mathcal{O} \xrightarrow{d} \omega \ .$$

Hence we find $H^0(C^\bullet) = \mathbb{C}$ and

$$H^1(C^\bullet) \approx \omega \ / d\mathcal{O} \ ,$$

a vector space that has, according to the formula of Buchweitz and Greuel [5], the *Milnor number* of C as dimension. So we find that the lagrangian deformation space has dimension μ, rather than τ as in the ordinary case.

There is a very beautiful geometrical explanation of this fact, which uses the theory of the *period mapping*, [28], [19]. Let us denote by Λ the semi-universal base for lagrangian deformations and consider an appropriate representative of a Milnor fibre C_t, $t \in \Lambda$, of C. For each homology cycle $\gamma \in H_1(C_t)$ one can consider the period $\int_\gamma pdq$, which is equal to the area of a 2-disc bounded by γ.



By moving around t, these integrals can be varied *independently* and in this way we get an identification of the lagrangian deformation space near t with $H^1(C_t) = \mathbb{C}^\mu$. The intersection of cycles provides $H^1(C_t)$ with a skew-symmetric form, which produces, via this identification, a closed two-form on Λ. If the curve singularity is *irreducible*, the intersection-form is non-degenerate and leads to a symplectic form on Λ. We refer to [28] for more details and [10] for a partial generalisation.

The following theorem guarantees the finite dimensionality of cohomology in important situations.

Definition 3.3. We say L satisfies *condition* P [24] or is *pyramidal* [10] if

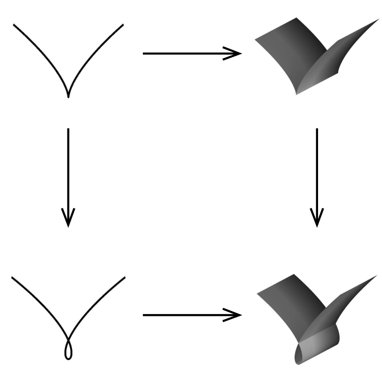
$$\dim S_k(L) \leq k$$

for all k.

Example 3.4. All examples we gave satisfy condition P. A lagrangian surface in $\mathbb{C}^4$ satisfies condition P if embedding dimension four occurs isolated at most. Points of embedding dimension three should occur along curves at most. The total space of a non-trivial one-parameter family of space curve singularities in $\mathbb{C}^4$ would have a one-parameter singular locus of points of embedding dimension four, hence does not satisfy condition P.

Theorem 3.5. *[24] If L satisfies condition P, then* $\dim_{\mathbb{C}}(H^p(C^\bullet)) < \infty$ *for all p.*

The mechanism is the *principle of propagation of deformations*: at points q where (L, q) exhibits a product structure, the deformed space must exhibit a similar product structure:



In fact, it is natural to consider a representative L of the germ and consider the cohomology *sheaves* $\mathcal{H}^p(C^\bullet)$ on L. If condition P holds, these are *constructible sheaves* on L. The following result is useful to establish rigidity for a large class of examples.

Theorem 3.6. *[25] Let L be a (contractible Stein representative of a) lagrangian singularity satisfying condition P, and let $T \subset L$ a closed subset of L. If*

1. $H^1_T(\delta\mathcal{O}\ ) = 0$,
2. $H^0(L \setminus T, \mathcal{H}^1(C^\bullet)) = 0$,

then L is rigid.

The sheaf $\delta\mathcal{O}$ sits in an exact sequence $0 \to \mathbb{C} \to \mathcal{O} \to \delta\mathcal{O} \to 0$, so from the long exact cohomology sequence

$$\cdots \longrightarrow H^1_T(\mathcal{O}) \longrightarrow H^1_T(\delta\mathcal{O}) \longrightarrow H^2_T(\mathbb{C}) \longrightarrow \cdots$$

we see that $H^1_T(\delta\mathcal{O})$ has a part that is related to the depth of L along T and a part that is purely topological. The other group $H^0(L \setminus T, \mathcal{H}^1(C^\bullet))$ of sections over $L \setminus T$ of the constructible sheaf $\mathcal{H}^1(C^\bullet)$ involves monodromy data and is rather subtle. For example, for the open swallowtail L the sheaf $\mathcal{H}^1(C^\bullet)$ is a rank two local system on the smooth part of the singular locus, and can be checked to have no sections over $\mathrm{Sing}(L) \setminus \{0\}$, [23]. From this result and the theorem one can deduce the rigidity of all open swallowtails and many other singularities as well, [25].

Problem 3.7. Do there exist lagrangian singularities with obstructed deformations? I think the answer must be yes, but due to the non-$\mathcal{O}$-linearity of the deformation problem, computations are quite difficult. Part of the problem is that $H^2(C^\bullet)$ is not the correct obstruction space for the deformation theory. In any case, it would be good to have more explicit non-trivial examples of lagrangian deformations.

Problem 3.8. Maybe a more natural viewpoint is to consider $\mathcal{O}$ as a special type of $\mathcal{O}$-module rather than a ring. In case L is a complete intersection, $\mathcal{O}$ is resolved by the Kozsul-complex from which it follows that

$$C^p = Ext^p_{\mathcal{O}}(\mathcal{O}, \mathcal{O})$$

and there were differential $\delta : C^p \longrightarrow C^{p+1}$. Now one should try to define, in the general case, differentials

$$\delta : Ext^p_{\mathcal{O}}(\mathcal{O}, \mathcal{O}) \longrightarrow Ext^{p+1}_{\mathcal{O}}(\mathcal{O}, \mathcal{O})$$

that generalise the complete intersection case.

4 The δ-constant Stratum

Let us consider a plane curve singularity C and let $n : \widetilde{C} \to C$ be the normalisation of C. The δ-invariant of C is the number $\delta(C) := \dim(\mathcal{O}_{\widetilde{C}}/\mathcal{O}_C)$ and is equal to the number of double points that appear in a generic deformation of the map n. The deformation space of this normalisation map $\mathrm{Def}(\widetilde{C} \to C)$ has a natural map to the deformation space $\mathrm{Def}(C)$ of the curve. The image

$$\mathrm{Im}\big(\mathrm{Def}(\widetilde{C} \to C) \longrightarrow \mathrm{Def}(C)\big) = B^\delta \subset \mathrm{Def}(C)$$

consists of those deformations of C for which the normalisation map can be lifted. It is a classical theorem of Teissier [26], that B^δ is exactly the δ-constant stratum of C, that is, the closure of those parameter values t for which the fibre C_t has exactly δ double points as singularities. Consequently, it has codimension equal to δ.

Now suppose that the plane curve singularity C is irreducible. As explained above, the lagrangian deformation space $\Lambda = \mathrm{DefLag}(C)$ has dimension $\mu = 2\delta$ and carries the structure of a symplectic manifold. Let $L^\delta := \pi^{-1}(B^\delta) \subset \Lambda$ be the preimage of the δ constant deformation, where $\pi : \Lambda \to \mathrm{Def}(C)$ is the map that forgets the 2-form $dp \wedge dq$. It is known that L is lagrangian with respect to the symplectic form defined above. This can be understood as follows: a curve C_t with δ double points arises from contraction of δ disjoint homology cycles which consequently span a lagrangian subspace in the cohomology, [28]

Problem 4.1. What is the depth of B^δ or L^δ? Are they Cohen-Macaulay? If yes, the rigidity theorem of [25] quoted above could be applied to conclude the rigidity of L^δ in the lagrangian sense.

There is a mysterious relation between the δ-constant stratum and torsion-free modules on C. Let $R = \mathcal{O}_C$ be the local ring of C and $\widetilde{R} = \mathcal{O}_{\widetilde{C}}$ the local ring of the normalisation. Any torsion-free R-module M is isomorphic to a unique R-module sitting between $\widetilde{R}$ and the conductor $c := \mathrm{Ann}(\widetilde{R}/R)$

$$c \subset M \subset \widetilde{R}$$

such that $\dim_{\mathbb{C}}(M/c) = \dim(\widetilde{R}/M) = \delta$. Hence, M determines and is uniquely determined by a δ-dimensional subspace $M/c \subset \widetilde{R}/c$ of dimension 2δ. The space

$$RGP(C) := \{V \in \mathrm{Grass}(\delta, \widetilde{R}/c) \ \mid \ V \ \text{ is an } R\text{-module}\}$$

considered by Rego [22] and by Greuel and Pfister [15].

One has:

Theorem 4.2. *(Rego [22]) $RGP(C)$ is homeomorphic to $\overline{JC}$.*

Here $\overline{JC}$ is the (highly singular) *compactified jacobian* of a rational curve with a unique singular point isomorphic to C.

Theorem 4.3. *(Göttsche-Fantechi-van Straten [7])*

$$\chi(\overline{JC}) = \mathrm{mult}(B^\delta)$$

Here χ denotes the topological Euler-characteristic.

I would like to understand this relation between these two seemingly unrelated objects much better. For example, is there a way to read off, not only the Euler-characteristic, but even all the *betti numbers*

$$b_i(C) := \dim H^i(RGP(C))$$

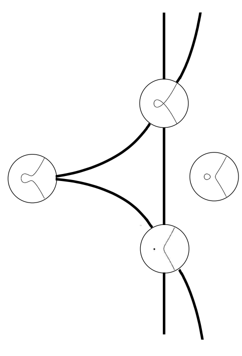
from the geometry δ-constant stratum?

Let me use this opportunity to make some rather wild conjectures.

Conjecture 4.4. Let C be an *irreducible* curve singularity. Then the space $RGP(C)$ has only cohomology in *even* dimensions: $b_{2i+1} = 0$.

There is some support for this from computations of T. Warmt [29] and in particular J. Piontkovski [20], who showed this to be the case for all irreducible quasi-homogeneous singularities $x^p + y^q = 0$ and for some series of cases with two Puiseux-exponents. These results depend on finding a cell-decomposition for $RGP(C)$.

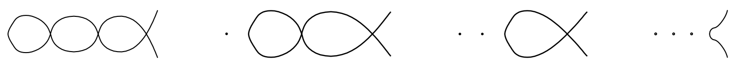
Example 4.5. Let us take a look at the A_2-singularity. The space $\overline{JC}$ can be identified with the cuspidal cubic in $\mathbb{P}^2$, which indeed is homeomorphic to $RGP(C) = \mathbb{P}^1$. Hence $\chi(RGP(C)) = 2$. Consider the versal deformation $\{y^2 = x^3 + ax + b\}$ of the A_2-singularity. The δ-constant stratum is a cuspidal curve $\{27b^2 + 4a^3 = 0\}$ in the a, b-plane, which has indeed multiplicity two.



One has

$$b_0(A_2) = b_2(A_2) = 1.$$

In the *real* geometry of the versal base we can see a *difference* between the two intersection points with the transversal $T = \{a = \text{const} < 0\}$: one of the intersection points correspond to a nodal curve with two real branches, the other to one with an isolated point. This generalises to other simple curve singularities. For A_{2k} one has $b_{2i}(A_{2k}) = 1$ for $i = 0, 1, \ldots, k$ and one can find a k-dimensional transversal T that intersects the δ-constant stratum in k real points that correspond to curves with $0, 1, \ldots, k-1$ isolated A_1 singularities.

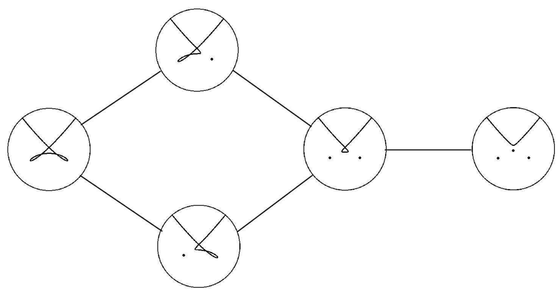


Intersection with transversal in A_6-case

For E_6 one has (see e.g. [29])

$$b_0 = b_2 = 1, b_4 = 2, b_6 = 1$$

and one finds a corresponding transversal:



Conjecture 4.6. Let C be a real irreducible curve singularity. Then there exists a δ-dimensional transversal T that intersects L^δ in $\mathrm{mult}(L^\delta)$ real points. (I call such transversals *good*.) I do not have much evidence for this conjecture, but it is true for the irreducible simple curve singularities.

Conjecture 4.7. The Poincaré-polynomial of $RGP(C)$ is given by

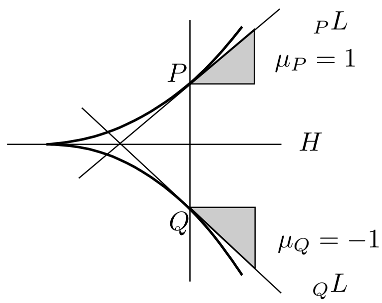
$$\sum_i \dim H^{2i}(RGP(C))t^i = \sum_{p \in \ {}^{\delta}\cap T} t^{i_p}$$

where T is a good transversal and i_p is the number of real isolated A_1-point of the corresponding fibre.

There is a way to formulate this purely in terms of the symplectic geometry of $L^\delta \subset \Lambda$. Let us recall the definition of the *Maslov index* (see e.g. [17])

$$\mu(L_1, L_2, L_3) \in \mathbb{Z}$$

of three lagrangian subspaces L_1, L_2, L_3 in a real symplectic vector space. It can be defined as the *signature* of the quadratic form q on the subspace $W := (L_1 + L_2) \cap L_3$ defined by the formula $q(z) := \omega(x, y)$, where one writes $z = x+y$, with $x \in L_1$ and $y \in L_2$. The Maslov index is totally anti-symmetric in its arguments and gives a purely lagrangian definition for the *Morse index* of a non-degenerate critical point of a function.



Conjecture 4.8.

$$2i_p = \delta(C) + \mu(T, H, T_pL)\,.$$

(Here one has to use the flat structure on the semi-universal base Λ to shift the linear spaces T and H in $T_0\Lambda$ to other tangent spaces $T_p\Lambda$.)

Acknowledgements

I would like thank C. Hertling, M. Garay, C. Sevenheck and A. Givental' for many discussions on lagrangian singularities over the years. Thanks also to O. Labs and T. Warmt for help with the pictures.

I also want to thank Gert-Martin for inviting me to Kaiserslautern in 1989 and teaching me the socratic truth that understanding nothing is the beginning of understanding anything. He seduced me to stay around in Kaiserslautern, which in fact was not so difficult because he had created an inspiring and excellent research environment. And there were great parties at his home.

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Topology, Geometry, and Equations of Normal Surface Singularities

Jonathan Wahl

To Gert-Martin Greuel on his sixtieth

Abstract

In continuing joint work with Walter Neumann, we consider the relationship between three different points of view in describing a (germ of a) complex normal surface singularity. The explicit equations of a singularity allow one to talk about hypersurfaces, complete intersections, weighted homogeneity, Hilbert function, etc. The geometry of the singularity could involve analytic aspects of a good resolution, or existence and properties of Milnor fibres; one speaks of geometric genus, Milnor number, rational singularities, the Gorenstein and $\mathbb{Q}$-Gorenstein properties, etc. The topology of the singularity means the description of its link, or equivalently (by a theorem of Neumann) the configuration of the exceptional curves in a resolution. We survey ongoing work ([15],[16]) with Neumann to study the possible geometry and equations when the topology of the link is particularly simple, i.e. the link has no rational homology, or equivalently the exceptional configuration in a resolution is a tree of rational curves. Given such a link, we ask whether there exist "nice" singularities with this topology. In our situation, that would ask if the singularity is a quotient of a special kind of explicitly given complete intersection (said to be "of splice type") by an explicitly given abelian group; on the topological level, this quotient gives the universal abelian cover of the link. Our major result gives a topological condition (i.e., a condition on the resolution graph) that there exists a singularity which arises in this way (and hence one whose equations can be written "explicitly"). T. Okuma ([18]) has recently proved our Conjecture that rational and minimally elliptic singularities are all "splice-quotients". We summarize first the well-studied case of plane curve singularities, to see what one might

1991 *Mathematics Subject Classification.* 32S50, 14J17 (*secondary:* 57N10, 57M25)

Key words. Surface singularity, Gorenstein singularity, rational homology sphere, complete intersection singularity, abelian cover, splice diagram.

mean about geometry, topology, and equations in that case. There follows an introductory discussion of normal surface singularities, before considering our recent work.

The purpose of the article is to survey the main ideas and directions, rather than to describe details, which can be found in other papers such as [15].

1 Introduction

To understand what we mean by "topology, geometry, and equations," we start with the germ at the origin of a complex irreducible (and reduced) plane curve singularity $C = \{f(x, y) = 0\} \subset \mathbb{C}^2$. Intersecting with a small 3-sphere gives a knot L in the 3-sphere. The embedded topology of the knot was studied by K. Brauner and E. Kähler in the 1920's (see [11] for some details). Their approach resulted in the topological description of the knot by iterated cabling on a torus knot. The description is given by a sequence of pairs of positive integers (p_i, r_i), the Puiseux pairs, which can be read off a fractional power series which parametrizes the curve; equivalently, a related approach produces the sequence of Newton pairs (p_i, q_i). A point is that the "link" of the singularity is intrinsically just a circle, so the "topology" of the situation should mean the embedded topology (i.e., knot type of L). In this case, from the equation $f(x, y) = 0$ one can iteratively read off the Puiseux pairs using Newton diagrams, and this data describes fully the topology. From a more geometric point of view, consider an embedded resolution of the curve by blowing up $\mathbb{C}^2$, until the reduced total transform of the curve has normal crossings. Again, the numerical data described above can be read off from the configuration of the exceptional curves and their self-intersections, plus the intersection with the transform of C. Alternatively, one may consider the *value semigroup* of the singularity; writing the integral closure of the local ring of C as $\mathbb{C}\{\{t\}\}$, one considers the collection of t-orders of all elements of the subring. Then this value semigroup is equivalent to the data of the Puiseux pairs.

What about recovering the equation of the curve from the above data? Of course, one has *equisingular* families for which the embedded topology is constant (i.e., same numerical data), but with analytically distinct individual curves. So one would ideally like to write down every plane curve singularity with given topological type. There are several ways to do this. For instance, in the Appendix to Zariski's book ([23]), B. Teissier considers a monomial curve given by generators of the value semigroup; this is known to be a complete intersection, and is weighted homogeneous. The versal deformation is smooth and also carries a $\mathbb{C}^*$-action; then the deformations of non-negative weight give all curves with the same value semigroup (though many of these curves are now no longer planar). Another approach is to write down the most general Puiseux series with given Puiseux pairs, as in [23] Chapitre III (cf. also [2], Appendix to Chapter 1). For one Puiseux pair (p, q) a family containing every analytic type is given by:

$$x^p + y^q + \sum t_{ij} x^i y^j = 0,$$

the sum over (i, j) such that

$$i/p \ + \ j/q \ > 1, \ \ 0 < i < p - 1, \ 0 < j < q - 1.$$

For a reducible plane curve, one must keep track not only of the topological type of each branch, but the linking numbers as well.

A new and important development in the study of curves appeared in the use of the *splice diagrams* of Siebenmann, by D. Eisenbud and W. Neumann [2]. These will be discussed below to study surface singularities. For this and other topics in the topology of plane curve singularities, we refer again to [11].

The moral for us is that at least for irreducible plane curve singularities, we know how to recover all the relevant (embedded) topological data from the equation, or a resolution, or the value semigroup; we can tell when a set of data comes from a singularity; and from the data we can write down equations of all plane curves of that topological type.

2 The Basics of Normal Surface Singularities

Working up a dimension from the case of curves, consider now the germ at the origin of an an isolated hypersurface singularity $Y = \{f(x, y, z) = 0\} \subset \mathbb{C}^3$. The *link* Σ of the singularity is the intersection with a small 5-sphere centered at the origin. Σ is a compact connected oriented 3-manifold, knotted somehow in S^5. The local topology of the pair $(Y, \mathbb{C}^3)$ is given by the topological cone over (Σ, S^5); in particular, Y is a topological manifold at the origin iff Σ is homeomorphic to the 3-sphere. As for knotting in the 5-sphere, one has (via the map $f/|f|$) the *Milnor fibration*; the complement of Σ in S^5 fibres over the circle, and the fibres are 4-manifolds with boundary Σ and with homology only in the middle dimension. The rank of this second homology group, called the *Milnor number* μ, is known to be computable as the colength of the Jacobian ideal (f_x, f_y, f_z) in $\mathbb{C}[x, y, z]$. This story is described in the classical book of Milnor [7].

The whole subject of the topology of normal surface singularities really began with an important discovery by D. Mumford in 1960. He showed that if the link Σ is simply-connected, then not only is it the 3-sphere (as the Poincaré Conjecture asserts), but it is unknotted in S^5, and in fact the origin is a non-singular point. Mumford's argument works not only in the hypersurface case. Suppose one has a germ of a normal surface singularity (from now on: NSS) $(Y, 0) \subset (\mathbb{C}^n, 0)$, and one considers the intersection with a small sphere $\Sigma = Y \cap S^{2n-1}$. The theorem asserts that if Σ is simply-connected, then Y is smooth at 0. Therefore, unlike in the case of curves, for a NSS the topology of the link can give you a huge amount of information about the singularity; and in the simply connected case, it tells you *everything* about the geometry of the point. So, from now on, by *the topology* of a NSS we shall mean simply the topological type of the 3-manifold Σ.

The natural way to see the topology of a NSS $(Y, 0)$ is via a "good" resolution $\pi : (\tilde{Y}, E) \to (Y, 0)$. Thus, $\tilde{Y}$ is smooth, π is proper and maps $\tilde{Y} - E$ isomorphically onto $Y - \{0\}$, and $E = \pi^{-1}(0)$ is a divisor consisting of smooth projective curves E_i, intersecting transversally (there is in fact a *minimal* good resolution, in an obvious sense). One can associate to E in the usual way the weighted resolution dual graph Γ: each irreducible component E_i of E gives a vertex, intersection points give edges of the graph, and each vertex is weighted by the degree of the normal bundle of the corresponding irreducible curve. In addition, the graph Γ is decorated at each vertex with the genus of the corresponding curve. The link Σ can be reconstructed from Γ (it is a *graph manifold*); this is because Σ may be viewed as the boundary of a tubular neighborhood of E on the smooth surface $\tilde{Y}$. A critical fact, noted originally by P. DuVal, is that the intersection matrix $(E_i \cdot E_j)$ is negative-definite. We conclude that the first betti number of Σ equals the number of cycles in the graph plus twice the sum of the genera of all the E_i.

If Σ is simply connected, then *a fortiori* $H_1(\Sigma; \mathbb{Q}) = 0$, i.e., Σ is a *rational homology sphere* (which we denote by $\mathbb{Q}$HS.) Thus, E is a tree of smooth rational curves. Mumford shows in this case how to compute the fundamental group in terms of the loops surrounding the exceptional components in $\tilde{Y}$. One must use simple connectivity to show that E can be contracted to a smooth point.

Mumford's method soon led to some generalizations. If $\pi_1(\Sigma)$ is finite, then the singularity itself is a quotient $\mathbb{C}^2/G$, where $G \subset GL(2, \mathbb{C})$ is a finite subgroup containing no pseudo-reflections. It also turns out that when $\pi_1(\Sigma)$ is solvable, one is in a very rigid (and well-understood) situation. But in analogy with the case of curves (and thus, e.g., of singularities defined by $z^n = f(x, y)$), one expects that rarely will Σ by itself determine the full analytic type of a singularity. In fact, H. Laufer ([4]), following some earlier results of G. Tjurina, gives a complete list of resolution dual graphs which have a unique analytic representative (he calls such singularities *taut*); they are all rational or minimally elliptic (see below for definitions).

One should mention that work of H. Grauert shows that every negative-definite weighted dual graph (with the genera included) does indeed arise from resolving some NSS $(Y, 0)$. One pastes together analytically a smooth surface with the desired curve configuration, and proves that you can blow-down the curve configuration to a point, necessarily on a normal analytic surface. (A general result of Hironaka proves that isolated analytic singularities are algebraic.) Of course, this purely existential result gives no indication whether such a singularity could have a nice property, like being a hypersurface or complete intersection (other than an obvious numerical condition that allows for a "canonical divisor" to exist — see below). An important result of W.Neumann [9] also shows that the homeomorphism type of Σ uniquely determines the graph Γ of the minimal good resolution (with the well-known exceptions of cyclic quotient singularities and "cusp" singularities, where orientation must be taken into account). The bottom line is that the topology of NSS's is reflected exactly by the dual graphs Γ.

So, a natural question is what statements can be made about the (many) singularities with given topology. One would not mind if the situation were similar to that of equisingularity for plane curves, i.e., if all such analytic types fit into nice topologically trivial (and geometrically similar) families. Unfortunately, that is not the case in general. For instance, recall that a germ $(Y, 0)$ is called *Gorenstein* if there exists a nowhere-0 holomorphic 2-form ω on $Y - \{0\}$ (and hence on $\tilde{Y} - E$). (There is also the traditional purely algebraic definition in terms of the local ring.) Such an ω extends meromorphically over E, hence gives rise to a "canonical divisor" K, an integral combination of the exceptional curves. Complete intersections are Gorenstein, as are quotient singularities $\mathbb{C}^2/G$ as above if and only if $G \subset SL(2, \mathbb{C})$. For a dual graph Γ to come from a Gorenstein singularity, it is necessary that there exist an integral divisor K satisfying the adjunction rules

$$K \cdot E_i + E_i \cdot E_i = 2g(E_i) - 2$$

for all i; we can call such a graph *numerically Gorenstein*. A key result of Laufer [5] shows that for any graph which does not correspond to a rational double point or minimally elliptic singularity, there always exists a non-Gorenstein singularity with that graph. So, if you consider the link of *any* hypersurface singularity in $\mathbb{C}^3$ which is not rational or minimally elliptic (e.g., has multiplicity at least 4), then there exists a non-Gorenstein singularity with the same link. It is completely unknown if a numerically Gorenstein graph always arises from a Gorenstein singularity, even when Γ is a rational tree.

Now, some non-Gorenstein singularities are still quite pleasant. We call a singularity $(Y, 0)$ $\mathbb{Q}$-*Gorenstein* if the canonical line bundle on $Y - \{0\}$ has finite order, i.e. some r-th tensor power has a nowhere-0 section. By taking cyclic covers, one sees equivalently that Y is a quotient of a Gorenstein singularity by a finite abelian group. For instance, all rational singularities are $\mathbb{Q}$-Gorenstein. One fully expects, as a generalization to Laufer's result above, that for any graph which is not rational or minimally elliptic, there exist non-$\mathbb{Q}$ -Gorenstein representatives. (However, we have not seen a proof of this statement.)

Finally, to illustrate how very different singularities can have the same link, consider the Brieskorn singularities

$$V(p,q,r) = \{x^p + y^q + z^r = 0\} \subset \mathbb{C}^3,$$

and their links $\Sigma(p,q,r)$. Then $\Sigma(3,4,12)$ is homeomorphic to $\Sigma(2,7,14)$ (both have as dual graph a single curve of genus 3, with weight -1). Yet these singularities have different multiplicities and different Milnor numbers (66 and 78, respectively). Further, if C is a smooth projective genus 3 curve, and $P \in C$, then

$$Y = \mathrm{Spec} \bigoplus \Gamma(\mathcal{O}_C(nP))$$

has the same link as these last 2, but is $\mathbb{Q}$-Gorenstein iff the degree 0 line bundle $\mathcal{O}(K - 4P)$ has finite order.

3 Some Cases when Topology Implies Nice Geometry

By the "geometry" of a singularity $(Y, 0)$, one is interested in analytic issues which go beyond the "mere" topology of the link. Relevant notions include: embedding dimension and Hilbert function; complete intersection, or Gorenstein, or $\mathbb{Q}$-Gorenstein; the geometric genus $p_g = \dim R^1\pi_*\mathcal{O}_{\tilde{Y}}$, where $\pi : \tilde{Y} \to Y$ is a resolution; nature of the defining equations and their syzygies. Complete intersection singularities have a simply connected Milnor fibre, and hence a Milnor number μ. By an old formula of Laufer [6], in the complete intersection case one has a relation between these invariants: $\mu - 12p_g$ is an explicit purely topological invariant. As already indicated, topologically equivalent germs could have very different geometry. But suppose one has a family $\mathcal{Y} \to T$, which has a simultaneous equitopological resolution $\tilde{\mathcal{Y}} \to \mathcal{Y} \to T$ (that is, one has a locally trivial deformation of the exceptional sets). Then the geometric genus is known to be constant; and quite generally, a small deformation of a complete intersection (or Gorenstein) singularity has the same property. On the other hand, in analogy with what is known from deforming space curves, one can easily have a topologically trivial and simultaneous resolution family of complete intersection surface singularities for which the embedding dimension jumps ([8] gives nice examples). Thus, in the general theory we should not be worried about jumping multiplicity or embedding dimension in "geometrically nice" families.

We review the two situations where one knows a great deal about the geometry of a singularity $(Y, 0)$ just from its topology (i.e., graph Γ). A good general reference for the following discussion is [21], Chapter 4. Consider the minimal good resolution $(\tilde{Y}, E) \to (Y, 0)$, and exceptional cycles $Z = \Sigma n_i E_i$. Among all such divisors, there is a minimal non-0 cycle Z_o with the property that $Z_o \cdot E_i \leq 0$, all i; this is the *fundamental cycle* (also known as the *numerical cycle* in [21]), which is easily (but not necessarily quickly) computed. The canonical line bundle K of $\tilde{Y}$ satisfies $K \cdot E_i + E_i \cdot E_i = 2g(E_i) - 2$ for every exceptional curve, hence one can make sense of K dotted with any cycle. In particular, Z_o and K are computed from the graph.

We say that Y is a *rational singularity* if $Z_o \cdot (Z_o + K) = -2$; a basic theorem says that this condition is equivalent to the vanishing of the geometric genus (which is a priori an analytic as opposed to topological invariant). Seminal work of M. Artin and E. Brieskorn, expanded upon by J. Lipman, shows the multiplicity of a rational Y is $m \equiv -Z_o \cdot Z_o$, and the embedding dimension is $m + 1$ (so the Hilbert function is $H(n) = mn + 1$). Y can be minimally resolved simply by a sequence of blowing-up the singular points (which are always themselves rational), and the exceptional divisor is a tree of smooth rational curves. The local Picard group (=divisor class group) has finite order. So the canonical line bundle on $Y - \{0\}$ has finite order, from which it follows that Y is $\mathbb{Q}$-Gorenstein (though it is Gorenstein only for the rational double points). It was proved in [22] that Y is defined by quadratic equations, and all the higher syzygies are "linear" in an appropriate sense. All finite quotient singularities $\mathbb{C}^2/G$ are rational; writing down the defining equations is a calculation in invariant theory. Up to now, given the graph of a rational singularity, there is no general method for writing down explicit equations of a corresponding rational singularity. The best results, due to De Jong and van Straten [3], show how to do this if the rational graph has reduced fundamental cycle.

Next, we say Y is *minimally elliptic* if on the minimal resolution $Z_o \equiv -K$. These singularities were introduced by H. Laufer in [5], though many of the results were discovered independently by Miles Reid (in unpublished notes, but see [21]). The definition is equivalent to Y being Gorenstein of geometric genus 1. Except for cones over elliptic curves and "cusp" singularities (whose resolution dual graph is a cycle of smooth rational curves), the minimal good resolution graph is a rational tree. When $m \equiv -Z_o \cdot Z_o$ is 1,2, or 3, one has a hypersurface of multiplicity 2, 2, or 3 respectively in $\mathbb{C}^3$; when $m \geq 4$, m is the multiplicity, the Hilbert function is $H(n) = mn$. Further, Y is defined by quadratic equations, with "linear syzygies" except at the last step [22]. Up to now, there is no general method to write down equations, given the minimally elliptic resolution graph.

We mentioned above that Laufer's result, and a presumed generalization, would imply that *any* resolution graph which is neither a rational nor minimally elliptic graph would arise from at least one non-$\mathbb{Q}$-Gorenstein singularity. In other words, we should be completely finished with the "nice" cases where the topology automatically forces some basic facts about the geometry.

But there is one more situation in which a great deal can be said from the topology — that is, if one additionally knows that $(Y, 0)$ is weighted homogeneous (that is, quasi-homogeneous, or admits a good $\mathbb{C}^*$-action). Then $Y = \text{Spec } A$, where A is a positively graded $\mathbb{C}$-algebra. It follows from early work of Orlik-Wagreich [19] that (except for cyclic quotient singularities) the exceptional divisor on the minimal good resolution consists of one central smooth curve (Proj A), and chains of smooth rational curves emanating from at least 3 points of this curve. Put another way, consider the weight filtration $\{I_n\}$ of A, where I_n is the ideal generated by elements of weight $\geq n$. Take the weighted blow-up $Z = \text{Proj } \oplus I_n \to Y = \text{Spec } A$ (the so-called *Seifert partial resolution*). Then Z is a normal surface with several cyclic quotient singularities along its exceptional divisor, which is isomorphic to Proj A. In particular, $(Y, 0)$ determines the following data: the isomorphism class of the central curve; its conormal line bundle; the location of the points on the curve at which Z has a singularity; and the data of the cyclic quotient singularities at these points. Conversely, it was shown by H. Pinkham [20] and independently by I. Dolgachev how to write down explicitly the graded algebra A from this data. In other words, this data uniquely determines the analytic type of the singularity.

Now suppose we have a weighted homogeneous singularity Y with *rational* central curve. Then the data you need to write down Y is numerical, contained in the graph Γ, except for the (analytically significant) location of the intersection points on the central rational curve. Such singularities all have $\mathbb{Q}$HS links, are rarely rational singularities. For instance, any Brieskorn hypersurface $V(p, q, r)$ for which p is relatively prime to qr give such examples; but if $p, q, r \geq 4$, then these are neither rational nor minimally elliptic. On the other hand, due to the grading, it is not too hard to find the numerical data that tell you when Y is Gorenstein; and it turns out that such Y are always $\mathbb{Q}$-Gorenstein. But much more is true; we need to explain first some general facts.

4 Universal Abelian Covers of Singularities with $\mathbb{Q}$HS Links

There is an alternate way to describe the equations of a weighted homogeneous singularity whose link is a $\mathbb{Q}$HS. Note that in general if the link Σ of a singularity is a $\mathbb{Q}$HS, then the first homology $H_1(\Sigma;\mathbb{Z})$ is a finite group computed directly from Γ; this *discriminant group* $D(\Gamma)$ is the cokernel of the intersection pairing $(E_i \cdot E_j)$ (so, the order is the absolute value of the determinant). The universal abelian covering $\tilde{\Sigma} \to \Sigma$ is finite, and it is important to note that it can be realized by a finite map of germs of NSS's $(X,0) \to (Y,0)$, unramified off the singular points. We abuse notation and refer to the map $X \to Y$ as the *universal abelian covering of the singularity*, or the UAC.

Recall that a Brieskorn complete intersection (or BCI) is a singularity $V(p_1,\ldots,p_t)$ defined by

$$\sum_{j=1}^{t} a_{ij} z_j^{p_j} = 0,\ i = 1,\ldots,t-2\,,$$

where $p_i \geq 2$ and every maximal minor of the matrix (a_{ij}) has full rank.

Theorem 4.1. *[10] Let $(Y,0)$ be a weighted homogeneous singularity whose link is a $\mathbb{Q}HS$. Then the UAC of $(X,0)$ is a BCI as above. The p_i and the diagonal action of the discriminant group on the ambient variables are explicitly computed from the graph Γ of Y.*

Of course, the values of the a_{ij} depend on the analytic class of t points on the central $\mathbb{P}^1$. Note that the Theorem allows one to write down "explicit" equations for the singularity $(Y,0)$. First, write down monomials generating the ring of invariants for the action of the discriminant group acting on $\mathbb{C}[z_1,\cdots,z_t]$; then, mod out by relations on these monomials; finally, divide by relations which follow from the BCI equations. While not as direct a method for writing equations as the aforementioned Pinkham-Dolgachev approach, this result actually generalizes to many other cases.

Example 4.2. The E_7 singularity (a rational double point) has graph Γ which is the Dynkin diagram for E_7. Its discriminant group has order 2, and the UAC is $V(2,3,4)$, which is the E_6 singularity; the group action on $x^2+y^3+z^4=0$ is given by $(x,y,z) \mapsto (-x,y,-z)$. So the quotient is generated by the invariants $A=x^2, B=xz, C=z^2, D=y$, with equations $AC-B^2=0$ from the group action, and $A+D^3+C^2=0$ from the equation. This yields the familiar (non-Brieskorn) equation for E_7:

$$B^2 + C(C^2+D^3) = 0.$$

There are two striking aspects of Theorem 4.1. First, the UAC turns out to be not only Gorenstein (which is clear), but even a complete intersection. Second, it is a complete intersection of very special type — a BCI (which need *not* have $\mathbb{Q}HS$ link).

For many years we wondered what is the most general natural context in which to view the previous theorem. It thus became clear one should study the UAC of singularities $(Y, 0)$ with $\mathbb{Q}HS$ link, but whose graph Γ has more than one node (unlike in the weighted homogenous case). If the UAC were to be a complete intersection, then Y must be $\mathbb{Q}$-Gorenstein. But what generalization of BCI would work as UAC's for a wide class of Y's, such as rational singularities?

One considers first the case that the link is a $\mathbb{Z}HS$, since then the UAC of the singularity is itself. This was the topic of several papers with Neumann [12, 16]. The discovery of the Casson invariant $\lambda(\Sigma)$ of a three-dimensional $\mathbb{Z}HS$ around 1986 and its subsequent calculation for certain examples led to the Conjecture of Neumann-Wahl:

Casson Invariant Conjecture. [12] For a complete intersection singularity with $\mathbb{Z}HS$ link, the Casson invariant is one-eighth the signature of the Milnor fibre.

Since the Casson invariant is topological, such a result would imply that for a complete intersection (a very strong geometric property), the $\mathbb{Z}HS$ topology determines the signature of the Milnor fibre (and hence, by well-known formulae, also the Milnor number and geometric genus). Implicit in the Conjecture (and what makes it provocative) is the prediction that the Milnor fibre itself is somehow canonically associated to the link (perhaps with some extra structure).

In spite of some progress on this Conjecture [16], and counterexamples showing it does not generalize to links of Gorenstein singularities (which might not even be smoothable) [8], the question remains open. However, it did raise the problem of trying to write down explicit examples of complete intersection singularities with $\mathbb{Z}HS$ links, beyond the BCI's $V(p_1, \ldots, p_t)$ (where the p_i are pairwise relatively prime). This ultimately led to the discovery of *complete intersections of splice type*, which play a role in the general problem.

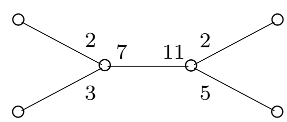
5 Splice Diagrams and Complete Intersections of Splice Type — $\mathbb{Z}$HS case

The usual topological description of a singularity link is via plumbing according to the resolution dual graph Γ. But when the link is a $\mathbb{Z}HS$ (i.e., the intersection matrix is unimodular), there is another topological construction, from a different kind of graph, which can be computed from Γ. The following discussion is largely taken from [16], itself depending heavily upon [2].

Suppose first that K_i is a knot in a $\mathbb{Z}HS$ $\Sigma_i, i = 1, 2$. Then one may "splice" the two three-manifolds together along the knots to form a new $\mathbb{Z}HS$:

remove from each Σ_i a tubular neighborhood of K_i, and then paste together along the boundaries (which are tori), but switching the roles of meridian and longitude. (Of course, orientation needs to be handled carefully).

A *splice diagram* is a finite tree with vertices only of valency 1 ("*leaves*") or ≥ 3 ("*nodes*") and with a collection of integer weights at each node, associated to the edges departing the node. The following is an example:



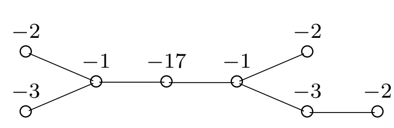
For an edge connecting two nodes in a splice diagram the *edge determinant* is the product of the two weights on the edge minus the product of the weights adjacent to the edge. Thus, in the above example, the one edge connecting two nodes has edge determinant $77 - 60 = 17$. This example is supposed to represent the result of splicing together the Brieskorn homology spheres $\Sigma(2, 3, 7)$ and $\Sigma(2, 5, 11)$ along the knots obtained by setting the last coordinate equal to 0 in the defining equations. Each leaf of a splice diagram corresponds to a knot on the corresponding $\mathbb{Z}HS$.

The splice diagrams that classify integral homology sphere singularity links satisfy the following conditions on their weights:

- the weights around a node are positive and pairwise coprime;
- the weight on an edge ending in a leaf is > 1;
- all edge determinants are positive.

Theorem 5.1 ([2]). *The integral homology spheres that are singularity links are in one-one correspondence with splice diagrams satisfying the above conditions.*

The splice diagram and resolution diagram for the singularity determine each other uniquely, and indicate how to construct the link by splicing or by plumbing. To go from resolution to splice diagram, one collapses all vertices of valency 2, and uses as weights the absolute value of the intersection matrices of certain subdiagrams. (Computing in the other direction is harder, and given in [2] or an appendix to [16]). The example above corresponds to the resolution diagram



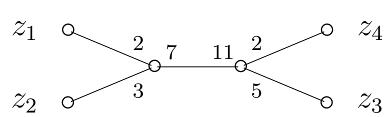
An important point is that the ends of the diagrams (in this $\mathbb{Z}HS$ case) correspond to certain natural isotopy classes of knots in the 3-manifold.

A surprising early discovery of the Neumann-Wahl collaboration was that if a singularity link is a $\mathbb{Z}HS$, and a certain condition on Γ is satisfied (the "semigroup condition"), then one can use the associated splice diagram Δ to write down explicit equations of a complete intersection singularity, whose link is what we started with. This works as follows: first, for each pair of distinct vertices v, v' of Δ, define the linking number $\ell_{vv'}$ to be the product of the weights adjacent to, but not on, the shortest path from v to v' (including weights around each vertex). To each leaf w, assign a variable z_w. To each node v, assign a weight ℓ_{vw} to the variable z_w, and assign a weight to v itself equal to the product of all the weights on the edges adjacent to v. Next, for each node v and adjacent edge e, choose *if possible* a monomial M_{ve} in the outer variables whose total weight is the weight of the node v. If the node v has valency δ_v, choose $\delta_v - 2$ equations by equating to 0 some $\mathbb{C}$-linear combinations of these monomials:

$$\sum_e a_{ie} M_{ve} = 0, \quad i = 1, \ldots, \delta_v - 2.$$

Repeating for all nodes, we get a total of $t-2$ equations. We require the coefficients a_{ie} of the equations be "generic" in the precise sense that all maximal minors of the $(\delta_v - 2) \times \delta_v$ matrix (a_{ie}) have full rank.

Example 5.2. For the Δ of the example above, we associate variables $z_1, \ldots, z_4$ to the leaves as follows:

$\Delta \quad =$ 

At the left node, the weights of the variables turn out to be (in order) 21, 14, 12, 30, and the total weight at the node is 42; so possible monomials for the left node are z_1^2, z_2^3, and z_3z_4. The monomials for the right node are z_3^5, z_4^2, and $z_1z_2^4$ or $z_1^3z_2$. Thus the system of equations might be

$$\begin{aligned} z_1^2 + z_2^3 + z_3z_4 &= 0\,, \\ z_3^5 + z_4^2 + z_1z_2^4 &= 0\,. \end{aligned}$$

The "semigroup condition" on Δ (or Γ) is exactly the ability to write down appropriate monomials at every node in every direction.

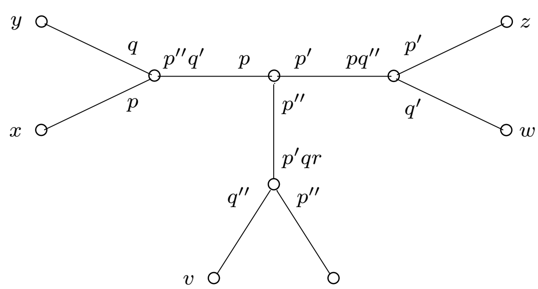
Theorem 5.3. *Let Δ be a splice diagram corresponding to a $\mathbb{Z}HS$ singularity link. Suppose Δ satisfies the semigroup conditions. Then the splice equations above describe a complete intersection singularity whose link is the $\mathbb{Z}HS$ associated to Δ. Further, each of the t coordinates, when set equal to 0, cuts out the knot on the link corresponding to the end ("leaf") of the diagram.*

So, we can summarize by saying that for all these topologies, there exists a very special kind of complete intersection singularity with the given link. This notion is a generalization of Brieskorn complete intersection $V(p_1, \cdots, p_t)$ already discussed. But in fact we can (and therefore should) generalize the above construction slightly by allowing one to add higher weight terms to each equation at each node. We then arrive at the notion of a *complete intersection of splice type*, or CIST. The only proof we know of this Theorem is as a special case of a much more general result, Theorem 6.4 below.

We also note that each node in the splice diagram corresponds to a valuation in the local ring of the CIST. For, the nodes give weights to the variables, and the nature of the defining equations means that the associated graded ring is an integral domain (follows from [15], Theorem 2.6). This parallels the role of the valuation for an irreducible curve, and the weight filtration for a weighted homogeneous singularity.

The following (Example 3 of [16]) shows that while the embedding dimension of a CIST is at most the number of ends of Δ, it could be considerably smaller.

Example 5.4. Let Δ be the splice diagram:



The integers p, q, p', q', p'', q'', r are ≥ 2 and must satisfy appropriate relative primeness conditions, as well as edge inequalities

$$q' > p'q, \quad q'' > p''q', \quad qr > pq''.$$

Associating variables x, y, z, w, u, v to the leaves in clockwise order starting from the left as shown, one may write splice equations:

$$x^p + y^q = z\,, \quad z^{p'} + w^{q'} = u\,, \quad u^{p''} + v^{q''} = x^r\,, \quad y + w = v\,.$$

These define the hypersurface singularity given by

$$((x^p + y^q)^{p'} + w^{q'})^{p''} + (y + w)^{q''} = x^r\,.$$

Given our general earlier warnings about NSS's, we can ask which of the analytic types $(Y, 0)$ for a given $\mathbb{Z}HS$ topology are so represented. Of course,

V must be Gorenstein if it is a CIST. But a very natural point of view is that one should be considering the "algebraic" nature not just of the link, but the t isotopy classes of knots which are part of the data of the link. We mentioned above that for a CIST, these knots are cut out by coordinate functions. Algebraically, this says that in the analytic local ring of the singular point, there are prime principal ideals which give topologically the knots in question. A converse statement holds:

Theorem 5.5. *Let $(Y, 0)$ be a NSS with $\mathbb{Z}HS$ link. Suppose each of the t knots in the link is represented by the vanishing of some function in the local ring. Then Y is a CIST; in particular, the link satisfies the semigroup condition.*

The method of the proof is as follows: choose an irreducible curve in the local ring cut out by the function corresponding to one of the knots. The other functions have a known order of vanishing along the normalization of this curve, hence contribute to the value semigroup. This subsemigroup, read off from the splice diagram, is shown to satisfy a certain inequality between its own δ invariant and the "Milnor number" of the curve itself. Applying now basic results of Buchweitz-Greuel [1], one proves that the subsemigroup is the full value semigroup of the curve, and these functions generate the maximal ideal. (Note that we did not need to assume the Gorenstein property at the beginning.)

These results clarify greatly the possible nice geometries for a NSS with given $\mathbb{Z}HS$ link; we even know how to write down explicit equations for singularities whose link satisfies the semigroup condition. On the other hand, one should keep in mind the following examples and open questions:

1. Does every complete intersection singularity with $\mathbb{Z}HS$ link satisfy the semigroup condition? Is every one a CIST? (This is related to a question about the Casson invariant.)

2. Does every CIST as above satisfy the Neumann-Wahl Casson Invariant Conjecture?

3. There exist Gorenstein singularities with $\mathbb{Z}HS$ link which do not satisfy the semigroup condition ([8], 4.5).

4. There exists a Gorenstein singularity, not a complete intersection, whose link is $\Sigma(2, 13, 31)$ ([8], 4.6).

The second item above is difficult simply because one knows of no good inductive way to compute the geometric genus of a complete intersection of splice type, even though the equations are quite explicit.

6 Generalized Splice Diagrams and CIST's

The results of the preceding section say a great deal about possible equations for a wide class of integral homology sphere links. But from one point of view, such links are not so common. Specifically, all rational singularities and nearly all minimally elliptic singularities have rational homology sphere link, but only a few have $\mathbb{Z}HS$ link. In the rational case, the only non-trivial example is the E_8 singularity $V(2,3,5)$ (this is a famous theorem of Brieskorn, usually stated in terms of trivial local divisor class group). Among minimally elliptics, one has $V(2,3,7)$ and $V(2,3,11)$ and their positive weight deformations. It is thus natural to try to extend the previous discussion of CIST's to say something about a NSS with $\mathbb{Q}HS$ link.

Given $(Y,0)$ with $\mathbb{Q}HS$ link and diagram Γ, one would like to get hold of the UAC $(X,0) \to (Y,0)$. If X is to be a complete intersection of a special type, one should try a generalization of the CIST's of the last section. In that case, one started with a splice diagram satisfying certain rules, and asked whether a certain "semigroup" condition was satisfied; then, one could write down equations of a complete intersection surface singularity, whose topology was what one wanted.

Let us consider a more general splice diagram Δ, where the weights around a node are positive, but are no longer required to be pairwise coprime. (For technical reasons, one should also allow 1 to be a weight on an edge leading to a leaf.) Then exactly as before, one can associate a variable to each of the t ends; for each node, assign weights to the variables and the node; choose (again, *if possible*) for each node and adjacent edge a monomial in the outer variables whose weight is that of the node; for each node, take generic (in a very specific sense) linear combinations of these monomials, giving weighted homogeneous polynomials for the node's weights; to each such polynomial, add terms of higher weight; set all these polynomials equal to 0. In other words, if the "semigroup" condition is satisfied for a general splice diagram Δ, one can as before produce subschemes $X(\Delta)$ of $\mathbb{C}^t$. Then a major result of [15] is

Theorem 6.1. *Suppose Δ is a generalized splice diagram satisfying the semigroup condition. Then $X(\Delta)$ has an isolated local complete intersection surface singularity.*

These singularities, which we still call CIST's (complete intersections of splice type), are the desired generalizations of Brieskorn complete intersections. In fact, when the splice diagram has one node, an $X(\Delta)$ is exactly a BCI but with higher weight terms possibly added to each equation. On the other hand, it is far from obvious how to prove that $X(\Delta)$ has an isolated singularity, at least with the very specific genericity condition we impose on the coefficient matrices. This is accomplished in [15] by an induction on the number of nodes.

Starting with a graph Γ representing a $\mathbb{Q}HS$, one can produce formally a generalized splice diagram Δ by the same procedure as in the $\mathbb{Z}HS$ case: collapse all vertices of valency 2, place certain subdeterminants as weights along every edge emanating from a node. The differences now are that the weights around a node need not be pairwise relatively prime if one did not start with a $\mathbb{Z}HS$; and, the constructed splice diagram has less obvious topological interpretation than in the earlier case. Further, it is easy to see that different Γ's can give rise to the same splice diagrams (this happens already in the weighted homogeneous case). However, we do have an unpublished result which indicates one is on the right track (and compares with Neumann's Theorem 4.1).

Theorem 6.2. *Suppose two $\mathbb{Q}HS$ links give rise to the same splice diagram. Then these two links have diffeomorphic universal abelian covers.*

Returning to our original singularity $(Y, 0)$, we have produced from the graph Γ a splice diagram Δ, and from it a class of isolated complete intersection surface singularities. We hope one of these could be the UAC of Y. But we need to bring into play the first homology group of the link; this "discriminant group" is computed from the intersection matrix $(E_i \cdot E_j)$, and will be denoted $D(\Gamma)$. If $\mathbb{E}$ denotes the free abelian group generated by the exceptional divisors on $\tilde{Y}$, then the intersection pairing gives an injective map of free $\mathbb{Z}$-modules

$$\mathbb{E} \to \mathbb{E}^* = \mathrm{Hom}(\mathbb{E}, \mathbb{Z}),$$

whose cokernel is the discriminant group.

Proposition 6.3. *Let $\{E_i\}$ be the exceptional curves, and $\{e_i\} \subset \mathbb{E}^*$ be the dual basis for the intersection pairing, i.e.*

$$e_i \cdot E_j = \delta_{ij}.$$

Then a faithful diagonal representation of the discriminant group on $\mathbb{C}^t$ (the vector space with basis the ends of the graph) is constructed as follows: $e \in \mathbb{E}^*$ acts on the coordinate z_i by multiplication by the root of unity $\exp(2\pi i(e \cdot e_i))$, where e_i is dual basis element corresponding to the end E_i.

In other words, one has a natural representation of the discriminant group on the polynomial ring in t variables, the ring from which the CIST's can be defined. So, we ought to look for *some* CIST on which the discriminant group acts equivariantly, i.e., for which every term of each defining equation transforms by the same character of the group. The semigroup condition guaranteed the existence of at least one "admissible" monomial for each node and adjacent edge; we need to be able to find one that transforms correctly. This translates easily into a condition on the original graph Γ, which we call the *congruence condition*. We can state the main theorem of [15].

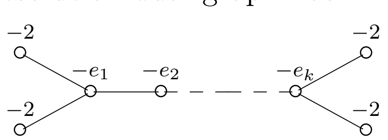
Theorem 6.4. *Let Γ be a graph of a $\mathbb{Q}HS$ link satisfying the semigroup and congruence conditions, with associated splice diagram Δ. Let $X(\Delta)$ be a complete intersection of splice type on which the discriminant group $D(\Gamma)$ acts equivariantly. Then*

1. *$D(\Gamma)$ acts freely on $X(\Delta)$ off the singular point at the origin*
2. *$Y \equiv X(\Delta)/D(\Gamma)$ is a germ of a NSS, whose resolution graph is Γ.*
3. *$X(\Delta) \to Y$ is the universal abelian covering.*

The bottom line is that if we are given a graph Γ satisfying the semigroup and congruence conditions, then we can "explicitly" write down the equations of a singularity with that link, in much the same way as discussed in the weighted homogeneous case following Citex.x. That is, we can write explicit equations of a complete intersection singularity (the UAC), and an explicit diagonal action of the discriminant group on that singularity. To see the actual equations of the desired singularity, one needs to do (perhaps very complicated) calculation of monomial invariants for the group action, find generators for the ideal of relations, and then deduce relations for these invariants which come from the splice equations. The easy case of Example 4.2 on the E_7 singularity as a quotient of the E_6 gives the general idea. We have already mentioned that in the weighted homogeneous cases, there are faster ways to get equations than the UAC method of Neumann's Theorem 4.1.

A singularity Y arising as in the Theorem is called a *splice quotient*. A natural question is to ask which singularities are of this type. We know that weighted homogeneous singularities with $\mathbb{Q}HS$ link are splice quotients. Theorem 5.5 gives an analytic necessary and sufficient condition for a singularity with $\mathbb{Z}HS$ link to be a splice quotient. On the other hand, an "equisingular" deformation of a splice quotient need not be of that type; even if the geometric genera for the singularities in a family are constant, the same need not be true for the geometric genera of the UAC's. An example of this phenomenon is found in [8].

Nonetheless, we conjectured about 7 years ago that rational and $\mathbb{Q}HS$ link minimally elliptic singularities are all splice quotients. (By the time [13] was written, we had intemperately generalized the conjecture to a point where it could not be correct, via [8].) The first non-trivial case was verified in [14] for the "quotient cusps," a class of log-canonical (and taut) rational singularities, whose resolution dual graph has 2 nodes:



$k \geq 2, \quad e_i \geq 2, \;\text{some } e_j > 2.$

Explicit equations for the UAC (which is a "cusp" singularity) and the action of the discriminant group are given in Section 5 of that paper.

The motivation for the general conjecture was not only the beauty of such a result, but because the rational and $\mathbb{Q}HS$ minimally elliptic singularities possess an important property of splice quotients analogous to that mentioned in Theorem 5.5 above. Recall that an *end-curve* on the minimal good resolution $\tilde{Y}$ is a rational curve that has just one intersection point with the rest of the exceptional divisor (so that it corresponds to a leaf in the splice diagram). The following is observed along the way to proving the Main Theorem above.

Proposition 6.5. *Let $(Y,0)$ be a splice quotient. Then for every end curve E_i on $\tilde{Y}$, there is a function $y_i : Y \to \mathbb{C}$ such that the proper transform on $\tilde{Y}$ of its zero-locus consists of one smooth irreducible curve C_i, which intersects E_i transversally at one point and intersects no other exceptional curve.*

Another way to state this property is that for every end-curve, there is a prime ideal in the analytic local ring of Y whose n_i-th symbolic power is a principal ideal (y_i), where y_i has the vanishing properties described above (i.e., its proper transform is n_iC_i.) Note that this integer n_i is the order of the image of the dual basis element e_i in the divisor class group.

Now, it is well-known that rational singularities have the "end-curves property" described in the Proposition; the same is true for $\mathbb{Q}HS$ link minimally elliptics ([21], p. 112). So, in an attempt to generalize Theorem 5.5 we have made the following

End-Curves Conjecture[1]. Suppose $(Y,0)$ is a NSS with $\mathbb{Q}HS$ link. Suppose to every end-curve on the minimal good resolution there exists a function as in Proposition 6.5. Then Y is a splice quotient.

Note that the assumptions about the end-curves are supposed to imply the semigroup and congruence conditions on the graph, as well as the $\mathbb{Q}$-Gorensteinness of the singularity.

This Conjecture is still open; but as we shall see, T. Okuma has recently proved that rational and minimally elliptic singularities are splice quotients.

7 Okuma's Theorem and Further Questions

Once we know that a graph Γ of a rational or $\mathbb{Q}HS$ minimally elliptic singularity satisfies the semigroup and congruence conditions, then it follows from Theorem 6.4 that there is at least one such singularity which is a splice quotient. For Γ with at most two nodes, these two conditions may be checked directly ([15],Section 11). But T. Okuma has proved in general

[1] *Remark in proof:* W. Neumann and the author have recently announced a proof of the End-Curves Conjecture.

Proposition 7.1. *[18] The graph of a rational or $\mathbb{Q}HS$ minimally elliptic singularity satisfies the semigroup and congruence conditions.*

Okuma's method is to give a condition on Γ that turns out to be equivalent to the semigroup and congruence conditions, and then to deduce this graph-theoretic property from a well-known stronger property for such rational or $\mathbb{Q}HS$ minimally elliptic graphs. Our own version of his result is found in [15], Section 13.

To get the strongest result, Okuma uses a precise description of the UAC of a NSS Y with $\mathbb{Q}HS$ link. Considering the MGR $\tilde{Y} \to Y$, he constructs a fairly explicit sheaf of algebras on $\tilde{Y}$ whose Spec is a partial resolution of the UAC, with only cyclic quotient singularities [17]. This is similar to the Esnault-Viehweg method for constructing cyclic coverings branched along normal crossings divisors. Using the preceding proposition, and the existence of appropriate end-curves, Okuma proves our old Conjecture about the UAC.

Theorem 7.2. *([18]) Every rational or $\mathbb{Q}HS$ minimally elliptic singularity is a splice quotient. In particular, one may write down explicit equations for it.*

We note that Okuma's original preprint does not specifically assert this Theorem in its full strength; one can find an explanation of why he has in fact obtained this result in [15], Section 13.

At this point, we now know many examples of singularities with $\mathbb{Q}HS$ links which are splice quotients — especially, rational, minimally elliptic, and weighted homogeneous. But there are many examples, even of hypersurface singularities, which could not be splice quotients. The next challenge is to try to understand better what is going on in the other cases — is there a nice theorem out there?

An interesting place to start is with some of the examples in [8]. For instance, we have found a hypersurface singularity which does not satisfy the semigroup condition; but nonetheless, the UAC is a complete intersection of splice type! These and related issues are currently being looked into.

Acknowledgements

Research supported under NSA grant no. H98320-04-1-0053.

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